Fixed points of completely contractive maps and convolution operators

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Convolution operators I

G – locally compact group
Prob(G) – probability measures on G
M(G) – bounded (Radon) measures on G

Given $\mu \in \mathrm{M}(G)$ define the convolution operator

$$(L_\mu f)(g) = \int_G f(hg) d\mu(h), \quad g \in G.$$

Such operators appear very naturally in the study of random walks on G, harmonic analysis on G, etc.. Our aim is to understand their fixed point spaces – spaces of μ-harmonic elements.

Convolution operators II

Recall

$$(L_\mu f)(g) = \int_G f(hg)d\mu(h), \quad g \in G.$$

Then $L_\mu : \mathrm{L}^\infty(G) \to \mathrm{L}^\infty(G)$; for $\mu \in \mathrm{Prob}(G)$, L_μ is unital, (completely) positive.

Consider $\Delta : \mathrm{L}^\infty(G) \to \mathrm{L}^\infty(G \times G) \approx \mathrm{L}^\infty(G)\overline{\otimes}\mathrm{L}^\infty(G)$

$$\Delta(f)(g,h) = f(gh), \quad g,h \in G$$

Then

$$L_\mu = (\mu \otimes \mathrm{id}) \circ \Delta.$$

Quantum groups

$\mathbb{G}$ – a locally compact quantum group – virtual object, described via von Neumann algebra $\mathsf{L}^\infty(\mathbb{G})$ and the coproduct $\Delta : \mathsf{L}^\infty(\mathbb{G}) \to \mathsf{L}^\infty(\mathbb{G}) \overline{\otimes} \mathsf{L}^\infty(\mathbb{G})$.

We can associate with $\mathbb{G}$ also C^*-algebras $\mathsf{C}_0(\mathbb{G})$ and $\mathsf{C}_0^u(\mathbb{G})$ (the universal object)

$$\mathsf{C}_0(\mathbb{G}) \subset \mathsf{L}^\infty(\mathbb{G}),$$

the space $\mathsf{M}(\mathbb{G}) = \mathsf{C}_0(\mathbb{G})^*$ and further convolution operators

$$L_\mu = (\mu \otimes \mathrm{id}) \circ \Delta$$

$\mathrm{Prob}(\mathbb{G})$ – states on $\mathsf{C}_0(\mathbb{G})$, $L^1(\mathbb{G}) = \mathsf{L}^\infty(\mathbb{G})_*$, $\mathsf{M}^u(\mathbb{G}) = \mathsf{C}_0^u(\mathbb{G})^*$.

Maps L_μ can be also defined for $\mu \in \mathsf{M}_u(\mathbb{G}) = \mathsf{C}_0^u(\mathbb{G})^*$; these are completely bounded normal maps whose preduals are $L^1(\mathbb{G})$-module maps (abstract Herz-Schur multipliers).

Convolution operators for group duals

Γ – locally compact group, $\mathbb{G} := \hat{\Gamma}$ – its quantum dual, so that $\mathrm{L}^\infty(\mathbb{G}) = \mathrm{VN}(\Gamma)$

$L^1(\mathbb{G}) = A(\Gamma)$ – Fourier algebra (a Banach algebra of functions on Γ, convolutions of pairs of $L^2(\Gamma)$ functions)

$\mathrm{M}_u(\mathbb{G}) = B(\Gamma)$ – Fourier-Stieltjes algebra (a Banach algebra of functions on Γ – span of positive definite functions)

Herz-Schur/Fourier multipliers: functions f on Γ such that the map

$$\varphi \mapsto f\varphi, \quad \varphi \in A(\Gamma)$$

is well-defined.
We then have $A(\Gamma) \subset B(\Gamma) \subset M_{cb}A(\Gamma)$.

Conditional expectations

M – von Neumann algebra

$E : \mathrm{M} \to \mathrm{M}$ – a ucp normal idempotent ($E \circ E = E$) and $\mathrm{Fix}\,(E) = E(\mathrm{M})$ a subalgebra of M

Then E is a conditional expectation ($E(\mathrm{M})$-bimodule map, etc.).

$P : \mathrm{M} \to \mathrm{M}$ – a normal ucp idempotent ($P \circ P = P$), and $\mathrm{Fix}\,(P) = P(\mathrm{M})$. This is only an *operator subsystem* of M, need not be an algebra!

