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Compact double difference of composition operators

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- D: unit disc in C.
- $\mathbf{T} = \partial \mathbf{D}$: unit circle in \mathbf{C} .
- $H(\mathbf{D})$: class of all holomorphic functions on \mathbf{D} .
- $H^{p}(\mathbf{D})$: Hardy spaces on **D**. Freely identified with $H^{p}(\mathbf{T})$.

$$f \in H^p(\mathbf{D}) \iff \|f\|_{H^p}^p = \sup_{0 < r < 1} \int_{\mathbf{T}} |f(r\zeta)|^p \, d\sigma(\zeta) < \infty$$

• $A^p_{\alpha}(\mathbf{D})$: Bergman space on **D**.

$$f \in A^p_{\alpha}(\mathbf{D}) \iff \|f\|^p_{A^p_{\alpha}(\mathbf{D})} = \int_{\mathbf{D}} |f(z)|^p \, dA_{\alpha}(z) < \infty$$

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$$f \in H^p(\mathbf{D}) \iff \|f\|_{H^p}^p = \sup_{0 < r < 1} \int_{\mathbf{T}} |f(r\zeta)|^p \, d\sigma(\zeta) < \infty$$

A^p_α(**D**): Bergman space on **D**.

$$f \in \mathcal{A}^p_{\alpha}(\mathbf{D}) \iff \|f\|^p_{\mathcal{A}^p_{\alpha}(\mathbf{D})} = \int_{\mathbf{D}} |f(z)|^p \, d\mathcal{A}_{\alpha}(z) < \infty$$

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Composition (Operator		

Composition Operator

For $\varphi:\Omega\to\Omega$ holomorphic self-map, composition operator is defined by

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$$C_{\varphi}f=f\circ\varphi.$$

Examples of Ω : **D**, **B**_n, **D**ⁿ, **C**ⁿ, strongly pseudoconvex domain.

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Question			
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For a smooth function g, we have

Let

 $T_{ij} = C_{\varphi_i} - C_{\varphi_j}$ so that $T_{ij}f(z) := f(\varphi_i(z)) - f(\varphi_j(z)),$

and

$$Tf(z) := T_{12}f(z) - T_{23}f(z) = f(\varphi_1(z)) - 2f(\varphi_2(z)) + f(\varphi_3(z)).$$

In view of this, can T behavior better than T_{12} ?

Double Difference Cancelation?

Can $(C_{\varphi_1} - C_{\varphi_2}) - (C_{\varphi_2} - C_{\varphi_3})$ be compact while both $(C_{\varphi_1} - C_{\varphi_2})$ and $(C_{\varphi_2} - C_{\varphi_3})$ are not compact?

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Let

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More generally,

Double Difference Cancelation?

Suppose $(C_{\varphi_1} - C_{\varphi_2}), (C_{\varphi_3} - C_{\varphi_4}), (C_{\varphi_1} - C_{\varphi_3})$ and $(C_{\varphi_2} - C_{\varphi_4})$ are all not compact.

Can $T := (C_{\varphi_1} - C_{\varphi_2}) - (C_{\varphi_3} - C_{\varphi_4}) = (C_{\varphi_1} - C_{\varphi_3}) - (C_{\varphi_2} - C_{\varphi_4})$ be compact?

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Results for $A^{p}_{\alpha}(\mathbf{D})$			
Theorem 1			

K-Wang (2015) Let $0 and <math>\alpha > -1$. Let $a_i \in \mathbf{C} \setminus \{0\}$ and assume C_{φ_i} is not compact on $A^p_{\alpha}(\mathbf{D})$ for each i = 1, 2, 3. Let $T := \sum_{i=1}^{3} a_i C_{\varphi_i}$.

If T compact on $A^p_{\alpha}(\mathbf{D})$, then one of the following holds:

• $T = a_i(C_{\varphi_i} - C_{\varphi_j} - C_{\varphi_k})$, where (i, j, k) is some permutation of (1, 2, 3)

•
$$T = a_1(C_{\varphi_1} - C_{\varphi_2}) + a_3(C_{\varphi_3} - C_{\varphi_2}).$$

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Results for $A^{p}_{\alpha}(\mathbf{D})$			
Theorem 2			

• Double difference

K-Wang (2015) Let $0 , <math>\alpha > -1$. Let $a, b \in \mathbf{C} \setminus \{0\}$ and $a + b \neq 0$. Assume C_{φ_i} is not compact on $A^p_{\alpha}(\mathbf{D})$ for each i = 1, 2, 3.

