

Connections between centrality and local monotonicity of certain functions on C^* -algebras

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Motivation

- Ogasawara¹ (1955): a C^* -algebra \mathcal{A} is commutative if and only if the map $x \mapsto x^2$ is monotonic increasing on the set of the positive elements of \mathcal{A}
- Pedersen² (1979): for any $p \in (1, \infty)$, the map $x \mapsto x^2$ may be replaced by $x \mapsto x^p$ in the above theorem
- Wu³ (2001): Ogasawara's result remains true when $x \mapsto x^2$ is replaced by $x \mapsto e^x$

¹T. Ogasawara, *A theorem on operator algebras*, J. Sci. Hiroshima Univ. Ser. A. **18** (1955), 307-309.

²G.K. Pedersen, *C^* -Algebras and Their Automorphism Groups*, London Mathematical Society Monographs, 14, Academic Press, Inc., London-New York, 1979.

³W. Wu, *An order characterization of commutativity for C^* -algebras*, Proc. Amer. Math. Soc. **129** (2001), 983–987.

Motivation

- Ji and Tomiyama⁴ (2003): let f be a continuous function on the positive axis which is monotonic increasing but is not matrix monotone of order 2. Then \mathcal{A} is commutative if and only if f is monotone increasing on the positive cone of \mathcal{A} .
- Molnár⁵ (2016): a positive element $x \in \mathcal{A}$ is central if and only if $x \leq y$ implies $e^x \leq e^y$
- Observe that Wu's theorem is an immediate consequence of Molnár's result
- Our goal is to provide "local" versions of the theorems of Ogasawara, Pedersen, Ji and Tomiyama

⁴G. Ji and J. Tomiyama, *On characterizations of commutativity of C^* -algebras*, Proc. Amer. Math. Soc. **131** (2003), 3845-3849.

⁵L. Molnár, *A characterization of central elements in C^* -algebras*, Bull. Austral. Math. Soc., to appear.

Basic notions

- the symbol \mathcal{A} stands for a unital C^* -algebra and I denotes its unit
- the spectrum of an element $A \in \mathcal{A}$ is denoted by $\sigma(A)$ and it is defined by

$$\sigma(A) = \{\lambda \in \mathbb{C} \mid \lambda I - A \text{ is not invertible}\}$$

- \mathcal{A}_s stands for the set of the self-adjoint elements of \mathcal{A} , and we say that $A \in \mathcal{A}_s$ is positive, if $\sigma(A) \subset [0, \infty)$
- the partial order \leq on \mathcal{A}_s is defined as follows: for any self-adjoint elements A and B , we have $A \leq B$ if and only if $B - A$ is positive
- denote by \mathcal{A}_+ (resp. \mathcal{A}_+^{-1}) the set of all positive (resp. positive invertible) elements of \mathcal{A}

The main theorem

Theorem (V., 2016)

Let $I = (\gamma, \infty)$ for some $\gamma \in \mathbb{R} \cup \{-\infty\}$ and let $f \in C^1(I)$ such that

- (i) $f'(x) > 0 \quad x \in I,$
- (ii) $x < y \Rightarrow f'(x) < f'(y) \quad x, y, \in I,$
- (iii) $\log(f'(tx + (1-t)y)) \geq$
 $t \log f'(x) + (1-t) \log f'(y) \quad x, y, \in I, t \in [0, 1].$

Let \mathcal{A} be a unital C^* -algebra and let $a \in \mathcal{A}$ be a self-adjoint element with $\sigma(a) \subset I$. The followings are equivalent.

- (1) a is central, that is, $ab = ba \quad b \in \mathcal{A},$
- (2) f is locally monotone at the point a , that is,
 $a \leq b \Rightarrow f(a) \leq f(b) \quad b \in \mathcal{A}_s.$

Examples

Example

Some intervals and functions satisfying the conditions given in the Theorem.

- $I = (0, \infty)$, $f(x) = x^p$ $p > 1$,
- $I = (-\infty, \infty)$, $f(x) = e^x$.

