Loewner's theorem in several variables

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Introduction

In this talk, E will denote a Hilbert space; n, k are integers, n denotes dimension of matrices, k denotes number of variables.

- $\mathbb{S}(E)$ denote the space of self-adjoint operators
- $\mathbb{S}_n$ is its finite n-by-n dimensional part
- $\mathbb{P} \subseteq \mathbb{S}$ denotes the cone of invertible positive definite and $\hat{\mathbb{P}}$ the cone of positive semi-definite operators
- $\mathbb{P}_n$ and $\hat{\mathbb{P}}_n$ denote the finite dimensional parts

$\mathbb{S}$ and $\mathbb{P}$ are partially ordered cones with the positive definite order:

$$A \leq B \text{ iff } B - A \text{ is positive semidefinite}$$

Loewner's theorem

Definition

A real function $f : (0, \infty) \mapsto \mathbb{R}$ is operator monotone, if $A \leq B$ implies $f(A) \leq f(B)$ for $A, B \in \mathbb{P}(E)$ and all E.

Theorem (Loewner 1934)

A real function $f : (0, \infty) \mapsto \mathbb{R}$ is operator monotone if and only if

$$f(x) = \alpha + \beta x + \int_0^\infty \frac{\lambda}{\lambda^2 + 1} - \frac{1}{\lambda + x} d\mu(\lambda),$$

where $\alpha \in \mathbb{R}$, $\beta \geq 0$ and μ is a unique positive measure on $[0, \infty)$ such that $\int_0^\infty \frac{1}{\lambda^2+1} d\mu(\lambda) < \infty$; if and only if it has an analytic continuation to the open upper complex half-plane $\mathbb{H}^+$, mapping $\mathbb{H}^+$ to $\mathbb{H}^+$.

Some real operator monotone functions on $\mathbb{P}$:

- x^t for $t \in [0, 1]$;
- $\log x$;
- $\frac{x-1}{\log x}$.

Theorem (a variant of Loewner's theorem)

A real function $f : (0, \infty) \mapsto [0, \infty)$ is operator monotone if and only if

$$f(x) = \alpha + \beta x + \int_0^\infty \frac{x(1+\lambda)}{\lambda + x} d\mu(\lambda),$$

where $\alpha, \beta \geq 0$ and μ is a unique positive measure on $(0, \infty)$.

Many different proofs of Loewner's theorem exists:

- Bendat-Sherman '55, Hansen '13, Hansen-Pedersen '82, Korányi-Nagy '58, Sparr '90, Wigner-von Neumann '54, ...
- According to Barry Simon, the *hard part* of Loewner's theorem is to obtain the analytic continuation.

Operator connections & means

Definition (Kubo-Ando connection)

A two-variable function M: $\mathbb{P} \times \mathbb{P} \mapsto \mathbb{P}$ is called an operator connection if

1. if $A \leq A'$ and $B \leq B'$, then $M(A, B) \leq M(A', B')$,
2. $CM(A, B)C \leq M(CAC, CBC)$ for all Hermitian C,
3. if $A_n \downarrow A$ and $B_n \downarrow B$ then $M(A_n, B_n) \downarrow M(A, B)$,

where $\downarrow$ denotes the convergence in the strong operator topology of a monotone decreasing net.

Theorem (Kubo-Ando 1980)

An $M : \mathbb{P}^2 \mapsto \mathbb{P}$ is an operator connection if and only if $M(A, B) = A^{1/2} f\left(A^{-1/2} B A^{-1/2}\right) A^{1/2}$ where $f : (0, \infty) \mapsto [0, \infty)$ is a real operator monotone function.

Operator connections & means: examples

Some operator connections on $\mathbb{P}^2$:

- Arithmetic mean: $\frac{A+B}{2}$
- Parallel sum: $A : B = \left(A^{-1} + B^{-1}\right)^{-1}$
- Geometric mean: $A \#_t B = A^{1/2}\left(A^{-1/2} B A^{-1/2}\right)^{t} A^{1/2}$ for $t \in [0, 1]$

The proof of Kubo-Ando's result relies on the original Loewner theorem.

