ON REDUCING SUB-MODULES OF HILBERT MODULES WITH $\mathfrak{S}_n$-INVARIANT KERNELS

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Abstract. Fix a bounded domain $\Omega \subseteq \mathbb{C}^n$ and a positive definite kernel K on Ω, both invariant under $\mathfrak{S}_n$, the permutation group on n symbols. Let $\mathcal{H} \subseteq \mathrm{Hol}(\Omega)$ be the Hilbert module determined by K. We show that $\mathcal{H}$ splits into orthogonal direct sum of subspaces $\mathbb{P}_{\boldsymbol{p}}\mathcal{H}$ indexed by the partitions $\boldsymbol{p}$ of n. We prove that each sub-module $\mathbb{P}_{\boldsymbol{p}}\mathcal{H}$ is a *locally free Hilbert module* of *rank* equal to square of the dimension $\chi_{\boldsymbol{p}}(1)$ of the irreducible representation corresponding to $\boldsymbol{p}$. Given two partitions $\boldsymbol{p}$ and $\boldsymbol{q}$, we show that if $\chi_{\boldsymbol{p}}(1) \neq \chi_{\boldsymbol{q}}(1)$, then the sub-modules $\mathbb{P}_{\boldsymbol{p}}\mathcal{H}$ and $\mathbb{P}_{\boldsymbol{q}}\mathcal{H}$ are not unitarily equivalent. We prove that the Taylor joint spectrum of the n-tuple of multiplication operators by elementary symmetric polynomials on $\mathbb{P}_{\boldsymbol{p}}\mathcal{H}$ is $\mathrm{clos}\big(\boldsymbol{s}(\Omega)\big)$, where $\boldsymbol{s} : \mathbb{C}^n \to \mathbb{C}^n$ is the symmetrization map. It is then shown that this commuting tuple of operators defines a contractive homomorphism of the ring of symmetric polynomials $\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}$ in n variables, equipped with the sup norm on $\mathrm{clos}\big(\boldsymbol{s}(\Omega)\big)$. We construct a large class of examples of $\mathfrak{S}_n$-invariant kernels. Moreover, restricting to the case of weighted Bergman modules $\mathbb{A}^{(\lambda)}(\mathbb{D}^n)$, we prove that the sub-modules $\mathbb{P}_{(n)}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$ and $\mathbb{P}_{(1,\ldots,1)}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$ corresponding to the trivial and the sign representations, are not unitarily equivalent. This shows that for $n = 2, 3$, the sub-modules of weighted Bergman module in this decomposition are not unitarily equivalent.

1. Introduction

Let $\Omega \subseteq \mathbb{C}^n$, be a bounded domain, K be a positive definite kernel on Ω, holomorphic in the first variable and anti-holomorphic in the second. It determines a Hilbert space $\mathcal{H}$ of holomorphic functions defined on Ω. The natural action of the permutation group $\mathfrak{S}_n$ on $\mathbb{C}^n$ is given by the formula:

$$(\sigma, \boldsymbol{z}) \mapsto \sigma \cdot \boldsymbol{z} := (z_{\sigma^{-1}(1)}, \ldots, z_{\sigma^{-1}(n)}),\ (\sigma, \boldsymbol{z}) \in \mathfrak{S}_n \times \mathbb{C}^n.$$

Throughout this paper the domain Ω is assumed to be invariant under $\mathfrak{S}_n$. We say a positive definite kernel K on an $\mathfrak{S}_n$-invariant domain Ω is $\mathfrak{S}_n$-invariant if $K(\sigma\cdot\boldsymbol{z}, \sigma\cdot\boldsymbol{w}) = K(\boldsymbol{z}, \boldsymbol{w})$ for $\sigma \in \mathfrak{S}_n$ and $\boldsymbol{z}, \boldsymbol{w} \in \Omega$. We assume that the kernel K is $\mathfrak{S}_n$-invariant as well. Let $\mathcal{H} \subseteq \mathrm{Hol}(\Omega)$ be a Hilbert module over the polynomial ring $\mathbb{C}[\boldsymbol{z}]$, determined by K. The module action is defined by the map

$$\mathfrak{m}_p(h) = p \cdot h,\ p \in \mathbb{C}[\boldsymbol{z}],\ h \in \mathcal{H},$$

where $p \cdot h$ is the point-wise multiplication.

