Quantum measure theory, quantum cardinals, and noncommutative Hardy spaces

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Abstract

The larger first part of the talk (joint work with Nik Weaver) is concerned with quantum measure theory (in the sense of generalizations of measure theory to von Neumann algebras and their projections). We make some new contributions here, obtaining new results for states on von Neumann algebras which are not normal but have other natural continuity properties We also develop some basics of the theory of quantum cardinals, using as tools e.g. the recent Kadison-Singer solution, and Farah and Weavers theory of quantum filters. As an application we characterize in terms of quantum measure theory the von Neumann algebras for which Uedas peak set theorem holds. Recently with Labuschagne we generalized this theorem in the context of (Arvesons) noncommutative Hardy spaces to von Neumann algebras possessing faithful states, using Haagerups reduction theory. As discussed in the Workshop, Uedas peak set theorem is needed currently to prove generalizations for subalgebras of σ-finite von Neumann algebras, of classical results about $H^\infty(\mathbb{D})$ such as uniqueness of predual, noncommutative versions of the Lebesgue decomposition, F and M Riesz theorem, etc.

*Note: Weaver and I have recently revised our ArXiV preprint, so this talk

Recall the nc Lebesgue decomposition: Functionals on a von Neumann algebra have a unique normal plus singular decomposition $\varphi = \varphi_n + \varphi_s$, and $\|\varphi\| = \|\varphi_n\| + \|\varphi_s\|$.

Here normal means weak* continuous, and singular for a state means that e.g. every nonzero projection dominates a nonzero projection in the kernel of the functional

Part I. Quantum measure theory

Quantum measure theory for us today is the theory of ‘measures’ and states on projection lattices of von Neumann algebras.

These projection lattices replace the σ-algebras of ordinary measure theory (which are of course Boolean algebras).

Question: To what extent do the basic properties from standard measure theory hold?

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Question: To what extent do the basic properties from standard measure theory hold?

- Many distinguished names associated with this topic.

Text: Quantum measure theory by J. Hamhalter (Springer), focussed in part on the impressive work of Bunce and Hamhalter

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• For example, there is an important intermediate condition between finite additivity and countable additivity:

We say that a state φ on a von Neumann algebra M is regular if for every sequence (q_n) of projections in M with $\varphi(q_n) = 1$ for all n, we have $\varphi(q) = 1$ for $q = \wedge_n q_n$ (infimum in M).

The results on this page are due to Bunce and Hamhalter or follow easily from their work.

Theorem For a state φ on a von Neumann algebra M TFAE:

(i) φ is regular.

(ii) $\varphi(\vee_n q_n) = 0$ for every increasing sequence of projections (q_n) in $\mathrm{Ker}(\varphi)$.

(iii) $\varphi(\vee_n q_n) = 0$ for every sequence of projections (q_n) in $\mathrm{Ker}(\varphi)$.

(iv) $\mathrm{Ker}(\varphi)_+$ is weak* sequentially closed (that is, if $0 \leq x_n \to x$ weak* with $x_n \in \mathrm{Ker}(\varphi)$ then $\varphi(x) = 0$).

(v) φ annihilates the support projection of any nonzero positive element in $\mathrm{Ker}(\varphi)$.

(vi) φ is not singular on every σ-finite von Neumann subalgebra of M.

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Theorem A pure state on a von Neumann algebra M is regular iff it is countably additive (as usual, on projections).

Remarkably, if M is a factor, or has no σ-finite direct summand, or is type III and σ-finite, then regularity is equivalent to the Jauch-Piron condition from quantum physics:

$$\varphi(q_1 \vee q_2) = 0 \text{ for all projections } q_1, q_2 \text{ in } \mathrm{Ker}(\varphi)$$

- Some new quantum measure theoretic results from [B-Weaver]:

Theorem Let κ be an uncountable cardinal and let ϕ be a $< \kappa$-additive (on projections) pure state on a von Neumann algebra M. Then the restriction of ϕ to any von Neumann subalgebra generated by $< \kappa$ elements is normal (we thank Ilijas Farah for pointing out that this version follows from a similar but weaker version of ours)

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- These are theorems in ordinary set theory. However they are 'not provable' in ZFC for non-pure states. Indeed assuming a positive solution to Banach's measure problem there is a singular countably additive state ϕ on $l^\infty(\mathbb{R})$, but $l^\infty(\mathbb{R})$ is countably generated, so by the above ϕ must be normal, a contradiction! But a positive solution to Banach's measure problem is generally believed to be consistent with ZFC.

