Determinacy for the complex moment problem via positive definite extensions

Dariusz Cichoń

December 2016, OTOA, Bangalore

Joint work with J. Stochel and F.H. Szafraniec

Introduction

Notations:
$\mathfrak{N} = \{(m, n) : m, n$ – integers such that $m \geqslant 0, n \geqslant 0\}$,
$\mathfrak{N}_+ = \{(m, n) : m, n$ – integers such that $m + n \geqslant 0\}$.

Question: when a sequence $\gamma = \{\gamma_{m,n}\}_{m,n \geqslant 0} \subset \mathbb{C}$ is a complex moment sequence? I.e. there exists a Borel measure μ on $\mathbb{C}$ such that

$$c_{m,n} = \int_{\mathbb{C}} z^m \bar{z}^n \mathrm{d}\mu(z), \quad m, n \geqslant 0.$$

An 'iff' criterion: PDE(γ) is nonempty, where

$$\mathrm{PDE}(\gamma) = \{\tilde{\gamma} \colon \tilde{\gamma} \text{ is a positive definite extension of } \gamma \text{ on } \mathfrak{N}_+\}$$

i.e. $\tilde{\gamma} = \{\tilde{\gamma}_{m,n}\}_{m+n \geqslant 0} \subset \mathbb{C}$ satisfies $\tilde{\gamma}|_{\mathfrak{N}} = \gamma$ and

$$\sum_{m+n \geqslant 0, p+q \geqslant 0} \lambda_{m,n} \bar{\lambda}_{p,q} \tilde{\gamma}_{m+q,n+p} \geqslant 0$$

for every finitely supported $\{\lambda_{m,n}\}_{m+n \geqslant 0}^{\infty} \subset \mathbb{C}$.

Introduction

Notations:
$\mathfrak{N} = \{(m,n) : m, n$ – integers such that $m \geqslant 0, n \geqslant 0\}$,
$\mathfrak{N}_+ = \{(m,n) : m, n$ – integers such that $m + n \geqslant 0\}$.

Question: when a sequence $\gamma = \{\gamma_{m,n}\}_{m,n\geqslant 0} \subset \mathbb{C}$ is a complex moment sequence? I.e. there exists a Borel measure μ on $\mathbb{C}$ such that

$$c_{m,n} = \int_{\mathbb{C}} z^m \bar{z}^n \mathrm{d}\mu(z), \quad m, n \geqslant 0.$$

An 'iff' criterion: $\mathrm{PDE}(\gamma)$ is nonempty, where

$$\mathrm{PDE}(\gamma) = \{\tilde{\gamma} \colon \tilde{\gamma} \text{ is a positive definite extension of } \gamma \text{ on } \mathfrak{N}_+\}$$

i.e. $\tilde{\gamma} = \{\tilde{\gamma}_{m,n}\}_{m+n\geqslant 0} \subset \mathbb{C}$ satisfies $\tilde{\gamma}|_{\mathfrak{N}} = \gamma$ and

$$\sum_{m+n\geqslant 0, p+q\geqslant 0} \lambda_{m,n} \bar{\lambda}_{p,q} \tilde{\gamma}_{m+q,n+p} \geqslant 0$$

for every finitely supported $\{\lambda_{m,n}\}_{m+n\geqslant 0}^{\infty} \subset \mathbb{C}$.

Notations:
$\mathfrak{N} = \{(m,n) : m, n -$ integers such that $m \geqslant 0, n \geqslant 0\}$,
$\mathfrak{N}_+ = \{(m,n) : m, n -$ integers such that $m + n \geqslant 0\}$.

Question: when a sequence $\gamma = \{\gamma_{m,n}\}_{m,n \geqslant 0} \subset \mathbb{C}$ is a complex moment sequence? I.e. there exists a Borel measure μ on $\mathbb{C}$ such that

$$c_{m,n} = \int_{\mathbb{C}} z^m \bar{z}^n \mathrm{d}\mu(z), \quad m, n \geqslant 0.$$

An 'iff' criterion: PDE(γ) is nonempty, where

$$\mathrm{PDE}(\gamma) = \{\tilde{\gamma} \colon \tilde{\gamma} \text{ is a positive definite extension of } \gamma \text{ on } \mathfrak{N}_+\}$$

i.e. $\tilde{\gamma} = \{\tilde{\gamma}_{m,n}\}_{m+n \geqslant 0} \subset \mathbb{C}$ satisfies $\tilde{\gamma}|_{\mathfrak{N}} = \gamma$ and

$$\sum_{m+n \geqslant 0, p+q \geqslant 0} \lambda_{m,n} \bar{\lambda}_{p,q} \tilde{\gamma}_{m+q,n+p} \geqslant 0$$

for every finitely supported $\{\lambda_{m,n}\}_{m+n \geqslant 0}^{\infty} \subset \mathbb{C}$.

Notations:
$\mathfrak{N} = \{(m,n) : m,n$ – integers such that $m \geqslant 0, n \geqslant 0\}$,
$\mathfrak{N}_+ = \{(m,n) : m,n$ – integers such that $m+n \geqslant 0\}$.

Question: when a sequence $\gamma = \{\gamma_{m,n}\}_{m,n\geqslant 0} \subset \mathbb{C}$ is a complex moment sequence? I.e. there exists a Borel measure μ on $\mathbb{C}$ such that

$$c_{m,n} = \int_{\mathbb{C}} z^m \bar{z}^n \mathrm{d}\mu(z), \quad m,n \geqslant 0.$$

An 'iff' criterion: PDE(γ) is nonempty, where

$$\mathrm{PDE}(\gamma) = \{\tilde{\gamma} \colon \tilde{\gamma} \text{ is a positive definite extension of } \gamma \text{ on } \mathfrak{N}_+\}$$

i.e. $\tilde{\gamma} = \{\tilde{\gamma}_{m,n}\}_{m+n\geqslant 0} \subset \mathbb{C}$ satisfies $\tilde{\gamma}|_{\mathfrak{N}} = \gamma$ and

$$\sum_{m+n\geqslant 0, p+q\geqslant 0} \lambda_{m,n}\bar{\lambda}_{p,q}\tilde{\gamma}_{m+q,n+p} \geqslant 0$$

for every finitely supported $\{\lambda_{m,n}\}_{m+n\geqslant 0}^{\infty} \subset \mathbb{C}$.

Why?

The main reason for this to work:

$$\mathfrak{N}_+ \text{ is semiperfect,}$$

i.e. every positive definite complex function on this semigroup can be represented via Borel measures. More specifically: if $\{\tilde{\gamma}_{m,n}\}_{m+n\geqslant 0}$ is positive definite on $\mathfrak{N}_+$, then there are Borel measures μ_1 on $\mathbb{C}^*$ (without 0) and μ_2 on $\mathbb{T}$ (the unit circle) such that

$$\tilde{\gamma}_{m,n} = \int_{\mathbb{C}^*} z^m \bar{z}^n \mathrm{d}\mu_1(z) + \underbrace{\delta_{m+n,0}}_{\text{the Dirac delta}} \int_{\mathbb{T}} z^m \bar{z}^n \mathrm{d}\mu_2(z).$$

The pair (μ_1, μ_2) will be called *representing* for $\tilde{\gamma}$.

Why?

The main reason for this to work:

$$\mathfrak{N}_+ \text{ is semiperfect,}$$

i.e. every positive definite complex function on this semigroup can be represented via Borel measures. More specifically: if $\{\tilde{\gamma}_{m,n}\}_{m+n\geqslant 0}$ is positive definite on $\mathfrak{N}_+$, then there are Borel measures μ_1 on $\mathbb{C}^*$ (without 0) and μ_2 on $\mathbb{T}$ (the unit circle) such that

$$\tilde{\gamma}_{m,n} = \int_{\mathbb{C}^*} z^m \bar{z}^n \mathrm{d}\mu_1(z) + \underbrace{\delta_{m+n,0}}_{\text{the Dirac delta}} \int_{\mathbb{T}} z^m \bar{z}^n \mathrm{d}\mu_2(z).$$

The pair (μ_1, μ_2) will be called *representing* for $\tilde{\gamma}$.