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Let $\widehat{\mathfrak{S}_n}$ denote the equivalence classes of all irreducible representations of $\mathfrak{S}_n$. It is well known that these are finite dimensional and they are in one-to-one correspondence with partitions $\boldsymbol{p}$ of n [14, Theorem 4.3]. Recall that a partition $\boldsymbol{p}$ of n is a decreasing finite sequence $\boldsymbol{p} = (p_1, \ldots, p_k)$ of non-negative integers such that $\sum_{i=1}^k p_i = n$. A partition $\boldsymbol{p}$ of n is denoted by $\boldsymbol{p} \vdash n$. Let $\boldsymbol{\pi_p}$ be a unitary representation of $\mathfrak{S}_n$ in the equivalence class of $\boldsymbol{p} \vdash n$, that is, $\boldsymbol{\pi_p}(\sigma) = \big(\big(\boldsymbol{\pi}_{\boldsymbol{p}}^{ij}(\sigma)\big)\big)_{i,j=1}^m \in \mathbb{C}^{m\times m}$, $\sigma \in \mathfrak{S}_n$, where $m = \chi_{\boldsymbol{p}}(1)$ and $\chi_{\boldsymbol{p}}(\sigma) = \operatorname{trace}\big(\boldsymbol{\pi_p}(\sigma)\big)$, $\sigma \in \mathfrak{S}_n$, is the character of the representation $\boldsymbol{\pi_p}$. These finite dimensional representations of the group $\mathfrak{S}_n$ define linear operators $\mathbb{P}_{\boldsymbol{p}}$ and $\mathbb{P}_{\boldsymbol{p}}^{ij}$ on the Hilbert space $\mathcal{H}$:

$$\begin{aligned} \mathbb{P}_{\boldsymbol{p}} f &= \frac{\chi_{\boldsymbol{p}}(1)}{n!} \sum_{\sigma \in \mathfrak{S}_n} \chi_{\boldsymbol{p}}(\sigma^{-1})(f \circ \sigma^{-1}),\ f \in \mathcal{H}; \\ \mathbb{P}_{\boldsymbol{p}}^{ij} f &= \frac{\chi_{\boldsymbol{p}}(1)}{n!} \sum_{\sigma \in \mathfrak{S}_n} \boldsymbol{\pi}_{\boldsymbol{p}}^{ji}(\sigma^{-1})(f \circ \sigma^{-1}),\ f \in \mathcal{H}. \end{aligned}$$

We show that $\mathbb{P}_{\boldsymbol{p}}$ and $\mathbb{P}_{\boldsymbol{p}}^{ii}$ are non-trivial orthogonal projections for $\boldsymbol{p} \vdash n$, $1 \le i \le \chi_{\boldsymbol{p}}(1)$. The Hilbert module $\mathcal{H}$, when considered as a module over the ring of symmetric polynomials $\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}$, admits the orthogonal decomposition:

$$\mathcal{H} = \bigoplus_{\boldsymbol{p} \vdash n} \mathbb{P}_{\boldsymbol{p}} \mathcal{H} = \bigoplus_{\boldsymbol{p} \vdash n} \bigoplus_{i=1}^{\chi_{\boldsymbol{p}}(1)} \mathbb{P}_{\boldsymbol{p}}^{ii} \mathcal{H}.$$

In this paper, we discuss the following questions.

(1) If the sub-modules $\mathbb{P}_{\boldsymbol{p}}\mathcal{H}$ and $\mathbb{P}_{\boldsymbol{q}}\mathcal{H}$ over $\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}$ are inequivalent for distinct partitions $\boldsymbol{p}$ and $\boldsymbol{q}$ of n?

(2) If the reducing sub-modules $\mathbb{P}_{\boldsymbol{p}}^{ii}\mathcal{H}$ and $\mathbb{P}_{\boldsymbol{q}}^{jj}\mathcal{H}$ over $\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}$ are inequivalent whenever $(\boldsymbol{p}, i) \neq (\boldsymbol{q}, j)$, where $\boldsymbol{p}$, $\boldsymbol{q}$ are partitions of n, $1 \le i \le \chi_{\boldsymbol{p}}(1)$ and $1 \le j \le \chi_{\boldsymbol{q}}(1)$?

(3) If the reducing sub-modules $\mathbb{P}_{\boldsymbol{p}}^{ii}\mathcal{H}$, $\boldsymbol{p}$ partition of n and $1 \le i \le \chi_{\boldsymbol{p}}(1)$, are minimal?

Let $\boldsymbol{s} : \mathbb{C}^n \to \mathbb{C}^n$ be the symmetrization map $\boldsymbol{s} = (s_1, \ldots, s_n)$, where $s_k(\boldsymbol{z}) = \sum_{1 \le i_1, \ldots, i_k \le n} z_{i_1} \cdots z_{i_k}$, $1 \le k \le n$. Let $\mathcal{Z}$ be the set of critical points of the proper map $\boldsymbol{s}$. For any partition $\boldsymbol{p}$ of n, we have shown, see Corollary 3.12, that the Hilbert modules $\mathbb{P}_{\boldsymbol{p}}\mathcal{H}$ are locally free of rank $\chi_{\boldsymbol{p}}(1)^2$ on an open subset of $\boldsymbol{s}(\Omega) \setminus \boldsymbol{s}(\mathcal{Z})$. Furthermore, using similar arguments, we show that the sub-modules $\mathbb{P}_{\boldsymbol{p}}^{ii}\mathcal{H}$, $1 \le i \le \chi_{\boldsymbol{p}}(1)$, are locally free of rank $\chi_{\boldsymbol{p}}(1)$. Therefore, if $\chi_{\boldsymbol{p}}(1) \neq \chi_{\boldsymbol{q}}(1)$, then the sub-modules $\mathbb{P}_{\boldsymbol{p}}^{ii}\mathcal{H}$ and $\mathbb{P}_{\boldsymbol{q}}^{jj}\mathcal{H}$ are not equivalent, see Theorem 3.14.