General ucp maps

$\Phi : \mathrm{M} \to \mathrm{M}$ – a normal ucp map. Then $\mathrm{Fix}\,(\Phi)$ – an *operator subsystem* of M. Of course $\mathrm{Fix}\,(\Phi) \subset \Phi(\mathrm{M})$

Theorem (Choi-Effros)

The operator system $\mathrm{Fix}\,(\Phi)$ has a (unique) structure of an (abstract!) von Neumann algebra, with the weak*-closed operator system structure inherited from M. It will be denoted H_Φ and called the Poisson boundary of Φ.

The product is constructed via an idempotent (non-normal!) projection $P_{\Phi,\mathcal{U}} : \mathrm{M} \to \mathrm{Fix}\,(\mathrm{M})$:

$$P_{\Phi,\mathcal{U}} = \lim_{\mathcal{U}} \frac{1}{N} \sum_{k=1}^{N} \Phi^k$$

where $\mathcal{U}$ is a fixed free ultrafilter, and convergence is point-weak*. Indeed,

$$x \circ y := P_{\Phi,\mathcal{U}}(xy), \quad x, y \in \mathrm{Fix}\,(\mathrm{M}).$$

In fact Arveson showed later that $x \circ y = \lim_{N\to\infty} \Phi^N(xy)$.

Positive convolution operators revisited

$\mu \in \mathrm{Prob}(G)$, $L_\mu : \mathrm{L}^\infty(G) \to \mathrm{L}^\infty(G)$

In fact L_μ always has a natural extension to a ucp map $\Phi_\mu : B(\mathrm{L}^2(G)) \to B(\mathrm{L}^2(G))$ – the unique map such that

$$\tilde{\Delta} \circ \Phi_\mu = (\mathrm{id} \otimes L_\mu) \circ \tilde{\Delta}$$

(where $\tilde{\Delta} : B(\mathrm{L}^2(G)) \to B(\mathrm{L}^2(G)) \overline{\otimes} B(\mathrm{L}^2(G))$ is a natural extension of the coproduct).

Questions to be asked:

1. when is L_μ an idempotent or conditional expectation? [answered in the earlier work]
2. what is the structure of H_{Φ_μ} or H_{L_μ}?
3. what changes when we pass to quantum groups?

Poisson boundaries and crossed product identification

Classically $H_{L_\mu} \approx \mathrm{L}^\infty(Y)$ for some measure space (equipped with a distinguished measure ν) – this is the famous classical Poisson boundary of Furstenberg, Kaimanovich, Vershik, etc..

We always get a natural $\mathbb{G}$-action α on H_{L_μ}. Moreover

Theorem (Kalantar, Neufang and Ruan)

$\mathbb{G}$ – locally compact quantum group, $\mu \in \mathrm{Prob}(\mathbb{G})$. Then

$$H_{\Phi_\mu} = \mathbb{G} \ltimes_\alpha H_{L_\mu}.$$

General convolution operators

Convolution operator L_μ can be defined for any $\mu \in \mathrm{M}(G)$. But then even full characterisation of idempotent measures ($\mu \star \mu = \mu$) is unknown!

We get much more information when $\mu \in \mathrm{M}(G)_1$ – then $L_\mu : \mathrm{L}^\infty(G) \to \mathrm{L}^\infty(G)$ is (completely) contractive!

We will see it leads us to considering ternary rings of operators.

W^*-TROs

Definition

A (concrete) W^*-TRO – ternary ring of operators – is a weak*-closed subspace $\mathrm{X} \subset B(\mathrm{H};\mathrm{K})$ such that

$$xy^*z \in \mathrm{X}, \quad x, y, z \in \mathrm{X}.$$

W^*-TROs can also be defined abstractly – they are always equipped with a triple product

$$\{\cdot,\cdot,\cdot\} : \mathrm{X} \times \mathrm{X} \times \mathrm{X} \mapsto \mathrm{X}$$

W^*-TROs vs von Neumann algebras

Concrete W^*-TROs arise as corners of von Neumann algebras: given $\mathsf{X} \subset B(\mathsf{H};\mathsf{K})$ we define the von Neumann algebra

$$\mathsf{R}_\mathsf{X} := \begin{pmatrix} \langle \mathsf{X}\mathsf{X}^* \rangle'' & \mathsf{X} \\ \mathsf{X}^* & \langle \mathsf{X}^*\mathsf{X} \rangle'' \end{pmatrix} \subset B(\mathsf{K} \oplus \mathsf{H}).$$

and call R_X the linking von Neumann algebra of X.

X is nondegenerately represented if $\langle \mathsf{X}\mathsf{H} \rangle = \mathsf{K}$, $\langle \mathsf{X}^*\mathsf{K} \rangle = \mathsf{H}$ (equivalently, if $I_{\mathsf{K}\oplus\mathsf{H}} \in \mathsf{R}_\mathsf{X}$).