Why?

The main reason for this to work:

$$\mathfrak{N}_+ \text{ is semiperfect,}$$

i.e. every positive definite complex function on this semigroup can be represented via Borel measures. More specifically: if $\{\tilde{\gamma}_{m,n}\}_{m+n\geqslant 0}$ is positive definite on $\mathfrak{N}_+$, then there are Borel measures μ_1 on $\mathbb{C}^*$ (without 0) and μ_2 on $\mathbb{T}$ (the unit circle) such that

$$\tilde{\gamma}_{m,n} = \int_{\mathbb{C}^*} z^m \bar{z}^n \mathrm{d}\mu_1(z) + \underbrace{\delta_{m+n,0}}_{\text{the Dirac delta}} \int_{\mathbb{T}} z^m \bar{z}^n \mathrm{d}\mu_2(z).$$

The pair (μ_1, μ_2) will be called *representing* for $\tilde{\gamma}$.

Why?

The main reason for this to work:

$$\mathfrak{N}_+ \text{ is semiperfect,}$$

i.e. every positive definite complex function on this semigroup can be represented via Borel measures. More specifically: if $\{\tilde{\gamma}_{m,n}\}_{m+n\geqslant 0}$ is positive definite on $\mathfrak{N}_+$, then there are Borel measures μ_1 on $\mathbb{C}^*$ (without 0) and μ_2 on $\mathbb{T}$ (the unit circle) such that

$$\tilde{\gamma}_{m,n} = \int_{\mathbb{C}^*} z^m \bar{z}^n \mathrm{d}\mu_1(z) + \underbrace{\delta_{m+n,0}}_{\text{the Dirac delta}} \int_{\mathbb{T}} z^m \bar{z}^n \mathrm{d}\mu_2(z).$$

The pair (μ_1, μ_2) will be called *representing* for $\tilde{\gamma}$.

Why?

The main reason for this to work:

$$\mathfrak{N}_+ \text{ is semiperfect,}$$

i.e. every positive definite complex function on this semigroup can be represented via Borel measures. More specifically: if $\{\tilde{\gamma}_{m,n}\}_{m+n\geqslant 0}$ is positive definite on $\mathfrak{N}_+$, then there are Borel measures μ_1 on $\mathbb{C}^*$ (without 0) and μ_2 on $\mathbb{T}$ (the unit circle) such that

$$\tilde{\gamma}_{m,n} = \int_{\mathbb{C}^*} z^m \bar{z}^n \mathrm{d}\mu_1(z) + \underbrace{\delta_{m+n,0}}_{\text{the Dirac delta}} \int_{\mathbb{T}} z^m \bar{z}^n \mathrm{d}\mu_2(z).$$

The pair (μ_1, μ_2) will be called *representing* for $\tilde{\gamma}$.

Main question

Is there any connection between the following two situations?

1. γ is a determinate complex moment sequence on $\mathfrak{N}$, i.e. the representing measure for γ is unique,
2. $\mathrm{PDE}(\gamma)$ is a singleton, i.e. $\mathrm{PDE}(\gamma) = \{\tilde{\gamma}\}$.

Remark: if (μ_1, μ_2) is representing for $\tilde{\gamma} \in \mathrm{PDE}(\gamma)$, then the measure $\mu_1 + \mu_2(\mathbb{T})\delta_0$ is representing for γ.

The natural condition appearing when dealing with determinacy of γ: p.d. extension $\tilde{\gamma} \in \mathrm{PDE}(\gamma)$ is called *semideterminate* if for any two representing pairs of measures (μ_1, μ_2) and (μ_1', μ_2') for $\tilde{\gamma}$ we have

$$\mu_1 = \mu_1' \text{ and } \mu_2 \circ \varphi^{-1} = \mu_2' \circ \varphi^{-1},$$

where $\varphi : \mathbb{T} \ni z \mapsto z^2 \in \mathbb{T}$. This happens if γ is determinate.

Main question

Is there any connection between the following two situations?

1. γ is a determinate complex moment sequence on $\mathfrak{N}$, i.e. the representing measure for γ is unique,
2. PDE(γ) is a singleton, i.e. PDE(γ) $= \{\tilde{\gamma}\}$.

Remark: if (μ_1, μ_2) is representing for $\tilde{\gamma} \in$ PDE(γ), then the measure $\mu_1 + \mu_2(\mathbb{T})\delta_0$ is representing for γ.

The natural condition appearing when dealing with determinacy of γ: p.d. extension $\tilde{\gamma} \in$ PDE(γ) is called *semideterminate* if for any two representing pairs of measures (μ_1, μ_2) and (μ_1', μ_2') for $\tilde{\gamma}$ we have

$$\mu_1 = \mu_1' \text{ and } \mu_2 \circ \varphi^{-1} = \mu_2' \circ \varphi^{-1},$$

where $\varphi : \mathbb{T} \ni z \mapsto z^2 \in \mathbb{T}$. This happens if γ is determinate.

Main question

Is there any connection between the following two situations?

1. γ is a determinate complex moment sequence on $\mathfrak{N}$, i.e. the representing measure for γ is unique,
2. PDE(γ) is a singleton, i.e. PDE$(\gamma) = \{\tilde{\gamma}\}$.

Remark: if (μ_1, μ_2) is representing for $\tilde{\gamma} \in$ PDE(γ), then the measure $\mu_1 + \mu_2(\mathbb{T})\delta_0$ is representing for γ.

The natural condition appearing when dealing with determinacy of γ: p.d. extension $\tilde{\gamma} \in$ PDE(γ) is called *semideterminate* if for any two representing pairs of measures (μ_1, μ_2) and (μ_1', μ_2') for $\tilde{\gamma}$ we have

$$\mu_1 = \mu_1' \text{ and } \mu_2 \circ \varphi^{-1} = \mu_2' \circ \varphi^{-1},$$

where $\varphi : \mathbb{T} \ni z \mapsto z^2 \in \mathbb{T}$. This happens if γ is determinate.

Main question

Is there any connection between the following two situations?

1. γ is a determinate complex moment sequence on $\mathfrak{N}$, i.e. the representing measure for γ is unique,
2. $\mathrm{PDE}(\gamma)$ is a singleton, i.e. $\mathrm{PDE}(\gamma) = \{\tilde{\gamma}\}$.

Remark: if (μ_1, μ_2) is representing for $\tilde{\gamma} \in \mathrm{PDE}(\gamma)$, then the measure $\mu_1 + \mu_2(\mathbb{T})\delta_0$ is representing for γ.

The natural condition appearing when dealing with determinacy of γ: p.d. extension $\tilde{\gamma} \in \mathrm{PDE}(\gamma)$ is called *semideterminate* if for any two representing pairs of measures (μ_1, μ_2) and (μ_1', μ_2') for $\tilde{\gamma}$ we have

$$\mu_1 = \mu_1' \text{ and } \mu_2 \circ \varphi^{-1} = \mu_2' \circ \varphi^{-1},$$

where $\varphi : \mathbb{T} \ni z \mapsto z^2 \in \mathbb{T}$. This happens if γ is determinate.

Main question

Is there any connection between the following two situations?

1. γ is a determinate complex moment sequence on $\mathfrak{N}$, i.e. the representing measure for γ is unique,
2. $\mathrm{PDE}(\gamma)$ is a singleton, i.e. $\mathrm{PDE}(\gamma) = \{\tilde{\gamma}\}$.

Remark: if (μ_1, μ_2) is representing for $\tilde{\gamma} \in \mathrm{PDE}(\gamma)$, then the measure $\mu_1 + \mu_2(\mathbb{T})\delta_0$ is representing for γ.

