Positive kernels and reproducing kernel spaces: a rich tapestry of settings and applications

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Bangalore
December 2016

I: The classical case

Given: Ω = set of points, $\mathcal{Y}$ = a Hilbert space, $\mathcal{B}(\mathcal{Y})$ = bounded linear operators on $\mathcal{Y}$, $K\colon \Omega \times \Omega \to \mathcal{B}(\mathcal{Y})$ = a function

Theorem (and Definition) 1:

We say that K is a positive kernel if any of the following equivalent conditions hold:

1. $\sum_{i,j=1}^{N} \langle y_i, K(\omega_i, \omega_j) y_j \rangle_{\mathcal{Y}} \geq 0 \;\; \forall \; y_1, \dots, y_n$ in $\mathcal{Y}$, $\omega_1, \dots, \omega_N$ in Ω for $N = 1, 2, \dots$
2. K is the reproducing kernel for a uniquely determined Reproducing Kernel Hilbert Space $\mathcal{H}(K)$: $k_{\omega,y} := K(\cdot, \omega) y \in \mathcal{H}(K)$ and $\langle k_{\omega,y}, f \rangle_{\mathcal{H}(K)} = \langle y, f(\omega) \rangle_{\mathcal{Y}}$
3. $\exists$ auxiliary Hilbert space $\mathcal{X}$ and function $H\colon \Omega \to B(\mathcal{X}, \mathcal{Y})$ so that $K(\zeta, \omega) = H(\zeta) H(\omega)^*$ (Kolmogorov decomposition)

Discussion of proof

- Property (2) = Reproducing property: For the case $\mathcal{Y} = \mathbb{C}$, Zaremba (1907): bdry-value problems for harmonic fctns
- The construction that (1) $\Rightarrow$ (2): Moore (1935), Aronszajn (systematic theory 1950) for the case $\mathcal{Y} = \mathbb{C}$
- Property (3): Kolmogorov in the context of covariance matrices

Sketch of (1) $\Rightarrow$ (2)

Proof of (1) $\Rightarrow$ (2)

Given $K(\zeta, \omega)$ satisfying (1), define

kernel elements $k_{\zeta,y} = K(\cdot, \zeta)y \colon \Omega \to \mathcal{Y}$

Define an inner product on $\mathcal{H}_0$ = span of kernel elements so that $\langle k_{\zeta,y'}, k_{\omega,y} \rangle_{\mathcal{H}_0} = \langle y', K(\zeta, \omega)y \rangle_{\mathcal{Y}} = \langle y', k_{\omega,y}(\zeta) \rangle_{\mathcal{Y}}$

(1) $\Rightarrow \langle \cdot, \cdot \rangle_{\mathcal{H}_0}$ positive semidefinite —even positive definite if $\mathcal{H}_0$ taken to be subspace of functions $f \colon \Omega \to \mathcal{Y}$

Let $\mathcal{H}(K)$ = Hilbert-space completion of $\mathcal{H}_0$: identify elements as still consisting of functions $f \colon \Omega \to \mathcal{Y}$ determined via reproducing property $\langle y, f(\omega) \rangle_{\mathcal{Y}} = \langle k_{\omega,y}, f \rangle_{\mathcal{H}(K)}$

Sketch of (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1)

Proof of (2) $\Rightarrow$ (3):

Take $\mathcal{X} = \mathcal{H}(K)$ and define $H\colon \Omega \to B(\mathcal{H}(K), \mathcal{Y})$ to be point evaluation: $H(\omega)\colon f \mapsto f(\omega)$. Then this works!

Proof of (3) $\Rightarrow$ (1):

Elementary computation: Assume (3). Then
$\sum_{i,j=1}^{N} \langle y_i, K(\omega_i, \omega_j) y_j \rangle_{\mathcal{Y}} = \sum_{i,j=1}^{N} \langle y_i, H(\omega_i) H(\omega_j)^* y_j \rangle_{\mathcal{Y}} = \sum_{i,j=1}^{N} \langle H(\omega_i)^* y_i, H(\omega_j)^* y_j \rangle_{\mathcal{Y}} = \| \sum_{j=1}^{N} H(\omega_j)^* y_j \|_{\mathcal{X}}^2 \geq 0.$

Converse: which functional Hilbert spaces are RKHSs?

Theorem 2:
Given $\mathcal{H}$ = Hilbert space consisting of functions $f\colon \Omega \to \mathcal{Y}$, TFAE:

1. There is a positive kernel $K\colon \Omega \times \Omega \to B(\mathcal{Y})$ so that $\mathcal{H} = \mathcal{H}(K)$
2. The point evaluations $ev(\omega)\colon f \mapsto f(\omega)$ are continuous

Sketch of proof
If $\langle y, f(\omega)\rangle_{\mathcal{Y}} = \langle k_{\omega,y}, f\rangle_{\mathcal{H}(K)}$ with $k_{\omega,y} \in \mathcal{H}(K)$, then $f \mapsto \langle y, f(\omega)\rangle_{\mathcal{Y}}$ continuous for each y . Then PUB $\Rightarrow f \mapsto f(\omega)$ continuous as well.

