Toeplitz and Asymptotic Toeplitz operators on $H^2(\mathbb{D}^n)$

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- To characterize asymptotically Toeplitz operators on $H^2(\mathbb{D}^n)$.
- To generalize some of the recent results of Chalendar and Ross to vector-valued Hardy space $H^2_{\mathcal{E}}(\mathbb{D})$ and as well as quotient spaces of $H^2(\mathbb{D}^n)$.

Notation

- Open unit polydisc $\mathbb{D}^n = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : |z_i| < 1, i = 1, \ldots, n\}$.
- Distinguished boundary of $\mathbb{D}^n$
  $\mathbb{T}^n = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : |z_i| = 1, i = 1, \ldots, n\}$.

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- Hardy space $H^2(\mathbb{D}) = \{f = \sum_{n=0}^{\infty} a_n z^n : \sum_{n=0}^{\infty} |a_n|^2 < \infty\}$.
- Vector-valued Hardy space
  $H^2_{\mathcal{E}}(\mathbb{D}) = \{f = \sum_{n=0}^{\infty} a_n z^n : a_n \in \mathcal{E}$ and $\sum_{n=0}^{\infty} \|a_n\|^2_{\mathcal{E}} < \infty\}$.
- $H^\infty(\mathbb{D}) = \{f = \sum_{n=0}^{\infty} a_n z^n : \sup\limits_{n \geq 0} |a_n| < \infty\}$.
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- $H^\infty(\mathbb{D}) = \{f = \sum_{n=0}^{\infty} a_n z^n : \sup_{n \geq 0} |a_n| < \infty\}$.
- M_z is the multiplication operator on $H^2(\mathbb{D})$ by the coordinate function z.
- Hardy space over polydisc $H^2(\mathbb{D}^n) = \left\{ f = \sum_{k \in \mathbb{N}^n} a_k z^k : \sum_{k \in \mathbb{N}^n} |a_k|^2 < \infty \right\}$,
  where $k = (k_1, \ldots, k_n) \in \mathbb{N}^n$ and $z^k = z_1^{k_1} \cdots z_n^{k_n}$.
- For $j = 1, \ldots, n$, M_{z_j} are the multiplication operators on $H^2(\mathbb{D}^n)$ by the j^{th} coordinate functions z_j.

Multiplication operator

- For $\phi \in L^\infty(\mathbb{T})$, define $M_\phi : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ by $M_\phi f = \phi f$ for $f \in L^2(\mathbb{T})$.

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- The matrix of M_ϕ with respect to the orthonormal basis $\{e^{in\theta}\}_{n=-\infty}^{\infty}$ of $L^2(\mathbb{T}) = H^2(\mathbb{D})^\perp \oplus H^2(\mathbb{D})$ is

$$M_\phi = \left[\begin{array}{cccc|cccc}
\ddots & \ddots & \ddots & & & & & \\
\ddots & & \phi_0 & \phi_{-1} & \phi_{-2} & & & \\
\ddots & & \phi_1 & \phi_0 & \phi_{-1} & \phi_{-2} & & \\
\hline
 & & \phi_2 & \phi_1 & \underline{\phi_0} & \phi_{-1} & \phi_{-2} & \\
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where $\phi = \sum_{n=-\infty}^{\infty} \phi_n e^{in\theta}$ is a Fourier expansion of ϕ.

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where $\phi = \sum_{n=-\infty}^{\infty} \phi_n e^{in\theta}$ is a Fourier expansion of ϕ.

- Toeplitz operator with symbol $\phi \in L^\infty(\mathbb{T})$ is the operator T_ϕ defined by

$$T_\phi f = P_{H^2(\mathbb{D})}(\phi f) \text{ for } f \in H^2(\mathbb{D}).$$

Toeplitz operator

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- Feintuch (1989) gives a remarkable characterization of asymptotically Toeplitz operators: A bounded linear operator T on $H^2(\mathbb{D})$ is asymptotically Toeplitz if and only if $T =$ compact $+$ Toeplitz.