The natural condition appearing when dealing with determinacy of γ: p.d. extension $\tilde{\gamma} \in \mathrm{PDE}(\gamma)$ is called *semideterminate* if for any two representing pairs of measures (μ_1, μ_2) and (μ_1', μ_2') for $\tilde{\gamma}$ we have

$$\mu_1 = \mu_1' \text{ and } \mu_2 \circ \varphi^{-1} = \mu_2' \circ \varphi^{-1},$$

where $\varphi : \mathbb{T} \ni z \mapsto z^2 \in \mathbb{T}$. This happens if γ is determinate.

Some answers

Theorem

The following are equivalent:

1. γ *is determinate on* $\mathfrak{N}$,
2. *there exists a unique* $\tilde{\gamma} \in \mathrm{PDE}(\gamma)$ *of Lebesgue type (i.e.* μ_2 *is a multiple of the Lebesgue measure on* $\mathbb{T}$*),*
3. *there exists a unique* $\tilde{\gamma} \in \mathrm{PDE}(\gamma)$ *of* δ_1 *type (i.e.* μ_2 *is a multiple of the Dirac measure* δ_1 *on* $\mathbb{T}$*)*

If γ is determinate with the representing measure μ such that $\mu(\{0\}) \neq 0$, then it follows that $\mathrm{PDE}(\gamma)$ is of cardinality continuum.

Reason: if (μ_1, μ_2) and (μ_1, μ_2') are representing for $\tilde{\gamma}$ and $\tilde{\gamma}' \in \mathrm{PDE}(\gamma)$, then (μ_1, ν) with ν – a convex combination of μ_2 and μ_2' is also representing for some new extension in $\mathrm{PDE}(\gamma)$.

Some answers

Theorem

The following are equivalent:

1. γ *is determinate on* $\mathfrak{N}$,
2. *there exists a unique* $\tilde{\gamma} \in \mathrm{PDE}(\gamma)$ *of Lebesgue type (i.e.* μ_2 *is a multiple of the Lebesgue measure on* $\mathbb{T}$*),*
3. *there exists a unique* $\tilde{\gamma} \in \mathrm{PDE}(\gamma)$ *of* δ_1 *type (i.e.* μ_2 *is a multiple of the Dirac measure* δ_1 *on* $\mathbb{T}$*)*

If γ is determinate with the representing measure μ such that $\mu(\{0\}) \neq 0$, then it follows that $\mathrm{PDE}(\gamma)$ is of cardinality continuum.

Reason: if (μ_1, μ_2) and (μ_1, μ_2') are representing for $\tilde{\gamma}$ and $\tilde{\gamma}' \in \mathrm{PDE}(\gamma)$, then (μ_1, ν) with ν – a convex combination of μ_2 and μ_2' is also representing for some new extension in $\mathrm{PDE}(\gamma)$.

Some answers

Theorem

The following are equivalent:

1. γ *is determinate on* $\mathfrak{N}$,
2. *there exists a unique* $\tilde{\gamma} \in \mathrm{PDE}(\gamma)$ *of Lebesgue type (i.e.* μ_2 *is a multiple of the Lebesgue measure on* $\mathbb{T}$*),*
3. *there exists a unique* $\tilde{\gamma} \in \mathrm{PDE}(\gamma)$ *of* δ_1 *type (i.e.* μ_2 *is a multiple of the Dirac measure* δ_1 *on* $\mathbb{T}$*)*

If γ is determinate with the representing measure μ such that $\mu(\{0\}) \neq 0$, then it follows that $\mathrm{PDE}(\gamma)$ is of cardinality continuum.

Reason: if (μ_1, μ_2) and (μ_1, μ_2') are representing for $\tilde{\gamma}$ and $\tilde{\gamma}' \in \mathrm{PDE}(\gamma)$, then (μ_1, ν) with ν – a convex combination of μ_2 and μ_2' is also representing for some new extension in $\mathrm{PDE}(\gamma)$.

Some answers

Theorem

The following are equivalent:

1. γ *is determinate on* $\mathfrak{N}$,
2. *there exists a unique* $\tilde{\gamma} \in \mathrm{PDE}(\gamma)$ *of Lebesgue type (i.e.* μ_2 *is a multiple of the Lebesgue measure on* $\mathbb{T}$*),*
3. *there exists a unique* $\tilde{\gamma} \in \mathrm{PDE}(\gamma)$ *of* δ_1 *type (i.e.* μ_2 *is a multiple of the Dirac measure* δ_1 *on* $\mathbb{T}$*)*

If γ is determinate with the representing measure μ such that $\mu(\{0\}) \neq 0$, then it follows that $\mathrm{PDE}(\gamma)$ is of cardinality continuum.

Reason: if (μ_1, μ_2) and (μ_1, μ_2') are representing for $\tilde{\gamma}$ and $\tilde{\gamma}' \in \mathrm{PDE}(\gamma)$, then (μ_1, ν) with ν – a convex combination of μ_2 and μ_2' is also representing for some new extension in $\mathrm{PDE}(\gamma)$.

Some answers

Theorem

The following are equivalent:

1. γ *is determinate on* $\mathfrak{N}$,
2. *there exists a unique* $\tilde{\gamma} \in \mathrm{PDE}(\gamma)$ *of Lebesgue type (i.e.* μ_2 *is a multiple of the Lebesgue measure on* $\mathbb{T}$*),*
3. *there exists a unique* $\tilde{\gamma} \in \mathrm{PDE}(\gamma)$ *of* δ_1 *type (i.e.* μ_2 *is a multiple of the Dirac measure* δ_1 *on* $\mathbb{T}$*)*

If γ is determinate with the representing measure μ such that $\mu(\{0\}) \neq 0$, then it follows that $\mathrm{PDE}(\gamma)$ is of cardinality continuum.

Reason: if (μ_1, μ_2) and (μ_1, μ_2') are representing for $\tilde{\gamma}$ and $\tilde{\gamma}' \in \mathrm{PDE}(\gamma)$, then (μ_1, ν) with ν – a convex combination of μ_2 and μ_2' is also representing for some new extension in $\mathrm{PDE}(\gamma)$.

Answers - continued

Determinacy implies uniqueness of p.d. extensions under an additional condition.

Theorem

If γ is determinate on $\mathfrak{N}$, μ is representing for γ and $\mu(\{0\}) = 0$, then $\mathrm{PDE}(\gamma) = \{\tilde{\gamma}\}$.

A more refined version:

Theorem

Let γ be a sequence defined on $\mathfrak{N}$. Then the following are equivalent:

1. *γ is a determinate moment sequence with a representing measure μ satisfying $\mu(\{0\}) = 0$,*
2. *$\mathrm{PDE}(\gamma) = \{\tilde{\gamma}\}$ and $\tilde{\gamma}$ is determinate.*

Answers - continued

Determinacy implies uniqueness of p.d. extensions under an additional condition.

Theorem

If γ is determinate on $\mathfrak{N}$, μ is representing for γ and $\mu(\{0\}) = 0$, then $\mathrm{PDE}(\gamma) = \{\tilde{\gamma}\}$.

A more refined version:

Theorem

Let γ be a sequence defined on $\mathfrak{N}$. Then the following are equivalent:

1. *γ is a determinate moment sequence with a representing measure μ satisfying $\mu(\{0\}) = 0$,*
2. *$\mathrm{PDE}(\gamma) = \{\tilde{\gamma}\}$ and $\tilde{\gamma}$ is determinate.*

Answers - continued

Determinacy implies uniqueness of p.d. extensions under an additional condition.

Theorem

If γ is determinate on $\mathfrak{N}$, μ is representing for γ and $\mu(\{0\}) = 0$, then $\mathrm{PDE}(\gamma) = \{\tilde{\gamma}\}$.

A more refined version:

Theorem

Let γ be a sequence defined on $\mathfrak{N}$. Then the following are equivalent:

1. *γ is a determinate moment sequence with a representing measure μ satisfying $\mu(\{0\}) = 0$,*
2. *$\mathrm{PDE}(\gamma) = \{\tilde{\gamma}\}$ and $\tilde{\gamma}$ is determinate.*

Answers - continued

Determinacy implies uniqueness of p.d. extensions under an additional condition.

