Transformations of the positive cone in operator algebras

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In this talk by an isometry between metric spaces we mean a mapping which preserves the distances of all pairs of elements.

Banach-Stone (1932, 1937) Let X, Y be compact Hausdorff spaces and $\phi : C(X) \to C(Y)$ a surjective linear isometry. Then there exists a homeomorphism $\varphi : Y \to X$ and a continuous scalar function τ on Y with values of modulus 1 such that

$$\phi(f) = \tau \cdot f \circ \varphi, \quad f \in C(X).$$

Recall: the transformations $f \mapsto f \circ \varphi$ are exactly the algebra isomorphisms between $C(X)$ and $C(Y)$.

Kadison (1951) Let $\mathcal{A}, \mathcal{B}$ be (unital) C^*-algebras and $\phi : \mathcal{A} \to \mathcal{B}$ a surjective linear isometry. Then there is a Jordan *-isomorphism $J : \mathcal{A} \to \mathcal{B}$ and a unitary element $U \in \mathcal{B}$ ($U^*U = UU^* = I$) such that ϕ is of the form

$$\phi(A) = U \cdot J(A), \qquad A \in \mathcal{A}.$$

Jordan homomorphism: If $\mathcal{A}, \mathcal{B}$ are complex algebras, then a linear map $J : \mathcal{A} \to \mathcal{B}$ is called a Jordan homomorphism if it satisfies $J(A^2) = J(A)^2$ for any $A \in \mathcal{A}$ or, equivalently, if it satisfies $J(AB + BA) = J(A)J(B) + J(B)J(A)$ for any $A, B \in \mathcal{A}$.

A Jordan *-homomorphism $J : \mathcal{A} \to \mathcal{B}$ between *-algebras $\mathcal{A}, \mathcal{B}$ is a Jordan homomorphism which preserves the involution in the sense that it satisfies $J(A^*) = J(A)^*$ for all $A \in \mathcal{A}$.

By a Jordan *-isomorphism we mean a bijective Jordan *-homomorphism.

Let $\mathfrak{A}$ be a real linear space partially ordered by a cone $\mathfrak{A}_+$.

Suppose there exists an order unit $u \in \mathfrak{A}_+$, i.e., for each $a \in \mathfrak{A}$ there is a positive number λ such that $-\lambda u \leq a \leq \lambda u$.

Also assume that $\mathfrak{A}$ is Archimedean, i.e., if $a \in \mathfrak{A}$ is such that $na \leq u$ for all positive integer n, then $a \leq 0$.

In that case we define the order unit norm

$$\|a\|_u = \inf\{\lambda > 0 \,|\, -\lambda u \leq a \leq \lambda u\}, \quad a \in \mathfrak{A}.$$

We have

$$\mathfrak{A}_+^\circ = \{a \in \mathfrak{A} \,|\, a \text{ is an order unit of } \mathfrak{A}\}.$$

For $a, b \in \mathfrak{A}_+^\circ$ let

$$M(a/b) = \inf\{\beta > 0 \,|\, a \leq \beta b\}.$$

Thompson metric

Hilbert's metric:

$$d_H(a,b) = \log M(a/b)M(b/a), \quad a,b \in \mathfrak{A}_+^\circ$$

Thompson's metric:

$$d_T(a,b) = \log \max\{M(a/b), M(b/a)\}, \quad a,b \in \mathfrak{A}_+^\circ$$

Originally, Hilbert defined his metric on open bounded sets in finite dimensional real linear spaces to obtain Finsler manifolds that generalize Klein's model of the real hyperbolic space. This played an important role in the solution of Hilbert's fourth problem.

The above approach to Hilbert's metric is due to Birkhoff. It has found numerous applications in the spectral theory of linear and nonlinear operators, ergodic theory, fractal analysis, etc.

Thompson's modification of Hilbert's metric is the prime example of a Banach-Finsler manifold (in case the order unit space is complete). Moreover, if the order unit space is the selfadjoint part of a C^*-algebra (more generally, a JB-algebra, then the Banach-Finsler manifold is symmetric and possesses features of nonpositive curvature. A lot of applications: spectral theory of operators on cones, geometry of spaces of positive operators, etc.

