Lectures 1 and 2: Operator algebras, etc (basic theory and real positivity)

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Abstract

Lecture 1: Operator algebras, etc

We briefly survey the existing theory of (possibly nonselfadjoint) algebras of operators on a Hilbert space, and unital linear spaces of such operators, and then turn to recent results involving real positivity.

The latter part may be subtitled: The quest for positivity in (non-selfadjoint) operator algebras. With Charles Read we have introduced and studied a new notion of (real) positivity in operator algebras, with an eye to extending/generalizing certain C*-algebraic results and theories to more general algebras. As motivation note that the completely' real positive maps on C*-algebras or operator systems are precisely the completely positive maps in the usual sense; however with real positivity one may develop a useful order theory for more general spaces and algebras. We have continued this work together with Read, and also with Matthew Neal and Narutaka Ozawa, and others.

In Lecture 2 we describe briefly several new variants of this theory of

Section I. Operator algebras and unital operator spaces

Ruan's theorem: An operator space (i.e. subspace of $B(H)$) 'is' a vector space X with a norm $\|\cdot\|_n$ on $M_n(X)$ for all $n \in \mathbb{N}$, such that

(R1) $\|axb\|_n \leq \|a\|\|x\|_n\|b\|$, for all $a, b \in M_n$,

(R2) $\left\|\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}\right\|_{m+n} = \max\{\|x\|_m, \|y\|_n\}$.

Operator spaces were ‘born’ in Arveson’s 1969 Acta paper (complete contractions, complete isometries, ...)

His spaces though were usually at least unital operator spaces:

Unital subspaces of C^*-algebras: $1_A \in X \subset A$

Such spaces have played a significant role in operator space theory in and since Arveson 1969 (one reason being that they include most known nonselfadjoint operator algebras, operator systems, etc)

Despite Ruan’s 1988 characterization of operator spaces, over the years there has been no abstract characterization of unital operator spaces until relatively recently

Here is our first answer to this question:

Notation: $$u_n = \begin{bmatrix} u & 0 & \cdots & 0 \\ 0 & u & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & u \end{bmatrix}$$

Theorem (B-Neal) If X is an operator space, $u \in X$, then (X, u) is a unital operator space if and only if

$$\|[u_n \ \ x]\| = \left\| \begin{bmatrix} u_n \\ x \end{bmatrix} \right\| = \sqrt{2},$$

for all $x \in M_n(X)$ of norm 1, and all $n \in \mathbb{N}$.

- This condition also characterizes 'unitaries in X'

We say an element v in an operator space X is unitary in X if there exists a complete isometry T from X into a C^*-algebra with $T(v)$ a unitary.

Similarly 'half of the conditions above' characterize isometries or coisometries

More recent characterization, which avoids 'matrices of matrices':

Theorem (B-Neal) If X is an operator space, $u \in X$, then (X, u) is a unital operator space (and u is unitary) if and only if

$$\max\{\|u_n + i^k x\| : k = 0, 1, 2, 3\} \geq \sqrt{1 + \|x\|}, \qquad x \in M_n(X),\ n \in \mathbb{N}.$$

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• There is a very very recent 'order-theoretic' characterization of unital operator spaces due to Travis Russell, in terms of the accretive elements (i.e. the x with $x + x^* \geq 0$). In a concrete operator space these form a cone, and conversely he gives conditions on a cone $\mathfrak{c}$ on an operator space X such that for a fixed element $u \in X$, we have X is a unital operator space with unit u and $\mathfrak{c}$ is the cone of accretive elements.

Arveson lemma: If X is a unital operator space then $X + X^*$ is a well defined operator system (independent of completely isometric unital representation of X)

In particular $A + A^*$ is a well defined operator system if A is a unital operator algebra

Theory of not necessarily selfadjoint operator algebras on a Hilbert space

Henceforth operator algebras

We are now going to take quite a bit of time surveying some of the main aspects of this theory

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I am in the thick of:

Several 'generalization projects', , in various stages of development: 1) Banach algebras (B-Ozawa 2015, 2016; an example of which I have been looking at recently being:), 2) Chris Phillips' L^p-operator algebras (subalgebras of $B(L^p)$), 3) Jordan operator algebras (collaboration in progress with Zhenhua Wang and Matt Neal), 4) operator algebras arising in noncommutative geometry, e.g. from spectral triples (collaboration in progress with Kaad and Mesland), 5) more C^*-theory to operator algebras to ... .

So for every statement made in the next N minutes, you should be asking: how does this generalize (particularly to 2, 3, 4)?

Several 'generalization projects', in various stages of development: 1) Banach algebras (B-Ozawa 2015, 2016, this is probably essentially done except:), 2) Chris Phillips' L^p-operator algebras (subalgebras of $B(L^p)$), 3) Jordan operator algebras (collaboration in progress with Zhenhua Wang and Matt Neal), 4) operator algebras arising in noncommutative geometry, e.g. from spectral triples (collaboration in progress with Kaad and Mesland), 5) more C^*-theory to operator algebras to ... .

