# Lecture 5: Peak sets for operator algebras and quantum cardinals

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## Abstract for Lecture 5

Abstract: We discuss noncommutative peak sets for operator algebras, and end with recent joint work with Weaver on quantum cardinals and quantum measure theory.

The latter half of the talk contains some background for my conference lecture next week.

FIRST, FINISH UP LECTURE 4:

(This material is joint work with L. Labuschagne; Lecture 5 mostly with Nik Weaver, some with Labuschagne)

One formulation (without mentioning peak sets) of our generalization of Ueda's peak set theorem to subalgebras of $\sigma$-finite von Neumann algebras:

Generalized Ueda peak set theorem Suppose $A$ is a subdiagonal subalgebra of a $\sigma$-finite von Neumann algebra $M$. For any singular state $\varphi$ on $M$, there is a sequence $(p_n)$ of projections in Ker $(\varphi)$ with sup in $M^{**}$ in $A^{\perp\perp}$, and sup in $M$ being $1$.

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The proof is far too technical to describe here. It uses the tools described above, together with the same basic strategy as Ueda' proof of his case, but becomes enormously more complicated technically. We will describe Ueda' original proof below, emphasizing where real positivity comes in. I will also explain why this is a theorem about noncommutative peak sets and reformulate it as such, explaining what noncommutative peak sets are

- All the other consequences found by Ueda of his peak set theorem, now go through in our more general case. It is convenient to phrase this as follows:

If $A$ is a weak* closed subalgebra of a von Neumann algebra $M$ then we say that $A$ is an Ueda algebra if Ueda's peak set theorem 'holds' for $A$.

Ueda's ideas then immediately give the following three generalizations of his beautiful results:

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Ueda's ideas then immediately give the following three generalizations of his beautiful results:

Theorem Suppose that $A$ is an Ueda algebra in a von Neumann algebra $M$. Write $A_s^*$ and $A_n^*$ for the set of restrictions to $A$ of singular and normal functionals on $M$. Each $\varphi \in A^*$ has a unique Lebesgue decomposition relative to $M$: $\varphi = \varphi_n + \varphi_s$ with $\varphi_n \in A_n^*$ and $\varphi_s \in A_s^*$. Moreover, $\|\varphi\| = \|\varphi_n\| + \|\varphi_s\|$.

Corollary Suppose that $A$ is an Ueda algebra in a von Neumann algebra $M$. Then the predual $A_*$ of $A$ is unique

**Theorem** (F. & M. Riesz type theorem) Suppose that $A$ is an Ueda algebra in a von Neumann algebra $M$. If $\varphi \in M^*$ annihilates $A$ (that is, $\varphi \in A^\perp$) then the normal and singular parts, $\varphi_n$ and $\varphi_s$, also annihilate $A$.

One may define an **F & M Riesz algebra** to be a weak* closed subalgebra $A$ of a von Neumann algebra $M$, such that if $\varphi \in A^\perp$ then the normal and singular parts, $\varphi_n$ and $\varphi_s$, also annihilate $A$. The F & M type theorem above says that any Ueda algebra is an F & M Riesz algebra.

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By proofs in [B-Labuschagne] (but using the F $\&$ M type theorem above instead of our original one) we have the following several results:

Corollary Suppose that $A$ is an F $\&$ M Riesz or Ueda algebra in a von Neumann algebra $M$ such that $A + A^*$ is weak* dense in $M$. Any Hahn-Banach extension to $M$ of a weak* continuous functional on $A$, is normal.

The last Corollary is related to the Gleason-Whitney theorem:

Lemma Suppose $A$ is a weak* closed subalgebra of a von Neumann algebra $M$. Then $A + A^*$ is weak* dense in $M$ iff there is at most one normal Hahn-Banach extension to $M$ of any weak* continuous functional on $A$.

