The limit spectrum of special random matrices

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[1] Joint work with Michał Wojtylak

Outline

- Wigner and Marchenko-Pastur theorems
- Generalized Wigner and Marchenko-Pastur matrices
- Stochastic domination
- Isotropic local law for Wigner and Marchenko-Pastur matrices
- Port-Hamiltonian matrices: Large perturbation of skew-hermitian matrix
- Deformation of large Wigner matrix
- Main Theorem

General problem

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where A_N is "almost" hermitian...

Wigner's theorem

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stands for the classical Wigner matrix, i.e. W_N is symmetric matrix such that

- x_{ij} are real, $\mathbb{E}x_{ij} = 0$,
- x_{ij} i.i.d. for $i < j$ (let $\mathbb{E}|x_{01}|^2 = 1$),
- x_{ii} i.i.d.,
- $\max\{\mathbb{E}|x_{00}|^k, \mathbb{E}|x_{01}|^k\} < +\infty \quad k = 1, 2, \dots$

Wigner's theorem

Let λ_i^N denote the (real) eigenvalues of a Wigner matrix W_N.
Let us consider the empirical distribution of the eigenvalues as the (random) probability measure on $\mathbb{R}$ defined by

$$L_N = \frac{1}{N+1} \sum_{i=0}^{N} \delta_{\lambda_i^N}.$$

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Theorem (Wigner)

The empirical measures L_N converges weakly, in probability, to the semicircle distribution $\sigma(x)dx$, where

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i.e. $\mathbb{P}\left(\left| \int f d\sigma - \int f dL_N \right| > \varepsilon \right) \to 0$, for any $\varepsilon > 0$ and $f \in C_b(\mathbb{R})$

Wigner's theorem

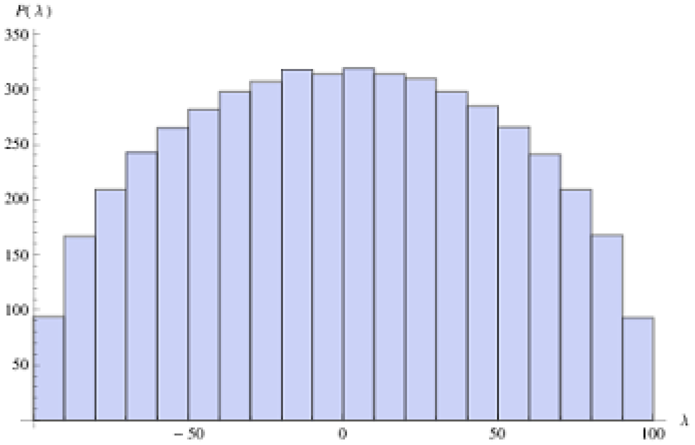

Figure: An empirical distribution of eigenvalues of Wigner matrix

Marchenko–Pastur law

Let

$$X_N = (N)^{-\frac{1}{2}}[x_{ij}] \in \mathbb{R}^{M \times N}$$

stands for a matrix such that

- $M/N \to y$, $y \in (0,1)$,
- x_{ij} are i.i.d.,
- $\mathbb{E}x_{ij} = 0$, $\mathbb{E}|x_{ij}|^2 = 1$.

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The matrix $X_N^* X_N$ is called Marchenko-Pastur matrix.

Marchenko–Pastur law

Let ν_i^N denote the (real) eigenvalues of $X_N^* X_N$, the Marchenko-Pastur matrix.
Now let us consider an empirical distribution of ν_i^N i.e.

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Theorem (Marchenko-Pastur)

The empirical measures L_N converges weakly, in probability, to the Marchenko-Pastur distribution μ with density

$$\frac{d\mu}{dx} = \frac{1}{2\pi xy} \sqrt{(x-a)(b-x)} \chi_{[a,b]},$$

where $a = (1-\sqrt{y})^2$ and $b = (1+\sqrt{y})^2$.