 $egin{aligned} T &:= a(C_{arphi_1}-C_{arphi_2})+b(C_{arphi_3}-C_{arphi_2}) ext{ is compact on } A^p_lpha(\mathbf{D}) \ &\Leftrightarrow \ both \ C_{arphi_1}-C_{arphi_2} ext{ and } C_{arphi_3}-C_{arphi_2} ext{ are compact on } A^p_lpha(\mathbf{D}). \end{aligned}$

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Results for $A^{p}_{\alpha}(\mathbf{D})$			
Notation			

Let

$$T := T_{12} - T_{34} = T_{13} - T_{24}, \quad T_{ij} = C_{\varphi_i} - C_{\varphi_j}.$$

We also put

$$ho_{ij}(z) =
ho_{arphi_i,arphi_j}(z) :=
ho ig(arphi_i(z), arphi_j(z) ig), \quad
ho(a,b) = igg| rac{a-b}{1-a\overline{b}}$$

and

$$M_{ij}(z)=M_{arphi_i,arphi_j}(z):=\left[rac{1-|z|}{1-|arphi_i(z)|}+rac{1-|z|}{1-|arphi_j(z)|}
ight]
ho_{ij}(z).$$

Finally, we put

$$M = M_{12} + M_{34}$$
 and $\widetilde{M} := M_{13} + M_{24}$.

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Theorem 3			

• General double difference

Choe-K-Wang (2017) $T := T_{12} - T_{34}$ is compact on $A^p_{\alpha}(\mathbf{D})$ if and only if

 $\lim_{|z|\to 1} M(z)\widetilde{M}(z) = 0.$

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• General double difference

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Hardy space case			
Questions for	$H^p(\mathbf{D})$		

• Component problems

Shapiro-Sundberg (1990)

• If $C_{\varphi} - C_{\psi}$ is compact, then do they belong to the same component?

 Is there non-compact C_φ which belongs to the component containing compact operators?

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Known results for	$H^p(\mathbf{D})$		

• Component

Moorhouse-Toews (2001), Bourdon(2003) There are C_{φ} and C_{ψ} which belong to the same component, but $C_{\varphi} - C_{\psi}$ is compact

Component

Gallardo-Gutierrez, Gonzalez, Nieminen-Saksman (2008)

- *H^p*(**D**): There is a non-compact C_φ which belongs to the component containing compact operators.
- $A^{p}_{\alpha}(\mathbf{D})$: The set of compact operators is a component.
- A^p_α(D): If the difference is compact, then they belong to the same component.

Component

Nieminen-Saksman (2004) $C_{\varphi} - C_{\psi}$ is compact on $H^{p}(\mathbf{D})$ for some $p \ge 1$, then for all $p \ge 1$.

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Questions for H^p	(D)		

• Component problems for $H^p(D)$

• Characterize components.

- Characterize the component containing compact operators.
- Characterize the compact difference, the joint Carleson measure.

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• Characterize the double difference compact operators.

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Boundedness				

Boundedness On Unit Disk

Weighted Bergman spaces

For p>0 and $\alpha\geq -1$, the weighted Bergman space $A^p_\alpha({\bf D})$ is the set of analytic functions f with

$$\|f\|^p := \int_{\mathbf{D}} |f(z)|^p dA_{\alpha}(z), \qquad dA_{\alpha}(z) := (1 - |z|^2)^{\alpha} dA(z).$$

Boundedness on weighted Bergman spaces

By Littlewood's Subordination Principle.

$$C_{\varphi}: A^p_{\alpha} \to A^p_{\alpha}.$$

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• Boundedness on weighted Bergman spaces

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Boundedness			
Subordination	Principle		

If g subharmonic and φ analytic with $\varphi(0) = 0$, then

$$\int_{0}^{2\pi} g \circ arphi(re^{i heta}) d heta \leq \int_{0}^{2\pi} g(re^{i heta}) d heta.$$

Proof) Let G = P(g), the Poisson integral of g.

$$\int_{0}^{2\pi} g \circ \varphi(re^{i\theta}) \frac{d\theta}{2\pi} \leq \int_{0}^{2\pi} G \circ \varphi(re^{i\theta}) \frac{d\theta}{2\pi}$$

$$= G \circ \varphi(0)$$

$$= \int_{0}^{2\pi} g(re^{i\theta}) \frac{d\theta}{2\pi}. \square$$

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$$\int_{0}^{2\pi} g \circ arphi(\mathsf{re}^{i heta}) d heta \leq \int_{0}^{2\pi} g(\mathsf{re}^{i heta}) d heta.$$