Theorem Any countably additive pure state on a von Neumann algebra M is sequentially weak* continuous on M.

- This again is a ZFC theorem.

We now turn to measurable cardinals

Reminder: Banach's measure problem: Is there a probability measure defined on all subsets of $[0, 1]$ which is zero on singletons? [If there is, then one can find another that extends Lebesgue measure.]

- You cannot prove an affirmitive answer in ZFC. It is generally believed that a positive answer is consistent with ZFC, thus in almost all of set theory it would be considered safe to add the affirmitive answer to Banach's measure problem as an extra axiom of set theory, if convenient (if you dont add CH too, or similar)

In set theory there is an elaborate hierarchy of large cardinal properties, some of which involve various notions of measurability. These are related to natural continuity properties for singular measures, which can be viewed as states on $l^{\infty}(\kappa)$ for the cardinal κ.

An uncountable cardinal κ is said to be

- measurable if there is a nonzero $< \kappa$-additive $\{0, 1\}$-valued measure on κ which vanishes on singletons
- real-valued measurable if there is a $< \kappa$-additive probability measure on κ which vanishes on singletons
- Ulam measurable if there is a nonzero countably additive $\{0, 1\}$-valued measure on κ which vanishes on singletons
- Ulam real-valued measurable if there is a countably additive probability measure on κ which vanishes on singletons. (cf. Banach's measure problem)

Here measures on κ are assumed defined on all subsets of κ, and "$< \kappa$-additive" means "additive on any family of fewer than κ disjoint sets".

Some of these cardinals are necessarily monstrous if they exist, others may possibly be the cardinality of $\mathbb{R}$.

• It is generally believed that the existence of such cardinals is consistent with ordinary set theory. They all have the same consistency strength.

• Each of these four kinds of measurability can be expressed in terms of states on $l^\infty(\kappa)$:

Proposition An uncountable cardinal κ is

(i) Ulam real-valued measurable if and only if there is a singular countably additive state on $l^\infty(\kappa)$

(ii) Ulam measurable if and only if there is a singular countably additive pure state on $l^\infty(\kappa)$

(iii) measurable iff there is a singular $<\kappa$-additive pure state on $l^\infty(\kappa)$

(iv) real-valued measurable iff there is a singular $<\kappa$-additive state on $l^\infty(\kappa)$

• It is natural to consider the quantum measure theory analogues, i.e. replace $l^\infty(\kappa)$ by $B(l^2(\kappa))$ or other von Neumann algebras in each of the above. We will do this.

Theorem If it is consistent that a measurable cardinal exists, then it is consistent that both $l^\infty(\mathbb{R})$ and $B(l^2(\mathbb{R}))$ have regular singular states, but neither algebra admits a countably additive singular state

The proof uses a relatively new result in set theory by Kumar and Kunen. The conclusion here is not consistent with the continuum hypothesis:

Theorem Assuming the continuum hypothesis, $l^\infty(\kappa)$ and $B(l^2(\kappa))$ have regular singular states if and only if κ is Ulam measurable.

Theorem Let κ be an uncountable cardinal. A von Neumann algebra M possesses a singular countably additive (resp. $< \kappa$-additive) state iff M has a collection of mutually orthogonal projections of cardinality the first real valued measurable cardinal (resp. the first real valued measurable cardinal $\geq \kappa$).

- If there is no real valued measurable cardinal $\geq \kappa$ this says that every $< \kappa$-additive state is normal.