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I will say more about some of that at the end, but really there is too much to say, and for some of these projects it is a bit premature (work in progress, and some are in relative infancy). The key point is that a huge amount works/can be done in some of these projects

Notation: ‘unital algebra’ means has an identity of norm 1, ‘approximately unital ’ means has a cai (contractive approximate identity).

The characterization of operator algebras (subalgebras of $B(H)$):

Completely isometric variant (B-Ruan-Sinclair): Assuming an identity or cai, up to completely isometric isomorphism the operator algebras are precisely the operator spaces with a multiplication such that

$$\|xy\| \; \leq \; \|x\| \, \|y\|,$$

for all matrices x, y with entries in the algebra (or equivalently for which the multiplication extends to a complete contraction $A \otimes_{\mathrm{h}} A \to A$ on the Haagerup tensor product).

Quotients and vice versa. An immediate

Corollary: A/I is an operator algebra, for any operator algebra A and closed ideal I.

Conversely, such quotient result is key to many facts about operator algebras, including their characterization, e.g. up to 'complete isomorphism'

Completely bounded characterization (B): Up to completely bounded isomorphism the operator algebras are precisely the operator spaces with an associative multiplication such that $\exists K \geq 0$ with

$$\|xy\| \ \leq \ K\|x\| \, \|y\|,$$

for all matrices x, y with entries in the algebra (that is, the multiplication extends to a completely bounded map $A \otimes_{\mathrm{h}} A \to A$).

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• It is a sad fact that we do not have a 'closed under quotients' result in two of the current generalization projects. Hence we do not have abstract characterizations there. (For one of these we do not have a counterexample either yet.)

• However quotients by approximately unital ideals certainly work in our current generalization projects

Characterization of dual operator algebras (weak* closed unital subalgebras of $B(H)$): these are just the operator algebras with an operator space predual (B, Le Merdy; in the nonunital case one also needs a result from B-Magajna)

- My favorite proof of the B-Ruan-Sinclair characterization of operator algebras uses the following:

Multipliers of operator spaces

[B, Werner, B-Paulsen, B-Effros-Zarikian]

$$\begin{array}{ccc} \text{Operator space} & \rightsquigarrow & \text{Operator algebra} \\ X & \rightsquigarrow & LM(X), \end{array}$$

and variants ... e.g.:

$$X \quad \rightsquigarrow \quad A_l(X),$$

where $A_l(X)$ is a C^*-algebra 'acting on the left' of X. Similarly for 'right'.

Consider the maps $T : X \to X$ such that for all $x, y \in X$:

$$\left\| \begin{bmatrix} Tx \\ y \end{bmatrix} \right\| \leq \left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\|$$

(And similarly for matrices $[x_{ij}], [y_{ij}] \in M_n(X)$)

These maps form the unit ball of $LM(X)$

$LM(X)$ is defined only in vector space and matrix norm terms.

Theorem If $T : X \to X$ TFAE:

(i) $T \in LM(X)$.

(ii) There exists c.i. embedding $X \hookrightarrow B(H)$ and $a \in B(H)$ with $Tx = ax$ for all $x \in X$.

(iii) ... in terms of 'noncommutative Shilov boundary' ...

(iv) ...

A few facts in passing:

- $LM(X)$ is always a unital operator algebra, indeed a dual operator algebra if X is a dual operator space
- It contains a related C^*-algebra $A_l(X)$, which is a von Neumann algebra if X is a dual operator space
- $LM(A) = A$ for a unital operator algebra
- $LM(A)$ is usual 'left multiplier algebra' for an approximately unital operator algebra
- $LM(Z)$ is the set of bounded module maps for a C^*-module Z.
- (The related algebra $A_l(Z)$ is the set of 'adjointable module maps')
- Thus we are getting algebra 'for free' from the norms

- Some applications of these facts:

- Can essentially recover the product on a unital operator algebra from its operator space structure.
- Using these, can get Banach-Stone-Kadison type theorems
- The point is the algebra $LM(X)$ is only defined in terms of operator space structure, yet often encodes important algebraic information.
- My main reason for studying multipliers originally was to 'improve on' characterizations of operator algebras and their modules. Indeed such 'improved' theorems follow in just a few lines from this approach.

• Proof of the B-Ruan-Sinclair theorem:
Let $\lambda : A \to B(A)$ be the homomorphism

$$\lambda(a)(b) = ab, \qquad x, y \in A$$

Let $a \in Ball(A)$, and $b, c \in A$:

$$\left\| \begin{bmatrix} ax \\ y \end{bmatrix} \right\| = \left\| \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x & 0 \\ y & 0 \end{bmatrix} \right\| \leq \left\| \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \right\| \left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\|$$
$$\leq 1 \left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\|$$

Thus $\lambda(a) \in Ball(LM(A))$. So $\lambda : A \to LM(A)$ and $\|\lambda\| \leq 1$. Conversely,

$$\|\lambda(a)\| \geq \|\lambda(a)(1)\| = \|a\|$$

So λ is an isometry. Similarly completely isometric. But $LM(X)$ is always an operator algebra! QED