Our main question:

What happens if we have multiple variables in general?

Free functions

Definition (Free function)

A several variable function $F : D(E) \mapsto \mathbb{S}(E)$ for a domain $D(E) \subseteq \mathbb{S}(E)^k$ defined for all Hilbert spaces E is called a *free* or *noncommutative function* (NC function) if for all E and all $A, B \in D(E) \subseteq \mathbb{S}(E)^k$

(1) $F(U^* A_1 U, \dots, U^* A_k U) = U^* F(A_1, \dots, A_k) U$ for all unitary $U^{-1} = U^* \in \mathcal{B}(E)$,

(2) $F\left(\begin{bmatrix} A_1 & 0 \\ 0 & B_1 \end{bmatrix}, \dots, \begin{bmatrix} A_k & 0 \\ 0 & B_k \end{bmatrix}\right) = \begin{bmatrix} F(A_1, \dots, A_k) & 0 \\ 0 & F(B_1, \dots, B_k) \end{bmatrix}$.

It follows: the domain $D(E)$ is closed under direct sums and element-wise unitary conjugation, i.e. $D = (D(E))$ is a *free set*.

Operator monotone, concave functions

Definition (Operator monotonicity)

An free function $F : \mathbb{P}^k \mapsto \mathbb{P}$ is operator monotone if for all $X, Y \in \mathbb{P}(E)^k$ s.t. $X \leq Y$, that is $\forall i \in \{1, \ldots, k\} : X_i \leq Y_i$, we have

$$F(X) \leq F(Y).$$

If this property is verified only (hence up to) $\dim(E) = n$, then F is n-monotone. Example: Karcher mean, ALM, BMP, etc.

Definition (Operator concavity & convexity)

A free function $F : \mathbb{P}^k \mapsto \mathbb{P}$ is operator concave if for all $X, Y \in \mathbb{P}(E)^k$ and $\lambda \in [0, 1]$, we have

$$(1 - \lambda)F(X) + \lambda F(Y) \leq F((1 - \lambda)X + \lambda Y)$$

Similarly we define n-concavity.

Operator monotone, concave functions: examples

- Karcher mean $\Lambda(A)$: for $A \in \mathbb{P}(E)^k$, $\Lambda(A)$ is the unique positive definite solution of $\sum_{i=1}^{k} \log(X^{-1}A_i) = 0$, if $\dim(E) < \infty$, then $\Lambda(A) = \arg\min_{X \in \mathbb{P}(E)} \sum_{i=1}^{k} d^2(X, A_i)$, where $d^2(X, Y) = \operatorname{tr}\{\log^2(X^{-1/2}YX^{-1/2})\}$
- Lambda-operator means $\Lambda_f(A)$: the unique positive definite solution of $\sum_{i=1}^{k} f(X^{-1}A_i) = 0$ for $A \in \mathbb{P}(E)^k$ and an operator monotone function $f : (0, \infty) \mapsto \mathbb{R}$, $f(1) = 0$.
- Matrix power means $P_t(A)$: for $A \in \mathbb{P}(E)^k$ and $t \in [0, 1]$, $P_t(A)$ is the unique positive definite solution of $\sum_{i=1}^{k} \frac{1}{k} X \#_t A_i = X$
- Inductive mean: $S(A) := \left(\cdots (A_1 \#_{1/2} A_2) \#_{1/3} \cdots \right) \#_{1/k} A_k$

Recent multivariable results

- For an operator convex free function $F : \mathbb{S}^k \mapsto \mathbb{S}$ that is rational - hence already free analytic and defined for general tuples of operators by virtue of non-commutative power series expansion - Helton, McCullogh and Vinnikov in 2006 proved a representation formula, that is superficially similar to our formula that we will obtain here later in full generality.
- For an operator monotone free function $F : \mathbb{S}^k \mapsto \mathbb{S}$ Agler, McCarthy and Young in 2012 proved a representation formula valid for commutative tuples of operators, assuming that F as a multivariable real function is continuously differentable. Using the formula they obtained the analytic continuation of the restricted F to $(H^+)^k$ mapping $(H^+)^k$ to H^+.