Given a unital C^*-algebra $\mathcal{A}$ ordered by the cone $\mathcal{A}_+$ of its positive (semidefinite) elements, the Thompson metric d_T on the set $\mathcal{A}_+^{-1}$ of its invertible positive elements (what we call the positive definite cone of $\mathcal{A}$) is given by

$$d_T(A, B) = \| \log A^{-1/2} B A^{-1/2} \|, \quad A, B \in \mathcal{A}_+^{-1}.$$

PROBLEM: What are the isometries?

M, 2009, PAMS: The structure of Thompson isometries for $\mathcal{A} = B(H)$. They are the transformations

$$A \longmapsto TAT^* \quad \text{or} \quad A \longmapsto TA^{-1}T^*,$$

where T is an invertible bounded either linear or conjugate-linear operator on H.

Approach: Try to find algebraic properties of Thompson isometries.

Mazur-Ulam (1932) Let X, Y be normed real linear spaces. Every surjective isometry $\phi : X \to Y$ is affine, i.e., respects the operation of convex combinations. (If ϕ is assumed to send 0 to 0, then it is linear.)

Of course, linearity does not tell too much in general, but having a look at the Thompson metric

$$d_T(A, B) = \| \log A^{-1/2} B A^{-1/2} \|, \quad A, B \in \mathcal{A}_+^{-1}$$

we see some sort of multiplicative background.

In fact, adapting the proof of Mazur-Ulam theorem (due to Väisälä) with, of course, a number of modifications, we find Thompson isometries preserve (not the arithmetic mean but) the geometric mean! This is the operation

$$A \# B = A^{1/2} (A^{-1/2} B A^{-1/2})^{1/2} A^{1/2}.$$

We were lucky, we had previously determined the structure of all bijective maps on the positive semidefinite cone in $B(H)$ (2009, PAMS) which preserve the geometric mean. Although this result could not be used directly, but still we could solve the isometry problem.

Most natural problems:
(1) what about general operator algebras, e.g., C^*-algebras;
(2) find general Mazur-Ulam theorems;

Non-commutative generalizations of Mazur-Ulam theorem; results on the algebraic behavior of isometries on more general algebraic structures, e.g., on groups. The operation of geometric mean is not appropriate in general groups, we cannot define it. But we have the Anderson-Trapp theorem:

For any given $A, B \in \mathcal{A}_+^{-1}$, the geometric mean $A \# B$ is the unique solution $X \in \mathcal{A}_+^{-1}$ of the equation $XA^{-1}X = B$.

The operation that appears on the left hand side is called the inverted Jordan triple product and it can be defined in any group!

We (Hatori, Hirasawa, Miura, M, 2012, TJM) proved: Under certain conditions, SURJECTIVE ISOMETRIES between groups (or certain subsets called twisted subgroups of groups) equipped with metrics compatible with the group operations necessarily PRESERVE locally(!) an ALGEBRAIC OPERATION, the inverted Jordan triple product $ab^{-1}a$ of elements. (Recall that the Jordan triple product of a and b is aba.)

Although, in general we have this preserver property only locally, in several important particular cases (Thompson metric is one of them!) we have that this product is in fact globally preserved. That means that in those cases the surjective isometries are sort of isomorphisms, inverted Jordan triple isomorphisms.

So in those cases in order to describe the structure of surjective isometries we can do the following: Determine the inverted Jordan triple isomorphisms (or the closely related Jordan triple isomorphisms: maps preserving the Jordan triple product *aba*) and select those which are in fact isometries.

This approach is not always useful. E.g., in the case of normed spaces (Mazur-Ulam theorem) we cannot determine the corresponding isomorphisms, i.e. the affine bijections, explicitly. But in many cases (especially in highly noncommutative structures) it does help a lot.

An abstract Mazur-Ulam type result follows. Presently, in a certain sense, this is the most general such result.

M, 2015, in a volume of the series "Operator Theory: Advances and Applications".

We shall need an abstract algebraic concept.
Manara and Marchi (1991): point-reflection geometry.

Definition 1

Let X be a set equipped with a binary operation $\diamond$ which satisfies the following conditions:

(a1) $a \diamond a = a$ holds for every $a \in X$;

(a2) $a \diamond (a \diamond b) = b$ holds for any $a, b \in X$;

(a3) the equation $x \diamond a = b$ has a unique solution $x \in X$ for any given $a, b \in X$.