A linear map $\alpha : \mathsf{X} \to \mathsf{Y}$ is a TRO morphism if it preserves the ternary product; it is said to be nondegenerate if the linear spans of $\alpha(\mathsf{X})\mathsf{Y}^*\mathsf{Y}$ and $\alpha(\mathsf{X})^*\mathsf{Y}\mathsf{Y}^*$ are weak*-dense respectively in Y and Y^*.

Hamana extensions...

Proposition

Let X and Y be W*-TROs and let $\alpha : \mathsf{X} \to \mathsf{Y}$ be a normal TRO morphism. Then there exists a unique normal *-homomorphism $\beta : \mathsf{R}_\mathsf{X} \to \mathsf{R}_\mathsf{Y}$ such that

$$\beta \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \alpha(x) \\ 0 & 0 \end{pmatrix}, \qquad x \in \mathsf{X}.$$

Moreover

1. (i) α is injective if and only if β is injective;
2. (ii) α is non-degenerate if and only if β is unital.

A C^*-version of this result is due to Hamana.

▸ Fixed points

...and group actions

Definition

G – locally compact group, X – W*-TRO. Aut(X) – set of all normal automorphisms of X (normal bijective TRO morphisms from X onto itself). A (continuous) *action* of G on X is a homomorphism $\alpha : G \to \mathrm{Aut}(X)$ such that for each $x \in X$ the map $\alpha^x : G \to X$ defined by

$$\alpha^x(s) = (\alpha(s))(x), \qquad s \in G,$$

is weak*-continuous.

Theorem

Let α be an action of a locally compact group G on a W*-TRO X. Then it possesses a unique extension to an action of G on R_X.

Group actions II

Theorem

There is a 1-1 correspondence between actions of G on X and injective, normal, non-degenerate TRO morphisms $\alpha : \mathsf{X} \to L^\infty(G)\overline{\otimes}\mathsf{X}$ such that

$$(\Delta \otimes \mathrm{id}_{\mathsf{X}}) \circ \alpha = (\mathrm{id}_{L^\infty(G)} \otimes \alpha) \circ \alpha.$$

This allows us to define the actions of quantum groups on W^*-TROs.

Definition

Let $\alpha : \mathsf{X} \to L^\infty(\mathbb{G})\overline{\otimes}\mathsf{X}$ be an action of a locally compact quantum group $\mathbb{G}$ on a W^*-TRO $\mathsf{X} \subset B(\mathsf{H};\mathsf{K})$. The space

$$\mathrm{w}^*\!-\mathrm{cl}\ \mathrm{Lin}\{(\mathrm{L}^\infty(\widehat{\mathbb{G}}) \otimes I_{\mathsf{K}})\alpha(\mathsf{X})\}$$

is a W^*-TRO in $B(\mathrm{L}^2(\mathbb{G}) \otimes \mathsf{H}; \mathrm{L}^2(\mathbb{G}) \otimes \mathsf{K})$, called the crossed product of X by α and denoted $\mathbb{G} \ltimes_\alpha \mathsf{X}$.

Group actions III

For classical group G we have $\mathrm{L}^\infty(\widehat{\mathbb{G}}) = \mathrm{VN}(G)$. TRO crossed products are corners in the von Neumann crossed products.

Proposition

Let $\alpha : \mathsf{X} \to L^\infty(\mathbb{G})\overline{\otimes}\mathsf{X}$ be an action of a locally compact quantum group $\mathbb{G}$ on a W^*-TRO X and let $\beta : \mathsf{R}_\mathsf{X} \to L^\infty(\mathbb{G})\overline{\otimes}\mathsf{R}_\mathsf{X}$ be its Hamana extension. Then β is an action of $\mathbb{G}$ on the von Neumann algebra R_X and

$$\mathsf{R}_{\mathbb{G}\ltimes_\alpha \mathsf{X}} = \mathbb{G} \ltimes_\beta \mathsf{R}_\mathsf{X}.$$

Completely contractive maps

$\Phi : \mathrm{M} \to \mathrm{M}$ – a normal completely contractive map, and $\mathrm{Fix}\,(\Phi) \subset \Phi(\mathrm{M})$ – a subspace of M.

Theorem

The subspace $\mathrm{Fix}\,(\Phi)$ has a (unique) structure of an (abstract!) W^*-TRO, with the weak* topology inherited from M. It will be denoted H_Φ and called the (generalised) Poisson boundary of Φ.

