Optimal domains for operators on function spaces

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Aim

* To present an overview of the concept of optimal domain for an operator on a function space.

* To discuss some of its applications: to classical inequalities (Sobolev, Hausdorff-Young), and to the extension of operators (Cesàro operator on Hardy spaces).

1. Representation versus extension

2. Three examples

Riesz theorem

Theorem (F. Riesz, 1909)

Let:

(a) *K be a compact Hausdorff space.*

(b) *$\mathcal{C}(K)$ be the space of continuous functions over K.*

(c) *$\Lambda\colon \mathcal{C}(K) \to \mathbb{C}$ be a positive linear functional.*

Then, there exists a regular, Borel measure μ on K such that

$$\Lambda(f) = \int_K f\, d\mu, \qquad f \in \mathcal{C}(K).$$

Riesz theorem revisited

Theorem (Extension version)

Let:

(a) K *be a compact Hausdorff space.*

(b) $\mathcal{C}(K)$ *be the space of continuous functions over* K.

(c) $\Lambda\colon \mathcal{C}(K) \to \mathbb{C}$ *be a positive linear functional.*

Then, there exists a unique, regular, Borel measure μ *on* K *such that*

$$\begin{tikzcd} \mathcal{C}(K) \arrow[rr, "\Lambda"] \arrow[d] & & \mathbb{C} \\ L^1(\mu) \arrow[urr, "I_\mu"] & & \end{tikzcd}$$

where $\mathcal{C}(K) \subseteq L^1(\mu)$ *and* I_μ *is the integration operator on* $L^1(\mu)$.

Vector analog

Theorem (Bartle, Dunford & Schwartz, 1955)

Let:

(a) *K be a compact Hausdorff space and $\mathcal{C}(K)$ be the space of continuous functions over K.*

(b) *X be a Banach space.*

(c) *$T\colon \mathcal{C}(K) \to X$ a weakly compact operator.*

Then, there exists

$$\nu\colon \mathcal{B}_o(K) \to X$$

a countably additive, X–valued measure on the Borel sets of K such that

$$Tf = \int_K f\,d\nu, \quad f \in \mathcal{C}(K).$$

Note: a vector integration theory is needed.

BDS theorem revisited

Theorem (Extension version)

Let:

(a) *K be a compact Hausdorff space and $\mathcal{C}(K)$ be the space of continuous functions over K.*

(b) *X be a Banach space.*

(c) *$T : \mathcal{C}(K) \to X$ a weakly compact operator.*

Then, there exists $\nu : \mathcal{B}_o(K) \to X$ such that

$$\begin{tikzcd} \mathcal{C}(K) \arrow[rr, "\Lambda"] \arrow[d] & & X \\ L^1(\nu) \arrow[urr, "I_\nu"] & & \end{tikzcd}$$

where $\mathcal{C}(K) \subseteq L^1(\nu)$ and I_ν is the BDS–integration operator on $L^1(\nu)$.

A general extension result

Theorem (C. & Ricker)

Let E be a Banach function space over finite measure space. If $T\colon E \to X$ a linear operator satisfying

$$f_n \uparrow f \text{ in } E \quad \Rightarrow \quad T(f_n) \to T(f) \text{ weakly in } X,$$

then, there exists an extension of T:

$$\begin{tikzcd} E \arrow[rr, "T"] \arrow[d, "j"'] & & X \\ L^1(\nu) \arrow[urr, "I_\nu"] & & \end{tikzcd}$$

where $L^1(\nu)$ is the largest domain where order bounded increasing sequences are norm convergent.

The L^1-space

- $L^1(\nu)$ is a Banach space of scalar measurable functions.
- Examples:
  - $L^1([0,1])$, $L^1(\mathbb{R}^n)$, weighted L^1-spaces.
  - All L^p-spaces for $1 \leq p < \infty$, **including L^2!**
  - Lorentz $L^{p,q}$-spaces, Orlicz spaces L^ϕ, Marcinkiewicz spaces $M(\varphi)$, Lorentz Λ-spaces, spaces Exp L^p of exponential integrability, ...

Theorem (C.)

Any Banach space of measurable functions where

order bounded increasing sequences are norm convergent

is L^1 of some vector measure.

Optimal extension

Extension diagram:

$$\begin{tikzcd} E \arrow[rr, "T"] \arrow[d, "j"'] & & X \\ Z \arrow[urr, "\tilde{T}"] & & \end{tikzcd}$$

Problem: to find the largest space Z within a certain class of spaces.

