# SPECTRAL ANALYSIS OF SEMIBOUNDED OPERATORS BY TRUNCATION

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# The scheme of presentation

- Introduction
- Preliminary Results
- Szego-type Theorem For Unbounded Selfadjoint Operators
- Arveson's Query
- Truncation method for semibounded operators.

## Abstract

In this lecture the infinite dimentional numerical linear algebraic method of William .B.Arveson is extended to semibounded, unbounded operators on complex separable Hilbert spaces.The idea is to apply the truncation method for bounded self- adjoint operators to a suitableÂă self-adjoint resolvent of the operator.Sufficient conditions are obtained for minimising computational difficulties that arise from inversion.An answer to 'Arveson's query' regarding distinguishability of transient points which are not in the spectrum is also given in one of the sections.. The techniques donot involve pseudospectral analysis via Hausdorff metric. This lecture reports a joint work with Prof.Kalyan B.Sinha,J.N. Centre for Advanced Scientific Research, Bangalore , India

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We start with the defenition of a semibounded operator.

**Definition**

A positive defenite linear operator **T** in **H** with dense domain **D(T)** is called *semibounded*, if there exists $\lambda \succ 0$ such that $< \mathbf{T}x, x > \geq \lambda < x, x >$ for all x in **D(T)**

Let A be a semibounded operator in **H** with domain **D(T)**. Let $\mathbf{H}_n$ be a sequence of $d_n$ dimentional subspaces of **D(A)** such that $\bigcup \mathbf{H}_n$ is dense in **H** and let $\mathbf{P}_n$ denote the orthogonal projection of **H** onto $\mathbf{H}_n$.If $\{e_n : n = 0, \pm 1, \pm 2, \ldots\}$ or $\{e_n : n = 0, 1, 2 \ldots\}$ are orthonormal bases in **H**, then one can take $\mathbf{H}_n = span\{e_k : k = 0, \pm 1, \pm 2, \ldots, \pm n\}$. or $\mathbf{H}_n$= span$\{e_k : k = 0, 1, 2, \ldots, n\}$. accordingly. Let $A_n = P_n A|_{\mathbf{H}_n}$ for each$n \geq 1$.

### Definition

A point $\lambda$ in $\mathbf{R}$ is called *essential* if for every neighbourhood $\mathbf{U}$ of $\lambda$,we have $\lim_{n\to\infty} N_n(\mathbf{U}) = \infty$ and *transient* if there exists a neighbourhood $\mathbf{U}$ of $\lambda$ such that $\lim_{n\to\infty} N_n(\mathbf{U}) < \infty$.The set of all essential points will be denoted by $\Lambda_e$ and the set of transient poits will be denoted by $\Lambda_t$.

As remarked in [6],one can easily see that $\lambda$ in $\mathbf{R}$ is nonessentail iff there is an open neighbourhood $U$of $\lambda$ and an infinite subsequence $\{n_k\}$ such that $N_{n_k}(\mathbf{U}) \leq \ M < \infty$ for every $k = 1,\ 2,\ 3,\ ....$. The following theorem is analogous to Theorem 2.3[6].

### Theorem

*Assume that the sequence* $A_1\ \ A_2\ \ A_3$ *.., arises from the operator A as before satisfy the additional condition that* $\lim_{n\to\infty} A_n x = A(x)$ *for all* $x$ *in the domain* $D(A)$ *of A.Then we have* $\sigma(A) \subseteq \Lambda$ *and* $\sigma_e(A) \subseteq \Lambda_e(A)$

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In [7] Arveson proves Szego-type theorem for bounded selfadjoint operators on seperable Hilbert spaces. It is well known that Szego-type results are very useful for localisation of spectrum.In this section similar results are proved for selfadjoint operators which are not necessarily bounded.

**Definition**

Let $A$ be a selfadjoint opearator on a seperable Hilbert space $\mathcal{H}$.Then a sequence $\{P_n\}$ of orthogonal projections on $\mathcal{H}$ is said to be *adapted* to $\mathcal{A}$ if it converges strongly to the identity operator $I$, increasing, is of finite rank for each $n$ and $P_n(\mathcal{H}) \subseteq Domain(\mathcal{A})$ for each $n$.