Marchenko–Pastur law

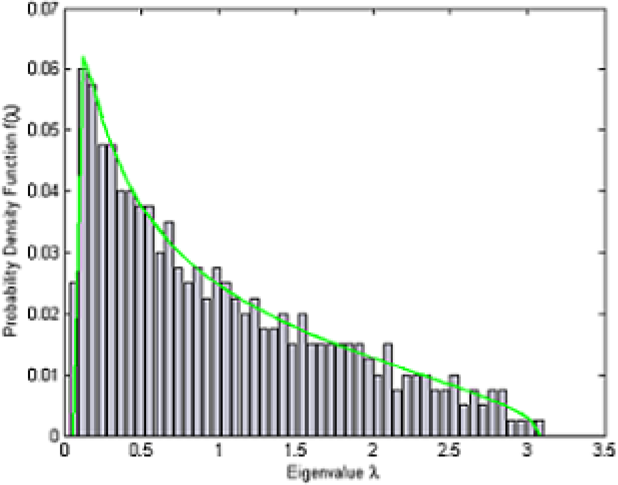

Figure: An empirical distribution of singular eigenvalues of Marchenko-Pastur matrix

Generalization of Wigner and Marchenko-Pastur matrices

Now

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- x_{ij} are independent for $i \leq j$,
- $\mathbb{E}x_{ij} = 0$,
- $cosnt \leq \mathbb{E}|x_{ij}|^2$,
- $\sum_j \mathbb{E}|x_{ij}|^2 = N$,
- $\mathbb{E}|x_{ij}|^p \leq const(p)$, for all $p \in \mathbb{N}$.

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Now

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will stand for a matrix such that

- $N^{\frac{1}{const}} \leq M(N) \leq N^{const}$,
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Stochastic domination

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Definition (see [1])

The family of nonnegative random variables
$\xi = \{\xi^{(N)}(z) : N \in \mathbb{N},\ z \in \mathbf{S}_N\}$ is stochastically dominated in z by $\zeta = \{\zeta^{(N)}(z) : N \in \mathbb{N},\ z \in \mathbf{S}_N\}$ if and only if for all $\varepsilon > 0$ and $\gamma > 0$ we have

$$\mathbb{P}\left\{\bigcap_{z \in \mathbf{S}_N} \{\xi^{(N)}(z) \leq N^{\varepsilon}\zeta^{(N)}(z)\}\right\} \geq 1 - N^{-\gamma}, \tag{1}$$

for large enough $N \geq N(\varepsilon,\gamma)$.

Stochastic domination

Example

Let $\mathbf{S}_N = \{0\}$, $\xi \sim \mathcal{N}(0,1)$ and $\zeta = \frac{1}{\log N}$. Thus for any $\varepsilon, \gamma > 0$ we have $\xi \leq \frac{N^\varepsilon}{\log N} = N^\varepsilon \zeta$ with probability greater than $1 - N^{-\gamma}$.

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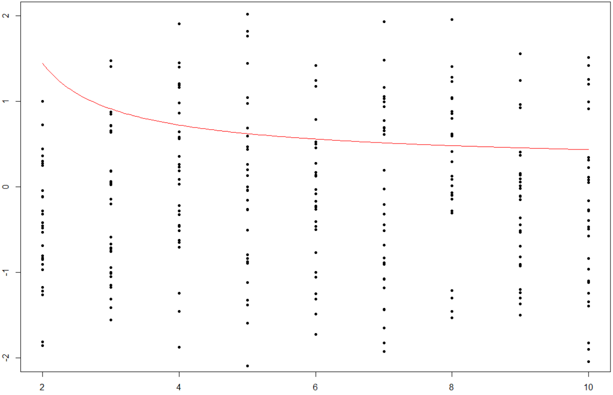