Recent multivariable results

- In 2013 Pascoe and Tully-Doyle proved that a free function $F : \mathbb{S}^k \mapsto \mathbb{S}$ that is free analytic, i.e. has a non-commutative power series expansion, thus already defined for general tuples of operators, is operator monotone if and only if it maps the upper operator poly-halfspace $\Pi(E)^k$ to $\Pi(E)$ for all finite dimensional E, where $\Pi(E) := \{X \in \mathcal{B}(E) : \frac{X - X^*}{2i} > 0\}$.

Our goal is to obtain a result that is valid *without* any additional assumptions, by establishing the *hard part* of Loewner's theorem, thus providing a full generalization.

Proposition

A concave free function $F : \mathbb{P}^k \mapsto \mathbb{S}$ which is locally bounded from below, is continuous in the norm topology.

Proposition (Hansen type theorem)

Let $F : \mathbb{P}^k \mapsto \mathbb{S}$ be a $2n$-monotone free function. Then F is n-concave, moreover it is norm continuous.

Corollary

An operator monotone free function $F : \mathbb{P}^k \mapsto \mathbb{S}$ is operator concave and norm continuous, moreover it is strong operator continuous on order bounded sets over separable Hilbert spaces E.

The reverse implication is also true if F is bounded from below:

Theorem

Let $F : \mathbb{P}^k \mapsto \mathbb{P}$ be operator concave (n-concave) free function. Then F is operator monotone (n-monotone).

Supporting linear pencils and hypographs

Definition (Matrix/Freely convex sets of Wittstock)

A graded set $C = (C(E))$, where each $C(E) \subseteq \mathbb{S}(E)^k$, is a bounded open/closed *matrix convex* or *freely convex* set if

(i) each $C(E)$ is open/closed;

(ii) C respects direct sums, i.e. if $(X_1, \ldots, X_k) \in C(N)$ and $(Y_1, \ldots, Y_k) \in C(K)$ and $Z_j := X_j \oplus Y_j$, then $(Z_1, \ldots, Z_k) \in C(N \oplus K)$;

(iii) C respects conjugation with isometries, i.e. if $Y \in C(N)$ and $T : K \mapsto N$ is an isometry, then $T^* YT = (T^* Y_1 T, \ldots, T^* Y_k T) \in C(K)$;

(iv) each $C(E)$ is bounded.

The above definition has some equivalent characterizations under slight additional assumptions.

Definition

A graded set $C = (C(E))$, where each $C(E) \subseteq \mathbb{S}(E)^k$, is *closed with respect to reducing subspaces* if for any tuple of operators $(X_1, \ldots, X_k) \in C(E)$ and any corresponding mutually invariant closed subspace $K \subseteq E$, the restricted tuple $(\hat{X}_1, \ldots, \hat{X}_k) \in C(K)$, where each $\hat{X}_i$ is the restriction of X_i to the invariant subspace K for all $1 \leq i \leq k$.

Lemma (Helton, McCullogh 2004)

Suppose that $C = (C(E))$ is a free set, where each $C(E) \subseteq \mathbb{S}(E)^k$, i.e. respects direct sums and unitary conjugation. Then:

(1) *If C is closed with respect to reducing subspaces then C is matrix convex if and only if each $C(E)$ is convex in the usual sense of taking scalar convex combinations.*

(2) *If C is (nonempty and) matrix convex, then $0 = (0, \ldots, 0) \in C(\mathbb{C})$ if and only if C is closed with respect to simultaneous conjugation by contractions.*

Given a set $A \subseteq \mathbb{S}(E)$ we define its *saturation* as

$$\mathrm{sat}(A) := \{X \in \mathbb{S}(E) : \exists Y \in A, Y \geq X\}.$$

Similarly for a graded set $C = (C(E))$, where each $C(E) \subseteq \mathbb{S}(E)$, its *saturation* $\mathrm{sat}(C)$ is the disjoint union of $\mathrm{sat}(C(E))$ for each E.

