Ando dilation and its applications

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OTOA - 2016
ISI Bangalore, December 20

(joint work with J. Sarkar and S. Sarkar)

Introduction

- $\mathbb{D}$: Open unit disc.
- $H^2_{\mathcal{E}}(\mathbb{D})$: $\mathcal{E}$-valued Hardy space over the unit disc.
- The shift operator on $H^2_{\mathcal{E}}(\mathbb{D})$ is denoted by M_z.
- For a contraction T, $\mathrm{D}_T := (I - TT^*)^{1/2}$ is the defect operator and $\mathcal{D}_T := \overline{\mathrm{Ran}\mathrm{D}_T}$ is the defect space of T.
- A contraction T on H is pure if $T^{*n} \to 0$ in S.O.T.

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- von Neumann inequality: For any polynomial $p \in \mathbb{C}[z]$,

$$\|p(T)\| \leq \sup_{z\in\mathbb{D}} |p(z)|.$$

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Definition

A variety $V = \{(z_1, z_2) \in \mathbb{D}^2 : p(z_1, z_2) = 0\}$ is a distinguished variety of the bidisc if

$$\overline{V} \cap \partial\mathbb{D}^2 = \overline{V} \cap (\partial\mathbb{D})^2.$$

Distinguished variety of the bidisc

Theorem (Agler and McCarthy)

V is a distinguished variety of the bidisc if and only if there is a matrix valued inner function Φ such that
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Theorem (Agler and McCarthy)

Let (T_1, T_2) be a pair of commuting strict matrices. Then there is a distinguished variety V of the bidisc such that

$$\|p(T_1, T_2)\| \leq \sup_{(z_1,z_2)\in V} |p(z_1, z_2)| \quad (p \in \mathbb{C}[z_1, z_2]).$$

Question and realization formula

Question: What are the commuting pair of contractions (T_1, T_2) which dilates to a pair of commuting isometries (M_z, M_Φ) on $H^2_{\mathcal{E}}(\mathbb{D})$ for some finite dimensional Hilbert space $\mathcal{E}$?

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- Is that all we need?
- Φ is an operator valued multiplier of $H^2_{\mathcal{E}}(\mathbb{D})$ if and only if there exist a Hilbert space H and an isometry $U = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ in $\mathcal{B}(\mathcal{E} \oplus H)$ such that

$$\Phi(z) = A + zB(I - zD)^{-1}C$$

for all $z \in \mathbb{D}$.

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- Consider the operator equality $(I - T_1 T_1^*) + T_1(I - T_2 T_2^*) T_1^* = T_2(I - T_1 T_1^*) T_2^* + (I - T_2 T_2^*)$.

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- $U : \{(\mathrm{D}_{T_1} h, \mathrm{D}_{T_2} h) : h \in H\} \to \{(\mathrm{D}_{T_1} T_2^* h, \mathrm{D}_{T_2} h) : h \in H\}$ defines an isometry defined by

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- Extend U to a unitary in $\mathcal{B}(\mathcal{D}_{T_1} \oplus \mathcal{D}_{T_2})$.

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- Extend U to a unitary in $\mathcal{B}(\mathcal{D}_{T_1} \oplus \mathcal{D}_{T_2})$.
- Let $\Phi \in H^\infty_{\mathcal{B}(\mathcal{D}_{T_1})}(\mathbb{D})$ be the matrix valued inner function corresponding to U^*. Then M_Φ^* is the co-isometric extension of T_2^*.

Theorem

Let (T_1, T_2) be a pair of commuting contractions on H with T_1 is pure and $\dim \mathcal{D}_{T_i} < \infty$, $i = 1, 2$. Then (T_1, T_2) dilates to (M_z, M_Φ) on $H^2_{\mathcal{D}_{T_1}}(\mathbb{D})$. Therefore, there exists a variety $V \subset \overline{\mathbb{D}^2}$ such that

$$\|p(T_1, T_2)\| \leq \sup_{(z_1,z_2)\in V} |p(z_1, z_2)| \quad (p \in \mathbb{C}[z_1, z_2]).$$

If, in addition, T_2 is pure then V can be taken to be a distinguished variety of the bidisc.

Berger-Coburn-Lebow representation

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Theorem (B-C-L)

A pure pair of commuting isometries (V_1, V_2) is unitary equivalent to a commuting pair of isometries (M_Φ, M_Ψ) on $H^2_{\mathcal{E}}$ for some Hilbert space $\mathcal{E}$ with

$$\Phi(z) = (zP^\perp + P)U^* \quad \text{and } \Psi(z) = U(zP + P^\perp)$$

where U is a unitary in $\mathcal{B}(\mathcal{E})$ and P is a projection in $\mathcal{B}(\mathcal{E})$.

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- $H^2_{\mathcal{E}}(\mathbb{D})$ is the model space of the pure isometry V_1V_2.
- Pure pair of commuting isometry $(V_1, V_2) \leftrightsquigarrow (\mathcal{E}, U, P)$.
- $M_z = M_\Phi M_\Psi$.

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A pure pair of commuting contractions (T_1, T_2) dilates to a pure pair of commuting isometries corresponding to a triple $(\mathcal{D}_{T_1} \oplus \mathcal{D}_{T_2}, U, P)$ where U is a unitary and P is a projection in $\mathcal{B}(\mathcal{D}_{T_1} \oplus \mathcal{D}_{T_2})$.

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Theorem

Let T be a pure contraction on H and let $T \cong P_{\mathcal{Q}}M_z|_{\mathcal{Q}}$ be the Sz.-Nagy and Foias representation of T. TFAE
(i) $T = T_1T_2$ for some commuting contractions T_1 and T_2 on H.
(ii) There exist $\mathcal{B}(\mathcal{D}_T)$-valued polynomial ϕ and ψ of degree ≤ 1 such that $\mathcal{Q}$ is a joint (M_ϕ^*, M_ψ^*)- invariant subspace,

$$P_{\mathcal{Q}}M_z|_{\mathcal{Q}} = P_{\mathcal{Q}}M_{\phi\psi}|_{\mathcal{Q}} = P_{\mathcal{Q}}M_{\psi\phi}|_{\mathcal{Q}}.$$

References

- J. Agler and J. E. McCarthy, *Distinguished Varieties*, Acta Math. **194** (2005), 133-153.
- C.A. Berger, L.A. Coburn and A. Lebow, *Representation and index theory for C^*-algebras generated by commuting isometries*, J. Funct. Anal. **27** (1978), 51-99.
- B. K. Das and J. Sarkar, *Ando dilations, von Neumann inequality, and distinguished varieties*, J. Funct. Anal. (to appear).
- B.K. Das, J. Sarkar and S. Sarkar, *Factorizations of contractions*, arxiv:1607.05815.