Compact double difference of composition operators

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Contents

Notation

- $\mathbf{D}$: unit disc in $\mathbf{C}$.
- $\mathbf{T} = \partial\mathbf{D}$: unit circle in $\mathbf{C}$.
- $H(\mathbf{D})$: class of all holomorphic functions on $\mathbf{D}$.
- $H^p(\mathbf{D})$: Hardy spaces on $\mathbf{D}$. Freely identified with $H^p(\mathbf{T})$.

$$f \in H^p(\mathbf{D}) \overset{def}{\iff} \|f\|_{H^p}^p = \sup_{0<r<1} \int_{\mathbf{T}} |f(r\zeta)|^p \, d\sigma(\zeta) < \infty$$

- $A^p_\alpha(\mathbf{D})$: Bergman space on $\mathbf{D}$.

$$f \in A^p_\alpha(\mathbf{D}) \overset{def}{\iff} \|f\|^p_{A^p_\alpha(\mathbf{D})} = \int_{\mathbf{D}} |f(z)|^p \, dA_\alpha(z) < \infty$$

where $\alpha > -1$ and $dA_\alpha(z) = c(1-|z|^2)^\alpha dA(z)$.

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Composition Operator

Composition Operator

For $\varphi : \Omega \to \Omega$ holomorphic self-map, composition operator is defined by

$$C_\varphi f = f \circ \varphi.$$

Examples of Ω: $\mathbf{D}, \mathbf{B}_n, \mathbf{D}^n, \mathbf{C}^n$, strongly pseudoconvex domain.

Question

For a smooth function g, we have

- $g(a+h) - g(a) = O(h)$.
- $g(a+h) - 2g(a) - g(a-h) = O(h^2)$.

Let

$$T_{ij} = C_{\varphi_i} - C_{\varphi_j} \quad \text{so that} \quad T_{ij} f(z) := f(\varphi_i(z)) - f(\varphi_j(z)),$$

and

$$Tf(z) := T_{12} f(z) - T_{23} f(z) = f(\varphi_1(z)) - 2f(\varphi_2(z)) + f(\varphi_3(z)).$$

In view of this, can T behavior better than T_{12} ?

Double Difference Cancelation?

Can $(C_{\varphi_1} - C_{\varphi_2}) - (C_{\varphi_2} - C_{\varphi_3})$ be compact while both $(C_{\varphi_1} - C_{\varphi_2})$ and $(C_{\varphi_2} - C_{\varphi_3})$ are not compact?

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Question

More generally,

Double Difference Cancelation?

Suppose $(C_{\varphi_1} - C_{\varphi_2}), (C_{\varphi_3} - C_{\varphi_4}), (C_{\varphi_1} - C_{\varphi_3})$ and $(C_{\varphi_2} - C_{\varphi_4})$ are all not compact.

Can $T := (C_{\varphi_1} - C_{\varphi_2}) - (C_{\varphi_3} - C_{\varphi_4}) = (C_{\varphi_1} - C_{\varphi_3}) - (C_{\varphi_2} - C_{\varphi_4})$ be compact?

Theorem 1

• Three sum

K-Wang (2015)
Let $0 < p < \infty$ and $\alpha > -1$. Let $a_i \in \mathbf{C} \setminus \{0\}$ and assume C_{φ_i} is not compact on $A^p_\alpha(\mathbf{D})$ for each $i = 1, 2, 3$. Let $T := \sum_{i=1}^3 a_i C_{\varphi_i}$.

If T compact on $A^p_\alpha(\mathbf{D})$, then one of the following holds:

- $T = a_i(C_{\varphi_i} - C_{\varphi_j} - C_{\varphi_k})$,
  where (i, j, k) is some permutation of $(1, 2, 3)$.
- $T = a_1(C_{\varphi_1} - C_{\varphi_2}) + a_3(C_{\varphi_3} - C_{\varphi_2})$.

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• Double difference

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$T := a(C_{\varphi_1} - C_{\varphi_2}) + b(C_{\varphi_3} - C_{\varphi_2})$ is compact on $A^p_\alpha(\mathbf{D})$
$\Longleftrightarrow$
both $C_{\varphi_1} - C_{\varphi_2}$ and $C_{\varphi_3} - C_{\varphi_2}$ are compact on $A^p_\alpha(\mathbf{D})$.

