Regular representations of completely bounded maps

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(Joint work with B.V.Rajarama Bhat and K.Sumesh)

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Basic definitions

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- A linear map $\varphi : \mathcal{A} \to \mathcal{B}$ between C^*-algebras is said to be
  - positive if $\varphi(a^*a) \geq 0$ for all $a \in \mathcal{A}$;
  - completely positive (CP) if $\varphi^{(k)} : M_k(\mathcal{A}) \to M_k(\mathcal{B})$ is positive for all $k \in \mathbb{N}$.
  $$\varphi^{(k)}(\begin{bmatrix} a_{ij} \end{bmatrix}) = \begin{bmatrix} \varphi(a_{ij}) \end{bmatrix}$$
  - completely bounded (CB) if $\|\varphi\|_{cb} := \sup_k \left\|\varphi^{(k)}\right\| < \infty$.

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  - completely bounded (CB) if $\|\varphi\|_{cb} := \sup_k \left\|\varphi^{(k)}\right\| < \infty$.
- Suppose E is a complex vector space which is also a right $\mathcal{A}$-module. E is said to be a Hilbert $\mathcal{A}$-module if there exists $\mathcal{A}$-valued inner-product $\langle \cdot, \cdot \rangle$ such that E is complete with respect to the norm $\|x\| := \|\langle x, x \rangle\|^{\frac{1}{2}}, \ x \in E$.
- $\mathcal{B}^a(E)$ denotes the space of bounded adjointable linear maps on E.

Motivation

- Stinespring's representation theorem for CP-maps (1955):

$$\varphi : \mathcal{A} \to \mathcal{B}(\mathcal{H}) \rightsquigarrow \begin{cases} \pi : \mathcal{A} \to \mathcal{B}(\mathcal{K}) \ \text{ unital } *\text{-homomorphism} \\ V \in \mathcal{B}(\mathcal{H}, \mathcal{K}) \\ \text{such that } \varphi(a) = V^* \pi(a) V. \end{cases}$$

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- **Idea:**
  - In Stinespring's representation theorem one requires $*$-homomorphisms to represent CP-maps.
  - For similar representation of CB-maps we need to consider homomorphisms which are not necessarily $*$-preserving.

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  $\tau : \mathbb{C} \to \mathcal{B}(\mathbb{C}^2)$ given by $a \mapsto \begin{bmatrix} \frac{a}{2} & \frac{a}{4} \\ a & \frac{a}{2} \end{bmatrix}$ is a J-homomorphism where $J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

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Proposition: Suppose $\tau : \mathcal{A} \to \mathcal{B}$ is a homomorphism. Then there exists a symmetry $J \in \mathcal{B}$ such that $\tau(a)^* = J\tau(a)J$ for all $a \in \mathcal{A}$ satisfying $\tau(a)^*\tau(1) = \tau(1)^*\tau(a)$ and $\tau(a)\tau(1)^* = \tau(1)\tau(a)^*$.

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- Let $v \in \mathcal{B}(\mathcal{H})$ be a non-scalar unitary. Define $\tau : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ by $\tau(a) = \begin{bmatrix} a & \sqrt{3}(va - av) \\ 0 & a \end{bmatrix}$.
  Then τ is a J-homomorphism with symmetry $J = \frac{1}{2}\begin{bmatrix} -1 & \sqrt{3}v \\ \sqrt{3}v^* & 1 \end{bmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$. But not τ is not regular.

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Theorem: Suppose $\tau : \mathcal{A} \to \mathcal{B}^a(E)$ is a $*$-nondegenerate, regular homomorphism. Then there exists a unique unital $*$-homomorphism $\vartheta : \mathcal{A} \to \mathcal{B}^a(E)$ such that

$$\tau(a) = \vartheta(a)\tau(1) = \tau(1)\vartheta(a)$$

for all $a \in \mathcal{A}$. Consequently τ is completely bounded with $\|\tau\|_{cb} = \|\tau(1)\|$.

Normal regular homomorphism

Theorem:

$$\tau : \mathcal{B}(\mathcal{H}) \xrightarrow[\textit{regular homo.}]{\textit{normal}} \mathcal{B}(\mathcal{K}) \rightsquigarrow \begin{cases} \mathcal{K}_\tau \ \text{ Hilbert space,} \\ T \in \mathcal{B}(\mathcal{K}_\tau) \ \text{ idempotent operator,} \\ V : \mathcal{H} \otimes \mathcal{K}_\tau \to \mathcal{K} \ \text{ isometry} \\ \text{such that } \tau(a) = V(a \otimes T)V^*. \end{cases}$$

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- We can have a similar structure theorem for regular homomorphisms $\tau : \mathcal{B}^a(E) \to \mathcal{B}^a(F)$ if E is a "full" Hilbert C^*-module and τ is continuous w.r.t the "strict" toplology.

Regular representation of CB-maps

Theorem:

$$\psi : \mathcal{A} \stackrel{cb}{\to} \mathcal{B}(\mathcal{H}) \rightsquigarrow \begin{cases} \tau : \mathcal{A} \to \mathcal{B}(\mathcal{K}) & \ast\text{-nondegenerate regular homomorphism} \\ W \in \mathcal{B}(\mathcal{H}, \mathcal{K}) \\ \text{such that } \psi(a) = W^* \tau(a) W. \end{cases}$$

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- In previous theorem, given any $t \in (1, \infty)$ we can choose τ and W such that τ satisfies

$$\tau(a)\tau(b)^*\tau(c) = t\tau(ab^*c) \quad \forall\ a, b, c \in \mathcal{A} \tag{$\dagger$}$$

and $\|\psi\|_{cb} \leq \sqrt{t}\, \|W\|^2$.

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Theorem:

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- Fundamental theorem for CB-maps:

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Other known representations of CB-maps

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- Suppose $(\mathcal{K}, \pi, T, V)$ is a commutant representation of a CB-map $\psi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$. Set $\widehat{\mathcal{K}} = \mathcal{K} \oplus \mathcal{K}$. Define $\tau : \mathcal{A} \to \mathcal{B}(\widehat{\mathcal{K}})$ by

$$\tau(a) = \begin{bmatrix} \pi(a) & (2T - I)\pi(a) \\ 0 & 0 \end{bmatrix} \text{ and set } W = \begin{bmatrix} \frac{V}{\sqrt{2}} \\ \frac{V}{\sqrt{2}} \end{bmatrix} \in \mathcal{B}(\mathcal{H}, \widehat{\mathcal{K}}).$$

  Then τ is a regular homomorphism and W is an isometry such that $\psi(\cdot) = W^* \tau(\cdot) W$

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- Suppose $(\mathcal{K}, \pi, T, V)$ is a commutant representation of a CB-map $\psi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$. Set $\widehat{\mathcal{K}} = \mathcal{K} \oplus \mathcal{K}$. Define $\tau : \mathcal{A} \to \mathcal{B}(\widehat{\mathcal{K}})$ by

$$\tau(a) = \begin{bmatrix} \pi(a) & (2T - I)\pi(a) \\ 0 & 0 \end{bmatrix} \text{ and set } W = \begin{bmatrix} \frac{V}{\sqrt{2}} \\ \frac{V}{\sqrt{2}} \end{bmatrix} \in \mathcal{B}(\mathcal{H}, \widehat{\mathcal{K}}).$$

  Then τ is a regular homomorphism and W is an isometry such that $\psi(\cdot) = W^*\tau(\cdot)W$
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- These relationships between various representations show the richness of the subject.
- For a given CB-map we may choose the representation of our liking.

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THANK YOU