In this case the pair $(X, \diamond)$ (or X itself) is called a point-reflection geometry.

Trivial example on the Euclidean plane: $a \diamond b$ = the reflection of b wrt a. In any linear space: $a \diamond b = 2a - b$.

A nontrivial example:

Let $\mathcal{A}$ be a C^*-algebra. For any $A, B \in \mathcal{A}_+^{-1}$ define $A \diamond B = AB^{-1}A$. In that way $\mathcal{A}_+^{-1}$ becomes a point-reflection geometry.
Indeed, the conditions (a1), (a2) above are trivial to check, validity of (a3) is just the Anderson-Trapp theorem.

And we can consider not only metrics but much more general distance measures (divergences)!

Definition 2

Given an arbitrary set X, the function $d : X \times X \to [0, \infty[$ is called a generalized distance measure if it has the property that for an arbitrary pair $x, y \in X$ we have $d(x, y) = 0$ if and only if $x = y$.

Theorem 3

Let X, Y be sets equipped with binary operations $\diamond, \star$, respectively, with which they form point-reflection geometries. Let $d : X \times X \to [0, \infty[$, $\rho : Y \times Y \to [0, \infty[$ be generalized distance measures. Pick $a, b \in X$, set

$$L_{a,b} = \{x \in X \, : \, d(a, x) = d(x, b \diamond a) = d(a, b)\}$$

and assume the following:

(b1) $d(c \diamond x, c \diamond x') = d(x', x)$ *holds for all* $c, x, x' \in X$*;*

(b1') $\rho(d \star y, d \star y') = \rho(y', y)$ *for all* $d, y, y' \in Y$*;*

(b2) $\sup\{d(x, b) \, : \, x \in L_{a,b}\} < \infty$*;*

(b3) *there exists a constant* $K > 1$ *such that* $d(x, b \diamond x) \geq Kd(x, b)$ *holds for every* $x \in L_{a,b}$*.*

Let $\phi : X \to Y$ *be a surjective map such that*

$$\rho(\phi(x), \phi(x')) = d(x, x'), \quad x, x' \in X.$$

Then we have

$$\phi(b \diamond a) = \phi(b) \star \phi(a).$$

Trivially includes the original Mazur-Ulam theorem.

We are going to present a general result on the structure of divergence preservers between positive definite cones in operator algebras.
Preparations.

By a symmetric norm on a C^*-algebra $\mathcal{A}$ we mean a norm N for which $N(AXB) \leq \|A\| N(X) \|B\|$ holds for all $A, X, B \in \mathcal{A}$ ($\|.\|$ stands for the original norm on $\mathcal{A}$).

Complete symmetric norms on $B(H)$ include the so-called (c, p)-norms and, in particular, the Ky Fan k-norms.

Whenever we speak about topological properties (convergence, continuity, etc.) on the positive definite cone of a C^*-algebra without specifying the topology we always mean the norm topology of $\|.\|$.

We define a large class of distance measures.

For a continuous function $f\colon]0,\infty[\to \mathbb{R}$ consider the following properties:

(c1) $f(y) = 0$ holds if and only if $y = 1$;

(c2) there exists a number $K > 1$ such that

$$|f(y^2)| \geq K|f(y)|$$

holds in a neighborhood of 1. (If f is differentiable at $y = 1$ with nonzero derivative, then this condition is satisfied.)

Let $\mathcal{A}$ be a C^*-algebra, N a norm on $\mathcal{A}$, $f\colon]0,\infty[\to \mathbb{R}$ a given continuous function with property (c1). Define $d_{N,f} : \mathcal{A}_+^{-1} \times \mathcal{A}_+^{-1} \to [0,\infty[$ by

$$d_{N,f}(A,B) = N(f(A^{1/2}B^{-1}A^{1/2})), \qquad A,B \in \mathcal{A}_+^{-1}. \tag{1}$$

It is apparent that $d_{N,f}$ is a generalized distance measure. Many particular examples, among them important ones even in the finite dimensional (matrix algebra) case.

Jordan triple isomorphisms, inverted Jordan triple isomorphisms.