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Notation

Let

$$T := T_{12} - T_{34} = T_{13} - T_{24}, \quad T_{ij} = C_{\varphi_i} - C_{\varphi_j}.$$

We also put

$$\rho_{ij}(z) = \rho_{\varphi_i,\varphi_j}(z) := \rho\big(\varphi_i(z), \varphi_j(z)\big), \quad \rho(a,b) = \left|\frac{a-b}{1-a\overline{b}}\right|$$

and

$$M_{ij}(z) = M_{\varphi_i,\varphi_j}(z) := \left[\frac{1-|z|}{1-|\varphi_i(z)|} + \frac{1-|z|}{1-|\varphi_j(z)|}\right]\rho_{ij}(z).$$

Finally, we put

$$M = M_{12} + M_{34} \quad \text{and} \quad \widetilde{M} := M_{13} + M_{24}.$$

Theorem 3

• General double difference

Choe-K-Wang (2017)
$T := T_{12} - T_{34}$ is compact on $A^p_\alpha(\mathbf{D})$ if and only if

$$\lim_{|z|\to 1} M(z)\widetilde{M}(z) = 0.$$

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Questions for $H^p(\mathbf{D})$

• Component problems

Shapiro-Sundberg (1990)

- If $C_\varphi - C_\psi$ is compact, then do they belong to the same component?
- Is there non-compact C_φ which belongs to the component containing compact operators?

Known results for $H^p(\mathbf{D})$

• Component

Moorhouse-Toews (2001), Bourdon(2003)
There are C_φ and C_ψ which belong to the same component, but $C_\varphi - C_\psi$ is compact

• Component

Gallardo-Gutierrez, Gonzalez, Nieminen-Saksman (2008)

- $H^p(\mathbf{D})$: There is a non-compact C_φ which belongs to the component containing compact operators.
- $A^p_\alpha(\mathbf{D})$: The set of compact operators is a component.
- $A^p_\alpha(\mathbf{D})$: If the difference is compact, then they belong to the same component.

• Component

Nieminen-Saksman (2004)
$C_\varphi - C_\psi$ is compact on $H^p(\mathbf{D})$ for some $p \geq 1$, then for all $p \geq 1$.

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Questions for $H^p(\mathbf{D})$

• Component problems for $H^p(\mathbf{D})$

- Characterize components.
- Characterize the component containing compact operators.
- Characterize the compact difference, the joint Carleson measure.
- Characterize the double difference compact operators.

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Boundedness On Unit Disk

Weighted Bergman spaces

For $p > 0$ and $\alpha \geq -1$, the weighted Bergman space $A^p_\alpha(\mathbf{D})$ is the set of analytic functions f with

$$\|f\|^p := \int_{\mathbf{D}} |f(z)|^p dA_\alpha(z), \qquad dA_\alpha(z) := (1-|z|^2)^\alpha dA(z).$$

- Boundedness on weighted Bergman spaces

By Littlewood's Subordination Principle.

$$C_\varphi : A^p_\alpha \to A^p_\alpha.$$

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Subordination Principle

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If g subharmonic and φ analytic with $\varphi(0) = 0$, then

$$\int_0^{2\pi} g \circ \varphi(re^{i\theta})d\theta \le \int_0^{2\pi} g(re^{i\theta})d\theta.$$

Proof) Let $G = P(g)$, the Poisson integral of g.

$$\begin{aligned}\int_0^{2\pi} g \circ \varphi(re^{i\theta})\frac{d\theta}{2\pi} &\le \int_0^{2\pi} G \circ \varphi(re^{i\theta})\frac{d\theta}{2\pi} \\ &= G \circ \varphi(0) \\ &= \int_0^{2\pi} g(re^{i\theta})\frac{d\theta}{2\pi}. \qquad \square\end{aligned}$$

Let $g = |f|^p$ to get the boundedness on $A^p_\alpha(\mathbf{D})$.

