Lecture 3: Modules over operator algebras

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Abstract

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We discuss the theory of operator modules over operator algebras, the generalization to operator algebras of Hilbert C^*-modules, the module Haagerup tensor product, and how some of these ideas are being used in the noncommutative geometry of spectral triples.

Why operator spaces?

Let A be your favourite unital subalgebra of a C^*-algebra, let $C_n(A)$ be the first column of $M_n(A)$, and consider the basic result from ring theory

$$M_n(A) \cong \mathrm{Hom}_A(C_n(A))$$

This relation breaks down when norms are placed on the spaces. That is, there is no sensible norm to put on $C_n(A)$ so that

$$M_n(A) \cong B_A(C_n(A)) \ \text{ isometrically}$$

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However, operator spaces save the day:

$$M_n(A) \cong CB_A(C_n(A)) \text{ completely isometrically}$$

• For C^*-algebras themselves this is OK, and the usefulness of operator spaces is less obvious, but will see later (must use 'completely bounded morphisms' in many later results, etc).

Section 1. Operator modules

A concrete left operator A-module is a linear subspace $X \subset B(K, H)$, which we take to be norm closed as always, together with a completely contractive homomorphism $\theta: A \to B(H)$ for which $\theta(A)X \subset X$. Such an X is a left A-module via θ.

There is also an abstract definition of operator modules.

Theorem [Christensen-Effros-Sinclair, B] Operator modules are just the operator spaces X which are modules over an operator algebra A, such that

$$\|ay\| \ \leq \ \|a\| \ \|y\|,$$

for all matrices $a \in M_n(A), y \in M_n(X), n \in \mathbb{N}$ (that is, the module action is completely contractive as a map on the Haagerup tensor product)

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• My favorite proof of the last theorem uses the operator space left multipliers mentioned in earlier talks

Recall: the maps $T : X \to X$ such that for all $x, y \in X$:

$$\left\| \begin{bmatrix} Tx \\ y \end{bmatrix} \right\| \ \leq \ \left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\|$$

(And similarly for matrices $[x_{ij}], [y_{ij}] \in M_n(X)$)

These maps form the unit ball of $LM(X)$

$LM(X)$ is defined only in vector space and matrix norm terms.

Theorem If $T : X \to X$ TFAE:

(i) $T \in LM(X)$.

(ii) There exists c.i. embedding $X \hookrightarrow B(H)$ and $a \in B(H)$ with $Tx = ax$ for all $x \in X$.

(iii) ... in terms of 'noncommutative Shilov boundary' ...

(iv) ...

• $LM(X)$ is always a unital operator algebra, indeed a dual operator algebra if X is a dual operator space

• $LM(X)$ is often what you would expect in operator algebraic situations. Thus we are getting algebra 'for free' from the matrix norms

Suppose that A, B are operator algebras

(1) If H, K are Hilbert spaces, and if $\theta: A \to B(H)$ and $\pi: B \to B(K)$ are completely contractive homomorphisms, then $B(K, H)$ is an operator A-B-bimodule (with the canonical module actions).

(2) Submodules of operator modules are clearly operator modules.

(3) Any operator space X is an operator $\mathbb{C}$-$\mathbb{C}$-bimodule.

(4) Any operator algebra A is an operator A-A-bimodule of course.

(5) Hilbert A-module: a Hilbert space H which is a left A-module whose associated homomorphism $\theta: A \to B(H)$ is completely contractive (or sometimes, completely bounded). Then H^c is an operator A-module.

- We define ${}_AHMOD$ to be the category of Hilbert A-modules, with bounded A-module maps as the morphisms (they are automatically completely bounded on the associated column Hilbert spaces).

The algebra of a bimodule: If X is an (operator) A-B-bimodule over algebras A and B, set D to be the algebra

$$\begin{bmatrix} a & x \\ 0 & b \end{bmatrix}$$

for $a \in A$, $b \in B$, $x \in X$. The product here is the formal product of 2×2 matrices, implemented using the module actions and algebra multiplications. This is an (operator) algebra.