Theorem

If γ is determinate on $\mathfrak{N}$, μ is representing for γ and $\mu(\{0\}) = 0$, then $\mathrm{PDE}(\gamma) = \{\tilde{\gamma}\}$.

A more refined version:

Theorem

Let γ be a sequence defined on $\mathfrak{N}$. Then the following are equivalent:

1. *γ is a determinate moment sequence with a representing measure μ satisfying $\mu(\{0\}) = 0$,*
2. *$\mathrm{PDE}(\gamma) = \{\tilde{\gamma}\}$ and $\tilde{\gamma}$ is determinate.*

Examples

Example

If $\{s_n\}_{n=0}^{\infty}$ is an indeterminate Hamburger sequence (i.e. it has non-unique representing Borel measure on $\mathbb{R}$) and $\gamma_{m,n} = s_{m+n}$, then $\mathrm{PDE}(\gamma)$ is of cardinality continuuum. This is due to the fact, that every indeterminate Hamburger moment problem has a representing measure with atom at 0.

Example

If in turn $\{s_n\}_{n=0}^{\infty}$ is a Herglotz moment sequence (i.e. representing measure is supported on the unit circle $\mathbb{T}$), then $\gamma_{m,n} = s_{m-n}$ is determinate with representing measure without atom at 0. Thus $\mathrm{PDE}(\gamma) = \{\tilde{\gamma}\}$. Reason: a representing measure supported on compact subset is always determinate, besides $0 \notin \mathbb{T}$ so there can be no atom at 0.

Examples

Example

If $\{s_n\}_{n=0}^{\infty}$ *is an indeterminate Hamburger sequence (i.e. it has non-unique representing Borel measure on* $\mathbb{R}$*) and* $\gamma_{m,n} = s_{m+n}$*, then* $\mathrm{PDE}(\gamma)$ *is of cardinality continuum. This is due to the fact, that every indeterminate Hamburger moment problem has a representing measure with atom at* 0*.*

Example

If in turn $\{s_n\}_{n=0}^{\infty}$ *is a Herglotz moment sequence (i.e. representing measure is supported on the unit circle* $\mathbb{T}$*), then* $\gamma_{m,n} = s_{m-n}$ *is determinate with representing measure without atom at* 0*. Thus* $\mathrm{PDE}(\gamma) = \{\tilde{\gamma}\}$*. Reason: a representing measure supported on compact subset is always determinate, besides* $0 \notin \mathbb{T}$ *so there can be no atom at* 0*.*

Examples

Example

If $\{s_n\}_{n=0}^{\infty}$ *is an indeterminate Hamburger sequence (i.e. it has non-unique representing Borel measure on* $\mathbb{R}$*) and* $\gamma_{m,n} = s_{m+n}$*, then* $\mathrm{PDE}(\gamma)$ *is of cardinality continuum. This is due to the fact, that every indeterminate Hamburger moment problem has a representing measure with atom at* 0*.*

Example

If in turn $\{s_n\}_{n=0}^{\infty}$ *is a Herglotz moment sequence (i.e. representing measure is supported on the unit circle* $\mathbb{T}$*), then* $\gamma_{m,n} = s_{m-n}$ *is determinate with representing measure without atom at* 0*. Thus* $\mathrm{PDE}(\gamma) = \{\tilde{\gamma}\}$*. Reason: a representing measure supported on compact subset is always determinate, besides* $0 \notin \mathbb{T}$ *so there can be no atom at* 0*.*

Example

If $\{s_n\}_{n=0}^{\infty}$ is an indeterminate Hamburger sequence (i.e. it has non-unique representing Borel measure on $\mathbb{R}$) and $\gamma_{m,n} = s_{m+n}$, then $\mathrm{PDE}(\gamma)$ is of cardinality continuum. This is due to the fact, that every indeterminate Hamburger moment problem has a representing measure with atom at 0.

Example

If in turn $\{s_n\}_{n=0}^{\infty}$ is a Herglotz moment sequence (i.e. representing measure is supported on the unit circle $\mathbb{T}$), then $\gamma_{m,n} = s_{m-n}$ is determinate with representing measure without atom at 0. Thus $\mathrm{PDE}(\gamma) = \{\tilde{\gamma}\}$. Reason: a representing measure supported on compact subset is always determinate, besides $0 \notin \mathbb{T}$ so there can be no atom at 0.

Question revised

The only case left to consider when γ on $\mathfrak{N}$ is indeterminate and <u>none</u> of its representing measures has atom at 0. Then the question of the cardinality PDE(γ) is not settled.

None of representing measures has atom at 0 – is it possible at all?

In case of Hamburger moment sequences - it is not.

It turns out that in the case of complex moment sequence the answer is in the affirmative. You may take line $\mathbb{R} + \mathrm{i}$ and define γ on $\mathfrak{N}$ via

$$\gamma_m, n = \int_{\mathbb{C}} z^m \bar{z}^n \mathrm{d}\mu(z), \quad m, n \geqslant 0,$$

with some indeterminate measure supported in $\mathbb{R} + \mathrm{i}$. Now apply the proposition: if there is a representing measure supported in a real algebraic subset A of $\mathbb{C}$, then all other representing measures are also supported in A.

Question revised

The only case left to consider when γ on $\mathfrak{N}$ is indeterminate and none of its representing measures has atom at 0. Then the question of the cardinality PDE(γ) is not settled.

None of representing measures has atom at 0 – is it possible at all?

In case of Hamburger moment sequences - it is not.

It turns out that in the case of complex moment sequence the answer is in the affirmative. You may take line $\mathbb{R} + \mathrm{i}$ and define γ on $\mathfrak{N}$ via

$$\gamma_m, n = \int_{\mathbb{C}} z^m \bar{z}^n \mathrm{d}\mu(z), \quad m, n \geqslant 0,$$

with some indeterminate measure supported in $\mathbb{R} + \mathrm{i}$. Now apply the proposition: if there is a representing measure supported in a real algebraic subset A of $\mathbb{C}$, then all other representing measures are also supported in A.

The only case left to consider when γ on $\mathfrak{N}$ is indeterminate and none of its representing measures has atom at 0. Then the question of the cardinality PDE(γ) is not settled.

None of representing measures has atom at 0 – is it possible at all?

In case of Hamburger moment sequences - it is not.

It turns out that in the case of complex moment sequence the answer is in the affirmative. You may take line $\mathbb{R} + \mathrm{i}$ and define γ on $\mathfrak{N}$ via

$$\gamma_m, n = \int_{\mathbb{C}} z^m \bar{z}^n \mathrm{d}\mu(z), \quad m, n \geqslant 0,$$

with some indeterminate measure supported in $\mathbb{R} + \mathrm{i}$. Now apply the proposition: if there is a representing measure supported in a real algebraic subset A of $\mathbb{C}$, then all other representing measures are also supported in A.

Question revised

The only case left to consider when γ on $\mathfrak{N}$ is indeterminate and none of its representing measures has atom at 0. Then the question of the cardinality $\mathrm{PDE}(\gamma)$ is not settled.

None of representing measures has atom at 0 – is it possible at all?

In case of Hamburger moment sequences - it is not.

It turns out that in the case of complex moment sequence the answer is in the affirmative. You may take line $\mathbb{R} + \mathrm{i}$ and define γ on $\mathfrak{N}$ via

$$\gamma_m, n = \int_{\mathbb{C}} z^m \bar{z}^n \mathrm{d}\mu(z), \quad m, n \geqslant 0,$$

with some indeterminate measure supported in $\mathbb{R} + \mathrm{i}$. Now apply the proposition: if there is a representing measure supported in a real algebraic subset A of $\mathbb{C}$, then all other representing measures are also supported in A.