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Carleson Measure

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For $0 < \delta < 1$, let $D(a) := D_\delta(a) := D(a, \delta(1-|a|))$. Then

$$\mu(D_\delta(a)) \lesssim (1-|a|)^{2+\alpha}$$

iff

$$\int_{\mathbb{D}} |f|^p d\mu \lesssim \int_{\mathbb{D}} |f|^p dA_\alpha.$$

$\Leftarrow$) Let $f_a(z) = \frac{1}{(1-z\bar{a})^n}$, then

$$\begin{aligned}\frac{A_\alpha \circ \varphi^{-1}(D(a))}{A_\alpha(D(a))} &\approx \frac{(1-|a|)^{np}}{(1-|a|)^{2+\alpha}} \int_{\varphi^{-1}(D(a))} \frac{1}{|1-\varphi(z)\bar{a}|^{np}} dA_\alpha(z) \\ &\lesssim \frac{\|f_a \circ \varphi\|^p}{\|f_a\|^p} \to 0 \text{ as } |a| \to 1.\end{aligned}$$

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Carleson Measure

$\Rightarrow$)

$$\begin{aligned}\int_{\mathbf{D}} |f|^p d\mu &\leq \int_{\mathbf{D}} \left(\frac{1}{A_\alpha(D_\delta(z))} \int_{D_\delta(z)} |f(w)|^p dA_\alpha(w) \right) d\mu(z) \\ &\leq \int_{\mathbf{D}} \left(\int_{\{z: w \in D_\delta(z)\}} d\mu(z) \right) \frac{|f(w)|^p}{(1-|w|)^{2+\alpha}} dA_\alpha(w) \\ &\lesssim \int_{\mathbf{D}} |f|^p dA_\alpha .\end{aligned}$$

Compact version: $\lim_{|a| \to 1} \frac{\mu(D(a))}{A_\alpha(D(a))} = 0.$

Carleson Measure

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$$
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\int_{\mathbb{D}} |f|^p d\mu &\leq \int_{\mathbb{D}} \left(\frac{1}{A_\alpha(D_\delta(z))} \int_{D_\delta(z)} |f(w)|^p dA_\alpha(w) \right) d\mu(z) \\
&\leq \int_{\mathbb{D}} \left(\int_{\{z : w \in D_\delta(z)\}} d\mu(z) \right) \frac{|f(w)|^p}{(1-|w|)^{2+\alpha}} dA_\alpha(w) \\
&\lesssim \int_{\mathbb{D}} |f|^p dA_\alpha .
\end{aligned}
$$

Compact version: $\lim_{|a|\to 1} \frac{\mu(D(a))}{A_\alpha(D(a))} = 0.$

Carleson Measure

$\Rightarrow)$

$$\begin{aligned}
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Compact version: $\lim_{|a| \to 1} \dfrac{\mu(D(a))}{A_\alpha(D(a))} = 0.$

Carleson Measure

Change of Variables

C_φ compact iff $A_\alpha \circ \varphi^{-1}$ is an α-Carleson measure.

Proof)

$$\int_{\mathbb{D}} |f \circ \varphi|^p dA = \int_{\mathbb{D}} |f|^p dA \circ \varphi^{-1}$$

where $A \circ \varphi^{-1}(E) := \int_{\varphi^{-1}(E)} dA$.

Carleson Measure

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Compactness

• Compactness on Bergman spaces

MacCluer and Shapiro (1986)
For $p > 0$, $\alpha > -1$, C_Φ is compact on A^p_α

$\iff \lim \frac{1-|z|}{1-|\varphi(z)|} = 0$ as $|z| \to 1^-$.

Remark: Julia-Caratheodory Theorem

φ has finite angular derivative at ζ.