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Now we define degree of an operator as in [7], but slightly different,as follows.

**Definition**

Let $A$ be a densely defined operator on a seperable Hilbertspace $\mathcal{H}$ and $\{P_n\}$ be a sequence of projections adapted to it. Then the degree of $A$ is defined as
$\tilde{deg}(A) = \sup rank(P_n(A)(I - P_n); n = 1, 2, 3, ..., )$

**Definition**

A selfadjoint operator $A$ adapted to $(P_n)$ as in the above defenition.Then $A$ is said to be *quasibounded* with respect to $(P_n)$ if

$$q(A) = sup\|[A, P_n]P_n\| : n = 1, 2, 3, ... < \infty \quad (1)$$

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$$q(A) = sup\|[A, P_n]P_n\| : n = 1, 2, 3, ... < \infty \tag{1}$$

Through this section the operator$A$ is assumed to be *quasibounded* with quasibound $q(A)$. Now we list a few properties of the above notion in what follows.
(1) Recall that for Arveson degree of $A$ i,denoted by $degA$ is ,

$$degA = suprank([P_n, A], n = 1, 2, 3....) \tag{2}$$

. It is easy to see that ,by orthogonality of ranges, $degA$ is twice $\tilde{deg}(A)$ when $A$ is selfadjoint.
(2) If $A$ is symmetric,then

$$\tilde{deg}(A) \leq \tilde{deg}(A^*) \tag{3}$$

(3) If $\{P_n\}$ is adapted to $A$ and $B$ then

$$\tilde{deg}(A+B) \leq \tilde{deg}(A) + \tilde{deg}(B) \tag{4}$$

For all nonzero scalars $\lambda$

$$\tilde{deg}(\lambda A) = \tilde{deg}(A) \tag{5}$$

(5) If $\mathcal{B}$ is bounded then

$$\tilde{deg}(BA) \leq \tilde{deg}(A) + \tilde{deg}(B) \tag{6}$$

Now we prove the main theorem of this section.

**Theorem**

*Let $P_n$ be adapted to a selfadjoint operator A and $\mathcal{A}$ be the Von-Neumann algebra affiliated to itthat admits a unique tracial state . For each Borel set $\triangle$ in $\mathcal{R}$,let $E_{\mathcal{A}}(\triangle)$ denote the associated spectral projection.*

$$\mu_A(\triangle) = \Gamma(E_\triangle) \tag{7}$$

*where $\Gamma(.)$ denote the unique tracial state on $\mathcal{A}$.Then$\Gamma$ is a probability measure on $\mathcal{R}$.*

**Theorem**

*For each positive integer n ,let* $A_n = P_n A P_n / \mathcal{H}_n$*,where* $\mathcal{H}_n$*=range*$P_n$*. Let* $\{\lambda_1, \lambda_2, \lambda_3, ...., \lambda_n\}$ *be the eigenvalues of* $A_n$*,repeated according to multiplicity. Then for each f in* $\mathcal{C}_c^1(\mathcal{R})$,

$$\frac{1}{d_n}\Sigma_{j=1}^{d_n} f(\lambda_j) \to \int_{\mathcal{R}} f(\lambda)d\mu \tag{8}$$

*as* $n \to \infty$*,where* $d_n$ *denotes the dimention of* $\mathcal{H}_n$ *for each n. In addition we have*

$$\frac{1}{d_n}\Sigma_{j=1}^{d_n} \exp(i\lambda_j) \to \int_{\mathcal{R}} \exp(it\lambda)d\mu \tag{9}$$

*as* $\quad n \to \infty$

# Proof

For every $T$ in $\mathcal{A}$ let

$$\Gamma_n(T) = \frac{1}{d_n} t_r(P_n(T)). \tag{10}$$

where $t_r(.)$ denotes the usual trace.Now,by Duhamel's formula,p.161,5.1,[10]

$$\begin{aligned}
&\Gamma_n(exp(itA) - \Gamma_n(\exp(itP_n(A)P_n) \\
&= \frac{1}{d_n} t_r[(\exp(itA)P_n - \exp(itP_nAP_n)] \\
&= \frac{1}{d_n} t_r \int_0^t (\exp(i(t-s)A))(AP_n) - (P_nAP_n)(\exp(isP_nAP_n))ds \\
&\leq \frac{|t| \|[A, P_n]P_n\| rank([A, P_n]P_n)}{d_n} \\
&\leq \frac{2|t|q(A)deg(A)}{d_n} \\
&\to 0 \quad as \quad n \to \infty.
\end{aligned}$$