Proof) Let G = P(g), the Poisson integral of g.

$$\begin{split} \int_{0}^{2\pi} g \circ \varphi(re^{i\theta}) \frac{d\theta}{2\pi} &\leq \int_{0}^{2\pi} G \circ \varphi(re^{i\theta}) \frac{d\theta}{2\pi} \\ &= G \circ \varphi(0) \\ &= \int_{0}^{2\pi} g(re^{i\theta}) \frac{d\theta}{2\pi}. \end{split}$$

	Background	Proof of Theorem	References
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Boundedness			
Subordination Pr	inciple		

If g subharmonic and φ analytic with $\varphi(0) = 0$, then

$$\int_{0}^{2\pi} g \circ arphi(extsf{re}^{ extsf{i} heta}) d heta \leq \int_{0}^{2\pi} g(extsf{re}^{ extsf{i} heta}) d heta.$$

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$$\int_{0}^{2\pi} g \circ \varphi(re^{i\theta}) \frac{d\theta}{2\pi} \leq \int_{0}^{2\pi} G \circ \varphi(re^{i\theta}) \frac{d\theta}{2\pi}$$
$$= G \circ \varphi(0)$$
$$= \int_{0}^{2\pi} g(re^{i\theta}) \frac{d\theta}{2\pi}.$$

	Background	Proof of Theorem	References
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Boundedness			
Subordination	Principle		

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	Background	Proof of Theorem	References
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Boundedness			
Subordination	Principle		

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$$= G \circ \varphi(0)$$

$$= \int_{0}^{2\pi} g(re^{i\theta}) \frac{d\theta}{2\pi}. \square$$
	Background	Proof of Theorem	References
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Boundedness			
Subordination	Principle		

• Littlewood's Subordination Principle

If g subharmonic and φ analytic with $\varphi(0) = 0$, then

$$\int_{0}^{2\pi} g \circ arphi(\mathsf{re}^{i heta}) d heta \leq \int_{0}^{2\pi} g(\mathsf{re}^{i heta}) d heta.$$

Proof) Let G = P(g), the Poisson integral of g.

$$\int_{0}^{2\pi} g \circ \varphi(re^{i\theta}) \frac{d\theta}{2\pi} \leq \int_{0}^{2\pi} G \circ \varphi(re^{i\theta}) \frac{d\theta}{2\pi}$$
$$= G \circ \varphi(0)$$
$$= \int_{0}^{2\pi} g(re^{i\theta}) \frac{d\theta}{2\pi}. \square$$

Let $g = |f|^{p}$ to get the boundedness on $A^{p}_{\alpha}(\mathbf{D})$.

	Background	Proof of Theorem	References
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Boundedness			
Carleson Mea	sure		

For
$$0<\delta<1$$
, let $D(a):=D_{\delta}(a):=D(a,\delta(1-|a|)).$ Then $\mu(D_{\delta}(a))\lesssim (1-|a|)^{2+lpha}$

$$\int_{\mathbf{D}} |f|^p d\mu \lesssim \int_{\mathbf{D}} |f|^p dA_{\alpha}.$$

 \Leftarrow) Let $f_a(z) = \frac{1}{(1-z\overline{a})^n}$, then

$$\begin{array}{ll} \displaystyle \frac{A_{\alpha} \circ \varphi^{-1}(D(a))}{A_{\alpha}(D(a))} &\approx & \displaystyle \frac{(1-|a|)^{np}}{(1-|a|)^{2+\alpha}} \int_{\varphi^{-1}(D(a))} \frac{1}{|1-\varphi(z)\overline{a}|^{np}} dA_{\alpha}(z) \\ &\lesssim & \displaystyle \frac{\|f_{a} \circ \varphi\|^{p}}{\|f_{a}\|^{p}} \to 0 \text{ as } |a| \to 1. \end{array}$$

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	Background	Proof of Theorem	References
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Boundedness			
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For
$$0<\delta<1$$
, let $D(a):=D_{\delta}(a):=D(a,\delta(1-|a|)).$ Then $\mu(D_{\delta}(a))\lesssim (1-|a|)^{2+lpha}$

iff

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 \Leftarrow) Let $f_a(z) = \frac{1}{(1-z\overline{a})^n}$, then

$$egin{aligned} rac{A_lpha \circ arphi^{-1}(D(a))}{A_lpha(D(a))} &pprox & rac{(1-|a|)^{np}}{(1-|a|)^{2+lpha}} \int_{arphi^{-1}(D(a))} rac{1}{|1-arphi(z)\overline{a}|^{np}} dA_lpha(z) \ &\lesssim & rac{\|f_a \circ arphi\|^p}{\|f_a\|^p} o 0 ext{ as } |a| o 1. \end{aligned}$$