$\iff \liminf_{z\to\zeta} \frac{1-|\varphi(z)|}{1-|z|} < \infty.$

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Compactness

Necessity

Let $\varphi(0) = 0$. By Schwartz Lemma, $D(0, r) \subset \varphi^{-1}(D(0, r))$ and

$$D_{\delta_1}(a) \subset \varphi^{-1}(D_\delta(b)), \qquad b = \varphi(a).$$

If not, then

$$\begin{aligned} 1 \approx \left(\frac{1 - |\varphi(a)|}{1 - |a|}\right)^{2+\alpha} &\lesssim \frac{A_\alpha(D_{\delta_1}(b))}{A_\alpha(D_\delta(a))} \\ &\leq \frac{A_\alpha \circ \varphi^{-1}(D_\delta(b))}{A_\alpha(D_\delta(a))} \\ &\approx \frac{A_\alpha \circ \varphi^{-1}(D_\delta(b))}{A_\alpha(D_\delta(b))} \end{aligned}$$

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Compactness

Sufficiency

$$
\begin{aligned}
& A_\alpha \circ \varphi^{-1}(D(a)) \\
= \quad & \int_{\varphi^{-1}(D(a))} \frac{(1-|z|)^{\alpha-\beta}}{(1-|\varphi(z)|)^{\alpha-\beta}}(1-|\varphi(z)|)^{\alpha-\beta} dA_\beta(z) \\
\leq \quad & \epsilon(1-|a|)^{\alpha-\beta} A_\beta \circ \varphi^{-1}(D(a)) \\
\lesssim \quad & \epsilon(1-|a|)^{\alpha-\beta} A_\beta(D(a)) \\
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Compact Difference:Joint Carleson measure

- Joint Carleson Measure(Saukko(2011), K-Wang(2014))

$C_\varphi - C_\psi$ compact on A^p_α iff μ is an α-Carleson where

$$\mu(E) = \int_{\varphi^{-1}(E)} \rho(\varphi,\psi)^p dA_\alpha + \int_{\psi^{-1}(E)} \rho(\varphi,\psi)^p dA_\alpha$$

where

$$\rho(z,w) := \left|\frac{z-w}{1-z\overline{w}}\right|.$$

Compact Difference:Joint Carleson measure

Necessity

Suppose $\frac{\mu(D(a_k))}{A_\alpha(D(a_k))} > c > 0$, and let

$$f_a = \frac{1}{(1 - z\bar{a})^n}.$$

Take test functions $f_k := f_{a_k}$ and $g_k = f_{b_k}$:

$$b_k := a_k(1 - N(1 - |a_k|)).$$

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Compact Difference:Joint Carleson measure

Sufficiency

Submeanvalue property:

$$|f(a)-f(b)|^p \leq |b-a|^p \sup_{[a,b]} |f'(z)|^p$$
$$\lesssim \frac{\rho(a,b)^p}{(1-|a)|)^{2+\alpha}} \int_{D_\delta(a)} |f(w)|^p dA_\alpha(w)$$

For $z \notin E = \{z : \rho < \epsilon\}$ let $a = \varphi(z)$ and $b = \psi(z)$, then

$$|(C_\varphi - C_\psi)f(z)|^p \lesssim \frac{\rho(\varphi(z),\psi(z))^p}{(1-|\varphi(z)|)^{2+\alpha}} \int_{D_\delta(\varphi(z))} |f(w)|^p dA_\alpha(w).$$

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Thus,

$$
\begin{aligned}
&\|(C_\varphi - C_\psi)f\|^p \\
\lesssim\ & \int_{\mathbb{D}\setminus E} \left(|C_\varphi(f)|^p + |C_\psi(f)|^p\right) dA_\alpha \\
+\ & \int_E \left(\frac{\rho(\varphi(z),\psi(z))^p}{(1-|\varphi(z)|)^{2+\alpha}} \int_{D_\delta(\varphi(z))} |f(w)|^p dA_\alpha(w) \right) dA_\alpha(z)
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Compact Difference:Characterization

- Moorhouse(2005)

$C_{\varphi_1} - C_{\varphi_2}$ is compact on A^p_α iff

$$\lim_{|\varphi_j(z)|\to 1} \rho(\varphi_1(z), \varphi_2(z)) \, \frac{1-|z|}{1-|\varphi_j(z)|} = 0.$$

Necessity

Adjoint action on kernels(Moorhouse for $p = 2$.)
Test function f_a(Choe-K-Park(2014)).

