On the C^*-algebras of solvable Lie groups

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OTOA
Bengaluru, 21.12.2016

Solvable Lie groups

- G connected closed subgroup of $\mathcal{T}^{\times}$, where $\mathcal{T} := \left\{ \begin{pmatrix} * & & * \\ 0 & \ddots & \\ 0 & 0 & * \end{pmatrix} \right\}$

- $\mathfrak{g}$ the linear subspace of $\mathcal{T}$ given by

$$X \in \mathfrak{g} \iff (\forall t \in \mathbb{R}) \quad \exp(tX) \in G$$

- G *exponential Lie group* $\iff$ $\exp\colon \mathfrak{g} \to G$ is a bijection

- G *nilpotent Lie group* $\iff$ $G \subseteq \left\{ \begin{pmatrix} 1 & & * \\ 0 & \ddots & \\ 0 & 0 & 1 \end{pmatrix} \right\}$

The C^*-algebra of a solvable Lie group G

- There is a Haar measure $\mathrm{d}x$ on G, invariant under left translations:
$$\int_G \varphi(a\cdot x)\mathrm{d}x = \int_G \varphi(x)\mathrm{d}x \text{ for } a\in G \text{ and } \varphi\in\mathcal{C}_c(G)$$

- *regular representation* $\lambda\colon \mathcal{C}_c(G)\to\mathcal{B}(L^2(G))$, $\lambda(f)\varphi = f*\varphi$,
where $(f*\varphi)(x)=\int_G f(a)\varphi(a^{-1}\cdot x)\mathrm{d}a$

- $\boxed{C^*(G) := \overline{\lambda(\mathcal{C}_c(G))}^{\|\cdot\|}} \subseteq \mathcal{B}(L^2(G))$

This is actually the *reduced* C^*-algebra of G, but G is an amenable group.

- $\widehat{G}$ = equivalence classes $[\pi]$ of unitary irreducible repres. $\pi\colon G\to U(\mathcal{H}_\pi)$
- Similar notation for C^*-alg. $\mathcal{A}$: $[\pi]\in\widehat{\mathcal{A}}$ for $*$-repres. $\pi\colon\mathcal{A}\to\mathcal{B}(\mathcal{H}_\pi)$
- $\widehat{G}\simeq\widehat{C^*(G)}$

Every $\pi\colon G\to U(\mathcal{H}_\pi)$ leads to $\pi\colon L^1(G)\to\mathcal{B}(\mathcal{H}_\pi)$, $\pi(\psi):=\int_G\psi(x)\pi(x)\mathrm{d}x$.

Basic question

What topology can $\widehat{G}$ have if G is a connected solvable Lie group?

- If G is an exponential solvable Lie group, then

$$\widehat{G} \text{ Hausdorff} \iff G \text{ commutative} \iff G = (\mathfrak{g}, +)$$

and if this is the case, then $\widehat{G} = \mathfrak{g}^*$, $C^*(G) \simeq \mathcal{C}_0(\widehat{G})$ via Fourier transf.

What else when G is an exponential solvable Lie group?

$\widehat{G} \ni [\pi] \rightsquigarrow \pi\colon C^*(G) \to \mathcal{B}(\mathcal{H}_\pi)$

- G type I $\implies \mathcal{K}(\mathcal{H}_\pi) \subseteq \boxed{\pi(C^*(G))} \simeq C^*(G)/\mathrm{Ker}\,\pi$
- $\{[\pi]\}$ is a locally closed (i.e., open $\cap$ closed) subset of $\widehat{G}$

On closed points of $\widehat{G}$

- $[\pi]$ closed point of $\widehat{G} \iff \pi(C^*(G)) = \mathcal{K}(\mathcal{H}_\pi)$ $\qquad (*)$

Now let G be a nilpotent Lie group.

