Matrix functions, matrix means, matrix inequalities

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Introduction

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A real valued function f on $(0, \infty)$ is operator monotone if whenever bounded invertible positive operators A, B satisfy $0 \leq A \leq B$, $f(A) \leq f(B)$. Both of functions $f(t) = t^s \quad (s \in [0, 1])$ and $f(t) = \log t$ are typical operator monotone.

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Similarily, f is operator convex if $f(tA + (1-t)B) \leq tf(A) + (1-t)f(B)$ for all bounded invertible positive operators A, B and for all numbers $0 \leq t \leq 1$. Both of functions $f(t) = t^s \ \ (s \in [1, 2])$ and $f(t) = t \log t$ are typical operator convex.

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When H is n-dimensional, that is, $B(H) = M_n$, n by n matrix algebra, they are called as matrix monotone functions of degree n, n-monotone in short (resp. matrix convex functions of degree n,

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(I) $A \leq C$ and $B \leq D$ imply $A\sigma B \leq C\sigma D$.

(II) $C(A\sigma B)C \leq (CAC)\sigma(CBC)$.

(III) If $A_n \searrow A$ and $B_n \searrow B$, then $A_n\sigma B_n \searrow A\sigma B$.

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They showed that there exists an affine order-isomorphism from the class of connections onto the class of positive operator monotone functions by

$$\sigma \mapsto f_\sigma(t)1 = 1\sigma(t1)$$
$$f \mapsto A\sigma_f B = A^{\frac{1}{2}} f(A^{-\frac{1}{2}} BA^{-\frac{1}{2}})A^{\frac{1}{2}}$$

for $A, B \in B(H)^{++}$.

Example

$$
\begin{aligned}
A\omega_l B &= A && \Longleftrightarrow f_{\omega_l}(t) = 1 \\
A\omega_r B &= B && \Longleftrightarrow f_{\omega_r}(t) = t \\
A : B &= (A^{-1} + B^{-1})^{-1} && \Longleftrightarrow f_{:}(t) = \frac{t}{1+t} \\
A!B &= 2(A^{-1} + B^{-1})^{-1} && \Longleftrightarrow f_{!}(t) = \frac{2t}{1+t} \\
A\sharp B &= A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}} && \Longleftrightarrow f_{\sharp}(t) = \sqrt{t} \\
A\nabla B &= \frac{A+B}{2} && \Longleftrightarrow f_{\nabla}(t) = \frac{1+t}{2}
\end{aligned}
$$

It is well-known that if $f : (0, \infty) \to (0, \infty)$ is operator monotone, the transpose $f'(t) = tf(\frac{1}{t})$, the adjoint $f^*(t) = \frac{1}{f(\frac{1}{t})}$, the dual $f^\perp = \frac{t}{f(t)}$ are also operator monotone (Hansen-Pedersen '80) and we call f symmetry if $f = f'$ and self-adjoint if $f = f^*$, respectively.

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It is shown in (Kubo-Ando '80) that if f is symmetry with $f(1) = 1$, then the corrsponding operator mean σ_f exists between the harmonic mean ! and the arithmetic mean ∇, that is, $! \leq \sigma_f \leq \nabla$.

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Moreover, if f is λ-weighted, that is, $\lambda\in[0,1]$ and $\dfrac{df}{dt}f(1)=\lambda$, then $!_\lambda\leq\sigma_f\leq\nabla_\lambda$, where $A\nabla_\lambda B=(1-\lambda)A+\lambda B$ and $A!_\lambda B=((1-\lambda)A^{-1}+\lambda B^{-1})^{-1}$.
Note that if f is symmetry, then $\dfrac{df}{dt}(1)=\dfrac{1}{2}$.

Let σ be a connection.

$$A\sigma' B := B\sigma A \qquad \text{(transpose)}$$
$$A\sigma^* B := (A^{-1}\sigma B^{-1})^{-1} \qquad \text{(ajoint)}$$
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Definition

- A connection σ is symmetry if $\sigma' = \sigma$.
- A connection σ is self-adjoint if $\sigma = \sigma^*$.