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Joint-Carleson measure criteria. Let $\rho(z) = \rho(\varphi_1(z), \varphi_2(z))$.

$$\begin{aligned}
&\int_{\varphi_j^{-1}(D(a))} \rho(z)^p dA_\alpha(z) \\
= &\int_{\varphi_j^{-1}(D(a))} \left[\rho(z)^p \left(\frac{1-|z|}{1-|\varphi_j(z)|}\right)^{\alpha-\beta}\right] (1-|\varphi_j(z)|)^{\alpha-\beta} dA_\beta(z) \\
\lesssim &\ \epsilon(1-|a|)^{\alpha-\beta}\ A_\beta \circ \varphi_j^{-1}(D(a)) \\
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Recall

Let

$$T := T_{12} - T_{34} = T_{13} - T_{24}$$

We also put

$$\rho_{ij}(z) = \rho_{\varphi_i,\varphi_j}(z) := \rho\big(\varphi_i(z), \varphi_j(z)\big)$$

and

$$M_{ij}(z) = M_{\varphi_i,\varphi_j}(z) := \left[\frac{1-|z|}{1-|\varphi_i(z)|} + \frac{1-|z|}{1-|\varphi_j(z)|}\right] \rho_{ij}(z).$$

Finally, we put

$$M = M_{12} + M_{34} \quad \text{and} \quad \widetilde{M} := M_{13} + M_{24}.$$

Theorem 3

T is compact on $A^p_\alpha(\mathbf{D}) \iff \lim_{|z|\to 1} M(z)\widetilde{M}(z) = 0.$

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$\varphi_1 = \varphi_4$

If $\varphi_1 = \varphi_4$, then we get

$$T := T_{12} - T_{34} = 2C_{\varphi_1} - C_{\varphi_2} - C_{\varphi_3}.$$

And

$$M = M_{12} + M_{34} = \widetilde{M} := M_{13} + M_{24}.$$

Thus, the following are equivalent.(K-Wang(2015))

- T is compact
- $\lim_{|z|\to 1}(M_{12}(z) + M_{13}(z)) = 0$
- $\lim_{|z|\to 1} M_{12}(z) = 0 = \lim_{|z|\to 1} M_{13}(z)$.
- T_{12}, T_{13} compact.

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$\varphi_1 = \varphi_4$

If $\varphi_1 = \varphi_2$, then we get

$$T := T_{12} - T_{34} = T_{43}.$$

And

$$M = M_{12} + M_{34} = M_{34}, \quad \widetilde{M} := M_{13} + M_{24} = M_{13} + M_{14}.$$

Thus, the following are equivalent.(Moorhouse(2005))

- T is compact
- $\lim_{|z|\to 1} M_{34}(z)(M_{13}(z) + M_{14}(z)) = 0$
- $\lim_{|z|\to 1} M_{34}(z) = 0$
- T_{34} compact.

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$\varphi_4 \equiv 0$

If $\varphi_4 \equiv 0$, then we get

$$T := T_{12} - T_{34} = C_{\varphi_1} - C_{\varphi_2} - C_{\varphi_3}, \rho_{j4} = \rho(\varphi_j(z), 0) = |\varphi_j(z)|.$$

And

$$M_{j4}(z) = \left[\frac{1-|z|}{1-|\varphi_j(z)|} + 1 - |z|\right] |\varphi_j(z)| = \frac{1-|z|}{1-|\varphi_j(z)|} - (1-|z|)[1-|\varphi_j(z)|]$$

Thus, the following are equivalent.(Moorhouse(2005))

- $C_{\varphi_1} - C_{\varphi_2} - C_{\varphi_3}$ is compact on $A^p_\alpha(\mathbf{D})$;
- $\lim_{|z|\to 1} \left[M_{12}(z) + \frac{1-|z|}{1-|\varphi_3(z)|}\right]\left[M_{13}(z) + \frac{1-|z|}{1-|\varphi_2(z)|}\right] = 0;$
- $F_1 = F_2 \cup F_3$, $F_2 \cap F_3 = \emptyset$ and $\lim_{z\to\zeta} M_{1j}(z) = 0$ for $\zeta \in F_j$.