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Corollary There exists a singular countably additive state (resp. $< \kappa$-additive) on $B(l^2(\kappa))$ if and only if κ is Ulam real valued measurable (resp. real valued measurable).

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• In our workshop talk we gave the proof, using a generalization of the recently solved Kadison-Singer problem (Marcus, Spielman and Srivastava, 2013), that if κ is measurable then there is a singular $< \kappa$-additive pure state on $B(l^2(\kappa))$. The converse follows from the following general von Neumann algebra result:

Theorem Let κ be an uncountable cardinal. If a von Neumann algebra M possesses a singular countably additive (resp. $< \kappa$-additive) pure state then M has a collection of mutually orthogonal projections of cardinality the first measurable cardinal (resp. the first measurable cardinal $\geq \kappa$).

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- So we have proved the earlier claimed result:

Theorem An uncountable cardinal κ is

(i) Ulam real-valued measurable if and only if there is a singular countably additive state on $B(l^2(\kappa))$

(ii) Ulam measurable if and only if there is a singular countably additive pure state on $B(l^2(\kappa))$

(iii) real-valued measurable if and only if there is a singular $< \kappa$-additive state on $B(l^2(\kappa))$

(iv) measurable if and only if there is a singular $< \kappa$-additive pure state on $B(l^2(\kappa))$.

Corollary Every countably additive state on a von Neumann algebra M is normal if M contains 'not too many' (in some kind of measurable cardinal sense) mutually orthogonal projections.

- This strengthens a similar theorem of Neufang for sequentially weak* continuous states/functionals.

A tool used in proofs: Farah and Weaver's theory of quantum filters

A quantum filter on a von Neumann algebra M is a family of projections $\mathcal{F}$ in M with the properties

(i) if $p \in \mathcal{F}$ and $p \leq q$ then $q \in \mathcal{F}$

(ii) if $p_1, \ldots, p_n \in \mathcal{F}$ then $\|p_1 \cdots p_n\| = 1$.

If ϕ is a state on M then

$$\mathcal{F}_\phi = \{p \in M : p \text{ is a projection and } \phi(p) = 1\}$$

is a quantum filter

- Pure states correspond to quantum ultrafilters

Some tools used in proofs (contd):

Lemma (Farah-W) Suppose that ϕ is a pure state on a von Neumann algebra M and that ψ is a state on M such that

$$\phi(p) = 1 \qquad \Rightarrow \qquad \psi(p) = 1$$

for any projection $p \in M$. Then $\phi = \psi$.

(Proof shown in workshop lecture)

Part II. (Application to Arveson's) noncommutative H^∞–for general von Neumann algebras (joint with Louis Labuschagne, ArXiV 2016).

• In several papers B-Labuschagne extended much of the theory of generalized H^p spaces for function algebras from the 1960s to the von Neumann algebraic setting of Arveson's subdiagonal algebras, a.k.a. noncomm. H^∞, inside 'finite' von Neumann algebras

• Subdiagonal algebras are certain unital weak* closed subalgebras A of a von Neumann algebra M, such that there exists a normal ($=$ weak* conts) conditional expectation $M \to A \cap A^*$ which is multiplicative on A.

$A = M$ is OK, so we are again in a situation generalizing both the classical function theory, and von Neumann algebras (and nc L^p-spaces)

• Earlier, we worked in the setting that M possesses a faithful normal tracial state, as Arveson mostly did too.

(Actually we should say maximal subdiagonal algebra above, and everywhere, but for brevity we will abusively just call them subdiagonal)

• Ueda followed our work by removing a hypothesis involving a dimensional restriction on $A \cap A^*$ in four or five of our results (e.g. F. & M. Riesz and Gleason-Whitney theorems), and also establishing several other beautiful theorems such as the fact that such an A has a unique predual, all of which followed from his very impressive noncommutative peak set theorem.