Old fact: every point $[\pi] \in \widehat{G}$ is closed

Theorem

$\pi\colon G \to \mathcal{B}(\mathcal{X})$ irreducible, $\|\pi(x)\| \le C$ for some $C > 0$ and all $x \in G$, $\mathcal{X}$ reflexive Banach space $\implies \overline{\pi(L^1(G))}^{\|\cdot\|} = \mathcal{K}(\mathcal{X})$

Proof: $\mathcal{X}_\infty := \{x \in \mathcal{X} \mid \pi(\cdot)x \in \mathcal{C}^\infty(G, \mathcal{X})\}$ Fréchet space, dense in $\mathcal{X}$
$\mathcal{S}(G) :=$ the space of Schwartz functions, is a $*$-subalgebra of $L^1(G)$
$\pi_{\mathcal{S}}\colon \mathcal{S}(G) \to \mathrm{End}\,(\mathcal{X}_\infty), \quad \pi_{\mathcal{S}}(\varphi) := \pi(\varphi)|_{\mathcal{X}_\infty}$
Step 1. $\pi_{\mathcal{S}}$ is an algebraically irreducible representation
Step 2. There is a unique unirrep $\pi_0\colon G \to \mathcal{B}(\mathcal{H})$ with

$$\mathcal{H}_\infty \simeq \mathcal{X}_\infty \text{ as topological } \mathcal{S}(G)\text{-modules}$$

Step 3. Use $(*)$ for $\subseteq$ and reflexivity of $\mathcal{X}$ for $\supseteq$.

When does $\widehat{G}$ have dense points?

C^*-algebra $\mathcal{A}$; $\{[\pi_0]\}$ dense in $\widehat{\mathcal{A}} \iff \operatorname{Ker}\pi_0 = \{0\}$

Def. $\mathcal{A}$ is *primitive* if it has faithful irreducible representations

G exponential solvable Lie group

- $C^*(G)$ is not primitive, unless $G = \{\mathbf{1}\}$
- G nilpotent $\iff (\forall[\pi] \in \widehat{G}) \quad \mathcal{K}(\mathcal{H}_\pi) = \pi(C^*(G))$

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  - Thus there is no dense point of $\widehat{G}$ in this case.
  - Yet, there always exist non-closed points of $\widehat{G}$, unless G is nilpotent.
- Groupoids $\rightsquigarrow$ "many" examples of connected solvable Lie groups G (besides $\mathbb{C} \rtimes \mathbb{C}^*$) for which $\widehat{G}$ has a dense point.
  Let $\mathfrak{s} = \mathfrak{n}^- \dotplus \mathfrak{h} \dotplus \mathfrak{n}^+$ be a complex simple Lie alg. with its Cartan subalgebra $\mathfrak{h}$.
  Borel subalgebra is $\mathfrak{g} := \mathfrak{h} \ltimes \mathfrak{n}^+$, and its group G acts algebraically on $\mathfrak{g}$ and on $\mathfrak{g}^*$.
  Fact: G has a nonempty open (dense) orbit in $\mathfrak{g}^*$

  $\iff \mathfrak{s} \in \{\mathrm{so}(2\ell+1, \mathbb{C}), \mathfrak{sp}(2\ell, \mathbb{C}), \mathrm{so}(2\ell, \mathbb{C})$ with $\ell \in 2\mathbb{N}, E_7, E_8, F_4, G_2\}$

Examples

Example 1 ($ax + b$ group)

$$G = \text{ set of all } \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \text{ with } a > 0,\ b \in \mathbb{R}$$

- $0 \to \mathbb{C}^2 \otimes \mathcal{K}(L^2(\mathbb{R})) \to C^*(G) \to \mathcal{C}_0(\mathbb{R}) \to 0$
- $\{2 \text{ open points}\} \sqcup \mathbb{R} \simeq \widehat{G}$

Example 2 (Heisenberg group)

$$\mathbb{H}_{2n+1} = \text{ set of all } \begin{pmatrix} 1 & \mathbf{x} & z \\ & \ddots & \mathbf{y} \\ & & 1 \end{pmatrix} \text{ with } \mathbf{x}, \mathbf{y} \in \mathbb{R}^n,\ z \in \mathbb{R},\ 0 \text{ elsewhere}$$