Three examples

We present 3 examples of application of optimal domains:

* The Sobolev embedding
* The Hausdorff-Young inequality
* The Cesàro operator on Hardy spaces

1. Optimal Sobolev embeddings

Sobolev embedding

Theorem (Sobolev, 1938)

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain and $1 \leq p < n$. There exist $C > 0$ such that

$$\|u\|_{L^q(\Omega)} \leq C \| \, |\nabla u| \, \|_{L^p(\Omega)}, \quad u \in \mathcal{C}_0^1(\Omega),$$

where $q := \frac{np}{n-p}$. Note that: $q > p \Rightarrow \; L^q(\Omega) \subsetneq L^p(\Omega)$.

- Problem: Is there a norm $\| \cdot \|_{Y(\Omega)}$, smaller than $\| \cdot \|_{L^p(\Omega)}$, such that

$$\|u\|_{L^q(\Omega)} \leq C \| \, |\nabla u| \, \|_{Y(\Omega)}, \quad u \in \mathcal{C}_0^1(\Omega)?$$

Refining the inequality $\equiv$ Extending the embedding

- We write the inequality as the Sobolev's embedding

$$W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega).$$

- Problem I: Find a space $Y(\Omega)$, larger than $L^p(\Omega)$, such that

$$\begin{tikzcd} W_0^{1,p}(\Omega) \arrow[rr] \arrow[d] & & L^q(\Omega) \\ W_0^1 Y(\Omega) \arrow[urr] & & \end{tikzcd}$$

- Problem II: Find the largest of such spaces.

Optimal embeddings

- Within the class of L^p-spaces

$$W_0^1 L^{\frac{nq}{n+q}}(\Omega) \hookrightarrow L^q(\Omega) \quad \text{is optimal.}$$

- Within the class of Lorentz $L^{p,q}$-spaces

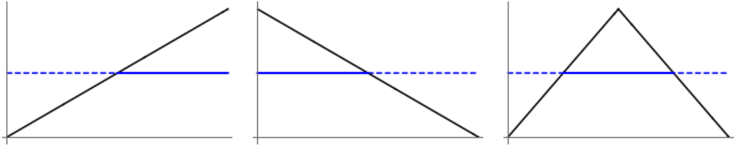
$$W_0^1 L^{\frac{nq}{n+q},q}(\Omega) \hookrightarrow L^q(\Omega) \quad \text{is optimal.}$$

- Within the class of Marzinkiewicz spaces

$$W_0^1 L^{\frac{nq}{n+q},\infty}(\Omega) \hookrightarrow L^{q,\infty}(\Omega) \quad \text{is optimal.}$$

A larger class of spaces

- Rearrangement invariant (r.i.) spaces: spaces of measurable functions where membership (of a function) is determined **by size, not by shape**



- Examples: L^p-spaces, Lorentz $L^{p,q}$-spaces, Orlicz spaces L^ϕ, Marcinkiewicz spaces $M(\varphi)$, Lorentz Λ-spaces, spaces, Exp L^p of exponential integrability, ...
- Is there a Sobolev type inequality

$$\|u\|_{X(\Omega)} \leq C \, \| \, |\nabla u| \, \|_{Y(\Omega)}, \quad u \in \mathcal{C}_0^1(\Omega)$$

obtained by substituting the spaces $L^p(\Omega)$ and $L^q(\Omega)$ by other **r.i. spaces** $X(\Omega)$, $Y(\Omega)$?

Reduction to a one dimensional problem

Theorem (Edmunds, Kerman & Pick)

Let X, Y be r.i. spaces. (*for example* $X = L^q$, $Y = L^p$)

Then:

$$\left.\begin{array}{c} \|u\|_{X(\Omega)} \leq C\| \, |\nabla u| \, \|_{Y(\Omega)} \\ \\ \text{for all } u \in C_0^1(\Omega) \end{array}\right\} \iff \left\{\begin{array}{c} \|Tf\|_X \leq K\|f\|_Y \\ \\ \text{for all } f \in Y \end{array}\right.$$

where T is the kernel operator associated with Sobolev's inequality

$$t \in [0,1] \longmapsto Tf(t) := \int_t^1 f(s) s^{\frac{1}{n}-1}\, ds, \qquad f \colon [0,1] \to \mathbb{R}.$$

Optimal Sobolev embedding

- We apply the optimal extension procedure to $T\colon Y \to X$:

$$
\begin{tikzcd}
Y \arrow[rr, "T"] \arrow[d, "j"'] & & X \\
{[T,X]^{\mathrm{ri}}} \arrow[urr, "T"] & &
\end{tikzcd}
$$

  $[T,X]^{\mathrm{ri}}$ is the optimal rearrangement invariant domain for T with values in X.