Corollary (Gleason-Whitney type theorem) Suppose that $A$ is an F & M Riesz or Ueda algebra in a von Neumann algebra $M$. Then $A + A^*$ is weak* dense in $M$ if and only if every normal functional on $A$ has a unique Hahn-Banach extension to $M$. This extension is normal.

Of course by our main theorem all of these hold when $A$ is a maximal subdiagonal subalgebra of a $\sigma$-finite von Neumann algebra $M$. Conversely these properties characterize maximal subdiagonal subalgebras (in terms of if and only if every normal functional on $A$ has a unique normal Hahn-Banach extension to $M$).

Summary of the last 3 pages: Any algebra satisfying the conclusions of Ueda's peak set theorem also has the F $\&$ M Riesz, Gleason-Whitney, unique predual, Lebesgue decomposition, etc. It also satisfies the earlier Kaplansky density theorem

The Lebesgue decomposition result generalizes the Lebesgue decomposition theorem for functionals on a von Neumann algebra, the unique predual result simultaneously generalizes the Ando-Wojtaszczyk result that $H^\infty(\mathbb{D})$ has a unique predual, and the Dixmier-Sakai result that von Neumann algebras have unique predual

# Section 4. Noncommutative peak sets and Ueda's theorem

(Classical) peak set for a uniform algebra $A \subset C(K)$:

Peak set: $E = f^{-1}(\{1\})$ for a norm 1 function $f$ in $A$. One may rechoose $f$ such that $|f| < 1$ on $E^c$ (replace with $\frac{1}{2}(1+f)$), in which case $f^n \to \chi_E$.

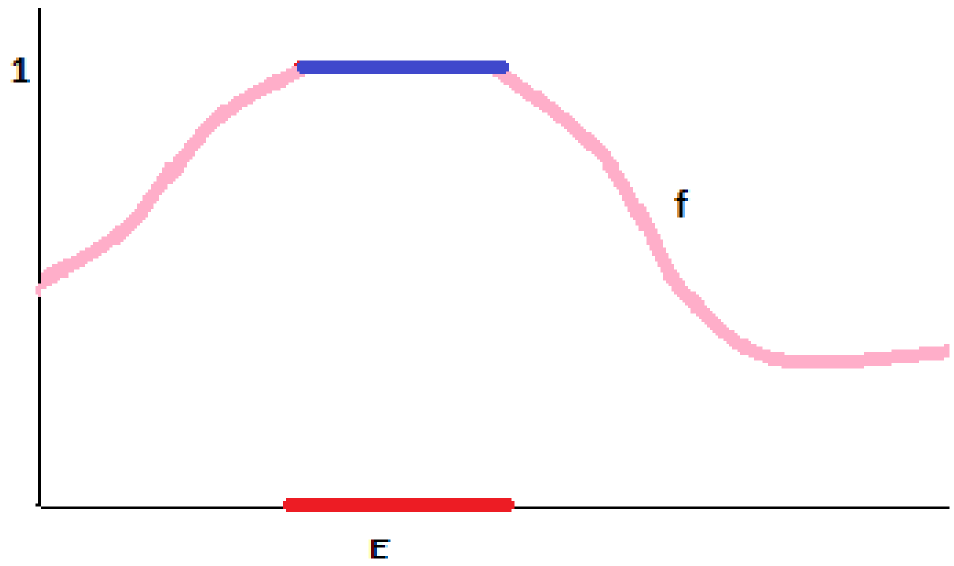


Figura 1: A peak set $E$

Recall that a peak set for a uniform algebra $A \subset C(K)$ is a closed set $E = f^{-1}(\{1\})$ for a norm 1 function $f$ in $A$. One may rechoose $f$ such that $|f| < 1$ on $E^c$, in which case $f^n \to \chi_E$.

$C^*$-algebraic variant:

$$u(x) = \chi_{\{1\}}(x) = \mathrm{w}^*\mathrm{lim}_n \, x^n, \qquad x \in \mathrm{Ball}(B)_+.$$

Here $B$ is a $C^*$-algebra.