Basic Definitions

- A closed subspace $\mathcal{S}$ of $\mathcal{H}$ is said to be invariant subspace of $T \in \mathcal{B}(\mathcal{H})$ if $T(\mathcal{S}) \subseteq \mathcal{S}$ and $\mathcal{S}$ is said to be co-invariant subspace if $T^*(\mathcal{S}) \subseteq \mathcal{S}$.

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- $H^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{D})$: the space of all operator valued bounded analytic functions on $\mathbb{D}$. A multiplier $\Theta \in H^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{D})$ is said to be inner if M_Θ is an isometry on $H^2_{\mathcal{E}}(\mathbb{D})$, where

$$(M_\Theta f)(w) = \Theta(w) f(w) \qquad (f \in H^2_{\mathcal{E}}(\mathbb{D}), w \in \mathbb{D}).$$

Model Space in $H^2(\mathbb{D})$

- *(Beurling Theorem (1948))* Let $\mathcal{S}$ be a non-zero shift invariant subspace of $H^2(\mathbb{D})$. Then $\mathcal{S} = \theta H^2(\mathbb{D})$ for some inner function $\theta \in H^\infty(\mathbb{D})$.

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- For an inner function θ, the model space K_θ is defined as $K_\theta = H^2(\mathbb{D}) \ominus \theta H^2(\mathbb{D})$. $\mathcal{K}_\theta$ is finite dimensional if θ is finite Blaschke product (that is, $\theta(z) = \prod_{k=1}^{n} \frac{z - z_k}{1 - \bar{z}_k z}$).

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  Let
  $$S_\theta = P_{\mathcal{K}_\theta} M_z|_{\mathcal{K}_\theta},$$
  where $P_{\mathcal{K}_\theta}$ denotes the orthogonal projection from $H^2(\mathbb{D})$ onto $\mathcal{K}_\theta$. S_θ is called a Jordan block.

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Theorem (Chalendar and Ross (2016))

Let $T \in \mathcal{B}(\mathcal{K}_\theta)$. Then
(i) $S_\theta^* T S_\theta = T$ if and only if $T = 0$
(ii)$\{S_\theta^{*m} T S_\theta^m\}_{m \geq 1}$ converges in norm if and only if T is compact.

Model Operator and Model Space

- Let $\mathcal{E}$ be a Hilbert space and $\Theta \in H^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{D})$ be an inner multiplier. Then the *model operator* S_Θ (see Garcia et al. (2016)) corresponding to Θ is the compression of M_z on the *model space* $\mathcal{K}_\Theta := H^2_{\mathcal{E}}(\mathbb{D}) \ominus \Theta H^2_{\mathcal{E}}(\mathbb{D})$, that is,

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  where $P_{\mathcal{K}_\Theta}$ denotes the orthogonal projection from $H^2_{\mathcal{E}}(\mathbb{D})$ onto $\mathcal{K}_\Theta$.
- Note that $\mathcal{K}_\Theta^\perp = \Theta H^2_{\mathcal{E}}(\mathbb{D})$ is an M_z-invariant subspace of $H^2_{\mathcal{E}}(\mathbb{D})$ and $S^*_\Theta = M^*_z|_{\mathcal{K}_\Theta} \in \mathcal{B}(\mathcal{K}_\Theta)$.

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Questions

- Characterize those $T \in \mathcal{B}(\mathcal{K}_\Theta)$ for which

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$$S_\Theta = P_{\mathcal{K}_\Theta} M_z|_{\mathcal{K}_\Theta},$$
where $P_{\mathcal{K}_\Theta}$ denotes the orthogonal projection from $H^2_{\mathcal{E}}(\mathbb{D})$ onto $\mathcal{K}_\Theta$.
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$$S_\Theta^* T S_\Theta = T.$$
- Characterize those $T \in \mathcal{B}(\mathcal{K}_\Theta)$ for which
$$S_\Theta^{*m} T S_\Theta^m \to A,$$
in norm, for some $A \in \mathcal{B}(\mathcal{K}_\Theta)$.

Results on Vector-valued Hardy space

Lemma 1(Böttcher and Silbermann)

Let A be a compact operator on a Hilbert space $\mathcal{H}$ and $R^{*m} \to 0$ in strong operator topology as $m \to \infty$, then $R^{*m}A \to 0$ in norm as $m \to \infty$.