Theorem (A more recent characterization of operator algebras, B-Neal) Let u be a coisometry in an operator space A (or equivalently, $\|[u_n \ \ x]\| = \sqrt{2}$ for $n \in \mathbb{N}$ and every matrix $x \in M_n(A)$ of norm 1). Suppose that $m : A \times A \to A$ is a bilinear map such that

$$\left\| \begin{bmatrix} m(x, a_{ij}) \\ b_{ij} \end{bmatrix} \right\| \leq \left\| \begin{bmatrix} a_{ij} \\ b_{ij} \end{bmatrix} \right\| \qquad [a_{ij}], [b_{ij}] \in M_n(A),$$

for $n \in \mathbb{N}$ and $x \in \mathrm{Ball}(A)$. We also suppose that $m(x, u) = x$ for all $x \in A$. Then m is an associative product, and A with this product is completely isometrically isomorphic to an operator algebra with a two-sided identity (namely, u). Conversely, every unital operator algebra satisfies all the conditions above.

• We emphasize that this uses a small (1×1) matrix (namely x) and one large $n \times (2n)$ matrix, and in particular uses at most one operation in each entry of the matrix, as opposed to the many operations (sums and products) that appear in the entries of a product of two large matrices.

• The multipliers above are related to and generalize a ‘classical’ and important theory in linear functional analysis:

Alfsen and Effros M-ideals (Annals, 1972), and Banach space multipliers. Eg. Two-sided closed ideals in an operator algebra may be characterized purely in Banach space terms.

Theorem The M-ideals in an operator algebra are the approximately unital closed ideals (Effros-Ruan). There is a similar fact for right M-ideals (B-Effros-Zarikian)

So if you want to generalize ideal theory of C^*-algebras to more general algebras ... you may find you really need M-ideals at times

Banach space multipliers: In any Banach space X the maps $T : X \to X$ such that $\|Tx + (I - T)y\| \leq \max\{\|x\|, \|y\|\}$ have some profound properties

If T is idempotent these are the M-projections. An M-ideal is a subspace $J \subset X$ such that $J^{\perp\perp}$ is the range of an M-projection. In a given Banach space these ideals are usually highly significant; and they have the ideal theory of C^*-algebras.

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These ideals are 'controlled' by these commuting projections, which generate a certain commutative von Neumann algebra in the bidual

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Representation theory of general operator algebras (a.k.a. Hilbert modules). Even the very basic theory has some nontrivial facts that use special properties of Hilbert space, which gives problems in some of our generalizations-in-progress

Second duals of operator algebras are operator algebras

Diagonal $\Delta(A) = A \cap A^*$, is independent of $B(H)$

$\Delta(A^{**})$ is a von Neumann algebra

- A completely contractive homomorphism from a operator algebra A into a weak* closed operator algebra M extends uniquely to a weak* continuous completely contractive homomorphism $\tilde{\pi} : A^{**} \to M$

Meyer's theorem A contractive or completely contractive homomorphism extends to such on any unitization A^1

Corollary Uniqueness of unitization of any operator algebra (even up to completely isometric isomorphism)

• There is a much simpler version for operator algebras with cai

Corollary A_{sa} makes sense, and is independent of H

These are the hermitian elements in A (that is, $\|\exp(ith)\| = 1$ for real t; or equivalently $\varphi(h) \in \mathbb{R}$ for all states φ of A, where by 'state' we mean a unital contractive functional).

Similarly, $\mathfrak{r}_A$, the accretive elements in A, may be defined by Re $\varphi(h) \geq 0$ for all states φ of A, and this is equivalent to all the other usual conditions

Then there are relations between A and the C^*-algebras it generates

Universal C^*-algebras, such as the noncommutative Shilov boundary or C^*-envelope $C^*_e(A)$

• Many important facts about operator algebras are derived by looking at this noncommutative Shilov boundary

This is related to the injective envelope $I(A)$ with its powerful 'rigidity' and 'essential' (and 'injective') properties. We have $A \subset C^*_e(A) \subset I(A)$

Theory of contractive approximate identities (cai's)

First ancient fact: such exist iff A^{**} unital

Cohen's factorization theorem and corollaries

Multiplier algebra $M(A)$, and $LM(A)$

Approximate identities and states There are crucial connections between states on A and states on A^1, which seem to break down for Banach algebras

E.g.: For an approximately unital Banach algebra A are the states on A^1 the convex hull of 0 and the state extensions of states of A? Equivalently: is every real positive linear $\varphi : A \to \mathbb{C}$ a nonnegative multiple of a state? Seems OK for most commonly encountered Banach algebras.

Section II. Real positivity: The basic idea, definitions, motivation, and some general results

• In [JFA 2011], Charles Read and I began to study real positivity in not necessarily selfadjoint operator algebras (and unital operator spaces)

... $\mathfrak{F}_A = \{x \in A : \|1 - x\| \leq 1\}$ plays a pivotal role, and the cone $\mathbb{R}^+ \mathfrak{F}_A$.

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Proposition $\overline{\mathbb{R}^+\mathfrak{F}_A} = \mathfrak{r}_A$ where the latter is the cone of elements with positive real part (or Re $\varphi(x) \geq 0$ for every state φ on unitization of A).

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Purely metric descriptions, e.g. x is real positive iff $\|1-tx\| \leq 1+t^2\|x\|^2$ for all $t > 0$.