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	Background	Proof of Theorem	References
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	Background	Proof of Theorem	References
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Carleson Measure			

 \Rightarrow)

$$\begin{split} \int_{\mathbf{D}} |f|^{p} d\mu &\leq \int_{\mathbf{D}} \left(\frac{1}{A_{\alpha}(D_{\delta}(z))} \int_{D_{\delta}(z)} |f(w)|^{p} dA_{\alpha}(w) \right) d\mu(z) \\ &\leq \int_{\mathbf{D}} \left(\int_{\{z: w \in D_{\delta}(z)\}} d\mu(z) \right) \frac{|f(w)|^{p}}{(1-|w|)^{2+\alpha}} dA_{\alpha}(w) \\ &\lesssim \int_{\mathbf{D}} |f|^{p} dA_{\alpha}. \end{split}$$

Compact version:
$$\lim_{|a| \to 1} \frac{\mu(D(a))}{A_{\alpha}(D(a))} = 0.$$

	Background	Proof of Theorem	References
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Carleson Measure			

$$\Rightarrow)$$

$$\begin{split} \int_{\mathbf{D}} |f|^{p} d\mu &\leq \int_{\mathbf{D}} \left(\frac{1}{A_{\alpha}(D_{\delta}(z))} \int_{D_{\delta}(z)} |f(w)|^{p} dA_{\alpha}(w) \right) d\mu(z) \\ &\leq \int_{\mathbf{D}} \left(\int_{\{z:w \in D_{\delta}(z)\}} d\mu(z) \right) \frac{|f(w)|^{p}}{(1-|w|)^{2+\alpha}} dA_{\alpha}(w) \\ &\lesssim \int_{\mathbf{D}} |f|^{p} dA_{\alpha}. \end{split}$$

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Carleson Measure			

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Carleson Measure			

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Compact version:
$$\lim_{|a|\to 1} \frac{\mu(D(a))}{A_{\alpha}(D(a))} = 0.$$

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Carleson Measure			
Carleson Measure	2		

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Change of Variables

 C_{φ} compact iff $A_{\alpha} \circ \varphi^{-1}$ is an α -Carleson measure.

Proof)

$$\int_{\mathbf{D}} |f \circ \varphi|^p dA = \int_{\mathbf{D}} |f|^p dA \circ \varphi^{-1}$$

here $A \circ \varphi^{-1}(E) := \int_{\varphi^{-1}(E)} dA$.

Introduction	Background	Proof of Theorem	References
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Carleson Measure			
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Change of Variables

 C_{φ} compact iff $A_{\alpha} \circ \varphi^{-1}$ is an α -Carleson measure.

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	Background	Proof of Theorem	References
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Carleson Measure			

• Compactness on Bergman spaces

Compactness

MacCluer and Shapiro (1986) For p > 0, $\alpha > -1$, C_{Φ} is compact on A^{p}_{α}

$$\iff \lim rac{1-|z|}{1-|arphi(z)|} = 0 \ {
m as} \ |z|
ightarrow 1^-.$$

Remark: Julia-Caratheodory Theorem

 φ has finite angular derivative at ζ .

$$\iff \liminf_{z \to \zeta} \frac{1 - |\varphi(z)|}{1 - |z|} < \infty.$$

	Background	Proof of Theorem	References
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Carleson Measure			
Compactness			

• Compactness on Bergman spaces

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	Background	Proof of Theorem	References
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Compactness			

Let arphi(0)=0. By Schwartz Lemma, $D(0,r)\subset arphi^{-1}(D(0,r))$ and

$$D_{\delta_1}(a) \subset \varphi^{-1}(D_{\delta}(b)), \qquad b = \varphi(a).$$

If not, then

$$1 \approx \left(\frac{1 - |\varphi(a)|}{1 - |a|}\right)^{2 + \alpha} \lesssim \frac{A_{\alpha}(D_{\delta_{1}}(b))}{A_{\alpha}(D_{\delta}(a))}$$
$$\leq \frac{A_{\alpha} \circ \varphi^{-1}(D_{\delta}(b))}{A_{\alpha}(D_{\delta}(a))}$$
$$\approx \frac{A_{\alpha} \circ \varphi^{-1}(D_{\delta}(b))}{A_{\alpha}(D_{\delta}(b))}$$