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Theorem 2

Let $\mathcal{E}$ be a Hilbert space and $T \in \mathcal{B}(H^2_{\mathcal{E}}(\mathbb{D}))$. Then T is a Toeplitz operator if and only if $M_z^* T M_z = T$.

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Theorem 3

Let $T, A \in \mathcal{B}(H^2_{\mathbb{C}^p}(\mathbb{D}))$ and $M_z^{*m} T M_z^m \to A$ in norm. Then A is a Toeplitz operator and $(T - A)$ is compact. Conversely, if A is a Toeplitz operator and $T - A$ is a compact operator, then T is asymptotically Toeplitz.

Results on Vector-valued Hardy space

Proposition 4

Let $\Theta \in H^{\infty}_{\mathcal{B}(\mathcal{E})}(\mathbb{D})$ be an inner multiplier and $T \in \mathcal{B}(\mathcal{K}_{\Theta})$. Assume that $\Theta(e^{i\theta})$ is invertible a.e. Then $S^*_{\Theta} T S_{\Theta} = T$ if and only if $T = 0$.

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Let $\Theta \in H^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{D})$ be an inner multiplier and $T \in \mathcal{B}(\mathcal{K}_\Theta)$. Assume that $\Theta(e^{i\theta})$ is invertible a.e. Then $S_\Theta^* T S_\Theta = T$ if and only if $T = 0$.

Theorem 5

Let $\Theta \in H^\infty_{\mathcal{B}(\mathbb{C}^p)}(\mathbb{D})$ be an inner multiplier and $T \in \mathcal{B}(\mathcal{K}_\Theta)$. Assume that $\Theta(e^{i\theta})$ is invertible a.e. Then T is compact if and only if $\{S_\Theta^{*m} T S_\Theta^m\}_{m \geq 1}$ converges in norm.

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Theorem 6

Let $\Theta \in H^\infty_{\mathcal{B}(\mathbb{C}^p)}(\mathbb{D})$ be an inner multiplier and $\Theta(e^{i\theta})$ is invertible a.e. and $T \in \mathcal{B}(\mathcal{K}_\Theta)$. Then TFAE:

(i) $\{S_\Theta^{*m} T S_\Theta^m\}_{m\geq 1}$ converges in norm;

(ii) $S_\Theta^{*m} T S_\Theta^m \to 0$ in norm;

(iii) T is a compact operator.

Results on $H^2(\mathbb{D}^n)$

Theorem 7

Let $T \in \mathcal{B}\left(H^2(\mathbb{D}^n)\right)$. Then T is a Toeplitz operator if and only if $M_{z_j}^* T M_{z_j} = T$ for all $j = 1, \dots n$.

Theorem 7

Let $T \in \mathcal{B}\left(H^2(\mathbb{D}^n)\right)$. Then T is a Toeplitz operator if and only if $M_{z_j}^* T M_{z_j} = T$ for all $j = 1, \ldots n$.

Proof.

Let $\varphi \in L^\infty(\mathbb{T}^n)$ and $T = P_{H^2(\mathbb{D}^n)} M_\varphi|_{H^2(\mathbb{D}^n)}$. Then for $f, g \in H^2(\mathbb{D}^n)$ and $j = 1, \ldots n$, we have

$$\langle (M_{z_j}^* T M_{z_j}) f, g\rangle_{H^2(\mathbb{D}^n)} = \langle \varphi e^{i\theta_j} f, e^{i\theta_j} g\rangle_{L^2(\mathbb{T}^n)} = \langle \varphi f, g\rangle_{L^2(\mathbb{T}^n)},$$

that is,

$$\langle (M_{z_j}^* T M_{z_j}) f, g\rangle_{H^2(\mathbb{D}^n)} = \langle P_{H^2(\mathbb{D}^n)} M_\varphi|_{H^2(\mathbb{D}^n)} f, g\rangle_{H^2(\mathbb{D}^n)},$$

and therefore $M_{z_j}^* T M_{z_j} = T$ for all $j = 1, \ldots n$. $\square$

Proof Cont.