Question revised

The only case left to consider when γ on $\mathfrak{N}$ is indeterminate and <u>none</u> of its representing measures has atom at 0. Then the question of the cardinality $\mathrm{PDE}(\gamma)$ is not settled.

None of representing measures has atom at 0 – is it possible at all?

In case of Hamburger moment sequences - it is not.

It turns out that in the case of complex moment sequence the answer is in the affirmative. You may take line $\mathbb{R} + \mathrm{i}$ and define γ on $\mathfrak{N}$ via

$$\gamma_m, n = \int_{\mathbb{C}} z^m \bar{z}^n \mathrm{d}\mu(z), \quad m, n \geqslant 0,$$

with some indeterminate measure supported in $\mathbb{R} + \mathrm{i}$. Now apply the proposition: if there is a representing measure supported in a real algebraic subset A of $\mathbb{C}$, then all other representing measures are also supported in A.

Theorem

Let $P \in \mathbb{C}[z, \bar{z}]$ be a polynomial such that $Z_P \stackrel{\text{def}}{=} P^{-1}(\{0\})$ is the proper nonempty subset of $\mathbb{C}$. Let $0 \notin Z_p$ and γ on $\mathfrak{N}$ be a sequence with a representing measure supported in Z_P. Then

1. *all representing measures of γ are supported in Z_P,*
2. *if (μ_1, μ_2) is representing for some $\tilde{\gamma} \in \mathrm{PDE}(\gamma)$, then $\mu_2 = 0$,*
3. *if μ and ν are representing for γ, then $\mu \circ \psi_P^{-1} = \nu \circ \psi_P^{-1}$, where $\psi_P : Z_P \ni z \mapsto z^2|z|^{-2} \in \mathbb{T}$,*
4. *if the every Borel subset τ of Z_P is of the form $\tau = \psi_P^{-1}(\sigma)$ with some Borel $\sigma \subset \mathbb{T}$, then*
    - *the mapping $\mathcal{M}(\gamma) \ni \mu \mapsto \tilde{\gamma} \in \mathrm{PDE}(\gamma)$, where $\tilde{\gamma}_{m,n} = \int_{\mathbb{C}} z^m \bar{z}^n \mathrm{d}\mu(z)$, $m + n \geqslant 0$, is bijective,*
    - *every $\tilde{\gamma} \in \mathrm{PDE}(\gamma)$ is determinate,*
    - *$\mathrm{PDE}(\gamma)$ is of cardinality continuum provided γ is indeterminate.*

Theorem

Let $P \in \mathbb{C}[z, \bar{z}]$ be a polynomial such that $Z_P \stackrel{\text{def}}{=} P^{-1}(\{0\})$ is the proper nonempty subset of $\mathbb{C}$. Let $0 \notin Z_p$ and γ on $\mathfrak{N}$ be a sequence with a representing measure supported in Z_P. Then

1. *all representing measures of γ are supported in Z_P,*
2. *if (μ_1, μ_2) is representing for some $\tilde{\gamma} \in \mathrm{PDE}(\gamma)$, then $\mu_2 = 0$,*
3. *if μ and ν are representing for γ, then $\mu \circ \psi_P^{-1} = \nu \circ \psi_P^{-1}$, where $\psi_P : Z_P \ni z \mapsto z^2|z|^{-2} \in \mathbb{T}$,*
4. *if the every Borel subset τ of Z_P is of the form $\tau = \psi_P^{-1}(\sigma)$ with some Borel $\sigma \subset \mathbb{T}$, then*
   - *the mapping $\mathcal{M}(\gamma) \ni \mu \mapsto \tilde{\gamma} \in \mathrm{PDE}(\gamma)$, where $\tilde{\gamma}_{m,n} = \int_{\mathbb{C}} z^m \bar{z}^n \mathrm{d}\mu(z)$, $m + n \geqslant 0$, is bijective,*
   - *every $\tilde{\gamma} \in \mathrm{PDE}(\gamma)$ is determinate,*
   - *$\mathrm{PDE}(\gamma)$ is of cardinality continuum provided γ is indeterminate.*

Algebraic sets without 0

Theorem

Let $P \in \mathbb{C}[z,\bar{z}]$ be a polynomial such that $Z_P \stackrel{\text{def}}{=} P^{-1}(\{0\})$ is the proper nonempty subset of $\mathbb{C}$. Let $0 \notin Z_p$ and γ on $\mathfrak{N}$ be a sequence with a representing measure supported in Z_P. Then

1. *all representing measures of γ are supported in Z_P,*
2. *if (μ_1, μ_2) is representing for some $\tilde{\gamma} \in \mathrm{PDE}(\gamma)$, then $\mu_2 = 0$,*
3. *if μ and ν are representing for γ, then $\mu \circ \psi_P^{-1} = \nu \circ \psi_P^{-1}$, where $\psi_P : Z_P \ni z \mapsto z^2|z|^{-2} \in \mathbb{T}$,*
4. *if the every Borel subset τ of Z_P is of the form $\tau = \psi_P^{-1}(\sigma)$ with some Borel $\sigma \subset \mathbb{T}$, then*
   - *the mapping $\mathcal{M}(\gamma) \ni \mu \mapsto \tilde{\gamma} \in \mathrm{PDE}(\gamma)$, where $\tilde{\gamma}_{m,n} = \int_{\mathbb{C}} z^m \bar{z}^n \mathrm{d}\mu(z)$, $m+n \geqslant 0$, is bijective,*
   - *every $\tilde{\gamma} \in \mathrm{PDE}(\gamma)$ is determinate,*
   - *$\mathrm{PDE}(\gamma)$ is of cardinality continuum provided γ is indeterminate.*

Algebraic sets without 0

Theorem

Let $P \in \mathbb{C}[z, \bar{z}]$ be a polynomial such that $Z_P \stackrel{\text{def}}{=} P^{-1}(\{0\})$ is the proper nonempty subset of $\mathbb{C}$. Let $0 \notin Z_p$ and γ on $\mathfrak{N}$ be a sequence with a representing measure supported in Z_P. Then

1. *all representing measures of γ are supported in Z_P,*
2. *if (μ_1, μ_2) is representing for some $\tilde{\gamma} \in \mathrm{PDE}(\gamma)$, then $\mu_2 = 0$,*
3. *if μ and ν are representing for γ, then $\mu \circ \psi_P^{-1} = \nu \circ \psi_P^{-1}$, where $\psi_P : Z_P \ni z \mapsto z^2|z|^{-2} \in \mathbb{T}$,*
4. *if the every Borel subset τ of Z_P is of the form $\tau = \psi_P^{-1}(\sigma)$ with some Borel $\sigma \subset \mathbb{T}$, then*
   - *the mapping $\mathcal{M}(\gamma) \ni \mu \mapsto \tilde{\gamma} \in \mathrm{PDE}(\gamma)$, where $\tilde{\gamma}_{m,n} = \int_{\mathbb{C}} z^m \bar{z}^n \mathrm{d}\mu(z)$, $m + n \geqslant 0$, is bijective,*
   - *every $\tilde{\gamma} \in \mathrm{PDE}(\gamma)$ is determinate,*
   - *$\mathrm{PDE}(\gamma)$ is of cardinality continuum provided γ is indeterminate.*

Algebraic sets without 0

Theorem

Let $P \in \mathbb{C}[z,\bar{z}]$ be a polynomial such that $Z_P \stackrel{\text{def}}{=} P^{-1}(\{0\})$ is the proper nonempty subset of $\mathbb{C}$. Let $0 \notin Z_p$ and γ on $\mathfrak{N}$ be a sequence with a representing measure supported in Z_P. Then

