Subnormal weighted shifts on directed trees whose nth powers have trivial domain

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OTOA 2016
19.12.2016 - Bangalore

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- Let $\mathscr{T} = (V, E)$ be a directed tree.
- Let $\ell^2(V)$ be the space of all square summable function on V with a scalar products

$$\langle f, g\rangle = \sum_{u\in V} f(u)\overline{g(u)}, \quad f, g \in \ell^2(V).$$

- For $u \in V$, let us define $e_u \in \ell^2(V)$ by

$$e_u(v) = \begin{cases} 1 & \text{if } u = v, \\ 0 & \text{if } u \neq v. \end{cases}$$

- $\{e_u\}_{u\in V}$ is an orthonormal basis in $\ell^2(V)$.

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- For a family $\boldsymbol{\lambda} = \{\lambda_v\}_{v \in V^\circ} \subseteq \mathbb{C}$ let us define an operator $S_{\boldsymbol{\lambda}}$ in $\ell^2(V)$ by

$$\mathcal{D}(S_{\boldsymbol{\lambda}}) = \{f \in \ell^2(V) \colon \Lambda_{\mathscr{T}} f \in \ell^2(V)\},$$
$$S_{\boldsymbol{\lambda}} f = \Lambda_{\mathscr{T}} f, \quad f \in \mathcal{D}(S_{\boldsymbol{\lambda}}),$$

- where $\Lambda_{\mathscr{T}}$ is define on functions $f \colon V \to \mathbb{C}$ by

$$(\Lambda_{\mathscr{T}} f)(v) = \begin{cases} \lambda_v \cdot f\big(\operatorname{par}(v)\big) & \text{if } v \in V^\circ, \\ 0 & \text{if } v = \operatorname{root}. \end{cases}$$

- An operator $S_{\boldsymbol{\lambda}}$ is called a *weighted shift on a directed tree* $\mathscr{T}$ with weights $\{\lambda_v\}_{v \in V^\circ}$.

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- Is it true that for every integer $n \geqslant 1$, there exists a subnormal weighted shift on a directed tree whose nth power is densely defined and the domain of its $(n+1)$th power is trivial?
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Characterization

Theorem

Let $S_{\boldsymbol{\lambda}}$ be a w.s. on a countably infinite directed tree $\mathscr{T} = (V, E)$ with weights $\boldsymbol{\lambda} = \{\lambda_v\}_{v \in V^\circ}$. Suppose $\exists \{\mu_v\}_{v \in V}$ of Borel probability measures on $\mathbb{R}_+$ and $\{\varepsilon_v\}_{v \in V} \subseteq \mathbb{R}_+$ such that

$$\mu_u(\varDelta) = \sum_{v \in \mathrm{Chi}(u)} |\lambda_v|^2 \int_{\varDelta} \frac{1}{s} \mathrm{d}\mu_v(s) + \varepsilon_u \delta_0(\varDelta), \quad \varDelta \in \mathfrak{B}(\mathbb{R}_+),\ u \in V. \tag{1}$$

Then the following two assertions hold:

1. (i) *if $S_{\boldsymbol{\lambda}}$ is densely defined, then $S_{\boldsymbol{\lambda}}$ is subnormal,*
2. (ii) *if $n \in \mathbb{N}$, then $S_{\boldsymbol{\lambda}}^n$ is densely defined if and only if $\int_0^\infty s^n \,\mathrm{d}\mu_u(s) < \infty$ for every $u \in V$ such that $\mathrm{Chi}(u)$ has at least two vertices.*

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Lemma

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Suppose μ *is a finite Borel measure on* $\mathbb{R}_+$ *such that* $\int_0^\infty s^n \mathrm{d}\mu(s) < \infty$ *for some* $n \in \mathbb{N}$*. Then* $\int_0^\infty s^k \mathrm{d}\mu(s) < \infty$ *for every* $k \in \mathbb{N}$ *such that* $k \leqslant n$*.*

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Lemma

Let $S_{\boldsymbol\lambda}$ be a weighted shift on a directed tree $\mathscr{T} = (V, E)$ with weights $\boldsymbol\lambda = \{\lambda_v\}_{v \in V^\circ}$ and let $n \in \mathbb{N}$. Then the following two conditions are equivalent:

(i) $\mathcal{D}(S_{\boldsymbol\lambda}^n) = \{0\}$,

(ii) $e_u \notin \mathcal{D}(S_{\boldsymbol\lambda}^n)$ *for every* $u \in V$.

Moreover, if there exist a family $\{\mu_v\}_{v \in V}$ *of Borel probability measures on* $\mathbb{R}_+$ *and a family* $\{\varepsilon_v\}_{v \in V} \subseteq \mathbb{R}_+$ *which satisfy* (1), *then* (i) *is equivalent to*

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Observation

If there exists a weighted shift S_λ on a directed tree $\mathscr{T}$ with nonzero weights such that

$$S_\lambda \text{ is densely defined and } \mathcal{D}(S_\lambda^2) = \{0\},$$

then the directed tree $\mathscr{T}$ is extremal.