Definition (Hypographs)

Let $F : \mathbb{P}^k \mapsto \mathbb{S}$ be a free function. Then we define its *hypograph* $\mathrm{hypo}(F)$ as the graded union of the saturation of its image, i.e.

$$\mathrm{hypo}(F) = (\mathrm{hypo}(F)(E)) := (\{(Y, X) \in \mathbb{S}(E) \times \mathbb{P}(E)^k : Y \leq F(X)\}).$$

Characterization of operator concavity

Theorem

Let $F : \mathbb{P}^k \mapsto \mathbb{S}$ *be a free function. Then its hypograph* $\mathrm{hypo}(F)$ *is a matrix convex set if and only if* F *is operator concave.*

Corollary

Let $F : \mathbb{P}^k \mapsto \mathbb{S}$ *be a free function. Then its hypograph* $\mathrm{hypo}(F)$ *is a matrix convex set if and only if* F *is operator monotone.*

Linear pencils

Definition (linear pencil)

A *linear pencil* for $x \in \mathbb{C}^k$ is an expression of the form

$$L_A(x) := A_0 + A_1 x_1 + \cdots + A_k x_k$$

where each $A_i \in \mathbb{S}(K)$ and $\dim(K)$ is the *size* of the pencil L_A. The pencil is *monic* if $A_0 = I$ and then L_A is a *monic linear pencil*. We extend the evaluation of L_A from scalars to operators by tensor multiplication. In particular L_A evaluates at a tuple $X \in \mathbb{S}(N)^k$ as

$$L_A(X) := A_0 \otimes I_N + A_1 \otimes X_1 + \cdots + A_k \otimes X_k.$$

We then regard $L_A(X)$ as a self-adjoint element of $\mathbb{S}(K \otimes N)$ and L_A becomes a free function.

Representation of supporting linear functionals

Suppose $C = (C(E)) \subseteq \mathbb{S}(E)^k$ is a norm closed matrix convex set that is closed with respect to reducing subspaces and $0 \in C(\mathbb{C})$. Then for each boundary point $A \in C(N)$ where $\dim(N) < \infty$, by the Hahn-Banach theorem there exists a continuous supporting linear functional $\Lambda \in (\mathbb{S}(N)^k)^*$ s.t.

$$\Lambda(C(N)) \leq 1 \text{ and } \Lambda(A) = 1$$

and since $\mathbb{S}(N)^* \simeq \mathbb{S}(N)$ we have that for all $X \in \mathbb{S}(N)^k$

$$\Lambda(X) = \sum_{i=1}^{k} \mathrm{tr}\{B_i X_i\}$$

for some $B_i \in \mathbb{S}(N)$.

Representation of supporting linear functionals

Proposition

Let $F : \mathbb{P}^k \mapsto \mathbb{P}$ *be an operator monotone function and let* N *be a Hilbert space with* $\dim(N) < \infty$. *Then for each* $A \in \mathbb{P}(N)^k$ *and each unit vector* $v \in N$ *there exists a linear pencil*

$$L_{F,A,v}(Y,X) := B(F,A,v)_0 \otimes I - vv^* \otimes Y + \sum_{i=1}^{k} B(F,A,v)_i \otimes (X_i - I)$$

of size $\dim(N)$ *which satisfies the following properties:*

(1) $B(F,A,v)_i \in \hat{\mathbb{P}}(N)$ *and* $\sum_{i=1}^{k} B(F,A,v)_i \leq B(F,A,v)_0$;

(2) *For all* $(Y,X) \in \mathrm{hypo}(F)$ *we have* $L_{F,A,v}(Y,X) \geq 0$;

(3) *If* $c_1 I \leq A_i \leq c_2 I$ *for all* $1 \leq i \leq k$ *and some fixed real constants* $c_2 > c_1 > 0$, *then* $\mathrm{tr}\{B(F,A,v)_0\} \leq \frac{F(c_2,\ldots,c_2)}{\min(1,c_1)}$.