• Real positives will play the role for us of positive elements in a C^*-algebra; the main goal of this program is to generalize certain nice C^*-algebraic results, or nice function space results, which use positivity or positive cai's.

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• An operator algebra or uniform algebra A may have no positive elements in the usual sense. However A has a contractive approximate identity (cai) iff there is a great abundance of real-positive elements (they span). This is also true for many classes of Banach algebras.

• Next we get a Kaplansky density type result in the bidual (note: all the standard variants of the Kaplansky density theorem follow easily from its bidual version)

Theorem (Kaplansky density type result) If A is an operator algebra then the ball of $\mathfrak{r}_A$ is weak* dense in the ball of $\mathfrak{r}_{A^{**}}$. Similarly for $\mathfrak{F}_A$.

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One application: to get a 'positive cai' in an algebra with cai: using this Kaplansky density to get a real positive cai by approximating $1_{A^{**}}$ in a standard way.

Corollary (Read's theorem) An operator algebra with cai has a real positive cai, indeed even a cai satisfying $\|1 - 2e_t\| \leq 1$

Variants of these work in Banach algebras satisfying the state extension property mentioned earlier [B-Ozawa]

Real positive maps

Real positive maps

(Below 1 is the identity of the unitization.)

Recall that $T : A \to B$ between C^*-algebras (or operator systems) is completely positive if $T(A_+) \subset B_+$, and similarly at the matrix levels

Definition (Bearden-B-Sharma) A linear map $T : A \to B$ between operator algebras or unital operator spaces is *real completely positive*, or RCP, if $T(\mathfrak{r}_A) \subset \mathfrak{r}_B$ and similarly at the matrix levels. (Later variant of a notion of B-Read.)

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(Extension and Stinespring-type) Theorem A linear map $T : A \to B(H)$ on an approximately unital operator algebra or unital operator space is RCP iff T has a completely positive (usual sense) extension $\tilde{T} : C^*(A) \to B(H)$

This is equivalent to being able to write T as the restriction to A of $V^*\pi(\cdot)V$ for a $*$-representation $\pi : C^*(A) \to B(K)$, and an operator $V : H \to K$.

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Theorem (A Banach-Stone type result) Suppose that $T : A \to B$ is a completely isometric surjection between approximately unital operator algebras. Then T is real completely positive if and only if T is an algebra homomorphism.

So now you know what to do ...

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... Run through C^*-theory, particularly where positivity and positive approximate identities are used, and also where completely positive maps appear, but for operator algebras ... the above is effective at generalizing some parts of the theory, but not others. The worst problem is that although we have a functional calculus, it is not as good. But frequently it is good enough.

• Quite often in a given C^*-subtheory this does not work. But sometimes it does work, or sometimes one has to look a little closer and work a little harder, and this can be quite interesting.

• So we are developing this new notion of positivity in operator algebras. Indeed the ideas make sense and give results in much more general spaces than operator algebras. One current direction being pursued is how general can some of our ideas be taken.

• Simultaneously, we are developing applications of real positivity to operator algebras, for example to noncommutative topology (eg. noncommutative Urysohn and Tietze for general operator algebras), noncommutative peak sets and related noncommutative function theory, lifting problems, peak interpolation, comparison theory, conditional expectations, approximate identities, and to new relations between an operator algebra and the C^*-algebra it generates.

• Only time to mention a few things about real positivity in operator algebras.

The Cayley and $\mathfrak{F}$ transform:

• The Cayley transform κ is the operator form of the remarkable analytic mapping $(z-1)/(z+1)^{-1}$. It has several striking properties/uses in operator theory, including in the proof of Meyer's theorem above.

• Because of the power of the Cayley transform, the related $\mathfrak{F}$-transform $\mathfrak{F}(x) = \frac{1}{2}(1+\kappa(x)) = x(x+1)^{-1} = 1-(x+1)^{-1}$ is also quite useful.

Lemma For any operator algebra A, the $\mathfrak{F}$-transform maps $\mathfrak{r}_A$ bijectively onto the set of elements of $\frac{1}{2}\mathfrak{F}_A = \{a \in A : \|1-2a\| \leq 1$ of norm < 1

• So it is usually equivalent to use $\mathfrak{r}_A$ or $\mathfrak{F}_A$

Support projections If x is an operator or element of a C^*-algebra A one usually only defines the support projection if it is selfadjoint (else the left support $\neq$ right support). If $x \geq 0$ the support projection $s(x)$ is the weak* limit of $x^{\frac{1}{n}}$. Or it is $\chi_{\{0\}}(x)^{\perp}$ (spectral projection/Borel functional calculus).

- We consider it in A^{**} usually, in this series. If A is a nonselfadjoint algebra we should rather take x real positive. Then:

Lemma For $x \in A$ real positive the weak* limit $s(x)$ of $x^{\frac{1}{n}}$, equals the left and right supports. Also $s(x)$ is the identity of $(\mathrm{oa}(x))^{**}$

• There is a calculus (collection of nice algebraic formulae) for these projections that plays a big role in C^*-algebra theory (like $s(x) \vee s(y) = s((x+y)/2)$). This will give a matching calculus of closed ideals (even one-sided ideals), or of hereditary subalgebras (HSA's), or of open projections, of the C^*-algebra. These facts/results/ideas are huge, in C^*-theory

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How to generalize all this to nonselfadjoint operator algebras (and beyond)?