	Background	Proof of Theorem	References
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Compactness			
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Let arphi(0)=0. By Schwartz Lemma, $D(0,r)\subset arphi^{-1}(D(0,r))$ and

$$D_{\delta_1}(\mathsf{a})\subset arphi^{-1}(D_\delta(\mathsf{b})), \qquad \mathsf{b}=arphi(\mathsf{a}).$$

If not, then

$$1 \approx \left(\frac{1 - |\varphi(\mathbf{a})|}{1 - |\mathbf{a}|}\right)^{2 + \alpha} \qquad \lesssim \qquad \frac{A_{\alpha}(D_{\delta_1}(b))}{A_{\alpha}(D_{\delta}(a))}$$
$$\leq \qquad \frac{A_{\alpha} \circ \varphi^{-1}(D_{\delta}(b))}{A_{\alpha}(D_{\delta}(a))}$$
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	Background	Proof of Theorem	References
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$$egin{aligned} &1pprox \left(rac{1-|arphi(a)|}{1-|a|}
ight)^{2+lpha} &\lesssim &rac{A_lpha(D_{\delta_1}(b))}{A_lpha(D_\delta(a))} \ &\leq &rac{A_lpha\circarphi^{-1}(D_\delta(b))}{A_lpha(D_\delta(a))} \ &pprox &rac{A_lpha\circarphi^{-1}(D_\delta(b))}{A_lpha(D_\delta(b))} \end{aligned}$$

	Background	Proof of Theorem	References
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	Background	Proof of Theorem	References
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$$egin{aligned} & A_lpha \circ arphi^{-1}(D(a)) \ &= & \int_{arphi^{-1}(D(a))} rac{(1-|z|)^{lpha-eta}}{(1-|arphi(z)|)^{lpha-eta}} (1-|arphi(z)|)^{lpha-eta} dA_eta(z) \ &\leq & \epsilon(1-|a|)^{lpha-eta} A_eta \circ arphi^{-1}(D(a)) \ &\lesssim & \epsilon(1-|a|)^{lpha-eta} A_eta(D(a)) \ &pprox & \epsilon A_lpha(D(a)) \end{aligned}$$

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$$\begin{array}{ll} & A_{\alpha} \circ \varphi^{-1}(D(a)) \\ = & \int_{\varphi^{-1}(D(a))} \frac{(1-|z|)^{\alpha-\beta}}{(1-|\varphi(z)|)^{\alpha-\beta}} (1-|\varphi(z)|)^{\alpha-\beta} dA_{\beta}(z) \\ \leq & \epsilon(1-|a|)^{\alpha-\beta} A_{\beta} \circ \varphi^{-1}(D(a)) \\ \lesssim & \epsilon(1-|a|)^{\alpha-\beta} A_{\beta}(D(a)) \\ \approx & \epsilon A_{\alpha}(D(a)) \end{array}$$

	Background	Proof of Theorem	References
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$$\begin{array}{ll} & A_{\alpha} \circ \varphi^{-1}(D(a)) \\ = & \int_{\varphi^{-1}(D(a))} \frac{(1-|z|)^{\alpha-\beta}}{(1-|\varphi(z)|)^{\alpha-\beta}} (1-|\varphi(z)|)^{\alpha-\beta} dA_{\beta}(z) \\ \leq & \epsilon(1-|a|)^{\alpha-\beta} A_{\beta} \circ \varphi^{-1}(D(a)) \\ \lesssim & \epsilon(1-|a|)^{\alpha-\beta} A_{\beta}(D(a)) \\ \approx & \epsilon A_{\alpha}(D(a)) \end{array}$$

	Background	Proof of Theorem	References
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Compactness			
Compactness			

$$egin{aligned} &A_lpha \circ arphi^{-1}(D(a))\ &=& \int_{arphi^{-1}(D(a))} rac{(1-|z|)^{lpha-eta}}{(1-|arphi(z)|)^{lpha-eta}}(1-|arphi(z)|)^{lpha-eta} dA_eta(z)\ &\leq& \epsilon(1-|a|)^{lpha-eta}A_eta\circarphi^{-1}(D(a))\ &\lesssim& \epsilon(1-|a|)^{lpha-eta}A_eta(D(a))\ &pprox& \epsilon A_lpha(D(a)) \end{aligned}$$