• Any C^*-module Z is an operator module (just look at the linking C^*-algebra $\mathcal{L}(Z)$)

$$\begin{bmatrix} k & z \\ \bar{w} & a \end{bmatrix}$$

for $a \in A$, $z, w \in Z$, and k in the 'algebra of compacts' on Z. The product here is the formal product of 2×2 matrices, implemented using the module actions and algebra multiplications.

$$\|[y_{ij}]\|_n = \|[\sum_{k=1}^{n} \langle\, y_{ki} \,|\, y_{kj} \,\rangle]\|^{\frac{1}{2}}$$

- A right C^*-module Z which is also a left module over a different C^*-algebra A via a nondegenerate *-homomorphism $\theta : A \to \mathbb{B}(Z)$, is also a left operator module. [Indeed by looking at $M(\mathcal{L}(Z))$ one can see Z is a left operator module over $\mathbb{B}(Z)$, the adjointable maps, $=$ the 1-1-corner of $M(\mathcal{L}(Z))$.]

- Indeed most of the important modules in C^*-theory are operator modules

• Any bounded module map between C^*-modules is completely bounded, with $\|T\|_{cb} = \|T\|$. Thus we find ourselves in a situation where we do not have to insist on working only with completely bounded maps, rather we can exploit the fact that our maps already are completely bounded. [One proof: WLOG $Y = Z$, then look in $LM(\mathcal{L}(Z))$ where T becomes 'left multiplication by an operator'. Recall also that $B_A(Z) \cong LM(\mathbb{K}(Z))$ (Lin).]

- Indeed one can give a complete operator space view of C^*-modules, and this is often a useful and powerful perspective.

• Indeed one can give a complete operator space view of C^*-modules, and this is often a useful and powerful perspective.

For example, the basic tensor product $Y \otimes_\theta Z$ of C^*-modules, sometimes called the interior tensor product, turns out to coincide with the module Haagerup tensor product $Y \otimes_{hA} Z$. Here $\theta : A \to \mathbb{B}(Z)$ is an nondegenerate $*$-homomorphism.

• This is wonderful, because the module Haagerup tensor product has very strong algebraic properties. It is functorial, associative, 'injective', 'projective', can be expressed in terms of nice norm formulae, etc.

Key point: Often the operator space/module Haagerup tensor product allows one to treat theories involving C^*-modules much more like pure algebra. It 'facilitates 'continuity' for algebraic (ring-theoretic) structures. In particular, there is a 'calculus' of algebraic formulae (involving tensor products) that is very useful.

- For example, it gives the

Hom-Tensor relations: Let A and B be C^*-algebras. We have

(1) $\mathbb{K}_A(Y, \mathbb{K}_B(Z, N)) \cong \mathbb{K}_B((Y \otimes_\theta Z), N)$ completely isometrically, if Y, Z are right C^*-modules over A and B respectively, and Z, N are left and right operator modules over A and B respectively.

(2) $\mathbb{K}_A(X, \mathbb{K}_B(W, M)) \cong \mathbb{K}_B((W \otimes_\theta X), M)$ completely isometrically, if X, W are left C^*-modules over A and B respectively, and W, M are right and left operator modules over A and B respectively.

(3) $\mathbb{K}_A(Y, (N \otimes_{hB} M)) \cong N \otimes_{hB} \mathbb{K}_A(Y, M)$ completely isometrically, if Y is a right module over A, if M is a $B - A$ operator bimodule and if N is a right A-operator module.

(4) $\mathbb{K}_A(X, N \otimes_{hB} M) \cong \mathbb{K}_A(X, N) \otimes_{hB} M$ completely isometrically, if X is a left C^*-module over A, if N is an $A - B$ operator bimodule and if M is a left B-operator module.

(5) $\mathbb{K}_B(\mathbb{K}_A(Y, W), M) \cong Y \otimes_{hA} \mathbb{K}_B(W, M)$ completely isometrically, if Y is a right C^*-module over A, M is a left B-operator module, and W is a right A operator module which is a left B C^*-module.

(6) $\mathbb{K}_B(\mathbb{K}_A(X,Z),N) \cong \mathbb{K}_B(Z,N) \otimes_{hA} X$ completely isometrically, if X is a left C^*-module over A, N is a right B operator module, and Z is a left A operator module which is a right B C^*-module.

(7) $\mathbb{K}_A(X, \mathbb{K}_B(Z,W)) \cong \mathbb{K}_B(Z, \mathbb{K}_A(X,W))$ completely isometrically, if X, Z are left and right C^*-modules over A and B respectively, and if W is an $A - B$ operator bimodule.

• It is important to note here that some of these spaces of 'compacts' dont make sense without operator space theory, and on these you have to use the completely bounded norm

• There is also a famous and powerful variant of C^*-modules and their theory appropriate to modules over von Neumann algebras

A W^*-module is a Hilbert C^*-module over a von Neumann algebra which is 'selfdual' (i.e. the appropriate 'Riesz representation theorem' for 'functionals' works), or equivalently which has a Banach space predual (a result of Zettl, see Effros-Ozawa-Ruan,B for proofs).