- $0 \to \mathcal{C}_0(\mathbb{R} \setminus \{0\}) \otimes \mathcal{K}(L^2(\mathbb{R}^n)) \to C^*(\mathbb{H}_{2n+1}) \to \mathcal{C}_0(\mathbb{R}^{2n}) \to 0$
- $(\mathbb{R} \setminus \{0\}) \sqcup \mathbb{R}^{2n} \simeq \widehat{\mathbb{H}}_{2n+1}$

On open points of $\widehat{G}$

G exponential Lie group with its Lie algebra $\mathfrak{g}$ and the duality pairing $\langle \cdot, \cdot \rangle \colon \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$.
For any basis $\{X_1, \dots, X_m\}$ in $\mathfrak{g}$ define the polynomial function

$$P \colon \mathfrak{g}^* \to \mathbb{R}, \quad P(\xi) := \det(\langle \xi, [X_j, X_k] \rangle)_{1 \le j,k \le m}.$$

- $\widehat{G}$ has open points $\iff (\exists \xi \in \mathfrak{g}^* \setminus \{0\}) \quad P(\xi) \neq 0$
- The number of open points of $\widehat{G}$ is even.

On C^*-algebras of nilpotent Lie groups (1)

Theorem

G nilpotent Lie group $\implies$ $C^*(G)$ is a solvable C^*-algebra

That is, there are ideals $\{0\} = \mathcal{J}_0 \subseteq \mathcal{J}_1 \subseteq \cdots \subseteq \mathcal{J}_n = \mathcal{A} := C^*(G)$ with $\mathcal{J}_j/\mathcal{J}_{j-1} \simeq \mathcal{C}_0(\Gamma_j, \mathcal{K}(\mathcal{H}_j))$ for $j = 1, \dots, n$, and moreover

1. $\widehat{\mathcal{A}} = \Gamma_1 \sqcup \cdots \sqcup \Gamma_n$
2. $\dim \mathcal{H}_n = 1$ and $\Gamma_n \simeq [\mathfrak{g}, \mathfrak{g}]^\perp \subseteq \mathfrak{g}^*$
3. $\dim \mathcal{H}_j = \infty$ if $j < n$, Γ_j is open dense in $\widehat{\mathcal{A}} \setminus \widehat{\mathcal{J}}_{j-1}$
4. $\Gamma_j \simeq$ semi-algebraic cone in a finite-dimensional vector space, which is a Zariski open set for $j = 1$

Corollary: Real rank of $C^*(G) = \dim(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])$
(holds true for any exponential Lie group, with a different proof)

On C^*-algebras of nilpotent Lie groups (2)

A subquotient of a C^*-algebra $\mathcal{A}$ is any quotient $\mathcal{J}_2/\mathcal{J}_1$, where $\mathcal{J}_1 \subseteq \mathcal{J}_2$ are ideals of $\mathcal{A}$.

- $\mathrm{SQ}^{\mathrm{Tr}}(\mathcal{A}) :=$ the subquotients with continuous trace;
- $\mathrm{SQ}_0^{\mathrm{Tr}}(\mathcal{A}) :=$ the subquotients that are Morita equivalent to commutative C^*-algebras (i.e., whose Dixmier-Douady invariant is equal to zero)

$\rightsquigarrow \mathrm{SQ}_0^{\mathrm{Tr}}(\mathcal{A}) \subseteq \mathrm{SQ}^{\mathrm{Tr}}(\mathcal{A})$

Conjecture (R.L. Lipsman, I. Raeburn, J. Rosenberg, 1988, 1996)

$$\mathcal{A} = C^*(G) \implies \mathrm{SQ}_0^{\mathrm{Tr}}(\mathcal{A}) = \mathrm{SQ}^{\mathrm{Tr}}(\mathcal{A})$$

Motivation: noncommutative version of the Fourier isomorphism $C^*(\mathcal{V},+) \simeq \mathcal{C}_0(\mathcal{V}^*)$.