By Banach-Alaglou theorem,$\{\Gamma_n(.)\}$ has limit points and are all tracial states on $\mathcal{A}$. But by assumption,$\Gamma(.)$ is the only tracial state on $\mathcal{A}$. Finally ,let,$f$ belong to $L^1(R)$. Now let, $\tilde{f}(A) = \int_{-\infty}^{\infty} f(t)exp(itA)d\mu(t)$. Now we have

$$\begin{aligned} &\Gamma_n(\tilde{f}(A)) - \Gamma_n(\tilde{f}(P_nAP_n)) \\ &= \frac{1}{d_n} t_r[(\tilde{f}(A)P_n - \tilde{f}(P_nAP_n))] \\ &\leq \frac{2|t|q(A)deg(A)}{d_n}\|f\|_1. \\ &\to 0 \quad as \quad n \to \infty. \end{aligned}$$

# Remark

The above theorem shows that the Szego limit property holds for a large class of functions $\tilde{f}$ by which one can localise spectrum of $A$.

**Example**

Unbounded pseudodifferential operators.

.
The above theorem shows that for large values of $n$,

$$\Gamma_n(\tilde{f}(P_nAP_n)) = \int_{-\infty}^{\infty} \tilde{f}(t)d\mu_A + 0(\frac{\|[A, P_n]P_n\|}{d_n}). \qquad (11)$$

It is worth observing that the Szego-type formula becomes more useful when it is given in the remainder estimate form as given above.

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# Laptev- Safarov Szego- type Theorem

The following theorem due A. Laptev and Y. Safarov deals with (given an operator)problem of identifying an appropriate family family of orthogonal projections with respect to which the Szego distribution formula of eigenvalues holds.In fact the abstract theorem is applied to concrete cases such as pseudo differential operators on suitable manifolds.

**Theorem**

*[14] Let B be the multiplication operator by a real function b in $\mathcal{C}_0^{2m}(R^n)$ in $L_2(R^n)$ and let $\psi \varepsilon W_\infty^n(R^1)$.If $\psi(0) = 0$,then*

$$T_r P_\lambda \psi(P_\lambda B P_\lambda) P_\lambda = \int \psi(b(x)) e(x, x, \lambda) dx + O(\lambda^{\frac{n-1}{2m}})$$

$$= (2\pi)^{-n} \lambda^{n/2m} \int_{a_{2m}<1} d\xi \int \psi(b(x)) dx + O(\lambda^{\frac{n-1}{2m}})$$

If we restrict $\lambda$ to natural numbers one gets the usual statements.It is also possible to define filtrations with continuous parameter instead of integers!

# Szego-Hansen,[3]

In [2] Hansen obtains a generalisation of Arveson's Szego-type theorem to a certain class of nonselfadjoint operators using Brown measures in the setting finite Von Neumann algebras with unique tracial states.

# Haar System

**I**

t is to be noted that Szego,s theorem can fail if the filtrations arise from a 'wild orthonormal basis' uch as Haar system in $L_2[01]$. [17].Let $\phi_n$ be be the Haar system in $L_2[01]$ which is ordered Lexicographically.To be more precise each $\phi_n$ is give as follows;

$\phi_n(x) = -2^{m/2} \quad if \quad (r+1/2)/2^m \leq x \leq (r+1)/2^m$
$= 2^{m/2} \quad if \quad r/2^m \leq x \leq (r+1/2)/2^m$
$= 0,$ otherwise

where $n = 2^m + r \quad 0 \leq r \leq 2^m, m \geq 0$

# Propostion 3,page 148, [17]

Let $T_{\phi_1}$ be the multiplication operator on $L_2[0,1]$ with $\phi_1$ as multiplier. Then the asymptotic Szego type formula is not satisfied when the trigonometric basis is replaced by the lexicographically ordered Haar basis.

