Contractive Projections on Spaces of Vector Valued Continuous Functions

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Basics

Let E be a Banach space and $P : E \to E$ a projection.
(i.e. P is a bounded linear idempotent operator, $P^2 = P$)

Properties:

- $Ran(P)$ is a closed subspace of E
- $Ran(P) \oplus Ker(P) = E$
- $\|P\| \geq 1$ $(P \neq 0)$
- $I - P$ is a projection

Definition
P is a contractive projection if $\|P\| = 1$
P is a bi-contractive projection if $\|P\| = 1$ and $\|I - P\| = 1$

Examples

1. Orthogonal projections on a Hilbert space are bi-contractive
2. $P : C([0,1]) \to C([0,1])$ such that $P(f)(t) = (1-t)f(0) + tf(1)$ is a contractive projection. Not bi-contractive
3. $P : C([0,1]) \to C([0,1])$ such that $P(f)(t) = \frac{f(t)+f(1-t)}{2}$ is a bi-contractive projection.
4. $P : C(\Omega, E) \to C(\Omega, E)$ such that $P(f)(t) = P_E \frac{f(t)+f(\tau(t))}{2}$, with P_E a contractive projection and τ an order 2 homeomorphism of Ω, is also a contractive projection

Generalized Bi-Circular Projections

Definition Let E be a Banach space.

- (Stachó and Zalar, 2004) A projection $P : E \to E$ is bi-circular iff $P + \lambda P^{\perp}$ is an isometry, for every λ of modulus 1
- (Fŏsner, Ilišević and Li, 2007) A projection $P : E \to E$ is a generalized bi-circular projection (GBP) iff $P + \lambda P^{\perp}$ is an isometry, for some λ of modulus 1 ($\lambda \neq 1$)

(with J.Jamison, JMAA 08) GBP's on $C(\Omega)$ are of the form $\frac{I+T}{2}$ with T a surjective isometric reflection, i.e. $T^2 = I$

(with J.Jamison, Acta Sci 09) Similar representation were derived for the generalized bi-circular projections on

- Spaces of Lipschitz functions ($Lip^{\alpha}(X)$ and $lip^{\alpha}(X)$)
- Pointed spaces of Lipschitz functions ($Lip^{\alpha}(X; x_0)$ and $lip^{\alpha}(X; x_0)$), endowed with $\max\{L_{\alpha}(f), \|f\|_{\infty}\}$

Projections: Bi-Circular and Contractive

- Generalized bi-circular projections are bi-contractive

Bi-circular projections $\subsetneq$ generalized bi-circular projections

Theorem (Pei-Kee Lin, JMAA 2008) Let n be an integer $n \geq 2$ and $\lambda = e^{i\frac{2k\pi}{n}}$ with $k \leq n$. Then there is a complex Banach space X and GBP P on X such that $P + \lambda(I - P)$ is an isometry on X

$X = \mathbb{C} \oplus \mathbb{C}$ with a Minkowski-type norm supports GBPs that are not bi-circular

Decomposition of Contractive Projections

Theorem (Friedman and Russo, 1982) Let P be a contractive projection on $C(\Omega)$ then there exist:

- A **"maximal"** family of measures $\{\mu_i\}$ ($\mu_i \in extP^*(C(\Omega)^*_1)$, $\mu_i = |\mu_i|\varphi_i$ with $\varphi_i \in L_1(|\mu_i|)$) such that
  1. $\|\mu_i\| = 1$
  2. $S_{\mu_i} \cap S_{\mu_j} = \emptyset$, if $i \neq j$
  3. For each $f \in C(\Omega)$, $Qf \in C_b(\cup_i S_{\mu_i})$ and given by

$$Qf|_{S_{\mu_i}} = Pf|_{S_{\mu_i}}$$

  $(Qf|_{S_{\mu_i}} = (\int f \, d\mu_i)\overline{\varphi}_i, \ \ |\mu_i| - a.e. \ \ on \ \ S_{\mu_i})$
- An isometric simultaneous extension operator $T : Q(C(\Omega)) \to C(\Omega)$, such that

$P = TQ$

An Example

$P : C([0,1]) \to C([0,1])$ such that $P(f)(t) = (1-t)f(0) + tf(1)$ is a contractive projection

$P(C([0,1])) =$ space of all affine maps on $[0,1]$

$ext\, P^*(C([0,1])_1^*) = \pm\delta_0,\ \pm\delta_1$

$\delta_0 \in extP^*(C([0,1])_1^*) \leftrightarrow \mu$ (a Borel measure)

$\int_{[0,1]} fd\mu = f(0)$ and $P(f)(t) = f(0)$ μ-a.e.