Analogues of the results earlier in this section hold for these

Theorem Suppose Y is a Banach space (resp. operator space) and a right module over a C^*-algebra A. Then Y is a C^*-module, and the norm on Y (resp. the matrix norms on Y) coincides with the C^*-module's norm (resp. canonical operator space structure) if and only if there exists a net of positive integers $n(\alpha)$, and contractive (resp. completely contractive) A-module maps $\phi_\alpha : Y \to C_{n(\alpha)}(A)$ and $\psi_\alpha : C_{n(\alpha)}(A) \to Y$ with $\psi_\alpha \circ \phi_\alpha \to Id_Y$ strongly (that is, point-norm) on Y. In this case, for $y, z \in Y$, the norm limit $\lim_\alpha \phi_\alpha(y)^* \phi_\alpha(z)$ exists in A and equals the C^*-module inner-product.

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• This suggests the following generalization of C^*-modules: for an operator algebra A and a right A-module Y which is also an operator space, such that there exists a net of positive integers $n(\alpha)$, and contractive (resp. completely contractive) A-module maps $\phi_\alpha : Y \to C_{n(\alpha)}(A)$ and $\psi_\alpha : C_{n(\alpha)}(A) \to Y$ with $\psi_\alpha \circ \phi_\alpha \to Id_Y$ strongly on Y.

This works! These are called A-rigged modules, and their theory generalizes the theory of C^*-modules, as we see in the next section.

Section 2. Why generalize C^*-modules to nonselfadjoint algebras?

Here is a great example from noncommutative geometry for why you might want to, due to Bram Mesland

In noncommutative geometry, and in particular in KK-theory, a key role is played by Connes' spectral triples, which he developed to extend the Atiyah-Singer index theorem to 'noncommutative' spaces.

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One should think of the algebra $\mathcal{A}$ of smooth functions on a manifold acting on the Hilbert space of L^2-spinors, together with the associated Dirac operator. If e.g. one has a group acting on the manifold, or a foliation structure on the manifold, the $*$-algebra $\mathcal{A}$ involved is no longer commutative, but in examples there is still a 'Dirac operator': an unbounded densely defined self-adjoint operator D on H such that $[a, D] = aD - Da$ is bounded, etc.

A hugely important generalization of the spectral triples/unbounded K-cycles are the unbounded KK-cycles of Baaj and Julg. For appropriate C^*-algebras (A, B), and dense $*$-subalgebra $\mathcal{A}$ of A, one of their formulations of these involve:

H, now a (countably generated graded Hilbert) C^*-module over B with an appropriate left action of A, and an unbounded self-adjoint regular operator F on H with $a(1+F^2)^{-1}$ compact for $a \in A$, and $[a, F]$ bounded for a in a dense $*$-subalgebra $\mathcal{A}$ of A.

There is also an unbounded Kasparov product, a notoriously difficult way to pair (homotopy classes) of KK-cycles for (A, B) with KK-cycles for (B, C) to give a KK-cycles for (A, C).

Mesland developed a framework involving differentiable C^*-modules and smooth connections which use nonselfadjoint operator algebras, and a version of my 'rigged module' generalization of C^*-modules, and in particular the great algebraic properties of the module Haagerup tensor product on operator modules over nonselfadjoint operator algebras generalizing C^*-modules, with a goal to establish a more algebraic and computable formula for the unbounded Kasparov product in certain cases.

Suppose that one is given two unbounded KK-cycles $(\mathcal{A}, X, E)$ and $(\mathcal{B}, Y, F)$. Here $\mathcal{A}, \mathcal{B}$ are dense $*$-subalgebras of C^*-algebras A and B, and X is a right C^*-module over B which has an A-action, and E is the usual unbounded operator on X with $[a, E]$ bounded for $a \in \mathcal{A}$. Similarly for Y and F, but the left action on Y is by B, and the right action is by a third C^*-algebra.

Mesland starts the process by completing the subalgebra $\mathcal{B}$ in a suitable operator space norm. Namely, represent $\mathcal{B}$ in $\mathbb{B}(Y \oplus Y)$ by

$$\begin{bmatrix} b & 0 \\ bF - Fb & b \end{bmatrix} \subset \mathbb{B}(Y \oplus Y)$$

Since $\mathbb{B}(Y \oplus Y)$ is a C^*-algebra, this gives $\mathcal{B}$ a new (nonselfadjoint operator algebra) norm, which we can complete in.