Explicit LMI solution formula

Theorem

Let $F : \mathbb{P}^k \mapsto \mathbb{P}$ be an operator monotone function. Then for each $A \in \mathbb{P}(N)^k$ with $\dim(N) < \infty$ and each unit vector $v \in N$

$$
\begin{aligned}
F(A)v = {} & v^* B_{0,11}(F,A,v)v \otimes Iv + \sum_{i=1}^{k} v^* B_{i,11}(F,A,v)v \otimes (A_i - I)v \\
& - \left\{ (v^* \otimes I) \left[B_{0,12}(F,A,v) \otimes I + \sum_{i=1}^{k} B_{i,12}(F,A,v) \otimes (A_i - I) \right] \right. \\
& \times \left[B_{0,22}(F,A,v) \otimes I + \sum_{i=1}^{k} B_{i,22}(A,v) \otimes (A_i - I) \right]^{-1} \\
& \times \left. \left[B_{0,21}(F,A,v) \otimes I + \sum_{i=1}^{k} B_{i,21}(F,A,v) \otimes (A_i - I) \right] (v \otimes I) \right\} v
\end{aligned}
$$

and

$$\begin{aligned}
&\left\{\left[\overline{B}_{0,22}(A,v)\otimes I+\sum_{i=1}^{k}B_{i,22}(A,v)\otimes A_i\right]\right.\\
&\left.-\left[\overline{B}_{0,21}(A,v)\otimes I+\sum_{i=1}^{k}B_{i,21}(A,v)\otimes A_i\right]\right\}(v^*\otimes v)\\
&=\sum_{j\in\mathcal{I}}\left[\overline{B}_{0,22}(A,v)\otimes I+\sum_{i=1}^{k}B_{i,22}(A,v)\otimes A_i\right](e_j^*\otimes e_j),
\end{aligned}$$

where $\{e_j\}_{j\in\mathcal{J}}$ is an orthonormal basis of N and

$$\begin{aligned}
B_{i,11}(F,A,v) &:= vv^*B_i(F,A,v)vv^*,\\
B_{i,12}(F,A,v) &:= vv^*B_i(F,A,v)(I-vv^*),\\
B_{i,21}(F,A,v) &:= (I-vv^*)B_i(F,A,v)vv^*,\\
B_{i,22}(F,A,v) &:= (I-vv^*)B_i(F,A,v)(I-vv^*)
\end{aligned}$$

for all $0 \leq i \leq k$ and $x, y \in \{1, 2\}$.
Moreover if $c_1 I \leq A_i \leq c_2 I$ for all $1 \leq i \leq k$ and some fixed real constants $c_2 > c_1 > 0$, then

$$\mathrm{tr}\{B_0(A, v)\} \leq \frac{F(c_2, \ldots, c_2)}{\min(1, c_1)}.$$

Definition (Natural map)

A graded map $F : \mathbb{S}(K)^k \times K \mapsto K$ defined for all Hilbert space K is called a *natural map* if it preserves direct sums, i.e.

$$F(X \oplus Y, v \oplus w) = F(X, v) \oplus F(Y, w)$$

for $X \in \mathbb{S}(K_1)^k$, $v \in K_1$ and $Y \in \mathbb{S}(K_2)^k$, $w \in K_2$.