They proved the above only for 2-step nilpotent Lie groups G, i.e., $[\mathfrak{g},[\mathfrak{g},\mathfrak{g}]] = \{0\}$.

Theorem

There is additional evidence for the above conjecture:

- *a sequence of groups having arbitrarily high nilpotency step;*
- *a continuous family of 3-step nilpotent groups.*

Quasidiagonality

$\mathcal{A}$ separable C^*-algebra
• $\pi\colon \mathcal{A} \to \mathcal{B}(\mathcal{H})$ *quasidiagonal* $*$-representation if

$$(\forall a \in \mathcal{A}) \ \lim_{n\to\infty} \|\pi(a)P_n - P_n\pi(a)\| = 0$$

for some $P_n = P_n^* = P_n^2 \in \mathcal{B}(\mathcal{H})$, $\operatorname{rank} P_n < \infty$, $P_n \nearrow \mathbf{1}$
• $\mathcal{A}$ is *strongly quasidiagonal* if all its (irreducible) $*$-representations are quasidiagonal
Example: $\mathcal{K}(\mathcal{H})$ is strongly quasidiagonal
Rem.: Let π be a quasidiagonal $*$-representation and $a \in \mathcal{A}$.
With $E_n := P_n - P_{n-1}$ and $\|\pi(a)P_n - P_n\pi(a)\| < \frac{1}{2^n}$ one has

$$\pi(a) = \sum_{n\geq 1} E_n\pi(a)E_n + K \text{ with } K := \sum_{n\geq 1} E_n\pi(a)P_n^\perp + P_n^\perp\pi(a)E_n.$$

Here $K \in \mathcal{K}(\mathcal{H})$. So if $\pi(a)$ is a Fredholm operator, then $\operatorname{index}\pi(a) = 0$.

When is $C^*(G)$ strongly quasidiagonal?

G connected, simply connected, solvable Lie group of type I.
So $\mathcal{K}(\mathcal{H}_\pi) \subseteq \pi(C^*(G))$ for every $[\pi] \in \widehat{G}$

Theorem

Equivalent:

1. $C^*(G)$ *is strongly quasidiagonal*
2. $\mathfrak{g} \ni X \mapsto [A, X] \in \mathfrak{g}$ *has all its eigenvalues in* $\mathrm{i}\mathbb{R}$ *for all* $A \in \mathfrak{g}$
3. $\pi(C^*(G)) = \mathcal{K}(\mathcal{H}_\pi)$ *for every* $[\pi] \in \widehat{G}$ *(i.e., all points of* $\widehat{G}$ *are closed)*
4. *Fredholm operators in* $\pi(C^*(G))$ *have index zero for every* $[\pi] \in \widehat{G}$

Corollary

If G *is an exponential Lie group, then the above are further equivalent to:*

5. G *is a nilpotent Lie group*

Problem: For G exponential, non-nilpotent, determine the unirreps $\pi\colon G \to \mathrm{U}(\mathcal{H})$ for which there are Fredholm operators in $\pi(C^*(G))$ having nonzero index.

Idea of proof

If $\mathcal{K}(\mathcal{H}_\pi) \subsetneqq \pi(C^*(G))$ for some $[\pi] \in \widehat{G}$, then there is a closed normal subgroup $H \subset G$ such that

$$G/H \simeq S_2,\ S_3^\sigma \text{ with } \sigma \neq 0, \text{ or } S_4, \text{ where}$$

- $S_2 = \mathbb{R} \ltimes \mathbb{R}$, $(t,x)\cdot(t',x') = (t+t', x+e^t x')$
- $S_3^\sigma = \mathbb{R} \ltimes \mathbb{R}^2$, $(t,x)\cdot(t',x') = (t+t', x+A(t)x')$,
  $A(t) = e^{\sigma t}\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$
- $S_4 = \mathbb{R}^2 \ltimes \mathbb{R}^2$, $(s,t,x)\cdot(s',t',x') = (t+t', x+B(t,s)x')$
  $B(t,s) = e^{t}\begin{pmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{pmatrix}$