Notation. If φ and ψ are elements of some Hilbert space \mathcal{H} , then the symbol $\varphi \otimes \psi$ denotes the linear map $\mathcal{H} \ni \xi \mapsto \langle \xi, \psi \rangle \varphi \in \mathcal{H}$. The inner product is linear in its first variable.

The following Lemma is a key step of the proof of the Theorem.

Outline of the proof

Lemma

Suppose that $I = (\gamma, \infty)$ for some $\gamma \in \mathbb{R} \cup \{-\infty\}$ and $f \in C^1(I)$ satisfies the conditions (i), (ii) and (iii) given in the Theorem. Let \mathcal{K} be a two-dimensional Hilbert space, let $\{u, v\} \subset \mathcal{K}$ be an orthonormal basis. Let $x, y \in I$ and set $A := xu \otimes u + yv \otimes v$. The followings are equivalent.

- (I) $x \neq y$,
- (II) there exist $\lambda, \mu \in \mathbb{C}$ (with $|\lambda|^2 + |\mu|^2 = 1$) and $t_0 > 0$ such that using the notation $B = (u + v) \otimes (u + v)$ and $w = \lambda u + \mu v$ we have

$$\langle f(A)w, w \rangle - \langle f(A + t_0 B)w, w \rangle > 0.$$

Outline of the proof

Notation. For any fixed interval $I = (\gamma, \infty)$ and function $f \in C^1(I)$ with the properties (i), (ii) and (iii), and different numbers $x, y \in I$, the above Lemma provides a positive number $\langle f(A)w, w \rangle - \langle f(A + t_0 B)w, w \rangle$. Let us introduce

$$\delta_{I,f,x,y} := \langle f(A)w, w \rangle - \langle f(A + t_0 B)w, w \rangle.$$

Proof of the Lemma.

- the direction (II) \Rightarrow (I) is easy to see (by contraposition)
- to verify the direction (I) \Rightarrow (II) we recall the following useful formula for the derivative of a matrix function⁶

⁶F. Hiai and D. Petz, *Introduction to Matrix Analysis and Applications*, Hindustan Book Agency and Springer Verlag (2014)

Outline of the proof

- if $A = xu \otimes u + yv \otimes v$, then for any self-adjoint $C \in \mathcal{B}(\mathcal{K})$ we have

$$\begin{aligned} & \lim_{t \rightarrow 0} \frac{1}{t} (f(A + tC) - f(A)) \\ &= f'(x) \langle Cu, u \rangle u \otimes u + \frac{f(x) - f(y)}{x - y} \langle Cv, u \rangle u \otimes v \\ &+ \frac{f(y) - f(x)}{y - x} \langle Cu, v \rangle v \otimes u + f'(y) \langle Cv, v \rangle v \otimes v. \end{aligned}$$

- matrix formalism: $[A] = \text{diag}(x, y)$ and

$$\left[\lim_{t \rightarrow 0} \frac{1}{t} (f(A + tC) - f(A)) \right] = \begin{bmatrix} f'(x) & \frac{f(x) - f(y)}{x - y} \\ \frac{f(y) - f(x)}{y - x} & f'(y) \end{bmatrix} \circ [C]$$

Outline of the proof

- in particular, for $B = (u + v) \otimes (u + v)$ we have

$$L := \lim_{t \rightarrow 0} \frac{1}{t} (f(A + tB) - f(A)) = f'(x)u \otimes u + \frac{f(x) - f(y)}{x - y} u \otimes v + \frac{f(y) - f(x)}{y - x} v \otimes u + f'(y)v \otimes v. \quad (1)$$

- the determinant of the matrix

$$[L] = \begin{bmatrix} f'(x) & \frac{f(x) - f(y)}{x - y} \\ \frac{f(y) - f(x)}{y - x} & f'(y) \end{bmatrix}$$

is negative as $\text{Det}[L] < 0 \Leftrightarrow f'(x)f'(y) < \left(\frac{f(x) - f(y)}{x - y}\right)^2 \Leftrightarrow$

$$\Leftrightarrow \log f'(x) + \log f'(y) < 2 \log \left(\int_{t=0}^1 f'(tx + (1 - t)y) dt \right)$$