1. *all representing measures of γ are supported in Z_P,*
2. *if (μ_1, μ_2) is representing for some $\tilde{\gamma} \in \mathrm{PDE}(\gamma)$, then $\mu_2 = 0$,*
3. *if μ and ν are representing for γ, then $\mu \circ \psi_P^{-1} = \nu \circ \psi_P^{-1}$, where $\psi_P : Z_P \ni z \mapsto z^2|z|^{-2} \in \mathbb{T}$,*
4. *if the every Borel subset τ of Z_P is of the form $\tau = \psi_P^{-1}(\sigma)$ with some Borel $\sigma \subset \mathbb{T}$, then*
   - *the mapping $\mathcal{M}(\gamma) \ni \mu \mapsto \tilde{\gamma} \in \mathrm{PDE}(\gamma)$, where $\tilde{\gamma}_{m,n} = \int_{\mathbb{C}} z^m \bar{z}^n \mathrm{d}\mu(z)$, $m+n \geqslant 0$, is bijective,*
   - *every $\tilde{\gamma} \in \mathrm{PDE}(\gamma)$ is determinate,*
   - *$\mathrm{PDE}(\gamma)$ is of cardinality continuum provided γ is indeterminate.*

Algebraic sets without 0

Theorem

Let $P \in \mathbb{C}[z, \bar{z}]$ be a polynomial such that $Z_P \stackrel{\text{def}}{=} P^{-1}(\{0\})$ is the proper nonempty subset of $\mathbb{C}$. Let $0 \notin Z_p$ and γ on $\mathfrak{N}$ be a sequence with a representing measure supported in Z_P. Then

1. *all representing measures of γ are supported in Z_P,*
2. *if (μ_1, μ_2) is representing for some $\tilde{\gamma} \in \mathrm{PDE}(\gamma)$, then $\mu_2 = 0$,*
3. *if μ and ν are representing for γ, then $\mu \circ \psi_P^{-1} = \nu \circ \psi_P^{-1}$, where $\psi_P : Z_P \ni z \mapsto z^2|z|^{-2} \in \mathbb{T}$,*
4. *if the every Borel subset τ of Z_P is of the form $\tau = \psi_P^{-1}(\sigma)$ with some Borel $\sigma \subset \mathbb{T}$, then*
   - *the mapping $\mathcal{M}(\gamma) \ni \mu \mapsto \tilde{\gamma} \in \mathrm{PDE}(\gamma)$, where $\tilde{\gamma}_{m,n} = \int_{\mathbb{C}} z^m \bar{z}^n \mathrm{d}\mu(z)$, $m + n \geqslant 0$, is bijective,*
   - *every $\tilde{\gamma} \in \mathrm{PDE}(\gamma)$ is determinate,*
   - *$\mathrm{PDE}(\gamma)$ is of cardinality continuum provided γ is indeterminate.*

Algebraic sets without 0

Theorem

Let $P \in \mathbb{C}[z,\bar{z}]$ be a polynomial such that $Z_P \overset{\text{def}}{=} P^{-1}(\{0\})$ is the proper nonempty subset of $\mathbb{C}$. Let $0 \notin Z_p$ and γ on $\mathfrak{N}$ be a sequence with a representing measure supported in Z_P. Then

1. *all representing measures of γ are supported in Z_P,*
2. *if (μ_1, μ_2) is representing for some $\tilde{\gamma} \in \mathrm{PDE}(\gamma)$, then $\mu_2 = 0$,*
3. *if μ and ν are representing for γ, then $\mu \circ \psi_P^{-1} = \nu \circ \psi_P^{-1}$, where $\psi_P : Z_P \ni z \mapsto z^2|z|^{-2} \in \mathbb{T}$,*
4. *if the every Borel subset τ of Z_P is of the form $\tau = \psi_P^{-1}(\sigma)$ with some Borel $\sigma \subset \mathbb{T}$, then*
   - *the mapping $\mathcal{M}(\gamma) \ni \mu \mapsto \tilde{\gamma} \in \mathrm{PDE}(\gamma)$, where $\tilde{\gamma}_{m,n} = \int_{\mathbb{C}} z^m \bar{z}^n \mathrm{d}\mu(z)$, $m+n \geqslant 0$, is bijective,*
   - *every $\tilde{\gamma} \in \mathrm{PDE}(\gamma)$ is determinate,*
   - *$\mathrm{PDE}(\gamma)$ is of cardinality continuum provided γ is indeterminate.*

Algebraic sets without 0

Theorem

Let $P \in \mathbb{C}[z, \bar{z}]$ be a polynomial such that $Z_P \stackrel{\text{def}}{=} P^{-1}(\{0\})$ is the proper nonempty subset of $\mathbb{C}$. Let $0 \notin Z_p$ and γ on $\mathfrak{N}$ be a sequence with a representing measure supported in Z_P. Then

1. *all representing measures of γ are supported in Z_P,*
2. *if (μ_1, μ_2) is representing for some $\tilde{\gamma} \in \mathrm{PDE}(\gamma)$, then $\mu_2 = 0$,*
3. *if μ and ν are representing for γ, then $\mu \circ \psi_P^{-1} = \nu \circ \psi_P^{-1}$, where $\psi_P : Z_P \ni z \mapsto z^2|z|^{-2} \in \mathbb{T}$,*
4. *if the every Borel subset τ of Z_P is of the form $\tau = \psi_P^{-1}(\sigma)$ with some Borel $\sigma \subset \mathbb{T}$, then*
    - *the mapping $\mathcal{M}(\gamma) \ni \mu \mapsto \tilde{\gamma} \in \mathrm{PDE}(\gamma)$, where $\tilde{\gamma}_{m,n} = \int_{\mathbb{C}} z^m \bar{z}^n \mathrm{d}\mu(z)$, $m + n \geqslant 0$, is bijective,*
    - *every $\tilde{\gamma} \in \mathrm{PDE}(\gamma)$ is determinate,*
    - *$\mathrm{PDE}(\gamma)$ is of cardinality continuum provided γ is indeterminate.*

Proof of

3. if μ and ν are representing for γ, then $\mu \circ \psi_P^{-1} = \nu \circ \psi_P^{-1}$, where $\psi_P : Z_P \ni z \mapsto z^2|z|^{-2} \in \mathbb{T}$,

We know that

$$\int_{\mathbb{C}} z^m \bar{z}^n \mathrm{d}\mu(z) = \int_{\mathbb{C}} z^m \bar{z}^n \mathrm{d}\nu(z), \quad m+n \geqslant 0.$$

Take $m \geqslant 0$ and $n = -m$, then

$$\int_{\mathbb{C}} (z^2|z|^{-2})^m \mathrm{d}\mu(z) = \int_{\mathbb{C}} (z^2|z|^{-2})^m \mathrm{d}\nu(z), \quad m \geqslant 0.$$

But $\psi_P(z) = z^2|z|^{-2}$ so by the measure transport theorem we get

$$\int_{\mathbb{T}} w^m \mathrm{d}(\mu \circ \psi_P^{-1})(w) = \int_{\mathbb{T}} w^m \mathrm{d}(\nu \circ \psi_P^{-1})(w), \quad m \geqslant 0.$$

Now apply determinacy of Herglotz moment sequences.

Proof of (3)

Proof of

3. if μ and ν are representing for γ, then $\mu \circ \psi_P^{-1} = \nu \circ \psi_P^{-1}$, where $\psi_P : Z_P \ni z \mapsto z^2|z|^{-2} \in \mathbb{T}$,

We know that

$$\int_{\mathbb{C}} z^m \bar{z}^n \mathrm{d}\mu(z) = \int_{\mathbb{C}} z^m \bar{z}^n \mathrm{d}\nu(z), \quad m + n \geqslant 0.$$

Take $m \geqslant 0$ and $n = -m$, then

$$\int_{\mathbb{C}} (z^2|z|^{-2})^m \mathrm{d}\mu(z) = \int_{\mathbb{C}} (z^2|z|^{-2})^m \mathrm{d}\nu(z), \quad m \geqslant 0.$$

But $\psi_P(z) = z^2|z|^{-2}$ so by the measure transport theorem we get

$$\int_{\mathbb{T}} w^m \mathrm{d}(\mu \circ \psi_P^{-1})(w) = \int_{\mathbb{T}} w^m \mathrm{d}(\nu \circ \psi_P^{-1})(w), \quad m \geqslant 0.$$

Now apply determinacy of Herglotz moment sequences.