Converse: Riesz representation theorem and PUB

Construction of RKHS from Kolmogorov decomp. factor

Theorem 3

Given $H\colon \Omega \to B(\mathcal{X}, \mathcal{Y})$, define $\mathcal{H} = \{H(\cdot)x\colon x \in \mathcal{X}\}$ with norm $\|f\|^2_{\mathcal{H}} = \min\{\|x\|^2\colon f(\cdot) = H(\cdot)x\}$.
Then $\mathcal{H} = \mathcal{H}(K)$ isometrically, where $K(\zeta, \omega) = H(\zeta)H(\omega)^*$

Proof

Compute:

$\langle f(\omega), y\rangle_{\mathcal{Y}} = \langle H(\omega)x, y\rangle_{\mathcal{Y}} = \langle x, H(\omega)^*y\rangle_{\mathcal{X}} =$
$\langle P_{\ker M_H} x, H(\omega)^*y\rangle_{\mathcal{X}} = \langle H(\cdot)x, H(\cdot)H(\omega)^*y\rangle_{\mathcal{H}} = \langle f, K(\cdot,\omega)y\rangle_{\mathcal{H}}$
$\Rightarrow \mathcal{H} = \mathcal{H}(K)$

Direct proof of (3) $\Rightarrow$ (2) in Theorem 1

Application 1.

1. Function-theoretic operator theory

Given a Hilbert space of analytic functions $\mathcal{H}$ with an explicit computable inner product, e.g.

$$H^2(\mathbb{D}) = \{f \colon \mathbb{D} \underset{holo}{\to} \mathbb{C} \colon f(z) = \sum_{n=0}^{\infty} f_n z^n \text{ with } \|f\|_{H^2}^2 := \sum_{n=0}^{\infty} |f_n|^2 < \infty\}$$

Polarization $\Rightarrow \langle g, f \rangle_{H^2} = \sum_{n=0}^{\infty} \overline{g_n} f_n$ if $g(z) = \sum_{n=0}^{\infty} g_n z^n$

Then guess that $H^2(\mathbb{D})$ = RKHS with kernel = Szegő kernel $k_{\mathrm{Sz}}(z, w) = \frac{1}{1 - z\overline{w}}$:

Check: $\langle k_w, f \rangle_{H^2} = \sum_{n=0}^{\infty} w^n f_n = f(w)$

Operator algebra of interest: the multiplier algebra

Application 2.

2. Machine Learning/Support Vector Machines

Start with Ω = input data points

Cook up feature map (nonlinear change of variable)
$\Phi\colon \omega \mapsto \Phi(\omega) = k_{\omega,1} = H(\omega)^* 1 \in \mathcal{H}$ (big unknown Hilbert space).

Nevertheless: Assume $\langle \Phi(\omega), \Phi(\omega') \rangle_{\mathcal{H}} = K(\omega, \omega')$ known
(Choice of K $\Leftarrow$ heuristic arguments for particular problem)

Language: one says that K = the kernel having Φ as its feature map (i.e., having $\Phi(\omega) = H(\omega)^*$ as right factor in Kolmogorov decomposition: $K(\omega', \omega) = H(\omega')H(\omega)^* = \Phi(\omega')^* \Phi(\omega)$) and then $\mathcal{H} = \mathcal{H}(K)$ (the RKHS) as in Theorem 3 = the feature space

Application 2 continued

Learning algorithm: Solve for $f^* \in \mathcal{H}(K)$ which minimizes the regularized risk function:

$\inf_{f \in \mathcal{H}(K)} \lambda \|f\|^2_{\mathcal{H}} + \mathcal{R}_{L,D}(f)$

where $\mathcal{R}_{L,D}$ = the loss or error associated with choice of predicted-value function $x \mapsto f(x)$ based on training data set $\mathcal{D} = \{(x_i, y_i) \colon i = 1, \dots, N\}$.

Assumptions: L depends only on (y_i, f) , not on (x_i, y_i, f); $\mathcal{R}_{L,D}(f)$ convex in f and depends only on $f(x_i)$ $(i = 1, \dots, N)$ $\Rightarrow$ solution has the form $f^* = \sum_{i=1}^N c_i K(\cdot, x_i)$ and therefore is computable (kernel trick!) .

$\Rightarrow$ Good employment opportunities for Math grad students in operator theory, but very different questions:
no interest in multiplier algebras in machine learning literature

Source: Steinwart-Christmann, Support Vector Machines, Springer 2008

3: Quantum mechanics: coherent states

Assume we have a map $H\colon \Omega \to B(\mathbb{C}^N, \mathcal{Y})$
(Ω = locally compact Hausdorff space, $N \in \mathbb{N} \cup \aleph_0$ ($\mathbb{C}^{\aleph_0} = \ell^2$))
written out in terms of coordinates:
$H(\omega) = \begin{bmatrix} h_1(\omega) & h_2(\omega) & \cdots & h_n(\omega) & \cdots \end{bmatrix}$ where $h_n(\omega) \in \mathcal{Y}$
Then $\mathrm{Ran} M_H = \{H(\cdot)x \colon x \in \ell^2\}$ with lifted norm = RKHS with kernel $K(\zeta, \omega) = H(\zeta)H(\omega)^*$ as in Theorem 3

Then for $y \in \mathcal{Y}$, the functions $\{k_{\omega,y} \colon \omega \in \Omega, y \in \mathcal{Y}\}$ given by $k_{\omega,y}(\zeta) = K(\zeta,\omega)y = H(\zeta)H(\omega)^*y$ are called coherent states (CS) thought of as an overcomplete system of vectors indexed by ω, y

i.e., CS = kernel elements in terminology above

Application 3 continued.