$\boxed{\{f \in P_\infty^+((0,\infty)) \mid f(1)=1\}} \Leftrightarrow \boxed{\text{The set of operator means}}$

(Kubo-Ando Theorem 1980)

$\Leftrightarrow \boxed{\text{The set of monotone metrics with } \ \gamma_D(I,I) = \mathrm{Tr}(D^{-1})}$

(Petz 1996)

$\subset \boxed{\text{Quantum Information Theory}}$

Matrix monotone functions

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(Gaps)

$P_{n+1}(\mathbf{I}) \subsetneq P_n(\mathbf{I}) \qquad K_{n+1}(\mathbf{I}) \subsetneq K_n(\mathbf{I})$

Criteria for n-monotonicity and n-convexity

Definition (Divided differences)

Let f be a real valued function on an open interval $\mathbf{I}$ in $\mathbf{R}$. The divided differences with respect to a function f as

$$[t_1, t_2]_f = \frac{f(t_1) - f(t_2)}{t_1 - t_2} \quad \text{and inductively,}$$

$$[t_1, t_2, \ldots, t_{n+1}]_f = \frac{[t_1, t_2, \ldots, t_n]_f - [t_2, t_3, \ldots, t_{n+1}]_f}{t_1 - t_{n+1}}.$$

Criterion Ia (Löwner 1934)

Let f be a real valued function on an open interval $\mathbf{I}$ in $\mathbf{R}$

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Let f be a real valued function in $C^{2n-1}(\mathbf{I})$.

1. Then f is n-monotone if and only if the Hankel matrix $M_n(f;t) = \left(\frac{f^{(i+j-1)}(t)}{(i+j-1)!}\right)$ is positive semi-definite for every $t \in \mathbf{I}$.

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2. Suppose that there exist an interior point t_0 in $\mathbf{I}$ such that $M_n(t_0; f) > 0$. Then there a positive number δ such that f is n-monotone in the subinterval $(t_0 - \delta, t_0 + \delta)$.

Examples

1. Let $f(t) = t^p$ defined on any subinterval **I** of $(0, \infty)$. Then f is 2-monotone if and only if $0 \leq p \leq 1$. Moreover, in this case f is operator monotone. (Löwner-Heinz Inequality) Indeed, $\det M_2(t; f) = -\dfrac{1}{12} p^2 (p-1)(p+1) t^{2p-4}$.

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2. A function $f(t) = expt$ is not 2-monotone, because that

$$\begin{pmatrix} e^t & \frac{e^t}{2!} \\ \frac{e^t}{2!} & \frac{e^t}{3!} \end{pmatrix}$$

is not posiive semidefinite.

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2. Suppose that there exist an interior point t_0 in $\mathbf{I}$ such that $K_n(t_0; f) > 0$. Then there a positive number δ such that f is n-convex in the subinterval $(t_0 - \delta, t_0 + \delta)$.

For the converse implication we need the following local property theorem.

Theorem (Local Property theorem)

Let (α, β) and (γ, δ) be two overlapping open intervals, where $\alpha < \gamma < \beta < \delta$. Suppose a function *f* is *n*-monotone on these intervals, then *f* is n-monotone on the larger interval (α, δ)

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Let (α, β) and (γ, δ) be two overlapping open intervals, where $\alpha < \gamma < \beta < \delta$. Suppose a function f is n-monotone on these intervals, then f is n-monotone on the larger interval (α, δ)

On the other hand, the local property theorem for n-convex functions is proved only in the case $n = 2$ (as we shall see later) and at present we have been unable to prove the theorem even for 3-convex functions. For the moment, all we can say now is the following fact.

Example

Let $f(t) = t^p$ defined on any subinterval **I** of $(0, \infty)$. Then f is 2-convex if and only if either $1 \le p \le 2$ or $-1 \le p \le 0$ Indeed, $\det K_2(t; f) = -\frac{1}{12} p^2(p-1)(p+1)t^{2p-4}$.
Moreover, in this case f is operator convex. For examples, when $p = 2$ it follows from the obvious inequality $\left(\frac{A+B}{2}\right)^2 \le \frac{A^2+B^2}{2}$.
When $p = -1$ it follows from the inequality $\left(\frac{I+C}{2}\right)^{-1} \le \frac{I+C^{-1}}{2}$ for $C > 0$. If we set $C = A^{-1/2}BA^{-1/2}$, then we have $\left(\frac{A+B}{2}\right)^{-1} \le \frac{A^{-1}+B^{-1}}{2}$

Note that f is n-convex if and only if f satisfies $f(\frac{A+B}{2}) \le \frac{f(A)+f(B)}{2}$ for any $n \times n$ Hermitian matrices A and B.

Let $\mathbf{C}^+ = \{z \in \mathbf{C} : \mathrm{Im} z > 0\}$ be the upper half plane. An anlatic function $f : \mathbf{C}^+ \to \mathbf{C}$ is called a Pick function if the range $f(\mathbf{C}^+)$ is included in the closed half-plane $\{z \in \mathbf{C} : \mathrm{Im} z \geq 0\}$.