Outline of the proof

- this latter inequality is true as

$$\begin{aligned}\log f'(x) + \log f'(y) &= 2 \cdot \int_{t=0}^1 t \log f'(x) + (1-t) \log f'(y) dt \\ &\leq 2 \int_{t=0}^1 \log (f'(tx + (1-t)y)) dt \\ &< 2 \log \left(\int_{t=0}^1 f'(tx + (1-t)y) dt \right)\end{aligned}$$

- in the above computation, the first inequality holds because of the log-concavity of f' and the second (strict) inequality holds because the logarithm function is strictly concave and f' is strictly monotone increasing

Outline of the proof

- so, the operator L (defined in eq. (1)) has a negative eigenvalue, that is, there exist $\lambda, \mu \in \mathbb{C}$ (with $|\lambda|^2 + |\mu|^2 = 1$) such that with $w = \lambda u + \mu v$ we have

$$\langle Lw, w \rangle = \left\langle \lim_{t \rightarrow 0} \frac{1}{t} (f(A + tB) - f(A))w, w \right\rangle < 0$$

- therefore,

$$\lim_{t \rightarrow 0} \frac{1}{t} (\langle f(A + tB)w, w \rangle - \langle f(A)w, w \rangle) < 0,$$

and so there exists some $t_0 > 0$ such that
 $0 < \langle f(A)w, w \rangle - \langle f(A + t_0B)w, w \rangle$

Outline of the proof

The proof of the Theorem.

- the direction (1) \Rightarrow (2) is easy to verify
- to see the contrary, assume that $a \in \mathcal{A}_s$, $\sigma(a) \subset I$ and $aa' - a'a \neq 0$ for some $a' \in \mathcal{A}$
- then there exists an irreducible representation $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$ such that $\pi(aa' - a'a) \neq 0$, that is, $\pi(a)\pi(a') \neq \pi(a')\pi(a)$
- let us fix this irreducible representation π once and for all
- so, $\pi(a)$ is a non-central self-adjoint (and hence normal) element of $\mathcal{B}(\mathcal{H})$ with $\sigma(\pi(a)) \subset I$ (as a representation do not increase the spectrum)
- by the non-centrality, $\sigma(\pi(a))$ has at least two elements, and by the normality, every element of $\sigma(\pi(a))$ is an approximate eigenvalue

Outline of the proof

- let x and y be two different elements of $\sigma(\pi(a))$, and let $\{u_n\}_{n \in \mathbb{N}} \subset \mathcal{H}$ and $\{v_n\}_{n \in \mathbb{N}} \subset \mathcal{H}$ satisfy

$$\lim_{n \rightarrow \infty} \pi(A)u_n - xu_n = 0, \quad \lim_{n \rightarrow \infty} \pi(A)v_n - yv_n = 0,$$

and $\langle u_m, v_n \rangle = 0 \quad m, n \in \mathbb{N}$ (as $x \neq y$, the approximate eigenvectors can be chosen to be orthogonal)

- set $\mathcal{K}_n := \text{span}\{u_n, v_n\}$ and let E_n be the orthoprojection onto the closed subspace $\mathcal{K}_n^\perp \subset \mathcal{H}$
- let

$$\psi_n(a) := xu_n \otimes u_n + yv_n \otimes v_n + E_n\pi(a)E_n$$

- a direct computation shows that

$$\lim_{n \rightarrow \infty} \psi_n(a) = \pi(a)$$

in the operator norm topology

Outline of the proof

- we have fixed l, f, x and y
- by the Lemma, we have $\lambda, \mu \in \mathbb{C}$ (with $|\lambda|^2 + |\mu|^2 = 1$) and $t_0 > 0$ such that using the notation $B_n := (u_n + v_n) \otimes (u_n + v_n)$ and $w_n := \lambda u_n + \mu v_n$, we have

$$\langle f(\psi_n(a)) w_n, w_n \rangle - \langle f(\psi_n(a) + t_0 B_n) w_n, w_n \rangle = \delta_{l,f,x,y} > 0 \quad (2)$$