$\varphi_4 \equiv 0$

If $\varphi_4 \equiv 0$, then we get

$$T := T_{12} - T_{34} = C_{\varphi_1} - C_{\varphi_2} - C_{\varphi_3}, \rho_{j4} = \rho(\varphi_j(z), 0) = |\varphi_j(z)|.$$

And

$$M_{j4}(z) = \left[\frac{1-|z|}{1-|\varphi_j(z)|} + 1 - |z|\right] |\varphi_j(z)| = \frac{1-|z|}{1-|\varphi_j(z)|} - (1-|z|)[1-|\varphi_j(z)|]$$

Thus, the following are equivalent.(Moorhouse(2005))

- $C_{\varphi_1} - C_{\varphi_2} - C_{\varphi_3}$ is compact on $A^p_\alpha(\mathbf{D})$;
- $\lim_{|z|\to 1} \left[M_{12}(z) + \frac{1-|z|}{1-|\varphi_3(z)|}\right]\left[M_{13}(z) + \frac{1-|z|}{1-|\varphi_2(z)|}\right] = 0;$
- $F_1 = F_2 \cup F_3$, $F_2 \cap F_3 = \emptyset$ and $\lim_{z\to\zeta} M_{1j}(z) = 0$ for $\zeta \in F_j$.

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The following are equivalent.

(1) $\lim_{|z|\to 1} M(z)\widetilde{M}(z) = 0$.

(2) For any $\zeta \in \mathbf{T}$ and any $z_n \to \zeta$, there is z_{n_k} such that

$$\lim_{k\to\infty} M(z_{n_k}) = 0 \quad \text{or} \quad \lim_{k\to\infty} \widetilde{M}(z_{n_k}) = 0.$$

Proof of (1) $\implies$ (2)

Note that both $M(z)$ and $\widetilde{M}(z)$ are non-negative.

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Proof of (2) $\implies$ (1)

Recall

$$M_{ij}(z) = M_{\varphi_i,\varphi_j}(z) := \left[\frac{1-|z|}{1-|\varphi_i(z)|} + \frac{1-|z|}{1-|\varphi_j(z)|}\right]\rho_{ij}(z)$$

and

$$M = M_{12} + M_{34} \quad \text{and} \quad \widetilde{M} := M_{13} + M_{24}.$$

Thus, both $M(z)$ and $\widetilde{M}(z)$ are bounded.

If not (1), there is a sequence $\{z_n\}$ such that $M(z_n)\widetilde{M}(z_n) > \delta_0 > 0$.

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Proof of Sufficiency

Let

$$U_\epsilon = \{z : M(z) \le \epsilon\}, \quad \widetilde{U}_\epsilon = \{z : \widetilde{M}(z) \le \epsilon\}.$$

Then, by assumption $M\widetilde{M} \to 0$, for each $\zeta \in \mathbf{T}$

$$S(\zeta, \delta_\zeta) \subset U_\epsilon \cup \widetilde{U}_\epsilon$$

for some $\delta_\zeta(\epsilon) > 0$, since otherwise $M(z_\delta)\widetilde{M}(z_\delta) > \epsilon^2$, $z_\delta \to \zeta$.
Since $\mathbf{T}$ is compact, there is ζ_j such that

$$\mathbf{D} \setminus (1-r)\mathbf{D} \subset \bigcup_{j=1}^{N} S(\zeta_j, \delta_j), \qquad r := \min\{\delta_j\} > 0.$$

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Proof of Necessity

Suppose $M\widetilde{M} \not\to 0$. Pick a sequence $z_n \to \zeta$ so that

$$M(z_n) \geq c > 0 \quad \text{and} \quad \widetilde{M}(z_n) > c > 0.$$

This implies the following holds:

$$\max\{M_{12}(z_n), M_{34}(z_n)\} \geq c/2 \quad \text{and} \quad \max\{M_{13}(z_n), M_{24}(z_n)\} \geq c/2$$

Then, we have the following four possibilities:

(a) $\min\{M_{12}(z_n), M_{13}(z_n)\} \geq c/2$;

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Divide each cases into sever cases, and then take appropriate test functions to deduce a contradiction.

Proof of these are long and some parts are delicate.

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THANKS A LOT !!