Theorem (Nevanlinna)

A function $f : \mathbf{C}^+ \to \mathbf{C}$ is a Pick function if and only if there exist an $\alpha \in \mathbf{R}$, a $\beta \geq 0$ and a positive finite Borel measure ν on $\mathbf{R}$ such that

$$\begin{aligned} f(z) &= \alpha + \beta z + \int_{-\infty}^{\infty} \frac{1 + \lambda z}{\lambda - z} d\nu(z) \\ &= \alpha + \beta z + \int_{-\infty}^{\infty} \left(\frac{1}{\lambda - z} - \frac{\lambda}{\lambda^2 + 1} \right) d\mu(\lambda) \quad z \in \mathbf{C}^+, \end{aligned}$$

where μ is a Borel measure on $\mathbf{R}$ given by $d\mu(\lambda) = (\lambda^2 + 1) d\nu(\lambda)$.

Examples

1. For $0 \leq p \leq 1$ $f(z) = z^p$ is a Pick function:

$$z^p = \cos\frac{p\pi}{2} + \frac{\sin p\pi}{\pi}\int_{-\infty}^{0}\left(\frac{1}{\lambda - z} - \frac{\lambda}{\lambda^2+1}\right)|\lambda|^p d\lambda, z \in \mathbf{C}^+.$$

2. The principa branchi $f(z) = \mathrm{Log} z$ is a Pick function;

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Here is a famous Löwner's Theorem.

Theorem (Löwner 1934)

Let $-\infty \le a < b \le \infty$ and f be a real valued function on (a, b). Then f is operator monotone if and only if f is a Pick function.

Theorem

Let f be a continuous and nonnegative function on $[0, \infty)$. Then f is operator monotone if and only if there exists a positive finite Borel measure m on $[0, \infty]$ such that

$$f(t) = a + bt + \int_{(0,\infty)} \frac{t(1+\lambda)}{t+\lambda}, t \in [0, \infty),$$

where $a = m(\{0\})$ and $b = m(\{\infty\})$.

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Example (C. Davis 1961; Nakamura-Umegaki 1961; Furuta '00)

$f(t) = -t \log t$ is operator concave.

Truncated moment problem and gaps

Until examples were provided in [Hansen-Ji-Tomiyama '04] there were no explicit examples in $P_n(\mathbf{I})\backslash P_{n+1}(\mathbf{I})$ for $n \geq 3$. The only one example for the gap between $P_2(\mathbf{I})$ and $P_3(\mathbf{I})$ was consructed by [Sparr 1980].

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Theorem (Hansen-Tomiyama '07, Osaka-Silvestrov-Tomiyama '07)

Let $\mathbf{I}$ be a finite interval and let n and m in $\mathbf{N}$ with $n \geq 2$.
(1) If $m \geq 2n-1$, there exists an n-monotone polynomial $p_m : \mathbf{I} \to \mathbf{R}$ of degree m,
(2) If $m \geq 2n$ there exists an n-convex and n-monotone polynomials $p_m : \mathbf{I} \to \mathbf{R}$ of degree m. Likewise there exists an n-concave and n-monotone polynomial $q_m : \mathbf{I} \to \mathbf{R}$ of degree m,
(3) There are no n-monotone polynomials of degree m in $\mathbf{I}$ for $m = 2, 3, \ldots, 2n-2$,
(4) There are no n-convex polynomials of degree m in $\mathbf{I}$ for $m = 3, 4, \ldots, 2n-1$.

(Skech of the proof) When I and J are finite intervals in the same forms (open, closed etc), for each $n \in \mathbf{N}$ there is a correspondence between $P_n(\mathbf{I})$ and $P_n(\mathbf{J})$, and $K_n(\mathbf{I})$ and $K_n(\mathbf{J})$, respectively by linear transition functions, which are operator monotone and operator convex. So are their inverse. Hence we may find such a polynomial in each statement on some interval on $[0, \alpha)$.