If $\mathcal{A}$ is a C^*-algebra and $A, B \in \mathcal{A}_+^{-1}$, then ABA is called the Jordan triple product of A and B while $AB^{-1}A$ is said to be their inverted Jordan triple product.

If $\mathcal{B}$ is another C^*-algebra and $\phi : \mathcal{A}_+^{-1} \to \mathcal{B}_+^{-1}$ is a map which satisfies

$$\phi(ABA) = \phi(A)\phi(B)\phi(A), \qquad A, B \in \mathcal{A}_+^{-1},$$

then it is called a Jordan triple map.

If $\phi : \mathcal{A}_+^{-1} \to \mathcal{B}_+^{-1}$ fulfills

$$\phi(AB^{-1}A) = \phi(A)\phi(B)^{-1}\phi(A), \qquad A, B \in \mathcal{A}_+^{-1},$$

then ϕ is said to be an inverted Jordan triple map.

A bijective Jordan triple map is called a Jordan triple isomorphism and a bijective inverted Jordan triple map is said to be an inverted Jordan triple isomorphism.

Algebraic property of maps preserving generalized distance measures.

Applying our general Mazur-Ulam type result Theorem 3 we have the following

Theorem 4

Let $\mathcal{A}, \mathcal{B}$ be C^*-algebras with complete symmetric norms N, M, respectively. Assume N satisfies $N(|A|) = N(A)$ for all $A \in \mathcal{A}$.
Suppose $f, g \colon]0, \infty[\to \mathbb{R}$ are continuous functions both satisfying (c1) and f also fulfilling (c2).
Let $\phi : \mathcal{A}_+^{-1} \to \mathcal{B}_+^{-1}$ be a surjective map which respects the pair $d_{N,f}, d_{M,g}$ of generalized distance measures in the sense that

$$d_{M,g}(\phi(A), \phi(B)) = d_{N,f}(A, B), \qquad A, B \in \mathcal{A}_+^{-1}.$$

Then ϕ is a continuous inverted Jordan triple isomorphism, i.e., a continuous bijective map that satisfies

$$\phi(AB^{-1}A) = \phi(A)\phi(B)^{-1}\phi(A), \qquad A, B \in \mathcal{A}_+^{-1}.$$

Application concerning Thompson isometries of the positive definite cones of C^*-algebras.
Hatori and M, 2014, JMAA proved the above statement in the particular setting of Thompson metric and used it to describe the corresponding isometries.

Conclusion: any surjective Thompson isometry ϕ between the positive definite cones $\mathcal{A}_+^{-1}, \mathcal{B}_+^{-1}$ of C^*-algebras $\mathcal{A}, \mathcal{B}$ is of the form

$$\phi(A) = \phi(I)^{1/2} \left(PJ(A) + (1 - P)J(A^{-1}) \right) \phi(I)^{1/2}, \quad A \in \mathcal{A}_+^{-1},$$

where P is a central projection in $\mathcal{B}$ and $J : \mathcal{A} \to \mathcal{B}$ is a Jordan *-isomorphism.

How to prove it?

We know that ϕ is an inverted Jordan triple isomorphism. It follows that

$$\psi(.) = \phi(I)^{-1/2}\phi(.)\phi(I)^{-1/2}$$

is a Jordan triple isomoprhism and it is again a Thompson isometry.

Next we use the fact that

$$\frac{d_T(\exp(tA), \exp(tB))}{t} \to \|A - B\|$$

as $t \to 0$ to show that

$$\Psi(T) = \log \psi(\exp(T))$$

defines a a bijective map between the selfadjoint parts of $\mathcal{A}$ and $\mathcal{B}$, respectively which is a surjective isometry. The classical Mazur-Ulam theorem gives us that S is real linear, and we apply a theorem of Kadison describing the linear isometries between the selfadjoint parts of C^*-algebras. We have the structure Ψ and then we have the structure of ψ and finally that of ϕ.

This approach works only for the Thompson metric. If we want to obtain a much more general result (relating general distance measures), we need to do something different.

Task: describe the structure of all continuous inverted Jordan triple isomorphisms (or Jordan triple isomorphisms) between positive definite cones.

Unfortunately, we can not do this for general C^*-algebras, only for von Neumann factors.

