CP-Semigroups and Subproduct Systems, Dilations and Superproduct Systems

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Joint work with Orr Shalit (since 2009 ☹☺)

Remember from Orr's talk:

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CAREFUL! Monoids of semigroups and (sub)(super)PS are opposite!

- Implications
  - Sub not into super $\Rightarrow$ no dilation. (Example!)
  - $\mathcal{B}^a(E) = \mathcal{A}$ has to do with one of the (stronger and not necessarily achievable) minimality conditions: $E = \mathcal{A}p \rightsquigarrow \mathcal{B}^a(E) = \overline{\operatorname{span}}^s \mathcal{A}p\mathcal{A}$. ($\xi = p \in pA = E$.)
  - We will try to embed subPS into PS. ([BS00] in the one-parameter case
- We choose $\mathbb{S}^{op}$ for the semigroups because:
  - convenient as we work more in PS;
  - Ore $S^{-1}S = G$, necessary for certain constructive results. To make p increasing, the semigroup indexing T should be anti-Ore.
  - In the end, no problems in the abelian case, like $\mathbb{R}_+^d, \mathbb{N}_0^d$.
  - Also: Arveson-Stinespring (Daniel's talk) are contravariant with the subPS and, thus, anti-Ore.
- In the sequel:
  - Explain (repetition) the preceding slide.
  - Go into the construction of product systems (for CP-semigroups and not) in the multi-parameter cases.
  - Use this to solve discrete 2-parameter and procude a 1000 c.-ex.s.

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Example (Shalit-ms)

The exist $\mathcal{B}$ and a semigroup (c_n) in $\mathcal{B}$ over $\mathbb{N}_0$ such that the GNS-subPS of $T_t = c_t^* \bullet c_t$ does have non-adjointable $w_{s,t}$.

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- We don't know if there is a CP-semigroup that admits a weak dilation but no strong.

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Henceforth, we restrict our attention to strong dilation and mainly to the unital case. (Exception: Next slide.)

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Suppose we have PS $E^{i\odot} = \left(E^i_{t^i}\right)$ over $\mathbb{S}^i$, $i = 1, \ldots, d$.

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For Hilbert spaces: Yes, with the flip.

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What is remarkable? Conditions **empty** for $d = 2$!

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Both is established by using the $\mathfrak{F}_{j,i}$ to define a representation of S_n on $(E_1 \oplus \ldots \oplus E_d)^{\odot n}$ containing all the preceding tensor products with n factors E_i as subspaces and by the observation that the permutation doing the right thing to these subspaces is **unique** and it it does not matter in which order we obtain it from transpositions.

Dilations of CP-semigroups over $\mathbb{N}_0^2$

Corollary

Suppose we have $\xi_i \in E_i$ *fulfilling* $\mathcal{F}_{j,i}(\xi_i \odot \xi_j) = \xi_j \odot \xi_i$. *Then the the elements* $\xi_{(n_1,\dots,n_d)} := \xi_1^{\odot n_1} \odot \dots \odot \xi_d^{\odot n_d}$ *form a unit.*

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An THIS leads to a whole bunch of ....

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  - The minimalized version on $\mathcal{A}_\infty$ is not of the form (E', ϑ', ξ'). That is, it is not Arveson minimal.

About the case $\mathbb{R}_+^d$: Exponentiating PS over $\mathbb{N}_0^d$

(Apart from our general results about products of d.)

- The basic example of a PS over $\mathbb{R}_+$ is the time ordered system:

$$\Gamma_t(F) \;:=\; \omega\mathcal{B} \oplus \bigoplus_{n\in\mathbb{N}} \Delta_n L^2([0,t)^n, F^{\odot n})$$

where Δ_n is the indicator function of $\{t_n > \ldots > t_1 > 0\}$.

Theorem (Shalit-ms)

Let $F^{\odot}$ be PS $\mathbb{N}_0^d$ with marginals $\left(F_i^{\odot n}\right)_{n\in\mathbb{N}_0}$. Then

$$E_{(t_1,\ldots,t_d)} := \Gamma_{t_1}(F_1) \odot \ldots \odot \Gamma_{t_d}(F_d)$$

inherits the structure of a PS over $\mathbb{R}_+^d$. Moreover, units lift to (exponential) units.

Thank you!

Bibliography I

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