- Via the result of Edmunds, Kerman & Pick we obtain the optimal rearrangement invariant Sobolev inequality

$$\|u\|_{X(\Omega)} \leq C \|\,|\nabla u|\,\|_{[T,X]^{\mathrm{ri}}(\Omega)}, \quad u \in C_0^1(\Omega).$$

A compactness theorem

Theorem (Rellich 1930; Kondrachov 1945)

Let $1 \leq p < n$*. The embedding*

$$W_0^{1,p}(\Omega) \longrightarrow L^q(\Omega)$$

is compact whenever $q < \frac{np}{n-p}$

- Q: Under what conditions on the r.i. spaces X, Y we have compactness of the embedding

$$W_0^1 Y(\Omega) \hookrightarrow X(\Omega)?$$

- Q: When is it compact the optimal embedding

$$W_0^1 [T, X]^{ri}(\Omega) \hookrightarrow X(\Omega)?$$

An optimal compactness theorem

Theorem (Pustylnik / C. & Ricker / Kerman & Pick)

$$\left.\begin{array}{c} W_0^1 Y(\Omega) \hookrightarrow X(\Omega) \\ \textit{is compact} \end{array}\right\} \iff \left\{\begin{array}{l} T\colon Y \to X \\ \textit{is compact} \end{array}\right.$$

Corollary

The compactness of optimal rearrangement invariant Sobolev embedding

$$W_0^1[T,X]^{ri}(\Omega) \hookrightarrow X(\Omega)$$

depends on the relation between the fundamental function of the spaces X and $L^{n'}$

$$\varphi_X(t) := \|\chi_{[0,t]}\|_X, \quad \varphi_{L^{n'}} = t^{1/n'}.$$

2. Optimal Hausdorff-Young inequality

Hausdorff-Young inequality

Theorem (Young, 1913; Hausdorff, 1923)

Let $1 \leq p \leq 2$ *and* $1/p + 1/p' = 1$*. The Fourier transform maps*

$$\mathcal{F}\colon L^p(\mathbb{T}) \to \ell^{p'}(\mathbb{Z}),$$

and satisfies

$$\|\mathcal{F}(f)\|_{p'} \leq \|f\|_p, \qquad f \in L^p(\mathbb{T}).$$

- Can we extend $\mathcal{F}$ to a **larger** space of measurable functions $\mathbf{F}^p(\mathbb{T})$ such that

  $$\mathcal{F}\colon \mathbf{F}^p(\mathbb{T}) \to \ell^{p'}(\mathbb{Z})$$

  still is bounded?

Optimal Hausdorff-Young inequality

Theorem (Mockenhaupt & Ricker)

Let $1 < p \leq 2$ and $\mathcal{F}$ be the Fourier transform.

(a) *The vector measure m_p*

$$A \in \mathcal{B}_o(\mathbb{T}) \longmapsto m_p(A) := \mathcal{F}(\chi_A) \in \ell^{p'}(\mathbb{Z}),$$

is countably additive. Set $\mathbf{F}^p(\mathbb{T}) := L^1(m_p)$.

(b) *The Hausdorff-Young inequality can be extended to functions in $\mathbf{F}^p(\mathbb{T})$:*

$$\|\mathcal{F}(f)\|_{p'} \leq 4\|f\|_{\mathbf{F}^p(\mathbb{T})}, \quad f \in \mathbf{F}^p(\mathbb{T}).$$

(c) *$\mathbf{F}^p(\mathbb{T})$ is the largest Banach function space on $\mathbb{T}$ with the above properties (& order bounded increasing seq. are convergent).*

Problem

The space $\mathbf{F}^p(\mathbb{T})$ can be alternatively described as:

$$\mathbf{F}^p(\mathbb{T}) = \left\{ f \in L^1(\mathbb{T}) : \mathcal{F}(f\chi_A) \in \ell^{p'}(\mathbb{Z}), \text{ for all } A \in \mathcal{B}_o(\mathbb{T}) \right\}.$$

This solves a question by R. E. Edwards in 1967 in his book "Fourier Series":

> *"What can be said about the family $\Phi^p(\mathbb{T})$ of functions $f \in L^1(\mathbb{T})$ having the property that $\mathcal{F}(f\chi_A)$ lies in $\ell^{p'}(\mathbb{Z})$ for all $A \in \mathcal{B}_o(\mathbb{T})$?"*

The solution is

$$\Phi^p(\mathbb{T}) = \mathbf{F}^p(\mathbb{T}) = L^1(m_p).$$

Optimal Hausdorff-Young inequality

Relation between $L^p(\mathbb{T})$ and the extended space $\mathbf{F}^p(\mathbb{T})$:

- In general, $L^p(\mathbb{T}) \subseteq \mathbf{F}^p(\mathbb{T})$, for $1 < p \leq 2$.
- For $p = 2$ we have $L^2(\mathbb{T}) = \mathbf{F}^2(\mathbb{T})$.
- For $1 < p < 2$ we have

$$L^p(\mathbb{T}) \subsetneq \mathbf{F}^p(\mathbb{T}) \subsetneq L^1(\mathbb{T}).$$

The proof that $L^p(\mathbb{T}) \subsetneq \mathbf{F}^p(\mathbb{T})$ is highly technical (it is based on Fourier restriction theory and Salem measures).