Stochastic domination

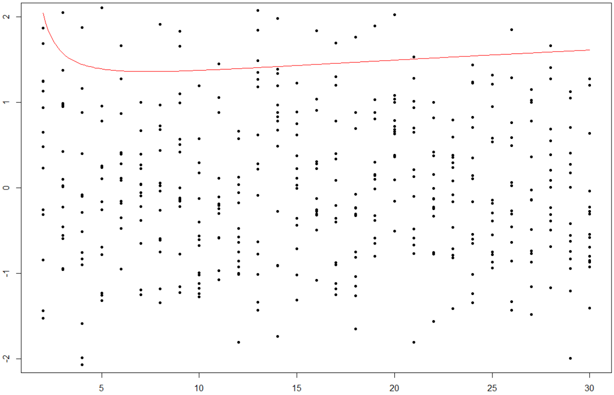

Figure: $N^{\varepsilon}/\log(N)$ vs. $\mathcal{N}(0,1)$

Stochastic domination

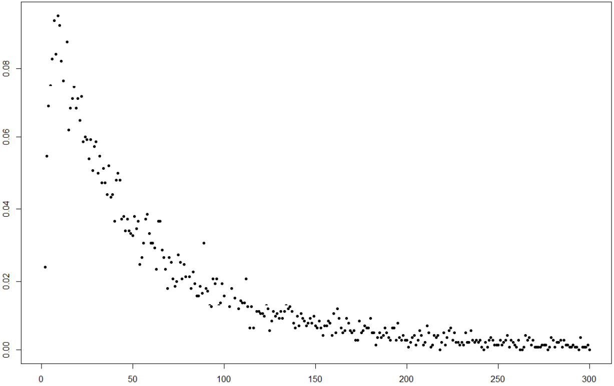

Figure: the empirical probability that $\xi \leq N^{\varepsilon}\zeta$

Isotropic local law for Wigner matrix

Let us denote by $m(z)$ a Stieltjes transform of Wigner semicircle distribution, i.e.

$$m(z) = \frac{-z + \sqrt{z^2 - 4}}{2}.$$

Let us consider a family of sets

$$\mathbf{S}_N = \left\{ z = x + \mathrm{i}\, y : |x| \le \omega^{-1},\ (\log N)^{-1+\omega} \le y \le \omega^{-1} \right\},$$

and a family of deterministic functions

$$\Psi(z) = \sqrt{\frac{\operatorname{Im} m(z)}{Ny}} + \frac{1}{Ny}.$$

Theorem (A. Knowles, J. Yin)

$$\|(W - z)^{-1} - m(z) I\|_{max} \prec \Psi(z)$$

Isotropic local law for Marchanko-Pastur matrix

Let us denote

$$\phi = M/N, \quad \gamma_\pm = \sqrt{\phi} + \frac{1}{\sqrt{\phi}} \pm 2, \quad K = \min(N, M).$$

Moreover, let us define the functions

$$m_\phi(z) = \frac{\phi^{1/2} - \phi^{-1/2} - z + \mathrm{i}\sqrt{(z-\gamma_-)(\gamma_+ - z)}}{2\phi^{-1/2} z}$$

on the sets

$$\mathbf{S}_N = \{z = x + \mathrm{i}\, y \in \mathbb{C} : (\log K)^{-1+\omega} \le |x| \le \omega^{-1},$$
$$(\log K)^{-1+\omega} \le y \le \omega^{-1},\ |z| \ge \omega\},$$

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Theorem (A. Knowles, J. Yin)

$\|(X^*X - z)^{-1} - m_\phi(z)I\|_{max} \prec \Psi(z)$

Nonhermitian case.

The main purpose of this talk is to show a limit behavior of eigenvalues of non-hermitian matrices.

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In the papers [2, 3] authors showed behavior of a non-real eigenvalue of the matrix HW_N, where W_N is a Wigner matrix, and $H = \mathrm{diag}(d, 1, 1, \ldots, 1)$, with $d < 0$.