Proof of

3. if μ and ν are representing for γ, then $\mu \circ \psi_P^{-1} = \nu \circ \psi_P^{-1}$, where $\psi_P : Z_P \ni z \mapsto z^2|z|^{-2} \in \mathbb{T}$,

We know that

$$\int_{\mathbb{C}} z^m \bar{z}^n \mathrm{d}\mu(z) = \int_{\mathbb{C}} z^m \bar{z}^n \mathrm{d}\nu(z), \quad m + n \geqslant 0.$$

Take $m \geqslant 0$ and $n = -m$, then

$$\int_{\mathbb{C}} (z^2|z|^{-2})^m \mathrm{d}\mu(z) = \int_{\mathbb{C}} (z^2|z|^{-2})^m \mathrm{d}\nu(z), \quad m \geqslant 0.$$

But $\psi_P(z) = z^2|z|^{-2}$ so by the measure transport theorem we get

$$\int_{\mathbb{T}} w^m \mathrm{d}(\mu \circ \psi_P^{-1})(w) = \int_{\mathbb{T}} w^m \mathrm{d}(\nu \circ \psi_P^{-1})(w), \quad m \geqslant 0.$$

Now apply determinacy of Herglotz moment sequences.

Proof of

3. if μ and ν are representing for γ, then $\mu \circ \psi_P^{-1} = \nu \circ \psi_P^{-1}$, where $\psi_P : Z_P \ni z \mapsto z^2|z|^{-2} \in \mathbb{T}$,

We know that

$$\int_{\mathbb{C}} z^m \bar{z}^n \mathrm{d}\mu(z) = \int_{\mathbb{C}} z^m \bar{z}^n \mathrm{d}\nu(z), \quad m + n \geqslant 0.$$

Take $m \geqslant 0$ and $n = -m$, then

$$\int_{\mathbb{C}} (z^2|z|^{-2})^m \mathrm{d}\mu(z) = \int_{\mathbb{C}} (z^2|z|^{-2})^m \mathrm{d}\nu(z), \quad m \geqslant 0.$$

But $\psi_P(z) = z^2|z|^{-2}$ so by the measure transport theorem we get

$$\int_{\mathbb{T}} w^m \mathrm{d}(\mu \circ \psi_P^{-1})(w) = \int_{\mathbb{T}} w^m \mathrm{d}(\nu \circ \psi_P^{-1})(w), \quad m \geqslant 0.$$

Now apply determinacy of Herglotz moment sequences.

Proof of

3. if μ and ν are representing for γ, then $\mu \circ \psi_P^{-1} = \nu \circ \psi_P^{-1}$, where $\psi_P : Z_P \ni z \mapsto z^2|z|^{-2} \in \mathbb{T}$,

We know that

$$\int_{\mathbb{C}} z^m \bar{z}^n \mathrm{d}\mu(z) = \int_{\mathbb{C}} z^m \bar{z}^n \mathrm{d}\nu(z), \quad m+n \geqslant 0.$$

Take $m \geqslant 0$ and $n = -m$, then

$$\int_{\mathbb{C}} (z^2|z|^{-2})^m \mathrm{d}\mu(z) = \int_{\mathbb{C}} (z^2|z|^{-2})^m \mathrm{d}\nu(z), \quad m \geqslant 0.$$

But $\psi_P(z) = z^2|z|^{-2}$ so by the measure transport theorem we get

$$\int_{\mathbb{T}} w^m \mathrm{d}(\mu \circ \psi_P^{-1})(w) = \int_{\mathbb{T}} w^m \mathrm{d}(\nu \circ \psi_P^{-1})(w), \quad m \geqslant 0.$$

Now apply determinacy of Herglotz moment sequences.

Condition (3) once more

A result similar to (3):

Proposition

If (μ_1, μ_2) *and* (μ_1', μ_2') *are representing for* $\tilde{\gamma} \in \mathrm{PDE}(\gamma)$*, then*

$$\mu_1 \circ \psi^{-1} + \mu_2 \circ \varphi^{-1} = \mu_1' \circ \psi^{-1} + \mu_2' \circ \varphi^{-1},$$

where

$$\psi : \mathbb{C}^* \ni z \mapsto z^2|z|^{-2} \in \mathbb{T}, \quad \varphi : \mathbb{T} \ni z \mapsto z^2 \in \mathbb{T}.$$

One may substitute a subset of $\mathbb{C}^*$ in place of $\mathbb{C}^*$ if we know that a representing measure is supported in this subset.

Condition (3) once more

A result similar to (3):

Proposition

If (μ_1, μ_2) *and* (μ_1', μ_2') *are representing for* $\tilde{\gamma} \in \mathrm{PDE}(\gamma)$*, then*

$$\mu_1 \circ \psi^{-1} + \mu_2 \circ \varphi^{-1} = \mu_1' \circ \psi^{-1} + \mu_2' \circ \varphi^{-1},$$

where

$$\psi : \mathbb{C}^* \ni z \mapsto z^2 |z|^{-2} \in \mathbb{T}, \quad \varphi : \mathbb{T} \ni z \mapsto z^2 \in \mathbb{T}.$$

One may substitute a subset of $\mathbb{C}^*$ in place of $\mathbb{C}^*$ if we know that a representing measure is supported in this subset.

Condition (3) once more

A result similar to (3):

Proposition

If (μ_1, μ_2) *and* (μ_1', μ_2') *are representing for* $\tilde{\gamma} \in \mathrm{PDE}(\gamma)$*, then*

$$\mu_1 \circ \psi^{-1} + \mu_2 \circ \varphi^{-1} = \mu_1' \circ \psi^{-1} + \mu_2' \circ \varphi^{-1},$$

where

$$\psi : \mathbb{C}^* \ni z \mapsto z^2 |z|^{-2} \in \mathbb{T}, \quad \varphi : \mathbb{T} \ni z \mapsto z^2 \in \mathbb{T}.$$

One may substitute a subset of $\mathbb{C}^*$ in place of $\mathbb{C}^*$ if we know that a representing measure is supported in this subset.

Condition (3) once more

A result similar to (3):

Proposition

If (μ_1, μ_2) *and* (μ_1', μ_2') *are representing for* $\tilde{\gamma} \in \mathrm{PDE}(\gamma)$*, then*

$$\mu_1 \circ \psi^{-1} + \mu_2 \circ \varphi^{-1} = \mu_1' \circ \psi^{-1} + \mu_2' \circ \varphi^{-1},$$

where

$$\psi : \mathbb{C}^* \ni z \mapsto z^2 |z|^{-2} \in \mathbb{T}, \quad \varphi : \mathbb{T} \ni z \mapsto z^2 \in \mathbb{T}.$$

One may substitute a subset of $\mathbb{C}^*$ in place of $\mathbb{C}^*$ if we know that a representing measure is supported in this subset.

Condition (4)

Recall

4. if the every Borel subset τ of Z_P is of the form $\tau = \psi_P^{-1}(\sigma)$ with some Borel $\sigma \subset \mathbb{T}$, then we are happy because we know all about PDE(γ).

Going back to the example of the line $\mathbb{R} + \mathrm{i}$ we see that every Borel subset of this line is of the form $\psi_P^{-1}(\sigma)$ with some Borel $\sigma \subset \mathbb{T}$. Here $P(z, \bar{z}) = z - \bar{z} + 2$.

Some other sets satisfy this condition, e.g.

$$P(x, y) = x^{2k} - (y - 1)^l,$$

where $l > 2k$ is odd (this may be written in variables z and $\bar{z}$).

Condition (4)

Recall

4. if the every Borel subset τ of Z_P is of the form $\tau = \psi_P^{-1}(\sigma)$ with some Borel $\sigma \subset \mathbb{T}$, then we are happy because we know all about PDE(γ).