Conversely, for each $k \in \mathbb{N}$, define $k_d \in \mathbb{N}^n$ by $k_d = (k, \ldots, k)$. From $M_{z_j}^* T M_{z_j} = T$, $j = 1, \ldots n$, we obtain that

$$M_z^{*k_d} T M_z^{k_d} = T \qquad (k \in \mathbb{N}).$$

Setting

$$A_k = M_{e^{i\theta}}^{*k_d} T P_{H^2(\mathbb{D}^n)} M_{e^{i\theta}}^{k_d} \qquad (k \geq 1),$$

we can prove that

$$\lim_{k \to \infty} \langle A_k f, g \rangle = \langle A_\infty f, g \rangle \qquad (f, g \in L^2(\mathbb{T}^n))$$

and $A_\infty M_{e^{i\theta_j}} = M_{e^{i\theta_j}} A_\infty$ for $j = 1, \ldots n$. Hence there exists φ in $L^\infty(\mathbb{T}^n)$ such that $A_\infty = M_\varphi$. Using the above condition, we also have $T = P_{H^2(\mathbb{D}^n)} A_\infty|_{H^2(\mathbb{D}^n)} = P_{H^2(\mathbb{D}^n)} M_\varphi|_{H^2(\mathbb{D}^n)}$, that is, T is a Toeplitz operator.

Results on $H^2(\mathbb{D}^n)$

Theorem 8

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Following Feintuch (1989) (and Barría and Halmos (1982)) one can now define asymptotic Toeplitz operator as follows:

Definition 9

A bounded linear operator T on $H^2(\mathbb{D}^n)$ is said to be asymptotic Toeplitz operator if there exists $A \in \mathcal{B}(H^2(\mathbb{D}^n))$ such that $M_{z_i}^{*m} T M_{z_i}^m \to A$ and $M_{z_i}^{*m}(T - A) M_{z_j}^m \to 0$ as $m \to \infty$ in norm, $1 \leq i, j \leq n$.

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Theorem 10

Let T be a bounded linear operator on $H^2(\mathbb{D}^n)$. Then T is an asymptotic Toeplitz operator if and only if T is a compact perturbation of Toeplitz operator.

Let $\mathcal{Q}$ be a joint $(M_{z_1}^*, \ldots, M_{z_n}^*)$-invariant subspace of $H^2(\mathbb{D}^n)$ and

$$C_{z_i} = P_{\mathcal{Q}} M_{z_i}|_{\mathcal{Q}}, \quad i = 1, \ldots, n.$$

Quotient spaces of $H^2(\mathbb{D}^n)$

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Theorem 11

Let $T, A \in \mathcal{B}(\mathcal{Q})$, $C_{z_i}^{*m} T C_{z_i}^m \to A$ and $C_{z_i}^{*m}(T - A) C_{z_j}^m \to 0$ in norm for all $i, j = 1, \ldots, n$. Then $T = A + K$, where $K \in \mathcal{B}(\mathcal{Q})$ is a compact operator and $C_{z_i}^* A C_{z_i} = A$ for all $i = 1, \ldots n$.

Quotient spaces of $H^2(\mathbb{D}^n)$

Proposition 12

Let $\Theta \in H^\infty(\mathbb{D}^n)$ be an inner function and $\mathcal{Q} = H^2(\mathbb{D}^n)/\Theta H^2(\mathbb{D}^n)$ and $A \in \mathcal{B}(\mathcal{Q})$. Then $C_{z_i}^* A C_{z_i} = A$ for all $i = 1, \ldots n$, if and only if $A = 0$.

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Proposition 12

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Summing up the above two results, we have the following generalization of Chalendar and Ross.

Theorem 13

Let $\Theta \in H^\infty(\mathbb{D}^n)$ be an inner function, and T and A be bounded linear operators on $\mathcal{Q} = H^2(\mathbb{D}^n)/\Theta H^2(\mathbb{D}^n)$. Then TFAE:
(i) $C_{z_i}^{*m} T C_{z_i}^m \to A$ and $C_{z_i}^{*m}(T - A) C_{z_j}^m \to 0$ in norm for all $i, j = 1, \dots, n$;
(ii) $C_{z_i}^{*m} T C_{z_i}^m \to 0$ in norm for all $i = 1, \dots, n$;
(iii) T is compact.

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Thank You