$Q : C([0,1]) \to C(\{0,1\})$ (essential part of P)

$T : C(\{0,1\}) \to C([0,1])$ isometric simultaneous extension.

Bi-contractive Projections

Theorem (Friedman and Russo, 1982) P is a bi-contractive projection on $C(\Omega)$ if and only if there exists an isometry T on $C(\Omega)$, of order 2, such that $P = \frac{I+T}{2}$ (generalized bi-circular projection or GBP)

A surjective isometry on $C(\Omega)$ is of the form $f \to \lambda f \circ \tau$, with $\lambda : \Omega \to \mathbb{S}^1$ continuous and τ a homeomorphism of Ω (Banach-Stone Theorem)

Homeomorphisms of [0, 1] of order 2 are id and 1-id

Are bi-contractive projections generalized bi-circular projections?

Contractive projections on closed subspaces of $C(\Omega)$

Proposition Let A be a closed subspace of $C(\Omega)$. Let $P : A \to A$ be a contractive projection. Then there exists a measure μ on $\mathcal{B}(\Omega)$ and $\psi : \Omega \to \mathbb{S}^1$ "in A" such that for every $f \in A$.

$$Pf = \left(\int f d\mu \right) \cdot \psi \quad (|\mu| - a.e.)$$

Sketch of the proof:

1. Every functional $\tau \in C(\Omega)^*$ is represented by a "unique" complex measure μ on Ω of bounded variation (with decomposition $\mu = \varphi|\mu|$) s.t.

$$\tau(f) = \int_\Omega f d\mu$$

   $\|\tau\| = |\mu|(\Omega) = \sup_{\mathcal{P}} \sum_{i=1}^n |\mu(\Omega_i)|$

2. Let μ be an extreme point of $P^*(A_1^*)$ (Krein-Milman Theorem). $Pf \cdot \varphi$ is constant ($|\mu|$ a.e.) (Atalla). Then $Pf \cdot \varphi = \int f d\mu$. Let $\psi = \bar{\varphi}$

Remarks

- A_1 is weak-* dense in A_1^{**} (Goldstine Theorem) then
$$|\varphi| P^{**}(\tau) = \left(\int_\Omega \tau \, d\mu \right) \bar{\varphi},$$
for all $\tau \in A^{**}$
- $P^* \mu = \mu$ implies
$$P^*(|\varphi| \cdot \nu) = \left(\int \bar{\varphi} \, d\nu \right) \mu,$$
for every $\nu \in A^*$
- Given two extreme points μ_1 and μ_2 either they differ by a scalar or they have disjoint supports

Examples: Contractive Projections on $C^1([0,1])$

Kawamura, Koshimizu and Miura, KKM-spaces of continuously differentiable functions (2016) $\left(C^1[0,1], \|\cdot\|_{<D>}\right)$,

$\|f\|_{<D>} = sup_{(r,s)\in D}(|f(r)| + |f'(s)|)$ D a compact connected

subset of $[0,1] \times [0,1]$ such that $p_1(D) \cup p_2(D) = [0,1]$

$C^1([0,1], \|\cdot\|_{<D>}) \hookrightarrow C(D \times \mathbb{S}^1)$ an isometric embedding with image $\mathcal{A}$

If $p_1(D) = [0,1]$, then $\mathcal{A}$ is a closed subalgebra of $C(\Omega)$ containing the constant functions.