Theorem 5

Assume $\mathcal{A}, \mathcal{B}$ are von Neumann algebras and $\mathcal{A}$ is a factor not of type I_2.
Let $\phi : \mathcal{A}_+^{-1} \to \mathcal{B}_+^{-1}$ be a continuous Jordan triple isomorphism.
*Suppose $\mathcal{A}$ is of infinite type. Then there is either an algebra *-isomorphism or an algebra *-antiisomorphism $\theta : \mathcal{A} \to \mathcal{B}$ and a number $c \in \{-1, 1\}$ such that*

$$\phi(A) = \theta(A^c), \qquad A \in \mathcal{A}_+^{-1}. \tag{2}$$

*Assume $\mathcal{A}$ is of finite type. Then there is either an algebra *-isomorphism or an algebra *-antiisomorphism $\theta : \mathcal{A} \to \mathcal{B}$, a number $c \in \{-1, 1\}$, and a real number d with $d \neq -c$ such that*

$$\phi(A) = e^{d\,\mathrm{Tr}(\log A)}\theta(A^c), \qquad A \in \mathcal{A}_+^{-1}. \tag{3}$$

True for factors of type I_2, too. Joint work with D. Virosztek, 2016, JMAA.

After this we easily obtain the structure of surjective maps between the positive definite cones of von Neumann factors which respect pairs of generalized distance measures.

Theorem 6

Let $\mathcal{A}, \mathcal{B}$ be von Neumann algebras with complete symmetric norms N, M, respectively. Assume $f, g\colon]0, \infty[\to \mathbb{R}$ are continuous functions both satisfying (c1) and f also fulfilling (c2). Suppose that $\mathcal{A}$ is a factor.
Let $\phi : \mathcal{A}_+^{-1} \to \mathcal{B}_+^{-1}$ be a surjective map which respects the pair $d_{N,f}, d_{M,g}$ of generalized distance measures in the sense that

$$d_{M,g}(\phi(A), \phi(B)) = d_{N,f}(A, B), \qquad A, B \in \mathcal{A}_+^{-1}. \tag{4}$$

*Suppose $\mathcal{A}$ is of infinite type. Then there is either an algebra *-isomorphism or an algebra *-antiisomorphism $\theta : \mathcal{A} \to \mathcal{B}$, a number $c \in \{-1, 1\}$, and an invertible element $T \in \mathcal{B}$ such that*

$$\phi(A) = T\theta(A^c)T^*, \qquad A \in \mathcal{A}_+^{-1}. \tag{5}$$

*Assume $\mathcal{A}$ is of finite type. Then there is either an algebra *-isomorphism or an algebra *-antiisomorphism $\theta : \mathcal{A} \to \mathcal{B}$, a number $c \in \{-1, 1\}$, an invertible element $T \in \mathcal{B}$, and a real number d with $d \neq -c$ such that*

$$\phi(A) = e^{d\,\mathrm{Tr}(\log A)}\,T\theta(A^c)T^*, \qquad A \in \mathcal{A}_+^{-1}. \tag{6}$$

Remark: Same structural result holds for geometric mean preservers.

An interesting consequence of the above theorem is that if the positive definite cones of two von Neumann factors are "isometric" in some very general sense (with respect a pair of generalized distance measures), then the underlying algebras are necessarily isomorphic or antiisomorphic as algebras. No such result on the full algebras equipped with general symmetric (or even unitary invariant) norms.

Natural question: What about general C^*-algebras? We do not know. Our method does not work in that setting.

The proof of the theorem is long, we mention only the main steps and the clue ideas in the case where $\mathcal{A}$ is not of type I_2.

First we prove that if $\phi : \mathcal{A}_+^{-1} \to \mathcal{B}_+^{-1}$ is a continuous Jordan triple map, then ϕ is a Lipschitz function in a neighborhood of the identity.

Next we show that every continuous Jordan triple map from $\mathcal{A}_+^{-1}$ into $\mathcal{B}_+^{-1}$ is of the form

$$\phi(A) = e^{f(\log A)}, \quad A \in \mathcal{A}_+^{-1}.$$

with some $f : \mathcal{A}_s \to \mathcal{B}_s$ which is a linear bijection that preserves commutativity.

