Automorphism Groups of Compact Quantum Groups

joint work with Alex Chirvasitu

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December 19, 2016

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Automorphism Groups of CQGs

Intro

Let $\mathbb{G}$ be a compact quantum groups. Then $\alpha : C(\mathbb{G}) \to C(\mathbb{G})$ is said to be a quantum group automorphism if α is C^*-isomorphism and $(\alpha \otimes \alpha) \circ \Delta_{\mathbb{G}} = \Delta_{\mathbb{G}} \circ \alpha$.

Generalises notion of group automorphisms.

Easy to see-

1. $h_{\mathbb{G}} \circ \alpha = h_{\mathbb{G}}$.
2. $(id \otimes \alpha)u$ is an irreducible representation of $\mathbb{G}$ when u is irreducible representation of $\mathbb{G}$. Thus α induces a permutation of $\mathrm{Irr}(\mathbb{G})$.
3. Extends to both $C(\mathbb{G})$ and $C_m(\mathbb{G})$.

History

First studied by Wang (PLMS, 95) who showed that if a discrete countable group Γ acts on $C_m(\mathbb{G})$, then the crossed product $C_m(\mathbb{G}) \rtimes_f \Gamma$ has a CQG structure.

Later studied by P. (IJM, 2013) in the context of normal subgroups, introduced "inner" automorphisms. Also studied in Fima, Mukherjee, P. (JNCG, 2016) in context of approximation properties.

Properties of (non-commutative) dynamical systems $(\mathbb{G}, \Gamma)$, were studied in Mukherjee and P. (2016).

Automorphism Groups

Let $\mathbb{G}$ be a CQG. We denote $\operatorname{Aut}(\mathbb{G})$ the group of quantum group automorphisms and topologise by pointwise norm topology (i.e. $\alpha_i \to \alpha \iff \alpha_i(a) \to \alpha(a)$ for all $a \in C(\mathbb{G})$).

$\alpha \in \operatorname{Aut}(\mathbb{G})$ is inner if it acts trivially on $\operatorname{Irr}(\mathbb{G})$.

Group of inner automorphisms is denoted as $\operatorname{Aut}_\chi(\mathbb{G})$, which is closed, normal and compact subgroup of $\operatorname{Aut}(\mathbb{G})$. The subgroup $\operatorname{Out}(\mathbb{G}) = \operatorname{Aut}(\mathbb{G})/\operatorname{Aut}_\chi(\mathbb{G})$ is totally disconnected.

Question Remained

For compact matrix quantum group $\mathbb{G}$, is $\mathrm{Out}(\mathbb{G})$ discrete?

For compact lie group G- Since the representation ring, $\mathbb{Z}[\mathrm{Irr}(G)]$ is finitely generated as a ring, the outer automorphism group is discrete.

Hence, key question- For compact matrix quantum group, is the representation ring $\mathbb{Z}[\mathrm{Irr}(\mathbb{G})]$ finitely generated?

Automorphism Groups of CQGs Reloaded

Journey to Greifswald

7th ECM satellite conference Compact Quantum Groups, organised by

ADAM SKALSKI

UWE FRANZ

MALTE GERHOLD

MORITZ WEBER

Enter Alex

Met my collaborator Alex Chirvasitu

Happy Ending

- We show that for a $\mathrm{Out}(\mathbb{G})$ for a compact matrix quantum group is indeed discrete.

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- For a compact matrix quantum group, $\mathrm{Aut}_{\chi}(\mathbb{G})$ is a compact lie group and $\mathrm{Aut}(\mathbb{G})$ is locally compact.
- If $\mathrm{Out}(\mathbb{G})$ is finite, then $\mathrm{Aut}(\mathbb{G})$ is a compact lie group. True for any deformation of simple compact lie groups.

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- Quatum version of Iwasawa's result- For any compact quantum group $\mathbb{G}$, we have $\mathrm{Aut}_0(\mathbb{G}) = (\mathrm{Aut}_\chi)_0(\mathbb{G})$.