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(1): We first introduce the polynomial p_m of degree m given by

$$p_m(t) = b_1 t + b_2 t^2 + \ldots + b_m t^m,$$

where

$$b_k = \int_0^1 t^{k-1} dt = \frac{1}{k}.$$

Then the ℓth derivative $p_m^{(\ell)}(0) = \ell! b_\ell$ for $\ell = 1, 2, \ldots, 2n-1$, and consequently

$$M_n(p_m; 0) = \left(\frac{p_m^{(i+j-1)}(0)}{(i+j-1)!} \right)_{i,j=1}^n = (b_{i+j-1})_{i,j=1}^n.$$

Now take a vector $c = (c_1, c_2, \ldots, c_n)$ in an n-dimensional Hilbert space, then

$$(M_n(p_m; 0)c|c) = \sum_{i,j=1}^{n} b_{i+j-1} c_j \bar{c}_i = \int_0^1 \left| \sum_{i=1}^{n} c_i t^{i-1} \right|^2 dt.$$

From this we can say that the matrix $M_n(p_m; 0)$ is positive definite, and then by the continuity of entries, we can find a positive number α such that $M_n(p_m; t)$ is positive in the interval $[0, \alpha)$. Hence by the criterion $p_m(t)$ becomes n-monotone.
(2) :Similarly, we can find a polynomial p_m such that both of $M_n(p_m; 0)$ and $K_n(p_m; 0)$ are positive definite. Hence we can find a positive number α such that p_m becomes both n-monotone and n-convex in the interval $[0, \alpha)$.

(3): Let f_m be an n-monotone polynomial of degree m on I with $2 \le m \le 2n-2$. We may assume as above that I contains 0 . Write $f_m(t) = b_0 + b_1 t + \ldots + b_m t^m$, where $b_m \neq 0$. We have then

$$f_m^{(m-1)}(0) = (m-1)! b_{m-1}, \quad f_m^{(m)}(0) = m! b_m, \quad f_m^{(m+1)}(0) = 0.$$

Consider the matrix $M_n(f_m; 0)$. We have to check two cases where $m = 2k$, even and $m = 2k-1$, odd. Note first that in both cases $k+1 \le n$. In the first case, the principal submatrix of $M_n(f_m; 0)$ consisting of the rows and columns with numbers k and $k+1$ is given by

$$\begin{pmatrix} b_{m-1} & b_m \\ b_m & 0 \end{pmatrix}$$

and it has determinant $-b_m^2 < 0$. In the latter case, we consider

$$\begin{pmatrix} b_{m-2} & b_m \\ b_m & 0 \end{pmatrix}$$

and this matrix also has determinant $-b_m^2 < 0$. Since $M_n(f_m; 0)$ is supposed to be positive semidefinite by the criterion we have in both cases a contradiction.

Corollary

Let $\mathbf{I}$ be a finite interval in $\mathbf{R}$. Then for any $n \in \mathbf{N}$ both of sets $P_n(\mathbf{I}) \backslash P_{n+1}(\mathbf{I})$ and $K_n(\mathbf{I}) \backslash K_{n+1}(\mathbf{I})$ are non-empty.

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Let $n \in \mathbf{N}$.
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Corollary

Let $\mathbf{I}$ be a non-trivil infinite interval.

1. For any $n \in \mathbf{N}$ the set $P_n(\mathbf{I}) \backslash P_{n+1}(\mathbf{I})$ is not empty.
2. For any $n \in \mathbf{N}$ the set $K_n(\mathbf{I}) \backslash K_{n+1}(\mathbf{I})$ is not empty.

Since the function $\dfrac{t}{1+t} : [0, \infty) \to [0, 1)$ is operator monotone, but operator concave. We need several observations to prove the statement $\boxed{2}$.

Double pilling structure

Theorem

Let $0 < \alpha \leq \infty$ and let f be a real valued continuous function in $I = [0, \alpha)$.
Consider the following three assertions.

(i) f is n-convex and $f(0) \leq 0$.
(ii) For a positive semidefinite n-matrix A with the spectrum in $[0, \alpha)$ and a contraction matrix C, the inequality $f(C^\star AC) \leq C^\star f(A)C$ holds.
(iii) The function $f(t)/t$ is n-monotone on the interval $(0, \alpha)$.

1. (Hansen-Pedersen 1982) If $n = \infty$, three assertions are equivalent.
2. (Osaka-Tomiyama 2009) (a) The assertions $(ii)_n$ and $(iii)_n$ are equivalent, (b) $(i)_n \prec (ii)_{n-1} \prec (i)_{[\frac{n}{2}]}$,

where the notation $(S)_m \prec (T)_n$ means that if (S) holds on M_m, the assetion (T) holds on M_n.