Its not easy ... a key point here is Hay's theorem. This does not seem to generalize easily beyond operator algebras, at least in the categories for which it is plausible we have not succeeded yet, but are trying.

Recall: E is an open set in a topological space K iff χ_E is a increasing (weak) limit of positive elements in $B = C_0(K)$ (a good word here is 'semicontinuous')

Similarly, an open projection is a projection in the bidual of a C^*-algebra B which is a increasing (weak) limit of positive elements in B

There are bijective correspondences between HSA's, left ideals, right ideals, and open projections. This is well known for C^*-algebras

Hay's theorem: two formulations: 1) The right ideals in A with left cai are in bijective correspondence with the projections $p \in A^{**}$ such that there is a net (a_t) in A with $a_t = pa_tp \to p$ weak* (note left-right symmetry here)

2) The right ideals in A with left cai are in bijective correspondence with the open projections in B^{**} which lie in $A^{\perp\perp}$

This allows the theory of HSA's in nonselfadjoint algebras to exist, and it behaves similarly to the selfadjoint case

Theorem The right ideals with left cai in an operator algebra A are exactly $\overline{EA}$ for some set E of real positive elements of A. One can find a real positive left cai here. Then the matching left ideal is $\overline{AE}$

- If A (or the right ideal) is separable can take E singleton

Theorem Let A be any operator algebra. The above class of right ideals in A, are precisely the closures of increasing unions of ideals of the form $\overline{xA}$, for $x \in \mathfrak{r}_A$.

(This says open projections are all sups of support projections. In the separable case they are support projections)

Similarly for hereditary subalgebras (HSA's): these are approximately unital subalgebras D with $DAD \subset D$

Compact and peak projections, noncommutative Glicksberg theorem, peak interpolation, etc, will be discussed in a later talk (or at least some of these). Suffice it to say here that again these are simultaneous generalizations of important C^*-algebra and function theory results. In the C^*-case positive elements are used; we use real positive elements.

The induced ordering on A is obviously $b \preccurlyeq a$ iff $a - b$ is real positive

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Theorem: If an approximately unital operator algebra A generates a C^*-algebra B, then A is *order cofinal* in B: given $b \in B_+$ there exists $a \in A$ with $b \preccurlyeq a$. Indeed can do this with $b \preccurlyeq a \preccurlyeq \|b\| + \epsilon$

(This and the next theorem are trivial if A unital)

Order theory in the unit ball

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Here is some order theory in the unit ball of an operator algebra. A feature of the first result is that having the order theory is possible iff there is a cai around

Theorem Let A be an operator algebra which generates a C^*-algebra B, and let $\mathcal{U}_A = \{a \in A : \|a\| < 1\}$. The following are equivalent:

(1) A is approximately unital.

(2) For any positive $b \in \mathcal{U}_B$ there exists real positive a with $b \preccurlyeq a$.

(2') Same as (2), but also $a \in \frac{1}{2}\mathfrak{F}_A$.

(3) For any pair $x, y \in \mathcal{U}_A$ there exist $a \in \frac{1}{2}\mathfrak{F}_A$ with $x \preccurlyeq a$ and $y \preccurlyeq a$.

(4) For any $b \in \mathcal{U}_A$ there exist $a \in \frac{1}{2}\mathfrak{F}_A$ with $-a \preccurlyeq b \preccurlyeq a$.

(5) For any $b \in \mathcal{U}_A$ there exist $x, y \in \frac{1}{2}\mathfrak{F}_A$ with $b = x - y$.

(6) $\mathfrak{r}_A$ is a generating cone (that is, $A = \mathfrak{r}_A - \mathfrak{r}_A$).

• In an operator algebra without any kind of approximate identity there is a biggest subalgebra having good order theory:

Theorem If operator algebra A has no cai then $D = \mathfrak{r}_A - \mathfrak{r}_A$ is the biggest subalgebra with a cai. It is a HSA (*hereditary subalgebra*).

• We recall that the positive part of the open unit ball $\mathcal{U}_B$ of a C^*-algebra B is a directed set, and indeed is a net which is a positive cai for B. The following generalizes this to operator algebras:

Corollary If A is an approximately unital operator algebra, then $\mathcal{U}_A \cap \frac{1}{2}\mathfrak{F}_A$ is a directed set in the $\preccurlyeq$ ordering, and with this ordering $\mathcal{U}_A \cap \frac{1}{2}\mathfrak{F}_A$ is an increasing cai for A.

Corollary If B is a C^*-algebra generated by approximately unital operator algebra A, and $b \in B_+$ with $\|b\| < 1$ then there is a 'nearly positive' increasing cai for A in $\frac{1}{2}\mathfrak{F}_A$, every term of which dominates b (in the $\preccurlyeq$ ordering).

There is a nonselfadjoint 'Tietze' extension theorem ([B-Ozawa]), which uses essentially a result of Smith et al, and this one generalizes to the case A is a Banach algebra.