# Concluding remarks

**V**

ictor Guillimen observed that the projections used in the classical theorem of Szego, are spectral projections of a second order differential operator.Following this idea,Laptev and Saffarov obtained Szego-type theorem for more general case using spectral projections of pseudo differential operators.Moreover the remainder estimate of this assymptotic formula is also obtained,which is very importand. In the other situations the proofs by and large relie heavily on Tauburian theorems to ensure existance of limits.Following W.Arveson's method we start with aribitray ,increasing sequence sequece of orthogonal projections that converges strongly to the identity operator.Then prove Szego-type results under the assumtion that the affiliated algebra has unique tracial state.

The problem remains to be investigated is to find conditions under which the affiliated algebra has uniqe traicial state !.

# Proof

**Proof.**

$T^{-1}$ being a uniform limit of band limitted operators in $DA(H)$, we may assume that $T^{-1}$ is band limitted and then the general case follows by continuity. For $A \in DL(H)$ $P_nAP_n$ is identified with the matrix of $P_nAP_n$ restricted to $H_n$ throughout the proof.
Let $T = (t_{ij})$ and $T^{-1} = (s_{ij})$. Let $N$ be a positive integer such that $s_{i,j} = 0$ for $|i - j| > N$.
Now

$$||(P_nTP_n)^{-1} - P_nT^{-1}P_n|| \leq ||(P_nTP_n)^{-1}|| \quad ||P_n - P_nTP_n\ P_nT^{-1}P_n||.$$

Here $||(P_nTP_n)^{-1}||$ is bounded uniformly.
We show that $||P_n - P_nTP_n\ P_nT^{-1}P_n|| \to 0$ as $n \to 0$.
Now

$$\begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ t_{21} & t_{22} & \cdots & t_{2n} \end{bmatrix}$$

W. Arveson studied the role of $C^*$-algebras in infinite dimensional numerical linear algebra wrote a series of articles articles[6, 7, 8]and dealt with determination of essential spectrum of bounded self adjoint operators on separable Hilbert spaces, using the method of truncation. He concludes his investigation by applying the theory to descretised Hamiltonion.

This section is based on an attempt to answer a query of Arveson during the general formulation his program. The following is a brief summary of the genesis of the problem.

For any $T \in B(\mathcal{H})$, $T$ self-adjoint, $\sigma(T) \subseteq \Lambda$

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# Query

How to distinguish between transient points which belong to $\sigma(T)$ and transient which do not belong to $\sigma(T)$.

**An answer to the query is as follows** Let $T \in B(H)$, be self adjoint.

$$\begin{aligned} m &= \text{infim } \sigma(T) \\ M &= \sup \sigma(T) \\ \nu &= \text{infim } \sigma_e(T) \text{ and} \\ \mu &= \sup \sigma_e(T). \end{aligned}$$

Let $\lambda \in \Lambda_t(T)$.

**Case 1.** $\lambda \notin [\nu, \mu]$.Let

$$\sigma(T) \cap [m, \nu] = \{\lambda_1^- \leq \lambda_2^- \cdots \leq \lambda_S^-\} \text{ and}$$
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$$\sigma(T) \cap [\mu, M] = \{\lambda_1^+ \geq \lambda_2^+ \cdots \geq \lambda_R^+\}$$

Then it is known that [**?**]

$$\lim_{n\to\infty} \lambda_{n+1-k}(T_n) = \begin{cases} \lambda_K^-, & \text{if } S = \infty \text{ or } 1 \le k \le S \\ \nu, & \text{if } S < \infty \text{ and } k \ge S+1. \end{cases}$$

$$\lim_{n\to\infty} \lambda_k(T_n) = \begin{cases} \lambda_K^+, & \text{if } R = \infty \text{ or } 1 \le k \le R \\ \mu, & \text{if } R < \infty \text{ and } K \ge R+1. \end{cases}$$