3. Extension of the Cesàro operator on Hardy spaces

Cesàro operator on Hardy spaces

* The Cesàro operator on analytic functions on the unit disc $\mathbb{D}$:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \longmapsto \mathcal{C}(f)(z) := \sum_{n=0}^{\infty} \Big(\frac{1}{n+1} \sum_{k=0}^{n} a_k \Big) z^n.$$

* The Hardy space $\mathcal{H}^2(\mathbb{D})$:

$$\mathcal{H}^2(\mathbb{D}) = \Big\{ f(z) = \sum_{n=0}^{\infty} a_n z^n : \sum_{n=0}^{\infty} |a_n|^2 < \infty \Big\}.$$

* Hardy's inequality (1920):

$$\sum_{n=1}^{\infty} \Big| \frac{1}{n} \sum_{k=1}^{n} a_k \Big|^2 \le 2^2 \sum_{n=1}^{\infty} |a_n|^2.$$

Problem

* Due to Hardy's inequality, the Cesàro operator is bounded on $\mathcal{H}^2(\mathbb{D})$:

$$\mathcal{C}\colon \mathcal{H}^2(\mathbb{D}) \to \mathcal{H}^2(\mathbb{D}).$$

* Can we extend $\mathcal{C}$ to a **larger** space of analytic functions $\mathbb{X}(\mathbb{D})$ such that

$$\mathcal{C}\colon \mathbb{X}(\mathbb{D}) \to \mathcal{H}^2(\mathbb{D})$$

still is bounded?

* Possible candidate:

$$\mathcal{H}^p(\mathbb{D}), 1 \leq p < 2, \quad \text{since} \quad \mathcal{H}^2(\mathbb{D}) \subsetneq \mathcal{H}^p(\mathbb{D}) \quad \& \quad \mathcal{C}\colon \mathcal{H}^p(\mathbb{D}) \to \mathcal{H}^p(\mathbb{D}).$$

Cesàro operator on Hardy spaces: extensions

- Weighted Hardy space: for $\psi \in \mathcal{H}(\mathbb{D})$ with $\log|\psi| \in L^1(\mathbb{T})$.

$$\mathcal{H}^2(\psi) := \left\{ f \in \mathcal{H}(\mathbb{D}) : f = \psi^{-1/2} \cdot g, \text{ for } g \in \mathcal{H}^2(\mathbb{D}) \right\}.$$

- $\mathcal{H}^2(\mathbb{D}) \subsetneq \mathcal{H}^2(\psi) \iff \psi$ is bounded, ψ^{-1} is unbounded.

Theorem (C. & Ricker)

$$\left. \begin{array}{c} \mathcal{C} \colon \mathcal{H}^2(\psi) \to \mathcal{H}^2(\mathbb{D}) \\ \textit{boundedly} \end{array} \right\} \iff \left\{ \begin{array}{c} z \in \mathbb{D} \mapsto \displaystyle\int_0^z \frac{\psi^{-1/2}(\xi)}{1-\xi}\, d\xi \\ \textit{belongs to BMOA.} \end{array} \right.$$

Cesàro operator on Hardy spaces: optimal extension

- Non-comparable extensions of $\mathcal{C}\colon \mathcal{H}^2(\mathbb{D}) \to \mathcal{H}^2(\mathbb{D})$.
- Does there exists the largest one?

Theorem (C. & Ricker)

The space

$$[\mathcal{C}, \mathcal{H}^2] := \left\{ f \in H(\mathbb{D}) : \int_0^{2\pi} \int_0^1 \frac{|f(re^{i\theta})|^2}{|1 - re^{i\theta}|^2} (1-r)\, dr\, d\theta < \infty \right\}.$$

is the optimal extension domain for the Cesàro operator on $\mathcal{H}^2(\mathbb{D})$:

$$\mathcal{C}\colon [\mathcal{C}, \mathcal{H}^2] \to \mathcal{H}^2(\mathbb{D})$$

boundedly and optimally.

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Thank you