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• There is a calculus (collection of nice algebraic formulae) for these that plays a big role in $C^*$-algebra theory. Indeed $u(x) = s(1-x)^\perp$ if $0 \leq x \leq 1$. Here $s(\cdot)$ is the support projection

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Theorem  If $A$ is a closed subalgebra of a $C^*$-algebra $B$ then $q$ is a peak projection for $A$ if and only if $q \in A^{\perp\perp}$ and $q$ is a peak projection for $B$.

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• But there is no time, so instead we discuss what we need for Lecture 4, peak projections in von Neumann's algebra, and Ueda's peak set theorem

• Recall that several famous theorems about $H^\infty$ of the disk, such as theuniqueness of predual (Ando-Wojtaszczyk), Lebesgue decomposition, F $\&$ M Riesz theorem, Gleason-Whitney, etc, can be shown to all follow from a peak set theorem.

• In the classical case of $H^\infty(\mathbb{D})$ this peak set theorem is due to Amar and Lederer: 'Any closed set of measure zero is contained in a peak set of measure zero'. (I am simplifying, namely where the sets live.)

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Ueda's (nc Amar-Lederer) peak set theorem may be phrased as saying that any singular support projection (i.e. the support of any singular state on $M$), is dominated by a peak projection $p$ for $A$ with $p$ in the 'singular part' of $M^{**}$ (that is, $p$ annihilates all normal functionals on $M$).

Lemma (Characterization of peak projections for subalgebras $A$ of a von Neumann algebra $M$, B+Labuschagne) A projection $q$ in $M^{**}$ is a peak projection for $A$ if and only if $q \in A^{\perp\perp}$ and $q = \wedge_n q_n$, the infimum in $M^{**}$ of a (decreasing if you wish) sequence $(q_n)$ of projections in $M$.

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- We can reformulate Ueda's peak set theorem using this lemma: the peak projection $p$ in Ueda's theorem in is in $A^{\perp\perp}$ and is the inf in $M^{**}$ of a sequence $(q_n)$ of projections in $M$. If $p$ dominates the support of $\varphi$ then this forces $\varphi(q_n) = 1$ for all $n$. To say $p$ annihilates all normal functionals on $M$ is easily seen to imply that the inf of $(q_n)$ in $M$ is $0$. And vice versa. This gives the formulation of Ueda's peak set theorem from the end of lecture 4.

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- Recall that Ueda proved his noncommutative peak set result in the case that $M$ has a faithful normal tracial state ('finite' vNA).

Restatement: Ueda's (nc Amar-Lederer) peak set theorem If $A$ is a subdiagonal subalgebra of a von Neumann algebra $M$ with a faithful normal tracial state. For a singular state $\varphi \in M^*$, there exists a contraction $a \in A$ and a projection $p \in M^{**}$ with

(1) $a^n \to p$ weak* in $M^{**}$.

(2) $a^n \to 0$ weak* in $M$.

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To see the equivalence, note the $p$ is the peak projection $u(a)$ by (1), and satisfies $\varphi(p) = 1$ by (3), so the support of $\varphi$ is dominated by this peak projection. For $\psi \in M_*$ we have $\psi(p) = \lim_n \psi(a^n) = 0$. So $p$ annihilates all normal functionals on $M$. And conversely.

Ueda's strategy for proving his original peak set theorem: A tale of two transforms:

Let $\varphi$ be a singular state. Zorn's lemma gives an increasing sequence of projections $(q_n)$ in Ker$(\varphi)$ with supremum $1$. So $p_n = q_n^{\perp} \searrow 0$ in $M$, and $p_n \searrow p$ say in $M^{**}$. So $\varphi(p) = \varphi(p_n) = 1$. Replacing by a subsequence if necessary, so that $\tau(p_n) < n^{-6}$ say, $g = \sum_n n\, p_n \in L^2(M)_+$.

Take the Hilbert transform of $g$ to get an accretive (real positive) element $g + iH(g)$ in noncommutative $H^2$ with real part $g$.