Hence $\lambda \notin \sigma(T) \cap \{[m,\nu] \cup [\mu, M]\}$ iff. and

$$\lim_{\epsilon\to 0} \lim_{n\to\infty} \Phi_{\epsilon,\lambda}(\lambda_k(T_n)) = 0 \text{ for all } k$$

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Note that in both the cases ,the *-convergence to 0 is exponential,which helps numerical verification computationally feasible.
where

$$\Phi_{\epsilon,\lambda}(x) = \frac{e^{-\frac{(x-\lambda)^2}{2\epsilon}}}{\sqrt{2\pi\epsilon}}$$

**Case 2.** Let $\lambda \in (\nu, \mu)$. This is more difficult to handle. In this case, $[\lambda - \epsilon, \lambda + \epsilon]$ is a spectral gap in $\sigma_e(T)$ for some $\epsilon \geq 0$.
Observe that $\lambda \in \sigma(T)$ iff

$$\|\Phi_{\epsilon,\lambda}(T)\| = \Phi_{\epsilon,\lambda}(\lambda).$$

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Now

$$\limsup_n \|\Phi_{\epsilon,\lambda}(T_n)\| \geq \|\Phi_{\epsilon,\lambda}(T)\|.$$

Let $\lambda^{(n)} \epsilon \sigma(T_n) \cap [\lambda - \epsilon, \lambda + \epsilon]$ be the closest to $\lambda$.
Hence $\lim_{n\to\infty} \Phi_{\epsilon,\lambda}(\lambda^{(n)}) = \Phi_{\epsilon,\lambda}(\lambda)$.
Now $\Phi_{\epsilon,\lambda}(\lambda^{(n)}) = \|\Phi_{\epsilon,\lambda}(T_n)\|$.

$$\text{Therefore} \quad \|\Phi_{\epsilon,\lambda}(T_n)\| \to \|\Phi_{\epsilon,\lambda}(\lambda)\|$$

Thus $\lambda \notin \sigma(T)$ if $\|\Phi_{\epsilon,\lambda}(T)\| < \Phi_{\epsilon,\lambda}(\lambda)$
iff. there exists $N$ large enough so that

$$\|\Phi_{\epsilon,\lambda}(T_n)\| < \Phi_{\epsilon,\lambda}(\lambda), foralln \geq N.$$

# Remark

It would be interesting to observe that one can consider $\Phi_{\epsilon,\lambda}(T - \lambda I)^2$ so that the gap arround $\lambda$ is opened.As a consequence,$\lambda$ belongs to the resolvent $\rho(T)$ iff 0 belongs to $\rho(\Phi_{\epsilon,\lambda}(T - \lambda I)^2)$.Observe that ,in this case,0 is below $\sigma_e(\Phi_{\epsilon,\lambda}(T))$.

# Truncation method for semibounded operators

**H**

ere truncation method for computing eigen values of semibounded operators is carried out by using known method for the bounded resolvent of it. However,care is taken to avoid actoual computation of resolvent by demanding certain feasible conditions for the original operator.

For a separable complex Hilbert space $H$, $DL(H)$ will denote the class of all densely defined closed linear operators in $H$, $SB(H)$ will denote the class of all semibounded linear operators in $H$ and $B(H)$ will denote the class of all bounded linear operators on $H$. Let $\{e_1, e_2, e_3, \dots\}$ be an orthonormal basis in $H$. Throughout this paper it is assumed that all operators $T$ will contain the above bases in its domain. For $T$ in $DL(H)$, $\sigma(T)$ and $\sigma_{\text{ess}(T)}$ will denote the spectrum and essential spectrum respectively. Also for each positive integer $k$, $S_k(T)$ will denote the $k$th singular value of $T$. Let $K(H)$ denote the class of all compact operators on $H$. For $T \in B(H)$ let

$$\|T\|_{\text{ess}} = \text{infimum}\,\{\|T - K\| : K \in k(H)\}.$$

The following remarkable theorem is due to I. Goghberg, S. Goldberg and M. A. Kaashoek [11].