Going back to the example of the line $\mathbb{R} + \mathrm{i}$ we see that every Borel subset of this line is of the form $\psi_P^{-1}(\sigma)$ with some Borel $\sigma \subset \mathbb{T}$. Here $P(z, \bar{z}) = z - \bar{z} + 2$.

Some other sets satisfy this condition, e.g.

$$P(x, y) = x^{2k} - (y-1)^l,$$

where $l > 2k$ is odd (this may be written in variables z and $\bar{z}$).

Condition (4)

Recall

4. if the every Borel subset τ of Z_P is of the form $\tau = \psi_P^{-1}(\sigma)$ with some Borel $\sigma \subset \mathbb{T}$, then we are happy because we know all about PDE(γ).

Going back to the example of the line $\mathbb{R} + \mathrm{i}$ we see that every Borel subset of this line is of the form $\psi_P^{-1}(\sigma)$ with some Borel $\sigma \subset \mathbb{T}$. Here $P(z, \bar{z}) = z - \bar{z} + 2$.

Some other sets satisfy this condition, e.g.

$$P(x, y) = x^{2k} - (y - 1)^l,$$

where $l > 2k$ is odd (this may be written in variables z and $\bar{z}$).

Condition (4)

Recall

4. if the every Borel subset τ of Z_P is of the form $\tau = \psi_P^{-1}(\sigma)$ with some Borel $\sigma \subset \mathbb{T}$, then we are happy because we know all about PDE(γ).

Going back to the example of the line $\mathbb{R} + \mathrm{i}$ we see that every Borel subset of this line is of the form $\psi_P^{-1}(\sigma)$ with some Borel $\sigma \subset \mathbb{T}$. Here $P(z, \bar{z}) = z - \bar{z} + 2$.

Some other sets satisfy this condition, e.g.

$$P(x, y) = x^{2k} - (y-1)^l,$$

where $l > 2k$ is odd (this may be written in variables z and $\bar{z}$).

Does uniqueness imply determinacy?

Assume that $\mathrm{PDE}(\gamma) = \{\tilde{\gamma}\}$. Does it follow that γ is determinate?

A partial solution:

Theorem

Let $P \in \mathbb{C}[z, \bar{z}]$ be a polynomial such that $\varnothing \neq Z_P \neq \mathbb{C}$. Let γ on $\mathfrak{N}$ admit a representing measure supported in Z_P. Suppose condition (4) holds for ψ_P, i.e. every Borel subset τ of Z_P is of the form $\tau = \psi_P^{-1}(\sigma)$ with some Borel $\sigma \subset \mathbb{T}$. If $\mathrm{PDE}(\gamma) = \{\tilde{\gamma}\}$, then γ is determinate.

(Recall that converse is true if there is no atom at 0.)

So the condition (4) becomes even more important. It holds on occasions, but it seems to fail in most cases. e.g. for the hiperbola $xy = 1$, the parabola $y = x^2 + 1$, the unit circle itself,

Does uniqueness imply determinacy?

Assume that $\mathrm{PDE}(\gamma) = \{\tilde{\gamma}\}$. Does it follow that γ is determinate? A partial solution:

Theorem

Let $P \in \mathbb{C}[z, \bar{z}]$ be a polynomial such that $\varnothing \neq Z_P \neq \mathbb{C}$. Let γ on $\mathfrak{N}$ admit a representing measure supported in Z_P. Suppose condition (4) holds for ψ_P, i.e. every Borel subset τ of Z_P is of the form $\tau = \psi_P^{-1}(\sigma)$ with some Borel $\sigma \subset \mathbb{T}$. If $\mathrm{PDE}(\gamma) = \{\tilde{\gamma}\}$, then γ is determinate.

(Recall that converse is true if there is no atom at 0.)

So the condition (4) becomes even more important. It holds on occasions, but it seems to fail in most cases. e.g. for the hiperbola $xy = 1$, the parabola $y = x^2 + 1$, the unit circle itself,

Does uniqueness imply determinacy?

Assume that $\mathrm{PDE}(\gamma) = \{\tilde{\gamma}\}$. Does it follow that γ is determinate? A partial solution:

Theorem

Let $P \in \mathbb{C}[z, \bar{z}]$ be a polynomial such that $\varnothing \neq Z_P \neq \mathbb{C}$. Let γ on $\mathfrak{N}$ admit a representing measure supported in Z_P. Suppose condition (4) holds for ψ_P, i.e. every Borel subset τ of Z_P is of the form $\tau = \psi_P^{-1}(\sigma)$ with some Borel $\sigma \subset \mathbb{T}$. If $\mathrm{PDE}(\gamma) = \{\tilde{\gamma}\}$, then γ is determinate.

(Recall that converse is true if there is no atom at 0.)

So the condition (4) becomes even more important. It holds on occasions, but it seems to fail in most cases. e.g. for the hiperbola $xy = 1$, the parabola $y = x^2 + 1$, the unit circle itself,

Does uniqueness imply determinacy?

Assume that $\mathrm{PDE}(\gamma) = \{\tilde{\gamma}\}$. Does it follow that γ is determinate? A partial solution:

Theorem

Let $P \in \mathbb{C}[z, \bar{z}]$ be a polynomial such that $\varnothing \neq Z_P \neq \mathbb{C}$. Let γ on $\mathfrak{N}$ admit a representing measure supported in Z_P. Suppose condition (4) holds for ψ_P, i.e. every Borel subset τ of Z_P is of the form $\tau = \psi_P^{-1}(\sigma)$ with some Borel $\sigma \subset \mathbb{T}$. If $\mathrm{PDE}(\gamma) = \{\tilde{\gamma}\}$, then γ is determinate.

(Recall that converse is true if there is no atom at 0.)

So the condition (4) becomes even more important. It holds on occasions, but it seems to fail in most cases. e.g. for the hiperbola $xy = 1$, the parabola $y = x^2 + 1$, the unit circle itself,

Does uniqueness imply determinacy?

Assume that $\mathrm{PDE}(\gamma) = \{\tilde{\gamma}\}$. Does it follow that γ is determinate? A partial solution:

Theorem

Let $P \in \mathbb{C}[z, \bar{z}]$ be a polynomial such that $\varnothing \neq Z_P \neq \mathbb{C}$. Let γ on $\mathfrak{N}$ admit a representing measure supported in Z_P. Suppose condition (4) holds for ψ_P, i.e. every Borel subset τ of Z_P is of the form $\tau = \psi_P^{-1}(\sigma)$ with some Borel $\sigma \subset \mathbb{T}$. If $\mathrm{PDE}(\gamma) = \{\tilde{\gamma}\}$, then γ is determinate.

(Recall that converse is true if there is no atom at 0.)

So the condition (4) becomes even more important. It holds on occasions, but it seems to fail in most cases. e.g. for the hiperbola $xy = 1$, the parabola $y = x^2 + 1$, the unit circle itself,

Does uniqueness imply determinacy?

Assume that $\mathrm{PDE}(\gamma) = \{\tilde{\gamma}\}$. Does it follow that γ is determinate? A partial solution:

Theorem

Let $P \in \mathbb{C}[z, \bar{z}]$ be a polynomial such that $\varnothing \neq Z_P \neq \mathbb{C}$. Let γ on $\mathfrak{N}$ admit a representing measure supported in Z_P. Suppose condition (4) holds for ψ_P, i.e. every Borel subset τ of Z_P is of the form $\tau = \psi_P^{-1}(\sigma)$ with some Borel $\sigma \subset \mathbb{T}$. If $\mathrm{PDE}(\gamma) = \{\tilde{\gamma}\}$, then γ is determinate.

(Recall that converse is true if there is no atom at 0.)

So the condition (4) becomes even more important. It holds on occasions, but it seems to fail in most cases. e.g. for the hiperbola $xy = 1$, the parabola $y = x^2 + 1$, the unit circle itself,


PRESENTATION IS FINISHED!
THANK YOU FOR YOUR ATTENTION!
risovach.ru