For a free function $F : \mathbb{S}^k \mapsto \mathbb{S}$ we define the natural map $\overline{F} : \mathbb{S}(K)^k \times K \mapsto K$ for any K by

$$\overline{F}(X, v) := F(X)v$$

for $X \in \mathbb{S}(K)^k$ and $v \in K$.
The function below is free, hence induces a natural map:

$$\begin{aligned} F(X) :=& v^* B_{0,11} v \otimes I + \sum_{i=1}^{k} v^* B_{i,11} v \otimes X_i \\ & - (v^* \otimes I) \left[B_{0,12} \otimes I + \sum_{i=1}^{k} B_{i,12} \otimes X_i \right] \\ & \times \left[B_{0,22} \otimes I + \sum_{i=1}^{k} B_{i,22} \otimes X_i \right]^{-1} \\ & \times \left[B_{0,21} \otimes I + \sum_{i=1}^{k} B_{i,21} \otimes X_i \right] (v \otimes I). \end{aligned}$$

Let $S(E) := \{v \in E : \|v\| = 1\}$ denote the unit sphere of the Hilbert space E. For fixed real constants $c_2 > c_1 > 0$, let

$$\begin{aligned} \mathbb{P}_{c_1,c_2}(E) &:= \{X \in \mathbb{P}(E) : c_1 I \leq X \leq c_2 I\}, \\ \Omega_{c_1,c_2} &:= \mathbb{P}_{c_1,c_2}(E)^k \times S(E) \end{aligned}$$

and let

$$\mathcal{H} := \bigoplus_{\dim(E)<\infty} \bigoplus_{\omega \in \Omega_{c_1,c_2}} E.$$

We equip $\mathcal{H}$ with the inner product

$$x^* y := \sum_{\dim(E)<\infty} \sum_{\omega \in \Omega_{c_1,c_2}} x(\omega)^* y(\omega).$$

Let $\mathcal{B}^+(\mathcal{H})^*$ denote the state space of $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}^+(\mathcal{H})_*$ is the normal part.

Definition

Let $F : \mathbb{P}^k \mapsto \mathbb{P}$ be an operator monotone function. Now let

$$\begin{aligned}\Psi_F(X) :=& B_{0,11} \otimes I + \sum_{i=1}^{k} B_{i,11} \otimes (X_i - I) \\ & - \left[B_{0,12} \otimes I + \sum_{i=1}^{k} B_{i,12} \otimes (X_i - I) \right] \\ & \times \left[B_{0,22} \otimes I + \sum_{i=1}^{k} B_{i,22} \otimes (X_i - I) \right]^{-1} \\ & \times \left[B_{0,21} \otimes I + \sum_{i=1}^{k} B_{i,21} \otimes (X_i - I) \right]\end{aligned}$$

where

$$B_{0,xy} := \bigoplus_{\dim(E)<\infty} \bigoplus_{(A,v)\in\Omega_{c_1,c_2}} B_{0,xy}(F,A,v),$$

$$B_{i,xy} := \bigoplus_{\dim(E)<\infty} \bigoplus_{(A,v)\in\Omega_{c_1,c_2}} B_{i,xy}(F,A,v)$$

for $1 \leq i \leq k$ and $x, y \in \{1, 2\}$.

Lemma

Let $F : \mathbb{P}^k \mapsto \mathbb{P}$ be an operator monotone function and let $\dim(E) < \infty$. Let $A_j \in \mathbb{P}_{c_1,c_2}(E)^k$ and $v_j \in S(E)$ for $j \in \mathcal{J}$ for some finite index set $\mathcal{J}$. Then there exists a $w \in S(\mathcal{H})$ such that

$$F(A_j)v_j = (w^* \otimes I)\Psi_F(A_j)(w \otimes I)v_j$$

for all $j \in \mathcal{J}$.