(This is probably easier to visualize by using Connes spectral triples, where Y is a Hilbert space)

This effectively moves the computations into the realm of nonselfadjoint operator algebras and their operator modules (generalizing C^*-modules) and module Haagerup tensor products, which as we shall see still behave nicely. The desired 'connection' maps into such a tensor product (over the nonselfadjoint operator algebra we just constructed) of a copy of X and $\mathbb{B}(Y)$, and is completely bounded. As mentioned, the great properties of the module Haagerup tensor product now help simplify that the resulting cycle indeed represents the Kasparov product.

The nonselfadjoint algebra occurring in the above construction has a natural completely isometric involution: it is a $*$-operator algebra in a sense considered by Kaad and Lesch

namely an operator algebra A with an involution $\dagger$ making it a $*$-algebra with $\|[a_{ji}^{\dagger}]\| = \|[a_{ij}]\|$.

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• Other examples of these occur in nc differential geometry, and in fact when one looks one sees them all over the place. Studying these with Kaad and Mesland:

Theorem For an operator algebra (not necessarily approximately unital). The following are equivalent:

(1) A is an operator $*$-algebra.

(2) There exists a C^*-algebra B containing a completely isometric homomorphic copy of A and a period 2 $*$-automorphism $\pi : B \to B$ with $\pi(A) = A^*$.

(3) Same as (2) but with $B = C^*_{\mathrm{e}}(A)$.

(4) A can be represented on a Hilbert space H such that there is a (even selfadjoint) unitary u on a Hilbert space $u^* A^* u = A$.

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• There are natural operator $*$-modules occurring in the work of Kaad et al, Mesland, etc, which we are studying

Summary: There are good reasons for generalizing C^*-modules and their theory to nonselfadjoint algebras. So lets do it:

Section 3. Rigged modules over operator algebras

• Recall: a rigged module: right A-module Y which is also an operator space, such that there exists a net of positive integers $n(\alpha)$, and completely contractive A-module maps $\phi_\alpha : Y \to C_{n(\alpha)}(A)$ and $\psi_\alpha : C_{n(\alpha)}(A) \to Y$ with $\psi_\alpha \circ \phi_\alpha \to Id_Y$ strongly on Y.

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• There are many alternative characterizations, e.g. in terms of an inner product on a containing C^*-module (which if you want can be chosen to be expressible in terms of the C^*-envelope ('minimal' C^*-algebra/nc Shilov boundary), or in terms of an approximate identity for $Y \otimes_{hA} X$, etc.

• Here X is the 'adjoint' module $\tilde{Y}$ of the rigged module Y, which may be viewed as a submodule of $CB_A(Y, A)$. It has many characterizations, for example as the maps $f \in CB_A(Y, A)$ with $f \circ \psi_\alpha \circ \phi_\alpha \to f$ (this is independent of the particular factorization nets)

• To get our 'primary definition' going; that is, to connect it to anything useful (e.g. things even remotely connected with e.g. the [B-Muhly-Paulsen] Morita equivalence ideas), one needs Hay's theorem discussed in Lecture 1; and the recent theory of hereditary subalgebras of operator algebras due to B-Hay-Neal (recall this uses some deep ideas from 'peak interpolation theory')

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• The operator space structure of a rigged module Y over M is given by $\|[y_{ij}]\|_{M_n(Y)} = \sup_\alpha \ \|[\phi_\alpha(y_{ij})]\|$ for $[y_{ij}] \in M_n(Y)$.

Theorem (2016) If Y is a rigged module over an operator algebra A, viewed as an operator space, and if $LM(Y)$ is the operator space left multiplier algebra of Y in the sense of Lecture 1, then $LM(Y) = CB(Y)_A$ completely isometrically isomorphically. This also equals the left multiplier algebra of $\mathbb{K}(Y)$, where the latter is the so-called compact maps on Y, namely the closure of the span of the maps on Y of form $y \mapsto y'(x, y)$ for some $y' \in Y$ and $x \in \tilde{Y}$

The last result should have many consequences. We give a couple of examples:

Corollary For any orthogonally complemented submodule W of a rigged module Y over an operator algebra A (that is, the range of a completely contractive A-module projection), there is a unique contractive linear projection from Y onto W.