Port-Hamiltonian: Large perturbation of skew-hermitian matrix

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Let $C \in \mathbb{C}^{k\times k}$ be a deterministic skew-hermitian matrix, i.e. $C = -C^*$. And let $P = P_N \in \mathbb{C}^{N\times k}$, $Q = Q_N \in \mathbb{C}^{k\times N}$ be the canonical embeddings, i.e. $P_N = \begin{bmatrix} I_k \\ 0 \end{bmatrix} \in \mathbb{C}^{N\times k}$,
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$$P_N C Q_N = \begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{C}^{N\times N}$$

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We wonder if

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By Woodbury matrix identity we have to check the matrix

$$(C - \frac{z}{2})^{-1} + Q(X^*X - \frac{z}{2})^{-1}P.$$

By isotropic local law (for $z \in S_N$):

$$\det\left((C - \frac{z}{2})^{-1} + Qm_\phi(\frac{z}{2})P\right) = \det\left((C - \frac{z}{2})^{-1} + m_\phi(\frac{z}{2})I_k\right) = 0.$$

Large perturbation of skew-hermitian matrix

Let

$$UCU^* = \operatorname{diag}(\underbrace{0, \ldots, 0}_{p_0}, \underbrace{\mathrm{i}\, t_1, \ldots, \mathrm{i}\, t_1}_{p_1}, \underbrace{-\mathrm{i}\, t_1, \ldots, -\mathrm{i}\, t_1}_{p_1}, \ldots, -\mathrm{i}\, t_k),$$

where $t_1, t_2, \ldots, t_k > 0$,

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$$\det\left((C - \frac{z}{2})^{-1} + m_\phi\left(\frac{z}{2}\right) I_k\right) = 0,$$

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$$\frac{1}{\mathrm{i}\, t - \frac{z}{2}} + m_\phi\left(\frac{z}{2}\right) = 0,$$

$$z_t := \frac{-1 + 3\,\mathrm{i}\, t + \sqrt{1 - 6\,\mathrm{i}\, t - t^2}}{2}.$$

Large perturbation of skew-hermitian matrix

Let us remain that

$$\|(X^*X - z)^{-1} - m_\phi(z)I\|_{max} \prec \Psi(z) \leq \left|\sqrt{\frac{\Im m(z)}{Ny}} + \frac{1}{Ny}\right|$$

$$\leq \sqrt{\frac{(\log N)^{1-\omega}}{N(\log N)^{-1+\omega}}} + \frac{1}{N(\log N)^{-1+\omega}} \leq N^{-\frac{1}{2}+\epsilon},$$

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Large perturbation of skew-hermitian matrix

then for any $j = 1, 2, \ldots, k$, p_j-closest eigenvalues of $PCQ + X^*X$ $\lambda_{j,1}, \lambda_{j,2}, \ldots, \lambda_{j,p_j}$ satisfy:

Theorem

$$|\lambda_{j,l} - z_{t_j}| \prec N^{-\frac{1}{2p_j}},$$

where $l \in \{1, 2, \ldots, p_j\}$.

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where $l \in \{1, 2, \ldots, p_j\}$.

i.e. for any $\gamma, \varepsilon > 0$ the probability that

$$|\lambda_{j,l} - z_{t_j}| \leq N^{-\frac{1}{2p_j}+\varepsilon}$$

is larger than $1 - N^{-\gamma}$.

Deformation of large Wigner matrix

Let us find non-real eigenvalues of the matrix $H_N W_N$, where W_N is a Wigner matrix and

$$H_N = \mathrm{diag}(d_1, d_2, \ldots, d_k, 1, 1, \ldots, 1) \in \mathbb{C}^{N \times N},$$

with $d_1, \ldots, d_k < 0$.