The Cayley transform of this gives an element $b \in \mathrm{Ball}(A)$ with an (unbounded) inverse, and set $a = \frac{1}{2}(1+b)$. This is real positive, so has a peak $u(x) = \mathrm{w}^*\mathrm{lim}\, a^n$, limit in $M^{**}$.

Then $a^n \to 0$ WOT by operator theory; since there is no nontrivial subspace on which $a$ acts isometrically. In turn this follows easily because, as we said, $b$ has an (unbounded) inverse.

A bit of work shows that $pa = p$, so $pa^n = p$ and in the weak* limit get

$$p \leq u(a) = \mathrm{w}^*\lim a^n,$$

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Again, why we wanted to describe this proof is to show the basic strategy for our proof of our very technical generalization of it to $\sigma$-finite von Neumann algebras; and to see more of the role of real positivity

- In our conference lecture we will give characterizations of the von Neumann algebras for which Ueda's theorem holds (taking $A = M$; i.e. the von Neumann algebras which are Ueda algebras): e.g. iff every collection of mutually orthogonal projections in $M$ has cardinality $<$ a fixed cardinal $\kappa_0$

It is possible that the cardinal $\kappa_0$ here is the continuum (cardinality of $\mathbb{R}$)

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Remark. If the Ueda peak set result fails then it fails for all subalgebras $A$ of $M$; so take $A = M$ henceforth.

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- Specializing our statement of Uedas peak set result from yesterdays lecture to the case $M = A = l^{\infty}(\mathbb{R})$, yields: For any singular state $\varphi$ on $M$, there is a sequence $(p_n)$ of projections in Ker $(\varphi)$ with sup in $M$ being $1$.

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- Projections $p_n$ in $l^{\infty}(\mathbb{R})$ correspond to subsets $E_n$ of $\mathbb{R}$, and $\varphi$ restricts to a finitely additive measure $\mu$ on $\mathbb{R}$ by $\mu(E) = \varphi(1_E)$. Saying that $\varphi$ is singular is saying that $\mu$ vanishes on singletons (since recall singular means that every nonzero projection dominates a nonzero projection in the kernel).

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- So Ueda's theorem in this case is implying that for every finitely additive probability measure $\mu$ defined on all subsets of $\mathbb{R}$ vanishing on singletons, there is a countable collection of $\mu$-null sets in $\mathbb{R}$ whose union is $\mathbb{R}$, which has measure $1$. Something smells fishy!

We are now partly moving into the realms of set theory, which I am not an expert on (consult my coauthor Nik Weaver)

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Why not disprovable? It is provable by a standard easy argument in set theory if you add the continuum hypothesis (feel free to ask for this argument )

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That is, they would think that it is safe to add e.g. the affirmitive answer to Banach's measure problem as an extra axiom of set theory, if convenient (and if you dont add CH too)

## Section 6. Quantum cardinals

In set theory there is an elaborate hierarchy of "large cardinal" properties, e.g. an uncountable cardinal $\kappa$ is said to be

- Ulam real-valued measurable if there is a countably additive probability measure on $\kappa$ which vanishes on singletons.
- Ulam measurable if there is a nonzero countably additive $\{0, 1\}$-valued measure on $\kappa$ which vanishes on singletons
- measurable if there is a nonzero $< \kappa$-additive $\{0, 1\}$-valued measure on $\kappa$ which vanishes on singletons
- real-valued measurable if there is a $< \kappa$-additive probability measure on $\kappa$ which vanishes on singletons

Here measures on $\kappa$ are assumed defined on all subsets of $\kappa$, and "$< \kappa$-additive" means "additive on any family of fewer than $\kappa$ disjoint sets".

Later, "$<\kappa$-additive" (for a state on $M$) means "additive on any family of fewer than $\kappa$ projections". Eg.

- Banach's measure problem is asking if $[0,1] = 2^{\aleph_0}$ is Ulam real-valued measurable.