Remark (Hoa-Osaka-Tomiyama 2013)

Let $0 < \alpha \leq \infty$ and let f be a real valued continuous function in $I = [0, \alpha)$.
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(i) f is n-concave and $f(0) \geq 0$.
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(iii) The function $t/f(t)$ is n-monotone on the interval $(0, \alpha)$.
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Then we have $(i)_n \prec (ii)_{n-1} \prec (i)_{[\frac{n}{2}]}$,

Question

For $n \in \mathbf{N}$ let $G(n) = \min\{k - n \mid (iii)_k \to (i)_n\}$. Is the set $\{G(n)\}_{n\in\mathbf{N}}$ bounded ?

Barbour transform

Let OM_+ be the set of positive operator monotone functions on $(0, \infty)$ and $OM_+^1 = \{f \in OM_+ \mid f(1) = 1\}$.

Definition

The transform $\hat{}\ : OM_+ \to OM_+^1$ by $f \mapsto \frac{t+f}{1+f}$ is called the Barbour transform.

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$$\widehat{f^\perp} = (\hat{f})^\perp$$
$$\widehat{f'} = (\hat{f})^*$$
$$\widehat{f^*} = (\hat{f})'$$
$$\hat{\omega}_l = \nabla, \quad \hat{\sharp} = \sharp, \quad \hat{\omega}_r = !$$

Recall that $f^\perp(t) = \dfrac{t}{f(t)}$, $f'(t) = tf\left(\dfrac{1}{t}\right)$, $f^*(t) = \dfrac{1}{f(\frac{1}{t})}$.

Properties of Barbour transform

The Barbour transform plays an important role

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Proposition (Kubo-Nakamura-Ohno-Wada 2011)

1. The Barbour transform is injective and $\widehat{OM_+} = OM_+^1 \backslash \{1, t\}$.
2. $\{f \in OM_+^1 \mid ! \leq \sigma_f \leq \nabla\} = \widehat{OM_+^1}$, where $! \leq \sigma_f \leq \nabla$ means that for any positive operators A and B $A!B \leq A\sigma_f B \leq A\nabla B$.

Self-adjoint means

Using the Barbour transform we can characterize the self-adjointness and the symmetricity in OM_+.

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Theorem (O-Wada 2015)

Let f be a positive cntinuous function on $(0, \infty)$. The folowings are equivalent.

1. $f \in OM_+^1 \backslash \{1, t\}$ and $f = f^*$.
2. There exists an operator monotone function $g \in OM_+$ such that $f = \sqrt{gg^*}$.
3. There exixts an operator monotone function $g \in OM_+$ such that
$$f = \frac{t + g + g'}{1 + g + g'}.$$

Remark

In (Kubo-Ando '80) they asked existence of self-adjoint operator means except trivial means ω_l, ω_r, the geometric mean $\sharp$, and σ_{t^p} ($p \in [0, 1]$), where $A\omega_l B = A$, $A\omega_r B = B$, $A\sharp B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}}$ for any positive operators A and B. Using the previous theorem we can construct many examples. For example, if $g(t) = \log(t+1)$, then corresponding operator means of functions $\sqrt{\log(t+1)/\log(t^{-1}+1)}$ and $\dfrac{t+\log(t+1)+t\log(t^{-1}+1)}{1+\log(t+1)+t\log(t^{-1}+1)}$ are self-adjoint. □

Symmetric means

Let σ be a n-connection. The *transpose* σ' is defined by $A\sigma' B = B\sigma A$. A connection is called *symmetric* if it equals to its transpose.

Denoted by Σ_n^{sym} is the image of the set of all symmetric n-connections by Σ_n, where $\Sigma_n : \{\sigma : n - connections\} \to P_n'$.

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Theorem (Hoa-O-Toan 2013)

For any natural number n there is an injective map Σ_n from the set of matrix connections of order n to $P_n^{'} \supset \mathcal{C}_{2n}$ associating each connection σ to the function f_σ such that $f_\sigma(t)I_n = I_n\sigma(tI_n)$ for $t > 0$. Furthermore, the range of this map contains $\mathcal{C}_{2n}$.

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Here P_n' is a set of all positive n-monotone functions on $(0, \infty)$.

Definition (Ameur-Kaijser-Silvestrov 2007)

A function $f : \mathbf{R}_+ \to \mathbf{R}_+$ is called an *interpolation function of order* n if for any $T, A \in M_n$ with $A > 0$ and $T^*T \leq 1$

$$T^*AT \leq A \implies T^*f(A)T \leq f(A).$$

We denote by $\mathcal{C}_n$ the class of all interpolation functions of order n on $\mathbf{R}_+$.