Introduction	Background	Proof of Theorem	References
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Compact Difference			

• Joint Carleson Measure(Saukko(2011), K-Wang(2014))

 $\mathcal{C}_{arphi}-\mathcal{C}_{\psi}$ compact on \mathcal{A}^{p}_{lpha} iff μ is an $lpha ext{-Carleson}$ where

$$\mu(E) = \int_{\varphi^{-1}(E)} \rho(\varphi, \psi)^{p} dA_{\alpha} + \int_{\psi^{-1}(E)} \rho(\varphi, \psi)^{p} dA_{\alpha}$$

where

$$\rho(z,w) := \left| \frac{z-w}{1-z\overline{w}} \right|.$$

	Background	Proof of Theorem	References
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Compact Difference			

Necessity

Suppose $\frac{\mu(D(a_k))}{A_{\alpha}(D(a_k))} > c > 0$, and let

$$f_{a}=\frac{1}{(1-z\overline{a})^{n}}.$$

Take test functions $f_k := f_{a_k}$ and $g_k = f_{b_k}$:

$$b_k := a_k(1 - N(1 - |a_k|)).$$

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Sufficiency

Submeanvalue property:

$$\begin{split} |f(a)-f(b)|^p &\leq |b-a|^p \sup_{[a,b]} |f'(z)|^p \\ &\lesssim \frac{\rho(a,b)^p}{(1-|a)|^{2+\alpha}} \int_{D_{\delta}(a)} |f(w)|^p dA_{\alpha}(w) \end{split}$$

For $z \notin E = \{z : \rho < \epsilon\}$ let $a = \varphi(z)$ and $b = \psi(z)$, then

$$|(C_{\varphi}-C_{\psi})f(z)|^p\lesssim rac{
ho(arphi(z),\psi(z))^p}{(1-|arphi(z)|)^{2+lpha}}\int_{D_{\delta}(arphi(z))}|f(w)|^p dA_{lpha}(w).$$

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Submeanvalue property:

$$egin{array}{rcl} |f(a)-f(b)|^p &\leq & |b-a|^p \, \sup_{[a,b]} |f'(z)|^p \ &\lesssim & rac{
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Compact	Difference: Joint Carleso	n measure	

Thus,

$$\begin{split} \| (C_{\varphi} - C_{\psi}) f \|^{p} \\ \lesssim \int_{\mathbf{D} \setminus E} \left(|C_{\varphi}(f)|^{p} + |C_{\psi}(f)|^{p} \right) dA_{\alpha} \\ + \int_{E} \left(\frac{\rho(\varphi(z), \psi(z))^{p}}{(1 - |\varphi(z)|)^{2 + \alpha}} \int_{D_{\delta}(\varphi(z))} |f(w)|^{p} dA_{\alpha}(w) \right) dA_{\alpha}(z) \end{split}$$

Compact	Difference: Joint Carleso	n measure	
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Introduction	Background	Proof of Theorem	References

Thus,

$$\begin{split} &\|(C_{\varphi}-C_{\psi})f\|^{p}\\ \lesssim &\int_{\mathbf{D}\setminus E}\left(|C_{\varphi}(f)|^{p}+|C_{\psi}(f)|^{p}\right)dA_{\alpha}\\ &+ &\int_{E}\left(\frac{\rho(\varphi(z),\psi(z))^{p}}{(1-|\varphi(z)|)^{2+\alpha}}\int_{D_{\delta}(\varphi(z))}|f(w)|^{p}dA_{\alpha}(w)\right)dA_{\alpha}(z) \end{split}$$

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Compact Difference:Characterization

• Moorhouse(2005)

$$\mathcal{C}_{arphi_1} - \mathcal{C}_{arphi_2}$$
 is compact on \mathcal{A}^{p}_{lpha} iff

$$\lim_{|\varphi_j(z)|\to 1}\rho(\varphi_1(z),\varphi_2(z))\,\,\frac{1-|z|}{1-|\varphi_j(z)|}=0.$$

Necessity

Adjoint action on kernels(Moorhouse for p = 2.) Test function f_a (Choe-K-Park(2014)).