Additional structure: Assume $\exists$ Resolution of the Identity:
$\exists$ Borel measure ν on Ω so that $\int_X H(\omega)^* H(\omega)\, \mathrm{d}\nu(\omega) = I_{\ell^2}$

Then the Reproducing Kernel is square-integrable in the sense that
$\int_X K(\omega, \zeta) K(\zeta, \omega')\, \mathrm{d}\nu(\zeta) = K(\omega, \omega')$

Proof uses associativity:

$$\int_X K(\omega, \zeta) K(\zeta, \omega')\, \mathrm{d}\nu(\zeta) = \int_X (H(\omega) H(\zeta)^*)(H(\zeta) H(\omega')^*)\, \mathrm{d}\nu(\zeta)$$
$$= \int_X H(\omega)(H(\zeta)^* H(\zeta)) H(\omega')^*\, \mathrm{d}\nu(\zeta)$$
$$= H(\omega) \left(\int_X H(\zeta)^* H(\zeta)\, \mathrm{d}\nu(\zeta) \right) H(\omega')^* = H(\omega) H(\omega')^*$$
$$= K(\omega, \omega')$$

Source: S.T. Ali, Reproducing Kernels in Coherent States, Wavelets, and Quantization , in: Part I Reproducing Kernel Hilbert Spaces (ed. F.H. Szafraniec), in:
Operator Theory, Volume 1 (ed. D. Alpay), Springer, 2015

The next step: Barreto-Bhat-Liebscher-Skeide (JFA 2004)

Given $K \colon \Omega \times \Omega \to B(\mathcal{A}, B(\mathcal{Y}))$ where $\mathcal{A} = C^*$-algebra

Thus, for $\zeta, \omega \in \Omega$ and $a \in \mathcal{A}$, $K(\zeta, \omega)(a) \in B(\mathcal{Y})$

We say that K as above is a completely positive (cp) kernel if any of the following equivalent conditions hold:

1. $\sum_{i,j=1}^{N} \langle y_i, K(\omega_i, \omega_j)(a_i^* a_j) y_j \rangle_{\mathcal{Y}} \geq 0 \ \forall\ \omega_1, \ldots, \omega_N$ in Ω, $a_1, \ldots, a_N$ in $\mathcal{A}$, $y_1, \ldots, y_N$ in $\mathcal{Y}$
2. The kernel $\mathfrak{K} \colon (\Omega \times \mathcal{A}) \times (\Omega \times \mathcal{A}) \to \mathcal{B}(\mathcal{Y})$ given by $\mathfrak{K}((\omega, a), (\omega', a')) = K(\omega, \omega')(a^* a')$ is a Moore-Aronszajn positive kernel
3. The mapping $K^{(n)} \colon [a_{ij}] \mapsto [K(\omega_i, \omega_j)(a_i^* a_j)]$ is a positive map from $\mathcal{A}^{n \times n}$ into $B(\mathcal{Y})^{n \times n}$ for any choice of $\omega_1, \ldots, \omega_n$ in Ω

BBLS version of Theorem 1

Theorem 1′

Given a kernel $K\colon \Omega \times \Omega \to B(\mathcal{A}, B(\mathcal{Y}))$, TFAE:

1. K is a cp kernel
2. K is the Reproducing Kernel for a Reproducing Kernel $(\mathcal{A}, \mathbb{C})$-correspondence: see next slide
3. K has a Kolmogorov decomposition: $\exists$ $(\mathcal{A}, \mathbb{C})$-correspondence $\mathcal{X}$ and function $H\colon \Omega \to B(\mathcal{X}, \mathcal{Y})$ so that $K(\zeta, \omega)(a) = H(\zeta)\sigma(a)H(\omega)^*$ where $\sigma(a)x = a \cdot x$ for $x \in \mathcal{X}$

Details on part 2 of Theorem $1'$

Reproducing Kernel $(\mathcal{A}, \mathbb{C})$-correspondence

Given a kernel K as above, $\mathcal{H}(K)$ is the associated unique $(\mathcal{A}, \mathbb{C})$ -correspondence means:

(i) Elements of $\mathcal{H}(K)$ are functions $f \colon \Omega \to B(\mathcal{A}, \mathcal{Y})$

(ii) $k_{\omega,a,y} \in \mathcal{H}(K)$ for any $\omega \in \Omega$,$a \in \mathcal{A}$, $y \in \mathcal{Y}$, where
$k_{\omega,a,y}(\zeta)(a') = K(\zeta, \omega)(a'a)y$

(iii) $k_{\omega,a,y}$ has the reproducing property:
$\langle k_{\omega,a,y}, f\rangle_{\mathcal{H}(K)} = \langle y, f(\omega)(a)\rangle_{\mathcal{Y}}$

(iv) for $a' \in \mathcal{A}$,
$(a' \cdot f)(\omega)(a) = f(\omega)(aa')$, or equivalently $a' \cdot k_{\omega,a,y} = k_{\omega,a'a,y}$

Proof of Theorem $1'$: functorial modification of proof of Theorem 1

cp global/nc kernels

Recall formulation (3) of $K\colon \Omega \times \Omega \to B(\mathcal{A}, B(\mathcal{Y}))$ is a cp kernel:

The mapping $K^{(n)}\colon [a_{ij}] \mapsto [K(z_i, z_j)(a_i^* a_j)]$ is a positive map from $\mathcal{A}^{n \times n}$ into $B(\mathcal{Y})^{n \times n}$ for any choice of $z_1, \dots, z_n$ in Ω

This suggests: Extend set of points Ω to its nc envelope $[\Omega]_{\rm nc}$ defined as follows . . .