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Theorem (Ameur-Kaijser-Silvestrov 2007)

A function $f\colon \mathbf{R}_+ \to \mathbf{R}_+$ belongs to $\mathcal{C}_n$ if and only if for every n-set $\{\lambda_i\}_{i=1}^n \subset \mathbf{R}_+$ there exists a P'-function h such that $f(\lambda_i) = h(\lambda_i)$ for $i = 1, \ldots, n$.

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Proposition (O-Wada 2015)

Let f be a positive continuous function on $(0, \infty)$. The followings are equivalent.

1. $f \in OM_+^1 \backslash \{1, t\}$ and $f = f'$.
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$$f = \frac{t + \sqrt{gg^*}}{1 + \sqrt{gg^*}}$$

Characterization of ∇ and !

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Lemma

Let $f\colon(0,\infty)\to(0,\infty)$ be a continuous function. The followings are equivalent.

1. $f\in OM_+$ and $f\geq f_\nabla$, that is $f(t)\geq\frac{1+t}{2}$ for $t\in(0,\infty)$.
2. There exists an operator monotone $g\in OM_+$ and nonnegative real number $a,b\geq\frac{1}{2}$ such that $\lim_{t\to 0}g(t)=0$, $\lim_{n\to\infty}\frac{g(t)}{t}=0$, and

$$f(t)=a+bt+g(t)\ (t\in(0,\infty)).$$

Lemma

Let $f\colon (0,\infty) \to (0,\infty)$ be a continuous function. The followings are equivalent.

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Non-symmetric means

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Lemma

Let f be a positive operator monotone function on $(0, \infty)$ with $f(1) = 1$. The followings are equivalent:

1. $\sigma_{\hat{f}}$ *is non-symmetric and* $! \leq \sigma_{\hat{f}} \leq \nabla$*,*
2. f *is non-self-adjoint.*

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Lemma

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1. *$\sigma_{\hat{f}}$ is non-self-adjoint and $! \leq \sigma_{\hat{f}} \leq \nabla$,*
2. *f is non-symmetry*

Proposition (O-Wada 2015)

$$\{f \mid f: \text{ non-symmetric}, f_! \leq f \leq f_\nabla\}$$
$$= \left\{\hat{f} \mid f: \text{ non-self-adjoint}\right\}$$
$$= \left\{\hat{\hat{f}} \mid f: \text{ non-symmetric}\right\}$$
$$\supset \left\{\hat{f} \mid f: \text{ symmetric}\right\} \setminus \{\sharp\}$$

Remark

From the previous Propositiona non-self-adjoint positive monotone functions f with $f(1) = 1$ give non-symmetric operator mean such that $! \leq \sigma_{\hat{f}} \leq \nabla$. For examples, let $-1 \leq p \leq 2$ and ALG_p be the corresponding function to the power diffrence mean defined by

$$\mathrm{ALG}_p(t) = \begin{cases} \dfrac{p-1}{p}\dfrac{1-t^p}{1-t^{p-1}} & t \neq 1 \\ 1 & t = 1 \end{cases}$$

and the Petz-Hasegawa function f_p which is defined by

$$f_p(t) = p(p-1)\frac{(t-1)^2}{(t^p-1)(t^{1-p}-1)}$$

are non-self-adjoint. Hence, $\sigma_{\widehat{ALG}_p}$ and $\sigma_{\widehat{f_p}}$ are non-symmetric operator means between $!$ and ∇.

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Corollary

Let $f \in OM_+$ such that $\sigma_f \geq \nabla$ and let $g \in OM_+$ such that $f(t) = a + bt + g(t)$ $(a, b \geq \frac{1}{2})$. Suppose that $a \neq b$. Then $\hat{\hat{f}}$ is not symmetric and $! \leq \sigma_{\hat{\hat{f}}} \leq \nabla$.

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Corollary

Let $f \in OM_+$ such that $\sigma_f \leq !$ and let $g \in OM_+$ such that $f(t) = \dfrac{t}{a + bt + g(t)}$ $(a, b \geq \frac{1}{2})$. Suppose that $a \neq b$. Then $\hat{\hat{f}}$ is not symmetric and $! \leq \sigma_{\hat{\hat{f}}} \leq \nabla$.

Theorem (O-Wada 2015)

Let σ_1, σ_2 *be symmetric operator means. If* $\sigma_1 \leq \sigma_2$ *and* $\sigma_1 \neq \sigma_2$*, then there exists a non-symmetric operator mean* σ *such that* $\sigma_1 \leq \sigma \leq \sigma_2$.

To prove the above theorem, we need the following lemma.