FACTS: No cardinal of these types can be proven to exist in ZFC, assuming ZFC is consistent. It is generally believed that the existence of such cardinals is consistent with ZFC. However, this, if true, could not be proven within ordinary set theory. They all have the same consistency strength, i.e., the consistency of any of the theories

ZFC + "a measurable cardinal exists"

ZFC + "a real-valued measurable cardinal exists"

ZFC + "an Ulam measurable cardinal exists"

ZFC + "an Ulam real-valued measurable cardinal exists"

implies the consistency of each of the others.

The basic relations among these notions:

- Every measurable cardinal is real-valued measurable, and among cardinals $> 2^{\aleph_0}$ the two notions coincide.

- No measurable cardinal can be $\leq 2^{\aleph_0}$, whereas if it is consistent that a measurable cardinal exists then it is consistent that there is a real-valued measurable cardinal $\leq 2^{\aleph_0}$.

- A cardinal is Ulam measurable if and only if it is greater than or equal to some measurable cardinal, and a cardinal is Ulam real-valued measurable if and only if it is greater than or equal to some real-valued measurable cardinal.

- Each of these four kinds of measurability can be expressed in terms of states on $l^\infty(\kappa)$: E.g.

Proposition An uncountable cardinal $\kappa$ is

(i) Ulam real-valued measurable if and only if there is a singular countably additive state on $l^\infty(\kappa)$

(ii) Ulam measurable if and only if there is a singular countably additive pure state on $l^\infty(\kappa)$

(iii) measurable iff there is a singular $< \kappa$-additive pure state on $l^\infty(\kappa)$

(iv) real-valued measurable iff there is a singular $< \kappa$-additive state on $l^\infty(\kappa)$

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(iii) measurable iff there is a singular $< \kappa$-additive pure state on $l^\infty(\kappa)$

(iv) real-valued measurable iff there is a singular $< \kappa$-additive state on $l^\infty(\kappa)$

A natural question is to replace $l^\infty(\kappa)$ by $B(l^2(\kappa))$ in each of the above, which presumably gives quantum versions of these four classes of measurable cardinals

Theorem The four quantum versions are the same as their classical versions. E.g. there exists a singular countably additive pure state on $B(l^2(\kappa))$ if and only if $\kappa$ is Ulam measurable.

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We will give new proofs of most of these in our conference talk, deducing these from a generalization from $B(l^2(\kappa))$ to all von Neumann algebras.

Essentially the only implication that we will not deduce in our conference talk from von Neumann algebra generalizations, we prove now, using a generalization of the recently solved Kadison-Singer problem (Marcus, Spielman and Srivastava, 2013):

Theorem (Kadison-Singer extension) Let $\kappa$ be an infinite cardinal. Then any pure state on the diagonal subalgebra of $B(l^2(\kappa))$ has a unique (necessarily pure) state extension to $B(l^2(\kappa))$.

Proof that there exists a singular countably additive pure state on $B(l^2(\kappa))$ if $\kappa$ is Ulam measurable:

If $\kappa$ is measurable then by the last Proposition there is a singular $<\kappa$-additive pure state $\phi$ on $l^\infty(\kappa)$. Let $\Phi$ be the conditional expectation from $B(l^2(\kappa))$ onto the diagonal subalgebra which is identified with $l^\infty(\kappa)$. Then $\psi = \phi \circ \Phi$ is a state on $B(l^2(\kappa))$. Since $\phi$ is pure, the Kadison-Singer extension above implies that $\psi$ is pure. It is easily seen to vanish on the compacts, so it is singular, and is $<\kappa$-additive since $E$ is normal.

Farah-Weaver: A quantum filter on a von Neumann algebra $M$ is a family of projections $\mathcal{F}$ in $M$ with the properties

(i) if $p \in \mathcal{F}$ and $p \leq q$ then $q \in \mathcal{F}$

(ii) if $p_1, \ldots, p_n \in \mathcal{F}$ then $\|p_1 \cdots p_n\| = 1$.