Main theorem

Theorem

Suppose $\mathscr{T} = (V, E)$ is an extremal directed tree and $n \in \mathbb{N}$. Then there exists a subnormal weighted shift $S_{\boldsymbol{\lambda}}$ on $\mathscr{T}$ with nonzero weights such that $S_{\boldsymbol{\lambda}}^n$ is densely defined and $\mathcal{D}(S_{\boldsymbol{\lambda}}^{n+1}) = \{0\}$.

Lemma

If A is an operator such that $\mathcal{D}(A^n) = \{0\}$ for some positive integer n, then A is injective.

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Main corollary

Corollary

For every $n \in \mathbb{N}$, there exists an unbounded subnormal composition operator C in an L^2-space over σ-finite measure space such that C^n is densely defined and $\mathcal{D}(C^{n+1}) = \{0\}$.

Lemma

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If $m \in \mathbb{N}$ and Δ is a countable subset of $\mathbb{R}_+$ such that $\sup \Delta = \infty$, then there exists a finite discrete Borel measure μ on $\mathbb{R}_+$ such that $\mathrm{At}(\mu) = \Delta$, $\int_0^\infty s^m \mathrm{d}\mu(s) < \infty$ and $\int_0^\infty s^{m+1} \mathrm{d}\mu(s) = \infty$.

Corollary

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If $m \in \mathbb{N}$, $\vartheta \in \mathbb{R}_+$ *and* E *is a countably infinite subset of* $\mathbb{R}_+$, *then there exists a finite discrete Borel measure* μ *on* $\mathbb{R}_+$ *such that* $\mathrm{At}(\mu)$ *is a countably infinite subset of* $[\vartheta, \infty)$, $E \cap \mathrm{At}(\mu) = \varnothing$, $\int_0^\infty s^m \mathrm{d}\mu(s) < \infty$ *and* $\int_0^\infty s^{m+1} \mathrm{d}\mu(s) = \infty$.

Lemma

Set $\mathscr{X} = \bigcup_{k=0}^{\infty} \mathscr{X}_k$, where $\mathscr{X}_k = \bigsqcup_{j=0}^{k} \mathbb{N}^j$ with $\mathbb{N}^0 = \{0\}$.

Lemma

If $n \in \mathbb{N}$ and $\vartheta \in \mathbb{R}_+$, then there exists a family $\{\nu_{\boldsymbol{x}}\}_{\boldsymbol{x} \in \mathscr{X}}$ of finite discrete Borel measures on $\mathbb{R}_+$ such that

(i) *$\{\mathrm{At}(\nu_{\boldsymbol{x}})\}_{\boldsymbol{x} \in \mathscr{X}}$ are pairwise disjoint countably infinite subsets of $[\vartheta, \infty)$,*

(ii) *$\sum_{\boldsymbol{x} \in \mathbb{N}^k} \int_0^\infty s^{k+n} \mathrm{d}\nu_{\boldsymbol{x}}(s) \leqslant 2^{-k}$ for all $k \in \mathbb{Z}_+$,*

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If $n \in \mathbb{N}$ and $\vartheta \in [1, \infty)$, then there exist a family $\{\Omega_{\boldsymbol{x}}\}_{\boldsymbol{x} \in \mathscr{X}}$ of countably infinite subsets of $[\vartheta, \infty)$ and a discrete measure $\nu \in \mathcal{P}_\vartheta(\mathbb{R}_+)$ such that

(i) $\mathrm{At}(\nu) = \Omega_0$,

(ii) $\Omega_0 = \bigsqcup_{j_1=1}^{\infty} \Omega_{j_1}$ *and* $\Omega_{j_1,\ldots,j_k} = \bigsqcup_{j_{k+1}=1}^{\infty} \Omega_{j_1,\ldots,j_k,j_{k+1}}$ *for all* $(j_1, \ldots, j_k) \in \mathbb{N}^k$ *and* $k \in \mathbb{N}$,

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Proof of Lemma

$$\Omega_0 = \bigsqcup_{\boldsymbol{x} \in \mathscr{X}} \Delta_{\boldsymbol{x}} = \bigsqcup_{k=0}^{\infty} \bigsqcup_{\boldsymbol{x} \in \mathbb{N}^k} \Delta_{\boldsymbol{x}} \text{ with } \Delta_{\boldsymbol{x}} = \mathrm{At}(\nu_{\boldsymbol{x}}) \text{ for every } \boldsymbol{x} \in \mathscr{X}$$

and

$$\nu = \sum_{k=0}^{\infty} \sum_{\boldsymbol{x} \in \mathbb{N}^k} \nu_{\boldsymbol{x}}.$$