The structure of those maps (commutativity preservers) between prime algebras not satisfying the standard polynomial identity of degree 4 is known due to Brešar. Purely algebraic result. We apply it and work further to finish the proof.

We note that there are a lot of important distance measures for which our result applies already in the case of matrix algebras: shortest path distances on positive definite matrices in certain Riemann and Finsler structures, Sra metric, Stein's loss, log determinat α-divergences, etc.

We conclude this part by some rather important recent developments.

Jordan operator algebras: Lemmens, Roelands amd Wortel, 2016, described the Thompson isometries in the case of JB-algebras.

Local version: Mankiewicz's theorem states that a surjective isometry between open connected subsets in real normed spaces can be extended to a surjective isometry between the whole spaces.
Hatori, 2016, proved a similar statement concerning Thompson isometries in C^*-algebras.

Recall that maps preserving the geometric mean played important role in the description of Thompson isometries.

We conclude the talk with a few recent results (to appear in a volume of Contemp. Math.) on maps transforming the geometric mean to the arithmetic mean and some of its consequences.

We remark that if $\mathcal{A}$ is a commutative C^*-algebra, then the map $A \mapsto \log A$ is clearly a bijective map from $\mathcal{A}_+^{-1}$ onto $\mathcal{A}_s$ which transforms the geometric mean to the arithmetic mean. What about the noncommutative case?

Theorem 7

Let $\mathcal{A}$ be a von Neumann factor, X be a Banach space. Let $\phi : \mathcal{A}_+^{-1} \to X$ be a continuous map such that

$$\phi(A\#B) = (1/2)(\phi(A) + \phi(B)) \quad A, B \in \mathcal{A}_+^{-1}.$$

If $\mathcal{A}$ is infinite, then ϕ is necessarily constant.
If $\mathcal{A}$ is of finite type, then there are vectors $x_0, y_0 \in X$ such that ϕ is of the form

$$\phi(A) = \mathrm{Tr}(\log A) \cdot x_0 + y_0, \quad A \in \mathcal{A}_+^{-1}.$$

In the proof we use the deep and famous result known as the solution of the Mackey-Gleason problem which concerns linear extensions of measures on projection lattices in von Neumann algebras.

Somewhat surprisingly, the theorem above also provides a common characterization of finite von Neumann factors, the trace functional and the logarithmic function. In fact, we have the following corollary.

Corollary 8

If $\mathcal{A}$ is a von Neumann factor, l is a nonzero continuous (real-)linear functional on $\mathcal{A}_s$ and f is a nonconstant continuous real function on the positive real numbers such that

$$l(f(A\#B)) = (1/2)(l(f(A)) + l(f(B))), \quad A, B \in \mathcal{A}_+^{-1},$$

i.e., the transformation $l \circ f$ sends the geometric mean to the arithmetic mean, then the algebra $\mathcal{A}$ is necessarily of finite type, l is a scalar multiple of the unique normalized trace on $\mathcal{A}$ and f is an affine function of the logarithmic function.

Another application of the argument used in the proof of the previous theorem is to give the following characterization of the so-called logarithmic product

$$A \bullet B = \exp(\log A + \log B), \quad A, B \in \mathcal{A}_+^{-1}$$

on the positive definite cone.

(This concept originally emerged from computational geometry but soon after serious applications have been found regarding the differential geometry of spaces of positive definite operators.)

Theorem 9

Let $\mathcal{A}$ be a von Neumann algebra without a type I_2 direct summand and X be a Banach space. Let $\bullet$ be a group operation on $\mathcal{A}_+^{-1}$ which coincides with the original product on commuting elements in $\mathcal{A}_+^{-1}$ and assume that there is an injective continuous map $\phi : \mathcal{A}_+^{-1} \to X$ such that

$$\phi(A \bullet B) = \phi(A) + \phi(B), \quad A, B \in \mathcal{A}_+^{-1}.$$

Then we have $A \bullet B = e^{\log A + \log B}$, $A, B \in \mathcal{A}_+^{-1}$.

Just for curiosity: The conclusion in the statement fails to be valid in the type I_2 case! Nontrivial, connection to Uhlhorn's version of Wigner's theorem on quantum mechanical symmetry transformations.