	Background		Proof of Theorem	References
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Compact Difference: Characterization

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		Background		Proof of Theorem	References
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Compact Difference: Characterization

Sufficiency

Joint-Carleson measure criteria. Let $\rho(z) = \rho(\varphi_1(z), \varphi_2(z))$.

$$\int_{\varphi_j^{-1}(D(a))} \rho(z)^p dA_\alpha(z)$$

$$= \int_{\varphi_j^{-1}(D(a))} \left[\rho(z)^p \left(\frac{1-|z|}{1-|\varphi_j(z)|} \right)^{\alpha-\beta} \right] (1-|\varphi_j(z)|)^{\alpha-\beta} dA_\beta(z)$$

$$\lesssim \epsilon (1-|a|)^{\alpha-\beta} A_\beta \circ \varphi_j^{-1}(D(a))$$

$$\lesssim \epsilon A_\alpha(D(a))$$

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	Background	Proof of Theorem	References
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Compact Difference			
Compact Differen	ce:Characterization		

Joint-Carleson measure criteria. Let $\rho(z) = \rho(\varphi_1(z), \varphi_2(z))$.

$$\begin{split} & \int_{\varphi_j^{-1}(D(a))} \rho(z)^p dA_\alpha(z) \\ = & \int_{\varphi_j^{-1}(D(a))} \left[\rho(z)^p \left(\frac{1 - |z|}{1 - |\varphi_j(z)|} \right)^{\alpha - \beta} \right] (1 - |\varphi_j(z)|)^{\alpha - \beta} dA_\beta(z) \\ \lesssim & \epsilon (1 - |a|)^{\alpha - \beta} \ A_\beta \circ \varphi_j^{-1}(D(a)) \\ \lesssim & \epsilon A_\alpha(D(a)) \end{split}$$

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	Background	Proof of Theorem	References
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Consequences of Theorem 3			
Recall			

Let

$$T := T_{12} - T_{34} = T_{13} - T_{24}$$

We also put

$$\rho_{ij}(z) = \rho_{\varphi_i,\varphi_j}(z) := \rho(\varphi_i(z),\varphi_j(z))$$

and

$$M_{ij}(z)=M_{arphi_i,arphi_j}(z):=\left[rac{1-|z|}{1-|arphi_i(z)|}+rac{1-|z|}{1-|arphi_j(z)|}
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ho_{ij}(z).$$

Finally, we put

$$M = M_{12} + M_{34}$$
 and $\widetilde{M} := M_{13} + M_{24}$.

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Theorem 3

T is compact on $A^p_{\alpha}(\mathbf{D}) \iff \lim_{|z| \to 1} M(z)M(z) = 0.$
	Background	Proof of Theorem	References
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Consequences of Theorem 3			

If
$$\varphi_1 = \varphi_4$$
, then we get

$$T := T_{12} - T_{34} = 2C_{\varphi_1} - C_{\varphi_2} - C_{\varphi_3}.$$

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$$M = M_{12} + M_{34} = \widetilde{M} := M_{13} + M_{24}.$$

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- T is compact
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	Background	Proof of Theorem	References
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Consequences of Theorem 3			
$\varphi_4 \equiv 0$			

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ho_{j4}=
ho(arphi_j(z),0)=|arphi_j(z)|.$$

And

$$M_{j4}(z) = \left[\frac{1-|z|}{1-|\varphi_j(z)|} + 1 - |z|\right] |\varphi_j(z)| = \frac{1-|z|}{1-|\varphi_j(z)|} - (1-|z|)[1-|\varphi_j(z)|]$$

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•
$$C_{\varphi_1} - C_{\varphi_2} - C_{\varphi_3}$$
 is compact on $A^p_{\alpha}(\mathbf{D})$;
• $\lim_{|z| \to 1} \left[M_{12}(z) + \frac{1 - |z|}{1 - |\varphi_3(z)|} \right] \left[M_{13}(z) + \frac{1 - |z|}{1 - |\varphi_2(z)|} \right] = 0$;
• $F_1 = F_2 \cup F_3, F_2 \cap F_3 = \emptyset$ and $\lim_{z \to \zeta} M_{1j}(z) = 0$ for $\zeta \in F_j$.

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ight]|arphi_j(z)| = rac{1-|z|}{1-|arphi_j(z)|} - (1-|z|)[1-|arphi_j(z)|]$$

Thus, the following are equivalent.(Moorhouse(2005))

•
$$C_{\varphi_1} - C_{\varphi_2} - C_{\varphi_3}$$
 is compact on $A^p_{lpha}(\mathbf{D})$;

•
$$\lim_{|z| \to 1} \left[M_{12}(z) + \frac{1 - |z|}{1 - |\varphi_3(z)|} \right] \left[M_{13}(z) + \frac{1 - |z|}{1 - |\varphi_2(z)|} \right] = 0;$$

• $F_1 = F_2 \cup F_3$, $F_2 \cap F_3 = \emptyset$ and $\lim_{z \to \zeta} M_{1j}(z) = 0$ for $\zeta \in F_j$.