Corollary Can lift real positives in quotients A/J to real positives in A (if J is nice)

- Just like in C^*-algebras

- There is a 'real positive version' of the Urysohn lemma.

B-Neal-Read noncommutative Urysohn lemma Let A be an operator algebra (unital for simplicity). Given p, q closed projections in A^{**}, with $pq = 0$ there exists $f \in \mathrm{Ball}(A)$ real positive and $fp = 0$ and $fq = q$.

- Can also do this with q closed in B^{**}, where B is the containing C^*-algebra, but now need an $\epsilon > 0$ (i.e. f 'close to zero' on p; that is $||fp|| < \epsilon$).

• In addition to the last Tietze theorem, we have a 'real positive version' of the Urysohn lemma.

B-Neal-Read noncommutative Urysohn lemma Let A be an operator algebra (unital for simplicity). Given p, q closed projections in A^{**}, with $pq = 0$ there exists $f \in \mathrm{Ball}(A)$ real positive and $fp = 0$ and $fq = q$.

• Can also do this with q closed in B^{**}, where B is the containing C^*-algebra, but now need an $\epsilon > 0$ (i.e. f 'close to zero' on p; that is $||fp|| < \epsilon$).

B-Read Strict noncommutative Urysohn lemma This is the variant where you want f above with also $0 < f < 1$ 'on' $q - p$.

• Generalizes both the topology strict Urysohn lemma, and the Brown-Pedersen strict noncommutative Urysohn lemma.

- One can generalize many of the results above to Banach algebras (B+Ozawa, 2015).

If I have time, I may come back and describe some of this

Examples of 'generalization' projects currently being worked on:

• Chris Phillips' L^p-operator algebras (subalgebras of $B(L^p)$): Currently this is almost all an example driven subject. We have started discussing this with Chris recently, and the possible use of real positivity.

We have somewhat 'cleared the undergrowth', at least as far as the topics I am most interested in, found out much of what does not work, and have isolated several good questions that need to be answered.

L^p-operator algebras are Arens regular, so some of the theory in [B-Ozawa] improves for these algebras. Indeed we can now answer some questions raised in [B-Ozawa].

Chris is looking for a class among the L^p-operator algebras which play exactly the role of C^*-algebras amongst the operator algebras. (It is looking like there are many possible C^*-like properties, but not one that rules them all.)

Mention a related interesting question of Dales: If A is a closed subalgebra of $C(K)$ such that every closed ideal in A has a cai, is A a C^*-algebra?

There is no usable characterization of L^p-operator algebras right now. This is related in one way to the fact that quotients of L^p-operator algebras are not necessarily L^p-operator algebras (although we have conditions under which they are).

It is not known if there is an Arveson-Wittstock extension theorem. The 'representation theory' has limitations. States don't appear to behave. On the other hand, there are some ways in which L^p-operator algebras are better than operator algebras–due to the rigidity in Lamperti's theorem.

• Jordan operator algebras (collaboration in progress with Zhenhua Wang and Matt Neal)

Definition Subspace of $B(H)$ closed under squares ($x^2 \in A$ if $x \in A$).

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There are interesting examples, that come up in operator space theory in key places

Example. $\{(T, [T]) \in B(H) \oplus (B(H)/K(H))^{\mathrm{op}} : T \in B(H)\}$

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Some limitations/things still to do: Although we do have an abstract 'completely isometric' characterization, it is not as useful as the characterization of operator algebras. We do not have an 'isomorphic' characterization. This is related in one way to the fact that quotients of Jordan operator algebras are probably not Jordan operator algebras (although we have conditions under which they are).

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We do not have an analogue of Hay's theorem. Nonetheless ...

- Operator algebras arising in noncommutative geometry, e.g. from spectral triples (collaboration in progress with Kaad and Mesland),

- Will say a little more about this in the next talk.

- More C^*-theory to operator algebras to ... .

- That is, to use real positivity to continue generalizing fundamental ideas/results from C^*-theory. In this series of talks there will be a few examples of this: e.g. in later lecture we will apply real positivity to obtain several fundamental results about Arvesons noncommutative H^∞ algebras, that is several generalizations of famous results about classical $H^\infty(\mathbb{D})$.

The following is an application of real positivity to generalize some results about projections on C^*-algebras

Section IV. Projections on operator algebras (with Matt Neal, 2015, 2016)

• In previous papers we studied completely contractive projections P (that is, idempotent linear maps) and conditional expectations on unital operator algebras. Assumed P unital (that is, $P(1) = 1$)

• Showed e.g. such a projection whose range is a subalgebra, is a 'conditional expectation' (i.e. $P(P(a)bP(c)) = P(a)P(b)P(c)$ for $a, b, c \in A$). This generalizes Tomiyama's theorem for C^*-algebras.

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Main idea here: to find operator algebra generalizations of certain deep results of Størmer, Friedman and Russo, Effros and Størmer, Robertson and Youngson, Youngson, and others, concerning projections and their ranges, assuming in addition that the map is real completely positive

• In particular we wish to investigate the 'bicontractive projection problem' and related problems (such as the 'symmetric projection problem' and the 'contractive projection problem') in the category of operator algebras with approximate identities.