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• [B-Labuschagne, 2016] generalized these five or six results (Ueda's peak set theorem plus the improved F. & M. Riesz and Gleason-Whitney theorems, etc) to subalgebras of σ-finite von Neumann algebras. See workshop talk (note: σ-finite does not mean a countable sum of finite ones).

• As reminder we state the function algebra case of some of these results first, then the matching von Neumann algebra results.

• $H^\infty(\mathbb{D})$ has a unique predual (Ando-Wojtaszczyk)/von Neumann algebras have unique predual (Dixmier-Sakai)

Related to the nc Lebesgue decomposition: Functionals on a von Neumann algebra have a unique normal plus singular decomposition $\varphi = \varphi_n + \varphi_s$, and $\|\varphi\| = \|\varphi_n\| + \|\varphi_s\|$.

Similar fact relative to H^∞: functionals on H^∞ have a unique normal plus singular decomposition on H^∞, and $\|\varphi\| = \|\varphi_n\| + \|\varphi_s\|$.

F. & M. Riesz reformulation For any functional φ on $L^\infty(\mathbb{T})$ annihilating $H^\infty(\mathbb{D})$, we have $\varphi_n, \varphi_s \perp H^\infty(\mathbb{D})$.

The latter is not exactly the classical F. & M. Riesz theorem, but it implies it easily (see e.g. Hoffman, Acta)

Gleason-Whitney type theorem Suppose that A is a weak* closed subalgebra of $M = L^\infty(\mathbb{T})$ satisfying the last result. Then $A + A^*$ is weak* dense in M if and only if every normal functional on A has a unique Hahn-Banach extension to M, and if and only if every normal functional on A has a unique normal Hahn-Banach extension to M.

- The main ingredient one may use to prove these, is a theorem about peak sets, in the classical case due to Amar and Lederer: 'Any closed set of measure zero is contained in a peak set of measure zero'.

Ueda's (nc Amar-Lederer) peak set theorem may be phrased as saying that any singular support projection (i.e. the support of any singular state on M), is dominated by a peak projection p for A with p in the 'singular part' of M^{**} (that is, p annihilates all normal functionals on M).

- Ueda proved this noncommutative peak set result in the case that M has a faithful normal tracial state ('finite' vNA).

In the workshop talk (1) we described the generalization of this, and hence all the consequences above, to von Neumann algebras with a faithful state (that is σ-finite vNa's). (2) We also dashed hopes of being able to prove the result in ZFC for all von Neumann algebras (or even commutative ones).

Theorem (B-Weaver, 2016) For a von Neumann M TFAE:

(i) Ueda's peak set result holds for M.

(ii) There exist no regular singular states on M (i.e. for all singular states φ of M, there is a sequence (q_n) of projections in Ker(φ) with $\varphi(\vee_n q_n) > 0$).

(ii)' For all singular states φ of M, there is a sequence (q_n) of projections in Ker(φ) with $\vee_n q_n = 1$.

(iii) Every collection of mutually orthogonal projections in M has cardinality $<$ a fixed cardinal κ, namely the first cardinal having a finitely additive 'regular' singular measure

It is possible (i.e. believed to be consistent with ZFC) that the cardinal here is the continuum (cardinality of $\mathbb{R}$, so not so threatening). This uses a theorem early in our talk, and is related to (but is not the same as) Banach's measure problem.

- Summary: Ueda's peak set theorem (case $A = M$) is largely about regular singular states on M, and about 'not too many' mutually orthogonal projections in M.

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On the other hand, we can show in ZFC that if κ is the first uncountable cardinal $\aleph_1$, then $l^\infty(\kappa)$ satisfies Ueda's theorem (see last slide workshop talk). Similarly for the second uncountable cardinal, etc.

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So there definitely are ('small') non σ-finite von Neumann algebras satisfying Ueda's theorem.

(This may suggest somebody should study variants of the notion of σ-finite von Neumann algebras strictly between σ-finite and 'every collection of mutually orthogonal projections has cardinality $< 2^{\aleph_0}$, assuming $\neg$CH)