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- For a compact matrix quantum group, $\mathrm{Aut}_{\chi}(\mathbb{G})$ is a compact lie group and $\mathrm{Aut}(\mathbb{G})$ is locally compact.
- If $\mathrm{Out}(\mathbb{G})$ is finite, then $\mathrm{Aut}(\mathbb{G})$ is a compact lie group. True for any deformation of simple compact lie groups.
- Quatum version of Iwasawa's result- For any compact quantum group $\mathbb{G}$, we have $\mathrm{Aut}_0(\mathbb{G}) = (\mathrm{Aut}_{\chi})_0(\mathbb{G})$.
- For any CQG $\mathbb{G}$, we have $\mathrm{Aut}(\mathbb{G})$ topologically isomorphic to $\mathrm{Aut}_m(\mathbb{G})$.

Twist in the tale

However, we give an example of a compact matrix quantum group, whose representation ring is not finitely generated as a ring.

This is obtained by taking a bicrossed product construction- $\mathbb{G} = C(\mathbb{Z}/2)\sharp A_u(n)$. On the tensor product algebra $C(\mathbb{Z}/2) \otimes A$, we define a coproduct using a coaction $\rho : A \to A \otimes C(\mathbb{Z}/2)$ and a map $\tau : A \to C(\mathbb{Z}/2)^{\otimes 2}$. This is a compact matrix quantum group but its representation ring is not finitely generated.

In this case, the (complex) representation ring of $\mathbb{G}$ surjects onto $\mathbb{C}\langle \alpha, \beta \rangle^{\mathbb{Z}/2}$ and this is not finitely generated.

Some Dynamics

CQG dynamical systems

Let now Γ act on $C(\mathbb{G})$ by quantum group automorphisms. We say $(\mathbb{G}, \Gamma)$ is a CQG dynamical system.

Studied in Mukherjee and P. (2016) from a dynamical perspective. Follows classical study of “algebraic actions” of groups on compact groups, a vast industry initiated by a paper of Halmos (1943).

In this paper, we study ergodicity, weak mixing, mixing, compactness, etc and get combinatorial conditions for these properties, in terms of the induced action of Γ on $\mathrm{Irr}(\mathbb{G})$. We study several examples, develop a notion of spectral measures for non-commutative group actions, connections to combinatorial group theory and finally study subgroup MASAs in QG von-Neumann algebras.

Compactness/Almost Periodicity

Definition

Let $(\mathbb{G}, \Gamma, \alpha)$ be a CQG dynamical system. Let $\|\cdot\|$ denote the C^*-norm on $A = C(\mathbb{G})$.

1. We say that the action is almost periodic if given any $a \in A$, the set $\{\alpha_\gamma(a) : \gamma \in \Gamma\}$ is relatively compact in A with respect to $\|\cdot\|$.
2. We say that the action is compact if given any $a \in A$, the set $\{\alpha_\gamma(a)\Omega_h : \gamma \in \Gamma\}$ is relatively compact in $L^2(A)$ with respect to the $\|\cdot\|_{2,h}$.
3. The extended action of Γ on $L^\infty(G)$ is compact if given any $a \in L^\infty(\mathbb{G})$, the set $\{\alpha_\gamma(a)\Omega_h : \gamma \in \Gamma\}$ is relatively compact in $L^2(A)$ with respect to the $\|\cdot\|_{2,h}$.

F.O.C.

Following characterization of compact actions was obtained in Mukherjee-P.

Theorem

Let $(\mathbb{G}, \Gamma, \alpha)$ *be a CQG dynamical system. TFAE:*
(i) *The action is almost periodic*;
(ii) *the action is compact*;
(iii) *the orbit of any irreducible representation in Irr*$(\mathbb{G})$ *is finite*;
(iv) *the extended action of* Γ *on* $L^\infty(\mathbb{G})$ *is compact.*

Inner $\Rightarrow$ Compact

Easily follows that an action by inner automorphisms is compact.

A converse

We can use discreteness of the outer automorphism of a compact matrix quantum group to show virtual innerness of compact actions.

Theorem

Let $\mathbb{G}$ *be a compact matrix quantum group. Let* $(\mathbb{G}, \Gamma, \alpha)$ *be a compact CQG dynamical system. Then the subgroup*

$$\Gamma_\chi := \{\gamma \in \Gamma : \alpha_\gamma \in \mathsf{Aut}_\chi(\mathbb{G})\}$$

of Γ *is of finite index.*

Thank You!