Preliminaries on nc sets and envelopes

Let $\mathcal{S}$ = a set. Define $\mathcal{S}_{\rm nc} = \amalg_{n=1}^{\infty} \mathcal{S}^{n \times n}$

where $\mathcal{S}^{n \times n}$ = $n \times n$ matrices with entries in $\mathcal{S}$

Suppose that $\mathcal{T} \subset \mathcal{S}_{\rm nc}$. Set $\mathcal{T}_n = \mathcal{T} \cap \mathcal{S}^{n \times n}$. Thus $\mathcal{T} = \amalg_{n=1}^{\infty} \mathcal{T}_n$

We say that $\mathcal{T}$ is a nc set if $Z \in \mathcal{T}_n$ and $W \in \mathcal{T}_m \Rightarrow$ $\left[\begin{smallmatrix} Z & 0 \\ 0 & W \end{smallmatrix}\right] \in \mathcal{T}_{n+m}$

For $\mathcal{T}$ = arbitrary subset of $\mathcal{S}_{\rm nc}$, define $[\mathcal{T}]_{\rm nc}$ = smallest nc subset containing $\mathcal{T}$ (noncommutative envelope of $\mathcal{T}$)

Suppose $\Omega \subset \mathcal{S} = (\mathcal{S}_{\rm nc})_1$. Then

$$[\Omega]_{\rm nc} = \amalg_{n=1}^{\infty} \left\{ \begin{bmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{bmatrix} : z_1, \dots, z_n \in \Omega \right\}$$

Extensions of kernels to nc envelopes

Given kernel $K\colon \Omega \times \Omega \to B(\mathcal{A}, B(\mathcal{Y}))$, extend K to
$\mathfrak{K}\colon [\Omega]_{\mathrm{nc},n} \times [\Omega]_{\mathrm{nc},m} \to B(\mathcal{A}^{n\times m}, B(\mathcal{Y})^{n\times m} \cong B(\mathcal{Y}^m, \mathcal{Y}^n))$ by

$$\mathfrak{K}\left(\begin{bmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{bmatrix}, \begin{bmatrix} w_1 & & \\ & \ddots & \\ & & w_m \end{bmatrix} \right)([a_{ij}]) = [K(z_i, z_j)(a_{ij})]$$

for any $n, m \in \mathbb{N}$

Then K being a cp kernel can be expressed more succinctly as:

for all $Z \in [\Omega]_{\mathrm{nc}}$, say $Z \in [\Omega]_{\mathrm{nc},n}$, $K(Z, Z)\colon \mathcal{A}^{n\times n} \to B(\mathcal{Y})^{n\times n}$ is a positive map

This suggests a more general formulaton . . .

Cp global kernels and global RK $(\mathcal{A}, \mathbb{C})$-Correspondences

Suppose Ω = nc subset of $\mathcal{S}_{\rm nc}$
(in particular, Ω not necessarily equal to $[\Omega_1]_{\rm nc}$)

Suppose $K\colon \Omega \times \Omega \to B(\mathcal{A}_{\rm nc}, B(\mathcal{Y})_{\rm nc})$.
We say that K is a global kernel if
(i) K is graded : $K\colon \Omega_n \times \Omega_m \mapsto B(\mathcal{A}^{n\times m}, B(\mathcal{Y})^{n\times m})$
(ii) K respects direct sums:

$$K\left(\begin{bmatrix} Z & 0 \\ 0 & \widetilde{Z} \end{bmatrix}, \begin{bmatrix} W & 0 \\ 0 & \widetilde{W} \end{bmatrix}\right)\left(\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}\right) = \begin{bmatrix} K(Z,W)(P_{11}) & K(Z,\widetilde{W})(P_{12}) \\ K(\widetilde{Z},W)(P_{21}) & K(\widetilde{Z},\widetilde{W})(P_{22}) \end{bmatrix}$$

We say that K is a cp global kernel if also for all $Z \in \Omega_n$,
$K(Z,Z)\colon \mathcal{A}^{n\times n} \to B(\mathcal{Y})^{n\times n}$ is a positive map , $n \in \mathbb{N}$ arbitrary

Cp global kernels and RK correspondences continued

Theorem 1′: first upgrade (Ball-Marx-Vinnikov JFA 2016)

Given Ω = nc subset of $\mathcal{S}_{\rm nc}$, $K \colon \Omega \times \Omega \to B(\mathcal{A}_{\rm nc}, B(\mathcal{Y})_{\rm nc})$, TFAE:

1. K is a cp global kernel
2. K is the RK for a global RK $(\mathcal{A}, \mathbb{C})$-correspondence —see next slide
3. K has a global Kolmogorov decomposition: $\exists$ a $(\mathcal{A}, \mathbb{C})$ -correspondence $\mathcal{X}$ and a global function $H \colon \Omega \to B(\mathcal{X}, \mathcal{Y})_{\rm nc}$ (see next slide) so that
   $K(Z, W)(P) = H(Z)(\mathrm{id}_{\mathbb{C}^{n \times m}} \otimes \sigma)(P) H(W)^*$
   for all $Z \in \Omega_n$, $W \in \Omega_m$, $P \in \mathcal{A}^{n \times m}$ where
   $\sigma(a)x = a \cdot x$ for $a \in \mathcal{A}$ and $x \in \mathcal{X}$ and
   $(\mathrm{id}_{\mathbb{C}^{n \times m}} \otimes \sigma)([P_{ij}]) = [\sigma(P_{ij})]$

Background material on global kernels and global functions

We say that $H\colon \Omega \to B(\mathcal{X}, \mathcal{Y})_{\rm nc}$ is a global function if
(i) H is graded : $Z \in \Omega_n \Rightarrow H(Z) \in B(\mathcal{X}, \mathcal{Y})^{n \times n} \cong B(\mathcal{X}^n, \mathcal{Y}^n)$
(ii) H respects direct sums: $H\left(\begin{bmatrix} Z & 0 \\ 0 & \widetilde{Z} \end{bmatrix} \right) = \begin{bmatrix} H(Z) & 0 \\ 0 & H(\widetilde{Z}) \end{bmatrix}$

$\mathcal{H}(K)$ = global RK $(\mathcal{A}, \mathbb{C})$-correspondence associated with cp global kernel K means:

(i) $\mathcal{H}(K)$ = $(\mathcal{A}, \mathbb{C})$-correspondence with elements f equal to global functions from Ω to $B(\mathcal{A}, \mathcal{Y})_{\rm nc}$
(so $f(Z) \in B(\mathcal{A}^n, \mathcal{Y}^n)$ for $Z \in \Omega_n$)