Projections on Vector Valued Function Spaces

Theorem (with J.Jamison, RM 10) Let E be a Banach space with the strong Banach Stone property. Then P is a generalized bi-circular projection on $C(\Omega, E)$ if and only it is of one the following forms:

1. $Pf = \frac{I+T}{2}$ with T an isometric reflection on $C(\Omega, E)$
2. $Pf = Q \cdot f$ with Q a generalized bi-circular projection on E

Banach spaces with the strong Banach Stone property include smooth spaces, strictly convex spaces, also reflexive spaces containing no nontrivial L_1 projections (Behrends)

The Vector Valued Case

Characterization of contractive projections on $C(\Omega, E)$ with Ω a compact Hausdorff topological space [RM, 2010]

Main ideas:

- Dual of $C(\Omega, E)$ can be identified with the space of regular and bounded variation vector measures on the σ-algebra of the Borel subsets of Ω with values in E^* (I. Singer)
- The form of the extreme points of the unit ball of the dual space $C(\Omega, E)^*$, $e^*\delta_x$, with $x \in \Omega$ and $e^* \in ext(E^*)$ (Arens-Kelley and Brosowski-Deutsch Theorems)
- If the range space is uniformly convex then every vector measure F has the decomposition $F = |F|\frac{dF}{d|F|}$ with $\frac{dF}{d|F|} : \Omega \to S_{E^*}$ a Bochner integrable function with respect to $|F|$ (Bogdanowicz-Kritt (1967) and Zimmer (2007))

The Vector Valued Case, cont.

$$
\begin{tikzcd}
\Omega \arrow[rr, "\frac{dF}{d|F|}"] \arrow[dr, swap, "\frac{d|F|}{dF}"] & & S_{E^*} \arrow[dl, "g"] \\
& S_E &
\end{tikzcd}
$$

g continuous s.t.
$g(u)(u) = 1, \; \forall\, u \in S_{E^*}$

- An extension of Atalla's Theorem for contractive projections on $C(\Omega, E)$

Atalla's Theorem Revisited

If P is a contractive projection of $C(\Omega, E)$, E is a uniformly convex Banach space, then for every extreme point F of $P^*(C(\Omega, E)^*_1)$

$$\langle Pf, \frac{dF}{d|F|} \rangle = \left(\int_\Omega f dF \right), \quad |F| - a.e.$$

If $\int_\Omega f dF \neq 0$ then $Pf = \left(\int_\Omega f dF \right) \frac{d|F|}{dF}$
Given F an extreme point of $P^*(C(\Omega, E)^*_1)$, it can be shown that

1. If $G \in \mathcal{M}(\Sigma(\Omega), E^*)$ then
$$P^* \left(G_{S(F)} \right) = \left(\int_\Omega \frac{d|F|}{dF} dG \right) F, \ \forall f \ s.t. \int_\Omega f dF \neq 0$$
2. Let F_1 and F_2 be two extreme points of $P^*(C(\Omega, E)^*_1)$. If $x \in S(|F_1|) \cap S(|F_2|)$ then $S(F_1) = S(F_2)$

Bi-contractive Projections

Let Ω be compact and E a uniformly convex space. Then P is a bi-contractive projection of $C(\Omega, E)$ if and only if P is of one of the following forms:

- There exists a continuous map $P_1 : \Omega \to \mathcal{BCP}(E)$ such that $(Pf)(x) = P_1(x)(f(x))$, for every $f \in C(\Omega, E)$ and $x \in \Omega$
- There exist a homeomorphism φ of Ω and map $U : \Omega \to \mathcal{U}(E)$ where $\mathcal{U}(E)$ denotes the surjective isometries of E such that $\varphi^2 = Id$, $U(w)U(\varphi(w)) = Id$ and

$$P(f)(x) = \frac{f(x) + U(x)f(\varphi(x))}{2}, \ \forall\ f \in C(\Omega, E) \text{ and } x \in \Omega$$

Other results on bi-contractive projections

- **R.Douglas** (1965) Contractive projections on L_1 spaces and conditional expectations
- **T.Ando** (1966) Contractive projections on L_p are isometrically equivalent to conditional expectations $(1 \leq p < \infty)$
- **S.Bernau** and **H.Lacey** (1977) Bi-contractive projections on L_p spaces $(1 \leq p < \infty$ and $p \neq 2)$ and L_1 predual spaces are GBPs
- **M.Baronti** and **P.Papini** (1989) Bi-contractive projections on sequences spaces (c_0)
- **A.Lima** (1978) Bi-contractive projections on real CL-spaces are GBPs
- **B.Randrianantoanina** (2011) Norm 1 projections in Banach spaces

Thank You