Guiding principle: 'real positivity' is often the right replacement in general algebras for positivity in C^*-algebras

• A well known result of Choi and Effros: the range of a completely positive projection $P : B \to B$ on a C^*-algebra B, is again a C^*-algebra with product $P(xy)$.

• The analogous result for unital completely contractive projections on unital operator algebras is true too, and is implicit in the proof of our generalization of Tomiyama's theorem above.

• However there is no analogous result for general nonunital completely contractive projections, or projections on nonunital operator algebras. Its a mess.

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• The 'guiding principle' above suggests to add the condition that P is also 'real completely positive'. Then the question does make good sense.

Theorem The range of a real completely positive projection $P : A \to A$ on an approximately unital operator algebra, is again an operator algebra with product $P(xy)$. Moreover P is a conditional expectation.

• uses: Proposition A real completely positive completely contractive projection (resp. linear map) on an approximately unital operator algebra A, extends to a unital completely contractive projection (resp. linear map) on the unitization A^1.

• (Unfortunately this proposition wont help us with the other 'completely-x projection problems'.)

Lemma: Suppose that E is a completely contractive completely positive projection on an operator system X. Then the range of E, with its usual matrix norms, is an operator system with matrix cones $E_n(M_n(X)_+) = M_n(X)_+ \cap \mathrm{Ran}(E_n)$, and unit $E(1)$.

The converse: Corollary Let A be an approximately unital operator algebra, with an approximately unital subalgebra B which is the range of a completely contractive projection P on A. Then P is real completely positive

Later we will need:

Theorem: Let $P : A \to A$ be a unital completely contractive projection on a unital operator algebra. If $P(A)$ generates A as an operator algebra, then $(I - P)(A) = \mathrm{Ker}(P)$ is an ideal in A. In any case, if D is the closed algebra generated by $P(A)$ then $(I - P)(D)$ is an ideal in D.

The symmetric projection problem

- We will say that an idempotent linear $P : X \to X$ is completely symmetric if $I - 2P$ is completely contractive

- We will say that an idempotent linear $P : X \to X$ is completely bicontractive if P and $I - P$ are completely contractive

- 'Completely symmetric' implies 'completely bicontractive'

The 'completely-x projection problem' 1) Characterize completely-x projections $P : X \to X$; or 2) characterize the range of such projections.

Theorem (c.f. Stormer/Friedman and Russo) On a unital C^*-algebra B the unital bicontractive projections are also symmetric, and are precisely $\frac{1}{2}(I+\theta)$, for a period 2 $*$-automorphism $\theta : B \to B$.

Theorem (c.f. Stormer/Friedman and Russo) On a unital C^*-algebra B the unital bicontractive projections are also symmetric, and are precisely $\frac{1}{2}(I+\theta)$, for a period 2 $*$-automorphism $\theta : B \to B$.

The possibly nonunital positive bicontractive projections P on a possibly nonunital C^*-algebra are of a similar form, and then $q = P(1)$ is a central projection in $M(B)$ with respect to which P decomposes into a direct sum of 0 and a projection of the above form $\frac{1}{2}(I+\theta)$, for a period 2 $*$-automorphism θ of qB. (This is where positivity needed, to reduce to unital case.)

Conversely, a map P of the latter form is automatically completely bicontractive, and the range of P is a C^*-subalgebra, and P is a conditional expectation.

- What from this theorem is true for general operator algebras A?

- Again our guiding principle leads us to use in place of the positivity there, the real positivity in the sense of the first part of the talk.

• The first thing to note is that now ‘completely bicontractive’ is no longer the same as ‘completely symmetric’

• The ‘completely symmetric’ case works beautifully, nicely generalizing the C^*-result above.

Here is our solution to the symmetric projection problem:

Theorem Let A be an approximately unital operator algebra, and $P : A \to A$ a real completely positive completely symmetric projection. Then the range of P is an approximately unital subalgebra of A. Moreover, $P^{**}(1) = q$ is a projection in the multiplier algebra $M(A)$ (so is both open and closed).

Set $D = qAq$, a hereditary subalgebra of A containing $P(A)$. There exists a period 2 surjective completely isometric homomorphism $\theta : A \to A$ such that $\theta(q) = q$, so that θ restricts to a period 2 surjective completely isometric homomorphism $D \to D$. Also, P is the zero map on $q^\perp A + Aq^\perp + q^\perp A q^\perp$, and

$$P = \frac{1}{2}(I + \theta) \quad \text{on } D.$$

In fact

$$P(a) = \frac{1}{2}(a + \theta(a)(2q - 1)), \qquad a \in A.$$

The range of P is the set of fixed points of θ (and these all are in D).

Conversely, any map of the form in the last equation is a completely symmetric real completely positive projection.

- For the more general class of completely bicontractive projections, much of the last paragraph no longer works in general.

- However our real completely positive assumption will allow us to reduce to the case of unital projections. There a closer look at the result, and at examples, reveals an interesting question.

• For the more general class of completely bicontractive projections, much of the last paragraph no longer works in general.