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Let us observe that

$$H_N W_N - z = H_N(W_N - H_N^{-1} z) = H_N(W_N - z - (H_N^{-1} - I)z).$$

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Let us observe that

$$H_N W_N - z = H_N(W_N - H_N^{-1} z) = H_N(W_N - z - (H_N^{-1} - I)z).$$

The polynomial $W_N - z - (H_N^{-1} - I)z$ has the form
$W_N - z - P_N C_N Q_N z$, where $Q_N^* = P_N = \begin{bmatrix} I_k \\ 0 \end{bmatrix} \in \mathbb{C}^{N\times k}$,
$C_N = \mathrm{diag}(\frac{1}{d_1} - 1, \frac{1}{d_1} - 1, \ldots, \frac{1}{d_k} - 1) \in \mathbb{C}^{n\times n}$.

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Theorem

$$|\lambda_{j,l} - z_{d_j}| \prec N^{-\frac{\beta}{2p_j}},$$

where p_j is a multiplicity of d_j and $l \in \{1, 2, \dots, p_j\}$.

Main Theorem

Theorem

Consider the following deterministic objects:

(d1') *sequences of matrices* $P_N \in \mathbb{C}^{N\times n}$, $Q_N \in \mathbb{C}^{n\times N}$ *satisfying*

$$\sup_N \max(\|P_N\|_2, \|Q_N\|_2) < \infty,$$

(d2) *sequences of matrix polynomials*

$$C_N(z) \in \mathbb{C}^{n\times n}[z], \quad P_N C_N(z) Q_N \in \mathbb{C}^{N\times N}[z],$$

and the following random object:

(r1) $W_N(z) \in \mathbb{C}^{N\times N}[z]$ *is a random matrix polynomial.*

Main Theorem

Theorem

We assume that $\mathbf{S}_N \subset \mathbb{C}$ *is a open set and that*

(a1) $\|W_N(z)^{-1} - M(z)\|_{max} \prec \Psi(z)$ *on the set* $\mathbf{S}_N$,

(a2) $C_N(z)$ *is invertible for* $z \in \mathbf{S}_N$,

(a3') $\sup_{z \in \mathbf{S}_N} |\Psi(z)| \leq N^{-\alpha}$ *for some* $\alpha > 0$,

(a4') $\|M_N(z)\|, \|W_N(z)^{-1}\|, \|C_N(z)^{-1}\| \leq (\log N)^\beta$ *on* $\mathbf{S}_N$ *for some* $\beta > 0$,

(a5') *the sequence* $Q_N M_N(z) P_N$ *is constant for any* $z \in \mathbf{S}_N$.

Main Theorem

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Further, let $z_0 \in \mathbf{S}_N$ *be such that*

$$\dim \ker(C_N(z_0)^{-1} + Q_N M_N(z_0) P_N) = p > 0. \tag{2}$$

Let the random variable λ_j *be define as* j*-th element of the set of eigenvalues* $\{\lambda \in \mathbb{C} : W_N(\lambda) + P_N C_N Q_N(\lambda) \text{ is not invertible } \}$ *in the radial lexicographic order centered in* z_0*, i.e. the order which firstly respects the absolute value* $|\lambda - z_0|$ *and secondary the argument* $\lambda - z_0$.

Then p*-closest eigenvalues (defined above) satisfy:*

$$|\lambda_j - z_0| \prec N^{-\frac{\alpha}{p}},$$

for any $j = 1, 2, \ldots, p$.

- A. Knowles, J. Yin, *The Isotropic Semicircle Law and Deformation of Wigner Matrices*, Comm. on Pure and Applied Mathematics, 66 (2013), 1663–1749.
- M. Wojtylak *On a class of H -selfadjont random matrices with one eigenvalue of nonpositive type*, Electron. Commun. Probab. 17 (2012), no. 45, 1–14.
- P. Pagacz, M. Wojtylak *On spectral properties of a class of H-selfadjoint random matrices and the underlying combinatorics*, Electron. Commun. Probab. 19 (2014), no. 7, 1–14.
- P. Pagacz, M. Wojtylak *Random perturbations of linear pencils*, Preprint 2016.

Thank you for your attention!