# Theorem

For $T \in B(H)$ let $|T| = (T^*T)^{1/2}$. Then $\sigma(|T|)\backslash[0, \|T\|_{\text{ess}}]$ is atmost countable and $\|T\|_{\text{ess}}$ is the only possible accumulation point, and all points of the set are eigen values of finite algebraic multiplicity of $|T|$. Let

$$\lambda_1(|T|) \geq \lambda_2(|T|) \geq \dots \tag{12}$$

be the eigen values in non increasing order (multiplicities taken into account) and let $N \in \{0, 1, 2, \dots\} \cup \{\infty\}$ be the number of terms in (12). Then

$$S_k(T) = \begin{cases} \lambda_k(|T|) & \text{if } N = \infty \text{ or } 1 \leq k \leq N, \\ \|T\|_{\text{ess}} & \text{if } N < \infty \text{ and } k \geq N + 1. \end{cases}$$

An attempt to get a semi bounded version of the above theorem was done in [**?**]. We recall this result and provide the proof so as to correct a misprint that went unnoticed.

Let $T \in SB(H)$ be such that $T$ and $T_n$ are invertible on $H$ and $H_n$ respectively. Also assume that

$$T_n^{-1} \to T^{-1} \text{ strongly (as } n \to \infty).$$

Then

$$\lim_{n\to\infty} \lim_{m\to\infty} S_k(P_n(P_m T P_m)P_n) = S_k(T^{-1})$$

for each $k$.

**Proof.**

Since $T \in SB(H)$, $T^{-1} \in B(H)$ and is self adjoint.
Since $(P_m T P_m)^{-1} \to T^{-1}$ strongly,

$$\lim_{m\to\infty} P_n (P_m T P_m)^{-1} P_n = P_n T^{-1} P_n \quad \text{(uniformly)}$$

for each $n$. Here

$$\lim_{m\to\infty} S_k(P_n (P_m T P_m)^{-1} P_n) = S_k(P_n T^{-1} P_n),$$

for each $k$ and for each $n$. Here

$$\lim_{n\to\infty} \lim_{m\to\infty} S_k(P_n (P_m T) P_m)^{-1} P_n) = S_k(T^{-1}) \text{ for each } k.$$

$\square$

# Remark

The drawback of the above result is the lack of computational feasibility that arises due to inversion followed by truncation of matrices of growing order. This reveals the significance of the **Problem** mentioned in the abstract namely when is it true that

$$\lim_{n\to\infty} S_k((P_nTP_n)^{-1}) = S_k(T^{-1}) \quad \text{for each } k \tag{13}$$

It is known that (13) is not true in general, even for bounded operators. See for instance example 1.12, p.34 [**?**]. How ever we present a modified version of it in what follows.

Let $\lambda = \{\lambda_n\}$ be a sequence of positive real numbers that decrease to zero and let $T$ be the block diagonal operator on $l^2$ whose $n$th block is given as

$$A_n = \begin{bmatrix} \epsilon + \lambda_n & \delta \\ \delta & \epsilon + \lambda_n \end{bmatrix}$$

where $\epsilon > 0$ and $\delta > 0$ to be chosen later. It is clear that $\sigma(A_n) = \{\epsilon + \lambda_n + \delta, \epsilon + \lambda_n - \delta\}$ for each $n$. Also, if $\epsilon \neq \delta$.

$$\sigma(A_n^{-1}) = \{\frac{1}{\epsilon + \lambda_n + \delta}, \frac{1}{\epsilon + \lambda_n - \delta}\} \quad \text{and}$$

$$\sigma(T^{-1}) = \text{closure } \cup_1^{\infty} \{\frac{1}{\epsilon + \lambda_n + \delta}, \frac{1}{\epsilon + \lambda_n - \delta}\}$$

Now choose $\delta > 0$ such that $\epsilon + \lambda_n < |\epsilon - \delta + \lambda_n|$ for all $n$.
Let $T_N$ denotes the corner $N$th truncation of $T$. Then it can be seen that $T_N$ is invertible for each $N$ and

$$\sigma(T_N^{-1}) = \cup_{n=1}^{N} \left\{ \frac{1}{\epsilon + \lambda_n + \delta}, \frac{1}{\epsilon + \lambda_n - \delta} \right\}, \text{ when } N \text{ is even,}$$
$$= \cup_{n=1}^{N-1} \left\{ \frac{1}{\epsilon + \lambda_n + \delta}, \frac{1}{\epsilon + \lambda_n - \delta}, \frac{1}{\epsilon + \lambda_N} \right\}, \text{ when } N \text{ is odd.}$$