(ii) for $W \in \Omega_m$, $v \in \mathcal{A}^{1 \times m}$, $y \in \mathcal{Y}^m$, $k_{W,v,y} \in \mathcal{H}(K)$ where
$k_{W,v,y}(Z)(u) = K(Z, W)(uv)y$ for $Z \in \Omega_n$, $u \in \mathcal{A}^n$

Global RK correspondence continued

(iii) $k_{W,v,y}$ has the reproducing property:
$\langle k_{W,v,y}, f \rangle_{\mathcal{H}(K)} = \langle y, f(W)(v^*) \rangle_{\mathcal{Y}}$

(iv) The left action of $\mathcal{A}$ on $\mathcal{H}(K)$ is given by
$(a \cdot f)(W)(u) = f(W)(ua)$ or equivalently $a \cdot k_{W,v,y} = k_{W,av,y}$

Stinespring representation theorem

Special case: $\Omega = [\Omega_1]_{\mathrm{nc}}$ and $\Omega_1 = \{\omega_0\}$ (singleton set)

Then $\Omega_n = \left\{ \begin{bmatrix} \omega_0 & & \\ & \ddots & \\ & & \omega_0 \end{bmatrix} \right\}$ (singleton set)

Suppose that $K\colon \Omega \times \Omega \to B(\mathcal{A}_{\mathrm{nc}}, B(\mathcal{Y})_{\mathrm{nc}})$ is a global kernel

Define $\varphi\colon \mathcal{A} \to B(\mathcal{Y})$ by $\varphi(a) = K(\omega_0, \omega_0)(a)$

Then $K\left(\begin{bmatrix} \omega_0 & & \\ & \ddots & \\ & & \omega_0 \end{bmatrix}, \begin{bmatrix} \omega_0 & & \\ & \ddots & \\ & & \omega_0 \end{bmatrix} \right)([a_{ij}])$

$= [K(\omega_0, \omega_0)(a_{ij})] = [\varphi(a_{ij})] = \varphi^{(n)}([a_{ij}])$

Conclude: K = cp global kernel $\Leftrightarrow \varphi\colon \mathcal{A} \to B(\mathcal{Y})$ = cp map

Kolmogorov decomposition for K $\Rightarrow$
$\varphi(a) = K(\omega_0, \omega_0)(a) = H(\omega_0)\sigma(a)H(\omega_0)^* = V^*\sigma(a)V$ where
$V = H(\omega_0)^*\colon \mathcal{Y} \to \mathcal{X}$ and $\sigma\colon \mathcal{A} \to B(\mathcal{X})$ = $*$-representation $\Rightarrow$
Steinspring representation for cp map φ

The next upgrade: noncommutative functions

Assume $\mathcal{S} = \mathcal{V}$ is a vector space, $\mathcal{V}_{\rm nc} = \amalg_{n=1}^{\infty} \mathcal{V}^{n \times n}$ is the associeted full nc set

Note: Vector spaces are bimodules over $\mathbb{C}$ $\Rightarrow$, for $[\alpha] \in \mathbb{C}^{k \times \ell}$, $[v] \in \mathcal{V}^{\ell \times m}$, $[\beta] \in \mathbb{C}^{m \times n}$, the product $[\alpha] \cdot [v] \cdot [\beta]$ makes sense via standard matrix multiplication

Suppose that $\mathcal{V}_0$ $=$ another vector space and $f \colon \Omega \to (\mathcal{V}_0)_{\rm nc}$
We say that f is a nc function if
(i) f is global, i.e. (i-a) f is graded: $f(Z) \in (\mathcal{V}_0)^{n \times n}$ if $Z \in \Omega_n$
and (i-b) f respects direct sums: $f\left(\begin{bmatrix} Z & 0 \\ 0 & \widetilde{Z} \end{bmatrix}\right) = \begin{bmatrix} f(Z) & 0 \\ 0 & f(\widetilde{Z}) \end{bmatrix}$

(ii) f respects similarities: $Z \in \Omega_n$, α invertible in $\mathbb{C}^{n \times n}$ such that $\alpha Z \alpha^{-1} \in \Omega_n$ $\Rightarrow$ $\alpha f(Z) \alpha^{-1} = f(\alpha Z \alpha^{-1})$

Noncommutative kernels

Suppose $K\colon \Omega \times \Omega \to B((\mathcal{V}_1)_{\rm nc}, (\mathcal{V}_0)_{\rm nc})$.

We say that K is a nc kernel if

(i) K is a global kernel, i.e., (i-a) K is graded: $Z \in \Omega_n$, $W \in \Omega_m$ $\Rightarrow K(Z, W) \in B((\mathcal{V}_1)^{n\times m}, (\mathcal{V}_0)^{n\times m})$
and (i-b) K respects direct sums:

$$K\left(\begin{bmatrix} Z & 0 \\ 0 & \widetilde{Z} \end{bmatrix}, \begin{bmatrix} W & 0 \\ 0 & \widetilde{W} \end{bmatrix}\right)\left(\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}\right) = \begin{bmatrix} K(Z,W)(P_{11}) & K(Z,\widetilde{W})(P_{12}) \\ K(\widetilde{Z},W)(P_{21}) & K(\widetilde{Z},\widetilde{W})(P_{22}) \end{bmatrix},$$

and

(ii) K respects similarities:
$Z, \widetilde{Z} \in \Omega_n$, $\alpha \in \mathbb{C}^{n\times n}$ invertible with $\widetilde{Z} = \alpha Z \alpha^{-1} \in \Omega_n$,
$W, \widetilde{W} \in \Omega_m$, $\beta \in \mathbb{C}^{m\times m}$ invertible with $\widetilde{W} = \beta W \beta^{-1} \in \Omega_m$,
$P \in \mathcal{V}_1^{n\times m} \Rightarrow K(\widetilde{Z}, \widetilde{W})(P) = \alpha\, K(Z, W)(\alpha^{-1} P \beta^{-1*})\, \beta^*$.

cp nc kernels

Restrict now to the case where $\mathcal{V}_1 = \mathcal{A}$ = a C^*-algebra, $\mathcal{V}_0 = \mathcal{B}(\mathcal{Y})$ for a Hilbert space $\mathcal{Y}$.