• However our real completely positive assumption will allow us to reduce to the case of unital projections. There a closer look at the result, and at examples, reveals an interesting question.

Question: given a real completely positive projection $P : A \to A$ which is completely bicontractive, when is the range of P a subalgebra of A? Equivalently, when is P a conditional expectation?

• This becomes the new 'bicontractive projection problem'—the right problem in the category of all operator algebras (without further restrictions)

• Three reductions of the completely bicontractive projection problem for an approximately unital operator algebra A, to put the problem in a 'standard position'.

First, we may assume that A is unital (otherwise look at P^{**} on A^{**})

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First, we may assume that A is unital (otherwise look at P^{**} on A^{**})

Second, we may assume that $P(1) = 1$:

Lemma Let A be a unital operator algebra, and $P : A \to A$ a completely contractive and bicontractive real positive projection. Then $P(1) = q$ is a projection (not necessarily central), and there exists a unital completely bicontractive (real completely positive) projection $P' : qAq \to qAq$ such that P is the zero map on $q^{\perp}A + Aq^{\perp}$, and $P = P'$ on qAq.

Theorem Let P be a completely bicontractive unital projection on a unital operator algebra A. Then A decomposes as $A = C \oplus B$, where $1_A \in B = P(A), C = (I-P)(A)$, and we have the relations $C^2 \subset B, CB+BC \subset C$.

The period 2 map $\theta : x+y \mapsto x-y$ for $x \in B, y \in C$ is a homomorphism (indeed an automorphism) on A iff $P(A)$ is a subalgebra of A.

This immediately clarifies the 'bicontractive projection problem', in terms similar to the C^*-algebra theorem above.

Third reduction, we may assume that $P(A)$ generates A as a Banach algebra. This is because of course $P(A)$ is a subalgebra if and only if $Q(D)$ is a subalgebra, where D is the closed subalgebra of A generated by $P(A)$, and $Q = P_{|D}$, which is a completely bicontractive unital projection on D. That is, we may as well replace A by the closed subalgebra generated by $P(A)$.

Lemma Let P be a completely bicontractive unital projection on a unital operator algebra A. Let D be the algebra generated by $P(A)$. Then $(I - P)(D) = \mathrm{Ker}(P_{|D})$ is an ideal in D and the product of any two elements in this ideal is zero.

- Solution to the ‘ bicontractive projection problem’ for uniform algebras:

Corollary Let $P : A \to A$ be a real positive bicontractive projection on an approximately unital function algebra. Then $P(A)$ is a subalgebra of A, and P is symmetric, and is a conditional expectation, and all the other good stuff from our ‘symmetric’ theorem holds.

- Solution to the ‘ bicontractive projection problem’ for uniform algebras:

Corollary Let $P : A \to A$ be a real positive bicontractive projection on an approximately unital function algebra. Then $P(A)$ is a subalgebra of A, and P is symmetric, and is a conditional expectation, and all the other good stuff from our ‘symmetric’ theorem holds.

Corollary Let $P : A \to A$ be a unital completely bicontractive projection on an operator algebra A. If A has no ideal of nilpotents, or if the algebra generated by $P(A)$ is semiprime, then $P(A)$ is a subalgebra of A.

• From the algebraic facts above, it is easy to see that to find a counterexample in the category of all operator algebras to the 'completely bicontractive projection problem' as we stated it, we need a unital operator algebra A with a nonzero ideal C with square zero, and a completely bicontractive projection onto C whose kernel generates A and contains 1_A.

• From the algebraic facts above, it is easy to see that to find a counterexample in the category of all operator algebras to the 'completely bicontractive projection problem' as we stated it, we need a unital operator algebra A with a nonzero ideal C with square zero, and a completely bicontractive projection onto C whose kernel generates A and contains 1_A.

Example:

$$\begin{bmatrix} \lambda & \nu & c & 0 & 0 \\ 0 & \lambda & \nu & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & \lambda & 2\nu \\ 0 & 0 & 0 & 0 & \lambda \end{bmatrix}, \qquad \lambda, \nu, c \in \mathbb{C},$$

with the projection being 'replacing the 1-3 entry' by 0.

• So the 'bicontractive projection problem' in this generality has a negative solution.

• What fixes this? I.e. get rid of such counterexamples. The following seems to be the right result:

• So the 'bicontractive projection problem' in this generality has a negative solution.

• What fixes this? I.e. get rid of such counterexamples. The following seems to be the right result:

Theorem If P is real positive completely bicontractive projection on an approximately unital operator algebra, such that the ideal mentioned above is also approximately unital (or equiv, is spanned by its real positive elements), then $P(A)$ is a subalgebra of A

• This generalizes the C^*-case

The following is another rather general condition under which the completely bicontractive projection problem is soluble.

Theorem Let A be a unital operator algebra, and $P : A \to A$ a completely bicontractive unital projection. Let D be the closed algebra generated by $P(A)$, and let $C = (I - P)(D)$.

Suppose further that either the left support projection of $P(A)C^*)$ is dominated by the right support projection of C, or the right support projection of $C^*P(A)$ is dominated by the left support projection of C. Then the range of P is a subalgebra of A and P is a conditional expectation.

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(This theorem also rules out the example above)