Now

$$S_1(T_N^{-1}) = \|T_N^{-1}\| = \frac{1}{\epsilon + \lambda_n - \delta} \qquad N \text{ even}$$
$$= \frac{1}{\epsilon + \lambda_N}, \qquad N \text{ odd.}$$

where as

$$\|T^{-1}\| = \frac{1}{\epsilon}.$$

$$S_k(T^{-1}) = \frac{1}{\epsilon} \quad \text{for all } k \text{ and}$$

$$\textit{Infactonecanseethat}\; S_k(T_N^{-1}) = \frac{1}{\epsilon + \lambda_{N-k}}, \quad N \text{ odd} \qquad \textit{Hence}$$

$$= \frac{1}{|\epsilon + \lambda_{N-k} - \delta|}, \quad N \text{ even.}$$

$\lim_{N\to\infty} S_k(T_N^{-1})$ does not exist for any $k$.

In this section we assume that $T \in SB(H)$, $T$ is invertible, $P_n T P_n$'s are invertible and $(P_n T P_n)^{-1} \to T^{-1}$ strongly as $n \to \infty$. First of all the notion of assymptotic diagonality is defined.

d̲efi $T \in B(H)$ is assymptotically band limitted diagonal w.r.t the bases chosen, if $[T] = (t_{ij})$ the matrix of $T$ w.r.t the bases is band limitted and $t_{\substack{m,n \\ m \neq n}} \to 0$ uniformly as $m, n \to \infty$. defi

$T$ is called assymptotically diagonal, if $T$ is a uniform limit of assymptotically band limitted diagonal operators.

$AD(H)$ and $D(H)$ will denote the class of all assymptotically diagonal and diagonal operators in $B(H)$. The following theorem gives a better picture of the class $AD(H)$.

$AD(H) = D(H) + K(H)$, where $K(H)$ denote the class of all compact operators on $H$. In particular $AD(H)$ is a $C^*$-algebra with identity and $AD(H)/K(H)$ is $*$-isomorphic to $l^\infty/C_0$, where $C_0$ is the set of all sequences of scalars converging to 0.

**Proof.**

Clearly $D(H) + K(H)$ is a $e^*$-subalgebra of $B(H)$. Let $T \in D(H) + K(H)$. Then $T = T_d + T_K$ where $T_d \in D(H)$ and $T_K \in K(H)$.
Now $T_d + P_n T_K P_n$ is band limitted, assymptotically diagonal for each $n$. Also

$$||T_d + T_K - (T_d + P_n T_K P_n)|| \to 0)$$

as $n \to 0$. Here $T_d + T_K = T \in DA(H)$.
Now let $T \in DA(H)$ and $[T] = (t_{i,j})$, matrix of $T$ w.r.t the concerned bases chosen.
Put

$$T_d = (d_{ij}),$$

# Remark

(1) It is to be remarked that fairly large class of semibounded operators have there inverses in $DA(H)$. However many important class of operators like Toeplitz operators and certain Hankel operators are not of this class.
(2) Example.
The condition that $\underset{i\neq j}{|t_{ij}|} \to 0$ uniformly is not sufficient for the operator $T$ where matrix $[T] = (t_{ij})$ to be in $DA(H)$. The following Hilbert matrix [**?**] is an example for that.

# Hilbert matrix

$$H(a) = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \cdots \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \cdots \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots & \cdots \\ \vdots & & & & \\ \frac{1}{n} & \frac{1}{n+1} & \frac{1}{n+2} & \cdots & \cdots \\ \vdots & \vdots & \vdots & & \end{bmatrix}$$

It is known that $H(a)$ is not compact . But its diagonal part,

$$H_d(a) = \begin{bmatrix} 1 & 0 & \cdots & \cdots \\ 0 & \frac{1}{3} & \cdots & \cdots \\ 0 & 0 & \frac{1}{5} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

is a compact operator. Hence its non-diagonal part can not be compact. Therefore $H(a)$ is not in $DA(H)$.
It is not clear whether

$$\lim_{n\to\infty} S_k(P_n\tilde{H}(a)P_n)^{-1} = S_k(\tilde{H}(a)^{-1})$$

where $\tilde{H}(a) = \beta I + H(a)$, $\beta > \|H(a)\|$ for each $k$. Observe that $\tilde{H}(a) \notin DA(H)$.