We say that K is a cp nc kernel if K is a cp global kernel which is also a nc kernel, i.e.,

(i) K is graded,

(ii) K respects direct sums, and

(iii) K respects similarities, and

(iv) $K(Z, Z)\colon \mathcal{A}^{n\times n} \to B(\mathcal{Y})^{n\times n}$ is a positive map for any $Z \in \Omega_n$

cp nc kernels and nc RKHSs

Theorem 1′: second upgrade (Ball-Marx-Vinnikov JFA 2016)

Assume that $K\colon \Omega \to B(\mathcal{A}_{\rm nc}, B(\mathcal{Y})_{\rm nc})$. Then TFAE:

1. K is a cp nc kernel.
2. K is the RK for a nc RK $(\mathcal{A}, \mathbb{C})$-correspondence —see next slide
3. K has a nc Kolmogorov decomposition: $\exists$ a $(\mathcal{A}, \mathbb{C})$ -correspondence $\mathcal{X}$ and a nc function $H\colon \Omega \to B(\mathcal{X}, \mathcal{Y})_{\rm nc}$ so that $K(Z, W)(P) = H(Z)({\rm id}_{\mathbb{C}^{n \times m}} \otimes \sigma)(P) H(W)^*$ for all $Z \in \Omega_n$, $W \in \Omega_m$, $P \in \mathcal{A}^{n \times m}$ where $\sigma(a) = a \cdot x$ for $a \in \mathcal{A}$ and $x \in \mathcal{X}$ and $({\rm id}_{\mathbb{C}^{n \times m}} \otimes \sigma)([P_{ij}]) = [\sigma(P_{ij})]$

nc Reproducing Kernel Correspondence

We say that $\mathcal{H}(K)$ = nc RK $(\mathcal{A}, \mathbb{C})$-correspondence associated with cp nc kernel K if:

(i) $\mathcal{H}(K)$ = $(\mathcal{A}, \mathbb{C})$-correspondence with elements f equal to nc functions from Ω to $B(\mathcal{A}, \mathcal{Y})_{\rm nc}$

(ii) for $W \in \Omega_m$, $v \in \mathcal{A}^{1 \times m}$, $y \in \mathcal{Y}^m$, $k_{W,v,y} \in \mathcal{H}(K)$ where $k_{W,v,y}(Z)(v) = K(Z,W)(uv)y$ for $Z \in \Omega_n$, $u \in \mathcal{A}^n$

(iii) $k_{W,v,y}$ has the reproducing property $\langle k_{W,v,y}, f \rangle_{\mathcal{H}(K)} = \langle y, f(W)(v^*) \rangle_{\mathcal{Y}}$

(iv) The left action of $\mathcal{A}$ on $\mathcal{H}(K)$ is given by $(a \cdot f)(v^*) = f(W)(v^* a)$ or equivalently $a \cdot k_{W,v,y} = k_{W,av,y}$

Global/nc RK correspondences: converse statements

Theorems $2'$ with First/ Second Upgrade:

Suppose $\mathcal{H}$ = Hilbert space with elements f equal to global/nc functions from Ω into $B(\mathcal{A}, \mathcal{L}(\mathcal{Y}))_{\rm nc}$ such that

(i) $W \in \Omega_m \Rightarrow f \mapsto f(W)$ bounded from $\mathcal{H}$ to $B(\mathcal{A}, \mathcal{Y})^{m \times m} \cong B(\mathcal{A}^m, \mathcal{Y}^m)$

(ii) $\sigma \colon \mathcal{A} \to B(\mathcal{H})$ given by $(\sigma(a)f)(W)(u) = f(W)(ua)$ defines a unital $*$-representation of $\mathcal{A}$

$\Rightarrow \exists$ cp global/nc kernel K so that $\mathcal{H} = \mathcal{H}(K)$ isometrically

Global/nc RK correspondences: converse statements cont.

Theorem 3 with Fist/Second Upgrade:

$\mathcal{X}$ = Hilbert space equipped with $*$ -rep $\sigma\colon \mathcal{A} \to B(\mathcal{X})$
$H\colon \Omega \to \mathcal{B}(\mathcal{X}, \mathcal{Y})_{\rm nc}$ = global/nc function

Define $\mathcal{H} = \{f(\cdot) = H(\cdot)x\colon x \in \mathcal{X}\}$ with
$\|f\|_{\mathcal{H}} = \min\{\|x\|_{\mathcal{X}}\colon f(\cdot) = H(\cdot)x\}$

Set $K(Z, W)(P) = H(Z)(\mathrm{id}_{\mathbb{C}^{n\times m}} \otimes \sigma(P)H(W)^*$ for $Z \in \Omega_n$, $W \in \Omega_m$, $P = [a_{ij}] \in \mathcal{A}^{n\times m}$.