Proof of Lemma

$$\Omega_0 = \bigsqcup_{j_1'=1}^{\infty} \Omega_{j_1'},$$

$$\text{if } l \geqslant 2, \text{ then } \Omega_{j_1,\ldots,j_{l-1}} = \bigsqcup_{j_l'=1}^{\infty} \Omega_{j_1,\ldots,j_{l-1},j_l'},$$

$$\left.\begin{array}{l} \{t_{j_1'}\}_{j_1'=1}^{\infty} \text{ is an injective sequence in } \Delta_0 \text{ such that } \Delta_0 = \{t_{j_1'} : j_1' \in \mathbb{N}\}, \\ \text{if } l \geqslant 2, \text{ then } \{t_{j_1,\ldots,j_{l-1},j_l'}\}_{j_l'=1}^{\infty} \text{ is an injective sequence in } \displaystyle\bigsqcup_{\boldsymbol{x} \in \mathscr{X}_{l-1}} \Delta_{\boldsymbol{x}} \\ \qquad \text{such that } \{t_{j_1,\ldots,j_{l-1}}\} \sqcup \Delta_{j_1,\ldots,j_{l-1}} = \{t_{j_1,\ldots,j_{l-1},j_l'} : j_l' \in \mathbb{N}\}, \end{array}\right\}$$

$$\Omega_{j_1,\ldots,j_l} = \{t_{j_1,\ldots,j_l}\} \sqcup \Delta_{j_1,\ldots,j_l} \sqcup \bigsqcup_{p=1}^{\infty} \bigsqcup_{(j_{l+1}',\ldots,j_{l+p}') \in \mathbb{N}^p} \Delta_{j_1,\ldots,j_l,j_{l+1}',\ldots,j_{l+p}'}.$$

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Let $\mathscr{T} = (V, E)$ *be an extremal directed tree. Suppose* $n \in \mathbb{N}$, $\vartheta \in [1, \infty)$ *and* $w \in V$*. Then there exist systems* $\{\lambda_v\}_{v \in \mathsf{Des}(w)^\circ} \subseteq (0, \infty)$ *and* $\{\mu_v\}_{v \in \mathsf{Des}(w)} \subseteq \mathcal{P}_\vartheta(\mathbb{R}_+)$ *such that for every* $u \in \mathsf{Des}(w)$,

$$\mu_u(\varDelta) = \sum_{v \in \mathsf{Chi}(u)} \lambda_v^2 \int_\varDelta \frac{1}{s} \mathrm{d}\mu_v(s) \text{ for every } \varDelta \in \mathfrak{B}(\mathbb{R}_+), \tag{2}$$

$$\int_0^\infty s^n \mathrm{d}\mu_u(s) < \infty \text{ and } \int_0^\infty s^{n+1} \mathrm{d}\mu_u(s) = \infty. \tag{3}$$

Proof of Lemma

Set $\mu_0 = \nu$. Then $\mu_0 \in \mathcal{P}_\vartheta(\mathbb{R}_+)$. For a given $k \in \mathbb{N}$ and $(j_1, \ldots, j_k) \in \mathbb{N}^k$, we define the Borel measure $\mu_{j_1,\ldots,j_k}$ on $\mathbb{R}_+$ and $\lambda_{j_1,\ldots,j_k} \in (0, \infty)$ by

$$\mu_{j_1,\ldots,j_k}(\varDelta) = \frac{\int_{\varDelta \cap \varOmega_{j_1,\ldots,j_k}} s^k \mathrm{d}\nu(s)}{\int_{\varOmega_{j_1,\ldots,j_k}} s^k \mathrm{d}\nu(s)}, \quad \varDelta \in \mathfrak{B}(\mathbb{R}_+),$$

$$\lambda_{j_1,\ldots,j_k} = \begin{cases} \sqrt{\int_{\varOmega_{j_1,\ldots,j_k}} s^k \mathrm{d}\nu(s)} & \text{if } k = 1, \\ \sqrt{\dfrac{\int_{\varOmega_{j_1,\ldots,j_k}} s^k \mathrm{d}\nu(s)}{\int_{\varOmega_{j_1,\ldots,j_{k-1}}} s^{k-1} \mathrm{d}\nu(s)}} & \text{if } k \geqslant 2. \end{cases}$$

Lemma

Lemma

Let $\mathscr{T} = (V, E)$ *be an extremal directed tree,* $w \in V^\circ$*,* $x = \mathrm{par}(w)$ *and* $n \in \mathbb{N}$*. Suppose that* $\{\lambda_v\}_{v \in \mathrm{Des}(w)^\circ} \subseteq (0, \infty)$ *and* $\{\mu_v\}_{v \in \mathrm{Des}(w)} \subseteq \mathcal{P}_1(\mathbb{R}_+)$ *satisfy* (2) *and* (3) *for every* $u \in \mathrm{Des}(w)$*. Then there exist* $\{\lambda_v\}_{v \in \mathrm{Des}(x)^\circ \setminus \mathrm{Des}(w)^\circ} \subseteq (0, \infty)$ *and* $\{\mu_v\}_{v \in \mathrm{Des}(x) \setminus \mathrm{Des}(w)} \subseteq \mathcal{P}_1(\mathbb{R}_+)$ *such that* $\{\lambda_v\}_{v \in \mathrm{Des}(x)^\circ}$ *and* $\{\mu_v\}_{v \in \mathrm{Des}(x)}$ *satisfy* (2) *and* (3) *for all* $u \in \mathrm{Des}(x)$*.*

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