Theorem (O-Wada 2015)

Let σ_1, σ_2 be symmetric operator means. If $\sigma_1 \leq \sigma_2$ and $\sigma_1 \neq \sigma_2$, then there exists a non-symmetric operator mean σ such that $\sigma_1 \leq \sigma \leq \sigma_2$.

To prove the above theorem, we need the following lemma.

Lemma

Let h_1 and h_2 be self-adjoint positive operator monotone functions on $(0, \infty)$ with $h_1 \neq h_2$ and $h_1(1) = h_2(1) = 1$. If $h_1(t) \leq h_2(t)$ for all $t < 1$ and $h_1(t) \geq h_2(t)$ for all $t > 1$, then there exists a non-self-adjoint positive operator monotone function h such that

$$
\begin{aligned}
h_1(t) \leq h(t) \leq h_2(t) \quad & \text{for all } t < 1, \\
h_1(t) \geq h(t) \geq h_2(t) \quad & \text{for all } t > 1.
\end{aligned}
$$

Let f_1, f_2 be positive operator monotone functions which correspond to σ_1, σ_2, respectively. If we define $h_1 := \check{f}_1, h_2 := \check{f}_2$, then h_1, h_2 satisfy the conditions appearing in Lemma.Thus, there exists a non-self-adjoint positive operator monotone function h such that

$$h_1(t) \leq h(t) \leq h_2(t) \quad \text{for all } t < 1$$

and

$$h_1(t) \geq h(t) \geq h_2(t) \quad \text{for all } t > 1.$$

By a simple calculation, we have

$$f_1 = \widehat{h_1} \leq \hat{h} \leq \widehat{h_2} = f_2$$

and

$$\hat{h} \neq (\widehat{h^*}) = (\hat{h})',$$

which means that $\sigma_{\hat{h}}$ is the desired mean

Generalized reverse Caucy inequality

(Reverse Caucy inequality)

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \leq \sqrt[n]{a_1 a_2 \cdots a_n} + \frac{1}{n} \sum_{1 \leq i,j \leq n} |a_i - a_j|.$$

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Theorem (Audenaert et al 2006)

For $0 \leq \nu \leq 1$ and $A, B \in B(H)^{++}$
$\mathrm{Tr}(A + B - |A - B|) \leq 2\mathrm{Tr}(A^\nu B^{1-\nu})$.

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For $0 \le \nu \le 1$ and $A, B \in B(H)^{++}$
$\mathrm{Tr}(A + B - |A - B|) \le 2\mathrm{Tr}(A^{\nu} B^{1-\nu})$.

Theorem (Hoa-Toan-O 2012)

$$\mathrm{Tr}(A + B - |A - B|) \le 2\mathrm{Tr}(f(A)^{\frac{1}{2}} g(B) f(A)^{\frac{1}{2}})$$

holds for any operator monotone function f on $[0, \infty)$ with $f((0, \infty))$ and $g(t) = \frac{t}{f(t)}$ $(t \in (0, \infty))$ and $g(0) = 0$.

Furuichi in 2011, however, showed that the trace inequality

$$\frac{1}{2}\mathrm{Tr}(A+B-|A-B|) \leq \mathrm{Tr}(A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}})$$

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is not true in general.
When $n = 2$ and $\nu = \frac{1}{2}$, a natural matrix form of the reverse Cauchy inequality for two positive definite matrices A and B could be written as

$$\frac{A+B}{2} \leq A\sharp B + \frac{|A-B|}{2},$$

where $A\sharp B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}}$ is the geometric mean of A, B.

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In general, the last inequality have the following form

$$\frac{A + B}{2} - A\sigma_f B \le \frac{|A - B|}{2},$$

where $A\sigma_f B = A^{\frac{1}{2}}f(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}$ is the operator mean corresponding the function f in the sense of Kubo and Ando.

Theorem (Hoa-Khue-O 2014)

Let f be a strictly positive operator monotone function on $[0, \infty)$ with $f((0, \infty)) \subset (0, \infty)$ and $f(1) = 1$. Then for any positive semidefinite matrices A and B with $AB + BA \geq 0$,

$$A + B - |A - B| \leq 2A\sigma_f B.$$

Proof

Since f is continuous, we may assume that A is invertible.
Let $A - B = P - Q$, where $P = (A - B)_+$ and $Q = (A - B)_-$ are, respectively, the positive and negative parts of $A - B$.
Since $2(AB + BA) = (A + B)^2 - |A - B|^2$ is positive, we have $|A - B| \leq A + B$, that is, $A - P$ is positive.