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	Background	Proof of Theorem	References
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Consequences of Theorem 3			
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$$\mathcal{T}:=\mathcal{T}_{12}-\mathcal{T}_{34}=\mathcal{C}_{arphi_1}-\mathcal{C}_{arphi_2}-\mathcal{C}_{arphi_3},
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And

= 0

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	Background	Proof of Theorem	References
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Proof			
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The following are equivalent.

(1)
$$\lim_{|z|\to 1} M(z)\widetilde{M}(z) = 0.$$

(2) For any $\zeta \in \mathbf{T}$ and any $z_n \to \zeta$, there is z_{n_k} such that

$$\lim_{k\to\infty}M(z_{n_k})=0\quad\text{or}\quad \lim_{k\to\infty}\widetilde{M}(z_{n_k})=0.$$

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Proof of $(1) \implies (2)$

Note that both M(z) and $\widetilde{M}(z)$ are non-negative.

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Proof of (1) \implies (2)

Note that both M(z) and $\widetilde{M}(z)$ are non-negative.

	Background	Proof of Theorem	References
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Proof			
Proposition			

Proof of (2) \implies (1)

Recall

$$M_{ij}(z)=M_{arphi_i,arphi_j}(z):=\left[rac{1-|z|}{1-|arphi_i(z)|}+rac{1-|z|}{1-|arphi_j(z)|}
ight]
ho_{ij}(z)$$

 and

$$M = M_{12} + M_{34}$$
 and $\widetilde{M} := M_{13} + M_{24}$.
Thus, both $M(z)$ and $\widetilde{M}(z)$ are bounded.
If not (1), there is a sequence $\{z_n\}$ such that $M(z_n)\widetilde{M}(z_n) > \delta_0 > 0$.

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	Background	Proof of Theorem	References		
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Proof					
Proof of Sufficiency					

$$U_{\epsilon} = \{z : M(z) \leq \epsilon\}, \quad \widetilde{U}_{\epsilon} = \{z : \widetilde{M}(z) \leq \epsilon\}.$$

Then, by assumption $MM \rightarrow 0$, for each $\zeta \in \mathbf{T}$

 $S(\zeta, \delta_{\zeta}) \subset U_{\epsilon} \cup \widetilde{U}_{\epsilon}$

for some $\delta_{\zeta}(\epsilon) > 0$, since otherwise $M(z_{\delta})\widetilde{M}(z_{\delta}) > \epsilon^2$, $z_{\delta} \to \zeta$. Since **T** is compact, there is ζ_j such that

$$\mathbf{D} \setminus (1-r)\mathbf{D} \subset \bigcup_{j=1}^N S(\zeta_j, \delta_j), \qquad r := \min\{\delta_j\} > 0.$$

Next, use standard argument with a sequence $\{f_n\}$ converging weakly to 0 and some weighted Carleson measure argument.

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	Background	Proof of Theorem	References		
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Proof					
Proof of Nec	Proof of Necessity				

$$M(z_n) \geq c > 0 \quad ext{and} \quad \widetilde{M}(z_n) > c > 0.$$

This implies the following holds:

 $\max\{M_{12}(z_n), M_{34}(z_n)\} \ge c/2$ and $\max\{M_{13}(z_n), M_{24}(z_n)\} \ge c/2$

Then, we have the following four possibilities:

(a)
$$\min\{M_{12}(z_n), M_{13}(z_n)\} \ge c/2;$$

- (b) $\min\{M_{12}(z_n), M_{24}(z_n)\} \ge c/2;$
- (c) $\min\{M_{34}(z_n), M_{13}(z_n)\} \ge c/2;$
- (d) $\min\{M_{34}(z_n), M_{24}(z_n)\} \ge c/2.$

Divide each cases into sever cases, and then take appropriate test functions to deduce a contradiction.

	Background	Proof of Theorem	References			
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- (d) $\min\{M_{34}(z_n), M_{24}(z_n)\} \ge c/2.$

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- (c) $\min\{M_{34}(z_n), M_{13}(z_n)\} \ge c/2;$
- (d) $\min\{M_{34}(z_n), M_{24}(z_n)\} \ge c/2.$

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Divide each cases into sever cases, and then take appropriate test functions to deduce a contradiction.

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THANKS A LOT !!