for any $n \in \mathbb{N}$

- that is, the left hand side of (2) is independent of n

- for any n , the operator B_n is a self-adjoint element of $\mathcal{B}(\mathcal{H})$ and \mathcal{K}_n is a finite dimensional subspace of \mathcal{H} , hence by Kadison's transitivity theorem, there exists a self-adjoint $b_n \in \mathcal{A}$ such that

$$\pi(b_n)|_{\mathcal{K}_n} = B_n|_{\mathcal{K}_n}$$

- observe that $B_n \mathcal{K}_n \subseteq \mathcal{K}_n$ and so $\pi(b_n) \mathcal{K}_n \subseteq \mathcal{K}_n$
- on the other hand, $\pi(b_n)$ is self-adjoint as b_n is self-adjoint, hence it follows that $\pi(b_n) \mathcal{K}_n^\perp \subseteq \mathcal{K}_n^\perp$
- therefore, the fact $B_n = \frac{1}{2} B_n^2$ implies that

$$\begin{aligned} \pi\left(\frac{1}{2} b_n^2\right)|_{\mathcal{K}_n} &= \left(\frac{1}{2} \pi(b_n)^2\right)|_{\mathcal{K}_n} \\ &= \frac{1}{2} \left(\pi(b_n)|_{\mathcal{K}_n}\right)^2 = \frac{1}{2} B_n^2|_{\mathcal{K}_n} = B_n|_{\mathcal{K}_n} \end{aligned}$$

- so we can rewrite (2) as

$$\langle f(\psi_n(a)) w_n, w_n \rangle - \left\langle f\left(\psi_n(a) + t_0 \pi\left(\frac{1}{2} b_n^2\right)\right) w_n, w_n \right\rangle = \delta_{l,f,x,y} \quad (3)$$

- a standard continuity argument — which is based on the fact that $\psi_n(a)$ tends to $\pi(a)$ in the operator norm topology,— shows that

$$\lim_{n \rightarrow \infty} \|f(\psi_n(a)) - f(\pi(a))\| = 0$$

and

$$\lim_{n \rightarrow \infty} \left\| f\left(\psi_n(a) + t_0 \pi\left(\frac{1}{2} b_n^2\right)\right) - f\left(\pi(a) + t_0 \pi\left(\frac{1}{2} b_n^2\right)\right) \right\| = 0.$$

Outline of the proof

- so for n large enough we have

$$\|f(\psi_n(a)) - f(\pi(a))\| < \frac{1}{4}\delta_{l,f,x,y}$$

and

$$\left\| f\left(\psi_n(a) + t_0\pi\left(\frac{1}{2}b_n^2\right)\right) - f\left(\pi\left(a + \frac{t_0}{2}b_n^2\right)\right) \right\| < \frac{1}{4}\delta_{l,f,x,y}.$$

- therefore, by (3), for n large enough the inequality

$$\langle f(\pi(a)) w_n, w_n \rangle - \langle f\left(\pi\left(a + \frac{t_0}{2}b_n^2\right)\right) w_n, w_n \rangle > \frac{1}{2}\delta_{l,f,x,y} > 0 \quad (4)$$

holds

Outline of the proof

- in other words,

$$f(\pi(a)) \not\leq f\left(\pi\left(a + \frac{t_0}{2}b_n^2\right)\right)$$

or equivalently,

$$\pi(f(a)) \not\leq \pi\left(f\left(a + \frac{t_0}{2}b_n^2\right)\right)$$

- any representation of a C^* -algebra preserves the semidefinite order, hence this means that

$$f(a) \not\leq f\left(a + \frac{t_0}{2}b_n^2\right),$$

despite the fact that $a \leq a + \frac{t_0}{2}b_n^2$



A final remark

Remark

Note that our theorem generalizes Molnár's result, and — as every "local" theorem easily implies its "global" counterpart — we recover the theorems of Ogasawara, Pedersen, and Wu, as well.

Thank you for your attention!