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Since $2(AB + BA) = (A + B)^2 - |A - B|^2$ is positive, we have $|A - B| \leq A + B$, that is, $A - P$ is positive.
Since $A - P = B - Q \leq B$, we have

$$A^{-\frac{1}{2}}(A - P)A^{-\frac{1}{2}} \leq A^{-\frac{1}{2}}BA^{-\frac{1}{2}}.$$

Proof

Since f is continuous, we may assume that A is invertible.
Let $A - B = P - Q$, where $P = (A-B)_+$ and $Q = (A-B)_-$ are, respectively, the positive and negative parts of $A - B$.
Since $2(AB + BA) = (A+B)^2 - |A-B|^2$ is positive, we have $|A - B| \leq A + B$, that is, $A - P$ is positive.
Since $A - P = B - Q \leq B$, we have

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Hence

$$\begin{aligned} A\sigma_f(A-P) &= A^{\frac{1}{2}}f(A^{-\frac{1}{2}}(A-P)A^{-\frac{1}{2}})A^{\frac{1}{2}} \\ &\leq A^{\frac{1}{2}}f(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}} \\ &= A\sigma_f B. \end{aligned}$$

Proof continued

We have, then,

$$\begin{aligned} A^{\frac{1}{2}} f(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{\frac{1}{2}} &= A\sigma_f B \\ &\geq A\sigma_f (A - P) \\ &\geq (A - P)\sigma_f (A - P) \\ &= A - P = \frac{1}{2}(A + B - |A - B|). \end{aligned}$$

Main Theorem

Theorem (O-Tsurumi-Wada 2015)

Let $\lambda \in [0,1]$ *and* σ *be operator mean. Suppose that* ϕ *is a non-negative operator convex with* $\phi(0) = 0$, $\phi(1) = 1$.*and* $\phi'_+(0) = 0$. *Then the following are equivalent:*

1. $\sigma = \nabla_\lambda$;
2. $\phi(A)\sigma\phi(B) \geq \phi(A\nabla_\lambda B) - \phi\left(r|A-B|\right)$ *for* $A, B \in B(H)^{++}$ *and nonnegative real number* r.

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Remark

1. When $r = \frac{1}{2}$ and $\phi(t) = t^2$, we have a counterexample of the pair of positive definite matrices which does not satisfy the inequality in the previous theorem.
2. When $r = \frac{1}{2}$ and $\phi(t) = t$, we need some extra conditions.

Corollary

Let σ be an operator mean. Suppose that $A\sigma B \geq \frac{A+B}{2} - \frac{|A-B|}{2}$ for $A, B \in B(H)^{++}$.

1. (Hoa 2015) If $\sigma = \sigma^{'}$, then $\sigma = \nabla$.
2. If $h_\sigma(0) = 0$, $\sigma = \omega_r$.

Remind that $A\sigma' B = B\sigma A$ and $A\omega_r B = B$.

Application

Theorem (O-Tsurumi-Wada 2015)

Let $\lambda \in [0,1]$ and σ be operator mean. Suppose that ϕ is a nonnegative operator convex with $\phi(0) = 0$, $\phi(1) = 1$. Then the following is equivalent:

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Application

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Corollary

Let ϕ be a nonnegative operator convex function on $[0,\infty)$ with $\phi(1) = 1$, $\lim_{x\to\infty} \phi(x) = +\infty$, and assume $\lambda \in [0,1]$. If an operator mean σ satisfies

$$\phi(A!_\lambda B) \geq \phi(A)\sigma\phi(B) \quad A, B \in B(H)^{++},$$

then $\sigma =!_\lambda$.

Remark

For $\lambda \in (0,1)$, let f be a continuous nonnegative function on $(0,\infty)$. Then the following conditions are equivalent:

1. f is operator monotone decreasing;
2. $f(A\nabla_\lambda B) \leq f(A)\sigma f(B)$ for all $A, B \in B(H)^{++}$ and for all operator means σ, and $f'_\sigma(1) = \lambda$;
3. $f(A\nabla_\lambda B) \leq f(A)\sharp_\lambda f(B)$ for all $A, B \in B(H)^{++}$;
4. $f(A\nabla_\lambda B) \leq f(A)\sigma f(B)$ for all $A, B \in B(H)^{++}$ and for some operator mean $\sigma \neq \nabla_\lambda$ with $f'_\sigma(1) = \lambda$,

where $A\sharp_\lambda B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\lambda A^{\frac{1}{2}}$.

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Thank you very much for your attention