The product is constructed via an idempotent (non-normal!) completely contractive projection $P_{\Phi,\mathcal{U}} : \mathrm{M} \to \mathrm{Fix}\,(\mathrm{M})$:

$$P_{\Phi,\mathcal{U}} = \lim_{\mathcal{U}} \frac{1}{N} \sum_{k=1}^{N} \Phi^k$$

where $\mathcal{U}$ is a fixed ultrafilter, and convergence is point-weak*. Indeed,

$$\{x, y, z\} := P_{\Phi,\mathcal{U}}(xy^*z), \quad x, y, z \in \mathrm{Fix}\,(\mathrm{M}).$$

The proof uses ideas of Choi and Effros, properties of completely contractive maps due to Youngson and results on abstract/concrete W^*-TROs due to Zettl.

Contractive convolution operators

$\omega \in \mathrm{M}(G)_1$, $L_\omega : \mathrm{L}^\infty(G) \to \mathrm{L}^\infty(G)$

In fact L_ω always has a natural extension to a completely contractive map $\Phi_\omega : B(\mathrm{L}^2(G)) \to B(\mathrm{L}^2(G))$

Questions to be asked:

1. when is L_ω an idempotent?
2. what is the structure of H_{Φ_ω} or H_{L_ω}?
3. what changes when we pass to quantum groups?

Contractive idempotent measures

$\omega \in \mathrm{M}(G)_1$, $L_\omega : \mathrm{L}^\infty(G) \to \mathrm{L}^\infty(G)$

Theorem (Greenleaf)

L_ω is an idempotent iff μ is an idempotent ($\omega \star \omega = \omega$) iff $\omega = h_H(\gamma \cdot)$, where h_H is the Haar measure on a compact subgroup H of G and γ is a character on H. Moreover $L_\omega(\mathrm{L}^\infty(G))$ is a subTRO of $\mathrm{L}^\infty(G)$.

For quantum groups the situation is more complicated: but if $\omega \in \mathrm{M}(\mathbb{G})_1$ is an idempotent functional, then its left/right absolute values are idempotent states and if these 'behave well', we get a full quantum version of the above result:

M. Neufang, P. Salmi, A. Skalski and N. Spronk, Contractive idempotents on locally compact quantum groups, *Indiana Univ. Math. J.* (2013)

Poisson boundaries and crossed product identification

$\omega \in \mathrm{M}(G)_1$, $L_\omega : \mathrm{L}^\infty(G) \to \mathrm{L}^\infty(G)$

We still get a natural G-action α on the W^*-TRO H_{L_ω}. Moreover

Theorem

$$H_{\Phi_\omega} = G \ltimes_\alpha H_{L_\omega}.$$

An analogous result holds for duals of classical groups (via the work of Anoussis, Katavolos and Todorov).

We do not know if it holds for all quantum groups!

A reformulation

The last theorem can be reformulated as follows.

Theorem

If $\omega \in \mathrm{M}(G)_1$, $L_\omega : \mathrm{L}^\infty(G) \to \mathrm{L}^\infty(G)$, $\Phi_\omega : B(\mathrm{L}^2(G)) \to B(\mathrm{L}^2(G))$ then

$$\mathrm{Fix}\,\Phi_\omega = \overline{\mathrm{Lin}}^{w*}\{\mathrm{Fix}\,L_\omega \cup \mathrm{VN}(G)\}$$

(after us shown independently by Anoussis, Katavolos and Todorov to hold for any $\omega \in M(G)$ if G is compact).

Theorem (Anoussis+Katavolos+Todorov, Salmi+AS)

Let $\omega \in B(\Gamma)_1$, $L_\omega : \mathrm{VN}(\Gamma) \to \mathrm{VN}(\Gamma)$, $\Phi_\omega : B(\mathrm{L}^2(\Gamma)) \to B(\mathrm{L}^2(\Gamma))$. Then $\mathrm{Fix}\,\Phi_\omega$, $\mathrm{Fix}\,L_\omega$ are W^*-TROs and

$$\mathrm{Fix}\,\Phi_\omega = \overline{\mathrm{Lin}}^{w*}\{\mathrm{Fix}\,L_\omega \cup \mathrm{L}^\infty(\Gamma)\}$$

Summary and comments

- study of fixed points of completely contractive maps leads naturally to W^*-TRO structures
- often one can borrow corresponding results from the von Neumann algebra theory, but this is not always straightforward!
- many (but again, not all!) results on fixed points of convolution operators pass to the C^*-setting
- the next natural step is the study of the more specific structure of the space of harmonic elements...
- and convergence of iterates.

Pekka Salmi and AS, Actions of locally compact (quantum) groups on ternary rings of operators, their crossed products and generalized Poisson boundaries, *Kyoto Journal of Mathematics*, to appear.