# Theorem

Let $T \in SB(H)$ be such that $T^{-1} \in DA(H)$, $P_nTP_n$ are invertible on $H_n$ for each $n$ and

$$(P_nTP_n)^{-1} \to T^{-1} \text{ strongly as } n \to \infty.$$

If

$$\sup_{\substack{i \neq j \\ i,j=1,\ldots}} |\langle T_{e_i,e_j}\rangle| < \infty \quad \text{then}$$

$$\lim_{n\to\infty} S_K((P_nTP_n)^{-1}) = S_K(T^{-1}) \quad \text{for each } n.$$

# Proof

**Proof.**

$T^{-1}$ being a uniform limit of band limitted operators in $DA(H)$, we may assume that $T^{-1}$ is band limitted and then the general case follows by continuity. For $A \in DL(H)$ $P_nAP_n$ is identified with the matrix of $P_nAP_n$ restricted to $H_n$ throughout the proof.
Let $T = (t_{ij})$ and $T^{-1} = (s_{ij})$. Let $N$ be a positive integer such that $s_{i,j} = 0$ for $|i - j| > N$.
Now

$$||(P_nTP_n)^{-1} - P_nT^{-1}P_n|| \leq ||(P_nTP_n)^{-1}|| \quad ||P_n - P_nTP_n\, P_nT^{-1}P_n||.$$

Here $||(P_nTP_n)^{-1}||$ is bounded uniformly.
We show that $||P_n - P_nTP_n\, P_nT^{-1}P_n|| \to 0$ as $n \to 0$.

$\square$

**Proof.**

Now

$$P_nTP_n = \begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ t_{21} & t_{22} & \cdots & t_{2n} \\ t_{31} & t_{32} & \cdots & t_{3n} \\ \vdots & & & \\ t_{m1} & t_{m2} & \cdots & t_{m,n}. \end{bmatrix}$$

□

**Proof.**

where $s_{i,j} = 0$ for $|i-j| > N$. Here $t_{i,j} = 0$ for $|i-j| > N$.
Put

$$Q_n = P_n S P_n \quad P_n S^{-1} P_n = (q_{ij}).$$

For $i \neq j$, $i, j \leq n$ we have

$$q_{i,j} = \sum_{\substack{|i-m| \leq N \\ |j-m| \leq N \\ m=1,2,\ldots}} t_{im}\, s_{m,j}.$$

Now let $n = N + k$, $k \geq 1$. Thus $q_{ij} = 0$ of $1 \leq i, j \leq k$. If $i = k + r$ and $j = k + s$, $r \geq s$

$$q_{i,j} = (-1) \sum_{k+N \leq m \leq N+k+s} t_{i,m}\, s_{m,j}.$$

Here $1 \leq r, s \leq N$. □

**Proof.**

Similarly when $i = j$

$$q_{i,j} = 1 \quad \text{if } 1 \leq i \leq k$$
$$= 1 - \sum_{k+N \leq m \leq N+k+s} t_{i,m}\, s_{m,j}.$$

Here $i = j = k + s$, $s > 1$.
Hence

$$P_n - P_n T P_n\, P_n T^{-1} P_n = R_n = (r_{i,j})$$

where $r_{ij} = 0$, $1 \leq i, j \leq k$.
Therefore each $R_n$ has atmost $\frac{(N-1)N}{2}$ non-zero entries for each $n$, which tends to zero uniformly as $n \to \infty$. Hence $||R_n|| \to 0$ as $n \to \infty$.
But
$\lim_{n\to\infty} S_K(P_n T^{-1} P_n) = S_k(T^{-1})$ for each $n$.
Hence
$\lim S_k((P_n T P_n)^{-1}) = S_k(T^{-1})$ for each $k \geq 1$. $\square$

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Thank You