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In the C^*-algebra case the modules with respect to which one can take a tensor product are often called correspondences. Rieffel called them B-rigged A-modules. As we saw, they are the right C^*-modules Y over B for which there exists a nondegenerate/essential $*$-homomorphism $\theta : A \to \mathbb{B}_B(Y)$ (so $\theta(e_t)y \to y$ for all $y \in Y$, where (e_t) is a cai for A)

Theorem Suppose that A, B are approximately unital operator algebras, and that Y is a right rigged B-module which is a nondegenerate left A-module via a homomorphism $\theta : A \to \mathbb{B}_B(Y) = M(\mathbb{K}(Y))$. Then with this action Y is a left operator A-module if and only if θ is completely contractive. If these hold then θ is essential/nondegenerate: there is a contractive approximate identity (e_t) for A with $e_t y \to y$ and $x e_t \to x$ for all $y \in Y, x \in \tilde{Y}$.

- This substantially simplifies my earlier definition of an A-B-correspondence.

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Interior tensor product: We define this to be the module Haagerup tensor product of a right A-rigged module and a right A-B-correspondence, both viewed as operator modules. We will write this tensor product as $X \otimes_\theta Y$, where θ is the left action as above.

It has the same wonderful properties as in the C^*-case, so I will not take the time to repeat them. We will just mention one:

For example, the assignment $X \mapsto X \otimes_\theta Y$ gives a strongly continuous completely contractive linear functor from the category of rigged modules over A to the category of rigged modules over B. The morphisms are the adjointable completely bounded M-module maps.

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The following Eilenberg-Watts result is the converse of the last fact:

Theorem (2016) Let A and B be approximately unital operator algebras, and suppose that F is a strongly continuous completely contractive linear functor from the category of rigged modules over A to the category of rigged modules over B Then there exists an A-B-correspondence Y such that F is naturally unitarily isomorphic to the interior ($=$ Haagerup module) tensor product with Y.

Summary: Rigged modules over a (nonselfadjoint) operator algebra generalize much of the (basic) theory of the *Hilbert* C^**-modules* over C^*-algebras.

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Note: they do not necessarily give a Morita equivalence, or 'countable stabilization' results, although we have a matching notion of Morita equivalence which constitute examples of rigged modules.

Eleftherakis has a stronger (i.e. more restrictive, but with much stronger consequences sometimes, such as 'countable stabilization') notion of Morita equivalence. His modules are examples of ours.

Section 4. Weak* rigged modules over dual operator algebras

Recall: A W^*-module is a Hilbert C^*-module over a von Neumann algebra which is 'selfdual' (i.e. the appropriate 'Riesz representation theorem' for 'functionals' works), or equivalently which has a Banach space predual

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The weak*-rigged or w^*-rigged modules, were introduced by [B+Kashyap], and are a generalization of W^*-modules to the setting of modules over a dual operator algebra. By the latter term we mean a unital weak* closed (nonselfadjoint) algebra of operators on a Hilbert space.

Definition Suppose that Y is a dual operator space and a right module over a dual operator algebra M. We say that Y is a weak* rigged module if there exists a net of positive integers $n(\alpha)$, and weak* continuous completely contractive M-module maps $\phi_\alpha : Y \to C_{n(\alpha)}(M)$ and $\psi_\alpha : C_{n(\alpha)}(M) \to Y$ with $\psi_\alpha(\phi_\alpha(y)) \to y$ in the weak* topology on Y for all $y \in Y$.

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Proposition (B-Kashyap, 2016) If Y is a weak* rigged module over M then the module action $Y \times M \to Y$ is separately weak* continuous.

- With Kashyap we generalized much of the von Neumann algebra theory of W^*-modules to weak* rigged modules

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We also obtain the HOM-TENSOR relations mentioned in the early part of the talk

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- This is again a w^*-rigged module over $M \overline{\otimes} N$.

The weak*-exterior tensor product has properties analogous to the interior tensor product, e.g. it is associative, 'injective', and commutes with direct sums:

Proposition Suppose that M, N are dual operator algebras. If $(Y_k)_{k\in I}$ is a family of right w^*-rigged modules over M, and Z is a right w^*-rigged module over N then we have

$$(\oplus_k^c Y_k)\bar{\otimes} Z \cong \oplus_k^c (Y_k \bar{\otimes} Z),$$

unitarily as right w^*-rigged modules

(Similar formula for exterior tensor product.)

Summary: Much of the basic theory of C^*-modules and W^*-modules generalizes 'functorially' to the case of nonselfadjoint operator algebras

- In addition to use for nonselfadjoint algebras, hopefully this will find more applications to C^*-theory such as those discussed, where a selfadjoint problem leads naturally to a nonselfadjoint operator algebra/module, and computations with these.