If $\phi$ is a state on $M$ then

$$\mathcal{F}_\phi = \{p \in M : p \text{ is a projection and } \phi(p) = 1\}$$

is a quantum filter

• Pure states correspond to quantum ultrafilters, that is the maximal quantum filters, a theorem of Farah-Weaver. To prove this one needs the following, which we really need later

**Lemma** (Farah-W) Suppose that $\phi$ is a pure state on a von Neumann algebra $M$ and that $\psi$ is a state on $M$ such that

$$\phi(p) = 1 \qquad \Rightarrow \qquad \psi(p) = 1$$

for any projection $p \in M$. Then $\phi = \psi$.

*Proof.* The multiplicative domain of $\phi$ is the set

$$D = \{x \in M : \phi(xy) = \phi(yx) = \phi(x)\phi(y) \text{ for all } y \in M\}.$$

A projection $p$ belongs to $D$ if and only if $\phi(p) = 0$ or $1$. Since $\phi(p) = 1$ implies $\psi(p) = 1$, we have

$$\phi(p) = 0 \quad \Rightarrow \quad \phi(1-p) = 1 \quad \Rightarrow \quad \psi(1-p) = 1 \quad \Rightarrow \quad \psi(p) = 0.$$

So $\phi(p) = \psi(p)$ for every projection $p \in D$. Anderson showed the span of the projections in $D$ is norm dense in $D$, so $\phi = \psi$ on $D$. Finally, since $\phi$ is pure it is the unique state extension to $M$ of its restriction to $D$. So $\phi = \psi$.
$\square$

**Quantum measure theory** Namely: the theory of 'measures' and states on projection lattices of von Neumann algebras.

These projection lattices replace the $\sigma$-algebras of ordinary measure theory (which are of course Boolean algebras).

In part of our conference talk we will explain some basics of quantum measure theory, and make some new contributions to this subject.

- In fact the validity of Ueda's theorem (particularly in the case $M = A$) is about existence of singular states with a certain continuity property: the theorem fails if there is a singular state $\varphi$ on $M$, so that if $\varphi(p_n) = 0$ for all $n \in \mathbb{N}$ then ... (we had something else here before but will improve it next time to $\varphi(\vee_n p_n) = 0$, a property known as regularity).

- Similar conditions on states were studied in the context of axiomatic von Neumann algebra quantum mechanics by e.g. Hamhalter and others.

- This motivated us to consider the general question of the existence of singular states or pure states on von Neumann algebras with various continuity properties, which we discuss in the later talk

Quantum filters need not be actual filters in the usual sense of being stable under finite meets. However:

Lemma Let $\phi$ be a regular state on a von Neumann algebra $M$. Then $\mathcal{F}_\phi$ is a $\sigma$-filter, i.e., it is stable under countable meets.

Proof. Let $\{p_n\}$ be a countable family in $\mathcal{F}_\phi$ and write $p_n^\perp = 1 - p_n$. Then

$$\phi\left(\bigwedge p_n\right) = \phi\left(1 - \bigvee p_n^\perp\right) = 1 - \phi\left(\bigvee p_n^\perp\right) = 1$$

where $\phi(\bigvee p_n^\perp) = 0$ by regularity. Thus $\bigwedge p_n \in \mathcal{F}_\phi$. $\square$

- Proof of Ueda's theorem for $l^\infty(\mathbb{R})$ if you add the continuum hypothesis:

Proof. The standard and reasonably simple proof that the first uncountable cardinal is not measurable, shows that given a finitely additive probability measure $m$ defined on the power set of $\kappa$ for which singleton sets are null, we can decompose $\kappa$ as a countable union $N_n$ of disjoint $m$-null sets. Let $A_n = \cup_{k=1}^n N_k$. Then $m(A_n) = 0$, and $\cup_{k=1}^\infty A_k = \kappa$. □