Theorem (Multivariable Loewner's theorem)

Let $F : \mathbb{P}^k \mapsto \mathbb{P}$ be an operator monotone function. Then there exists a state $\omega \in \mathcal{B}_1^+(\mathcal{H})^*$ such that for all $\dim(E) < \infty$ *and $X \in \mathbb{P}(E)^k$ we have*

$$
\begin{aligned}
F(X) =& (\omega \otimes I)(\Psi_F(X)) = \omega(B_{0,11}) \otimes I + \sum_{i=1}^{k} \omega(B_{i,11}) \otimes (X_i - I) \\
&- (\omega \otimes I) \Bigg\{ \left[B_{0,12} \otimes I + \sum_{i=1}^{k} B_{i,12} \otimes (X_i - I) \right] \\
&\times \left[B_{0,22} \otimes I + \sum_{i=1}^{k} B_{i,22} \otimes (X_i - I) \right]^{-1} \\
&\times \left[B_{0,21} \otimes I + \sum_{i=1}^{k} B_{i,21} \otimes (X_i - I) \right] \Bigg\} .
\end{aligned}
$$

The upper operator half-space $\Pi(E)$ consists of $X \in \mathcal{B}(E)$ s.t. $\Im X := \frac{X-X^*}{2i} > 0$.

Theorem (Multivariable Loewner's theorem cont.)

Let $F : \mathbb{P}^k \mapsto \mathbb{P}$ be a free function. Then the following are equivalent

(1) *F is operator monotone;*

(2) *F is operator concave;*

(3) *F is a conditional expectation of the Schur complement of a linear pencil $L_B(X) := B_0 \otimes I + \sum_{i=1}^k B_i \otimes (X_i - I)$ over some auxiliary Hilbert space $\mathcal{H}$ with $B_i \in \hat{\mathbb{P}}(\mathcal{H})$, $B_0 \geq \sum_{i=1}^k B_i$;*

(4) *F admits a free analytic continuation to the upper operator poly-halfspace $\Pi(E)^k$, mapping $\Pi(E)^k$ to $\Pi(E)$ for all E.*

Further results

Let $\mathcal{L}$ be a fixed Hilbert space.

Definition (Free function relaxed)

A several variable function $F : D(E) \mapsto \mathbb{S}(\mathcal{L} \otimes E)$ for a domain $D(E) \subseteq \mathbb{S}(E)^k$ defined for all Hilbert spaces E is called a *free* if for all E and all $A, B \in D(E) \subseteq \mathbb{S}(E)^k$

(1) $F(U^*A_1U, \ldots, U^*A_kU) = (I \otimes U^*)F(A_1, \ldots, A_k)(I \otimes U)$ for all unitary $U^{-1} = U^* \in \mathcal{B}(E)$,

(2) $F\left(\begin{bmatrix} A_1 & 0 \\ 0 & B_1 \end{bmatrix}, \ldots, \begin{bmatrix} A_k & 0 \\ 0 & B_k \end{bmatrix}\right) = \begin{bmatrix} F(A_1, \ldots, A_k) & 0 \\ 0 & F(B_1, \ldots, B_k) \end{bmatrix}$.

We may define operator monotonicity of F in the same way: $A \leq B$ implies $F(A) \leq F(B)$.

Theorem (Multivariable Loewner's theorem II)

Let $F : \mathbb{P}(E)^k \mapsto \mathbb{P}(\mathcal{L} \otimes E)$ be an operator monotone function. Then there exists a completely positive $\omega : \mathcal{B}(\mathcal{H}) \mapsto \mathcal{B}(\mathcal{L})$ such that for all $\dim(E) < \infty$ and $X \in \mathbb{P}(E)^k$ we have

$$\begin{aligned}
F(X) =& (\omega \otimes I)(\Psi_F(X)) = \omega(B_{0,11}) \otimes I + \sum_{i=1}^{k} \omega(B_{i,11}) \otimes (X_i - I) \\
& - (\omega \otimes I) \Bigg\{ \left[B_{0,12} \otimes I + \sum_{i=1}^{k} B_{i,12} \otimes (X_i - I) \right] \\
& \times \left[B_{0,22} \otimes I + \sum_{i=1}^{k} B_{i,22} \otimes (X_i - I) \right]^{-1} \\
& \times \left[B_{0,21} \otimes I + \sum_{i=1}^{k} B_{i,21} \otimes (X_i - I) \right] \Bigg\} .
\end{aligned}$$

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Thank you for your kind attention!