Then K is a global/nc kernel and $\mathcal{H} = \mathcal{H}(K)$ isometrically as a global/nc $(\mathcal{A}, \mathbb{C})$-correspondence

Applications of nc RK Correspondences

Applications to function-theoretic operator theory

(i) Nevanlinna-Pick interpolation theory for multipliers on nc RK correspondences

(ii) Notion of complete cp nc kernels

(iii) nc Schur-Agler class vs nc Schur class: skip

Ball-Marx-Vinnikov 2016: to appear in IWOTA 2015 Proceedings (Tbilsi, Georgia) (available on arXiv)

Applications to Machine Learning/Support Vector Machines

?

Applications to Math Physics (coherent states)

?

Recall Moore-Aronszajn

Moore-Aronszajn RKHS

Given $K\colon \Omega \times \Omega \to B(\mathcal{Y})$, TFAE:

1. K is a positive kernel: $\sum_{i,j=1}^{N} \langle y_i, K(\omega_i, \omega_j) y_j \rangle_{\mathcal{Y}} \geq 0 \ \forall$ ys, ωs, Ns
2. K = RK for RKHS $\mathcal{H}(K)$: $\langle y, f(\omega) \rangle_{\mathcal{Y}} = \langle k_{\omega,y}, f \rangle_{\mathcal{H}(K)}$
3. $\exists$ Hilbert space $\mathcal{X}$ and function $H\colon \Omega \to B(\mathcal{S}, \mathcal{Y})$ so that $K(\zeta, \omega) = H(\zeta) H(\omega)^*$

Moore-Aronszajn pos. kernel: op.-valued case reformulated

Given $K\colon \Omega \times \Omega \to B(\mathcal{Y})$, TFAE:

1. K is a positive kernel: $[K(\omega_i, \omega_j)]$ is positive in $B(\mathcal{Y})^{N \times N}$, or $\sum_{i,j=1}^N T_i^* K(\omega_i, \omega_j) T_j^* \succeq 0 \;\; \forall \; T_j$s, ω_js, Ns, $T_j \in B(\mathcal{Y})$
2. K $=$ RK for RK Hilbert module over $B(\mathcal{Y})$: $f(\omega) = \langle k_\omega, f \rangle_{\mathcal{H}(K)}$ where $k_\omega(\zeta) = K(\zeta, \omega)$ and where $f(\omega) \in B(\mathcal{Y})$
3. K has a Kolmogorov decomposition $K(\zeta) = H(\zeta)H(\omega)^*$: the same

Hilbert module version: replace $B(\mathcal{Y})$ by $\mathcal{B} = C^*$-algebra

Theorem A

Given $\mathcal{B} = C^*$-algebra and $K\colon \Omega \times \Omega \to \mathcal{B}$, TFAE:

1. K is a positive kernel:
   $[K(\omega_i, \omega_j)] \succeq 0$ in $\mathcal{B}^{N \times N}$ $\forall\, \omega_1, \dots, \omega_N \in \Omega$ $\forall\, N = 1, 2, \dots$,
   or $\sum_{i,j=1}^{N} b_i^* K(\omega_i, \omega_j) b_j^* \succeq 0$ $\forall\, \omega$s in Ω, bs in $\mathcal{B}$
2. K is the RK for a RK Hilbert module $\mathcal{H}(K)$ over $\mathcal{B}$:
   $f(\omega) = \langle k_\omega, f \rangle_{\mathcal{H}(K)}$ where $k_\omega(\zeta) = K(\zeta, \omega) \in \mathcal{B}$ and where $f(\omega) \in \mathcal{B}$
3. $\exists$ Hilbert $\mathcal{B}$-module $\mathcal{X}$ and $H\colon \Omega \to \mathcal{L}(\mathcal{X}, \mathcal{B})$ so that
   $K(\zeta, \omega) = H(\zeta) H(\omega)^*$

History:

Stinespring rep. with $\mathcal{L}(E)$ in place of $B(\mathcal{H})$: Kasparov 1980

(1) $\Leftrightarrow$ (3): Murphy 1997

Incorporate (2): Szafraniec 2010

Original form of Barreto-Bhat-Liebscher-Skeide (JFA 2004)

We say that $K\colon \Omega \times \Omega \to B(\mathcal{A}, \mathcal{B})$ is a cp kernel if
$\sum_{i,j=1}^{N} b_i^* K(\omega_i, \omega_j)(a_i^* a_j) b_j \succeq 0$
for all as in $\mathcal{A}$, bs in $\mathcal{B}$, ωs in Ω, $N = 1, 2, \dots$

or equivalently
$[K(\omega_i, \omega_j)]\colon \mathcal{A}^{N \times N} \to \mathcal{B}^{N \times N}$ is a positive map for all
$\omega_1, \dots, \omega_N \in \Omega$, $N = 1, 2, \dots$

cp kernels and RK $(\mathcal{A}, \mathcal{B})$-correspondences

Theorems A, A-first upgrade, A-second upgrade

Given $K \colon \Omega \times \Omega \to \mathcal{B}(\mathcal{A}, \mathcal{B})$, TFAE:

1. K is a cp (global/nc) kernel
2. $\exists$ (global/nc) $(\mathcal{A}, \mathcal{B})$-corresondence $\mathcal{H}(K)$ with RK equal to K —see next slide
3. K has a Kolmogorov decompostion: $\exists$ $(\mathcal{A}, \mathcal{B})$-correspondence $\mathcal{X}$ and (global/nc) function $H \colon \Omega \to \mathcal{L}(\mathcal{X}, \mathcal{B})$ so that $K(Z, W)(P) = H(Z)(\mathrm{id}_{\mathbb{C}^{n \times m}} \otimes \sigma)(P) H(W)^*$ where $\sigma(a)x = a \cdot x$, for $Z \in \Omega_n$, $W \in \Omega_m$, $P = [a_{ij}] \in \mathcal{A}^{n \times m}$

RK $(\mathcal{A}, \mathcal{B})$-correspondence associated with RK K

$\mathcal{H}(K)$ = RK $(\mathcal{A}, \mathcal{B})$-correspondence associated with cp (global/nc) kernel K means:

$\mathcal{H}(K)$ = $(\mathcal{A}, \mathcal{B})$-correspondence with elements f equal to (global/nc) functions $f\colon \Omega \to B(\mathcal{A}, \mathcal{B})_{\rm nc}$ such that

(i) For each $W \in \Omega_m$, $v \in \mathcal{A}^{1\times m}$, $k_{W,v} \in \mathcal{H}(K)$ where $k_{W,v}(Z)(u) = K(Z,W)(uv)$ for $Z \in \Omega_n$, $u \in \mathcal{A}^n$

(ii) $k_{W,v}$ has the reproducing property: $f(W)(v^*) = \langle k_{W,v}, f\rangle_{\mathcal{H}(K)}$

(iii) Left $\mathcal{A}$ -action on $\mathcal{H}(K)$ given by

$(a\cdot f)(W)(v^*) = f(W)(v^* a)$ for $a \in \mathcal{A}$, $W \in \Omega_m$, $v \in \mathcal{A}^{1\times m}$,

or equivalently $a \cdot k_{W,v} = k_{W,av}$ (Theorem A: $n = m = 1$ only)

History: (1) $\Leftrightarrow$ (3) in Theorem A: Barreto-Bhat-Liebscher-Skeide 2004

Incorporate (2): Ball-Marx-Vinnikov 2016

Converse theorems for Hilbert-module setting

Theorem B/B$'$ (Hilbert -module analogue of Theorem 2/2$'$) problematical, due to failure of Riesz representation theorem for linear functions $\mathcal{X} \to \mathcal{B}$ ($\mathcal{X}$ = Hilbert module over the C^*-algebra $\mathcal{B}$).

The fix: Let $\mathcal{A}$, $\mathcal{B}$ = W^*-algebras and let $\mathcal{X}$ be a self-dual W^*-$(\mathcal{A}, \mathcal{B})$- correspondence

Converse theorems for Hilbert-module setting continued

Theorem B′ (with First/Second Upgrade)

$\mathcal{A}$, $\mathcal{B} = W^*$-algebras

$\mathcal{H}$ = self-dual W^*-$(\mathcal{A}, \mathcal{B})$- correspondence whose elements f are (global/nc) functions from Ω to $B(\mathcal{A}, \mathcal{B})_{\rm nc}$ such that

(i) for $W \in \Omega_m$, $f \mapsto f(W)$ bounded from $\mathcal{H}$ to $B(\mathcal{A}, \mathcal{B})^{m \times m} \cong B(\mathcal{A}^m, \mathcal{B}^m)$, and

(ii) $(a \cdot f)(W)(u) = f(W)(ua)$ gives the left action of $\mathcal{A}$ on $\mathcal{H}$

Then $\exists$ a cp (global/nc) normal kernel K so that $\mathcal{H} = \mathcal{H}(K)$ isometrically as (global/nc) self-dual W^*-$(\mathcal{A}, \mathcal{B})$-correspondences

History: Self-dual W^*-HIlbert modules in general: Paschke 1973, Skeide 2000 & 2005

Theorem C

Given: $\mathcal{B}$ $= W^*$-algebra, E $=$ self-dual Hilbert module over $\mathcal{B}$

$\mathcal{X}$ $=$ self-dual W^*-module over $\mathcal{B}$,

H $=$ a function from Ω to $\mathcal{L}(\mathcal{X}, E)$

Define $\mathcal{H} = \{H(\cdot)x \colon x \in \mathcal{X}\}$ with $\|H(\cdot)x\| = \|P_{(\mathrm{Ker}\, M_H)^\perp} x\|$

($\mathcal{X}$ & $\mathcal{B}$ self-dual $\Rightarrow P_{(\mathrm{Ker}\, M_H)^\perp}$ exists)

Then $\mathcal{H} = \mathcal{H}(K)$ (self-dual RK Hilbert module over $\mathcal{B}$ consisting of functions $f \colon \Omega \to E$) with RK $K \colon \Omega \times \Omega \to \mathcal{L}(\mathcal{E})$ given by $K(\zeta, \omega) = H(\zeta)H(\omega)$

Interpretation: Let $\{e_n \colon n \in \mathbb{N}\}$ $=$ o.n.b. for E (over $\mathcal{B}$)

Then $\{H(\omega)^* e_n \colon \omega \in \Omega, n \in \mathbb{N}\}$ $=$ module-valued coherent states

Bhattacharyya-Roy 2012

Summary

- Choose one of three:
  Theorems 1/A: Given K , construct $\mathcal{H}(K)$ and H
  Theorems 2/B: Given $\mathcal{H}$ with ..., identify $\mathcal{H} = \mathcal{H}(K)$
  Theorem 3/C: Given $\omega \mapsto H(\omega)$, construct $\mathcal{H}(K)$
- Choose one of two:
  Without $'$: $\mathcal{A} = \mathbb{C}$ or no $\mathcal{A}$
  With $'$: general $\mathcal{A}$
- Choose one of two:
  Theorems 1, 2, 3: Target space of $K(\cdot,\cdot)$ or $K(\cdot,\cdot)(\cdot)$ is $B(\mathcal{Y})$
  Theorems A, B, C: Target space of $K(\cdot,\cdot)$ or $K(\cdot,\cdot)(\cdot)$ is $\mathcal{B}$
- Choose one of three:
  No upgrade: $f \in \mathcal{H}(K)$ = function
  First Upgrade: $f \in \mathcal{H}(K)$ = global function
  Second Upgrade: $f \in \mathcal{H}(K)$ = nc function

Conclusion: 1 theorem with 36 flavors!

Thanks for your attention!