Since the Hilbert module $\mathbb{P}_{\boldsymbol{p}}\mathcal{H}$, as well as the sub-modules $\mathbb{P}_{\boldsymbol{p}}^{ii}\mathcal{H}$, $1 \le i \le \chi_{\boldsymbol{p}}(1)$, are locally free on the open set of $(\Omega) \setminus \boldsymbol{s}(\mathcal{Z})$, it follows that these are in one to one correspondence with holomorphic hermitian vector bundles defined on some open subset of Ω. The rank of this vector bundle is an invariant, albeit a very weak one. However, it is the rank which is used to distinguish the sub-modules $\mathbb{P}_{\boldsymbol{p}}^{ii}\mathcal{H}$ in this paper. We give an explicit realization of a spanning holomorphic cross-section for the sub-modules $\mathbb{P}_{\boldsymbol{p}}^{ii}\mathcal{H}$. The existence of spanning holomorphic cross section for holomorphic hermitian vector bundles was established in [13], [29]. However, it is not clear how to construct these even in the simplest of examples of vector bundles of rank > 1, for instance, when the vector bundle is the direct sum of two line bundles. For all the sub-modules of the Hilbert module $\mathcal{H}$, we study in this paper, an algorithm for constructing the spanning sections is given. Also, this provides an invariant that we believe will be useful in our future work.

For any partition $\boldsymbol{p}$ of n, following arguments similar to the ones given in [4], we further show that the commuting n-tuple of multiplications $M_{\boldsymbol{s}}^{(\boldsymbol{p})} = (M_{s_1}, \ldots, M_{s_n})$ by the elementary symmetric functions $\boldsymbol{s}$ defined on the Hilbert space $\mathbb{P}_{\boldsymbol{p}}\mathcal{H}$, admit $\operatorname{clos}(\boldsymbol{s}(\Omega))$ as a spectral set. Here, we assume that Ω is polynomially convex and $\|\mathfrak{m}_p\| \leq \|p\|_{\infty,\operatorname{clos}(\Omega)}$, that is, the compact set $\operatorname{clos}(\Omega)$ is a spectral set for the commuting tuple $(M_{z_1}, \ldots, M_{z_n})$ of multiplication by the coordinate functions on $\mathcal{H}$. Since $\mathbb{P}_{\boldsymbol{p}}\mathcal{H}$ admits a further decomposition into a direct sum of the sub-modules $\mathbb{P}_{\boldsymbol{p}}^{ii}\mathcal{H}$, $1 \leq i \leq \chi_{\boldsymbol{p}}(1)$, it follows that the n-tuple $M_{\boldsymbol{s}}^{(\boldsymbol{p})}$ acting on these reducing subspaces has the same property, which is Theorem 3.22 of this paper. What is more, we have shown that the Taylor joint spectrum of each of these n-tuples is $\operatorname{clos}(\boldsymbol{s}(\Omega)$ and thus, in these examples, the spectrum is a spectral set.

In section 3 (see Example 3.4), we exhibit a large class of $\mathfrak{S}_n$-invariant kernel. As a particular case, we note that the action of $\mathfrak{S}_n$ on Ω is bi-holomorphic, and hence it follows that the Bergman kernel B of Ω is automatically $\mathfrak{S}_n$-invariant whenever the domain Ω is $\mathfrak{S}_n$-invariant. Furthermore if $\lambda > 0$ is chosen such that the kernel B^λ, defined by polarizing $B(w,w)^\lambda$, is positive definite, then B^λ is also $\mathfrak{S}_n$-invariant. Thus, all these spaces have similar decompositions in to reducing submodules. In the last section of this paper, we discuss the important special case (motivated from appendix A) of the weighted Bergman modules $\mathbb{A}^{(\lambda)}(\mathbb{D}^n)$, $\lambda > 1$, of square integrable holomorphic functions defined on the polydisc $\mathbb{D}^n$ with respect to the measure

$$\Big(\prod_{i=1}^{n}(1-|z_i|^2)^{\lambda-2}\Big)dV(\boldsymbol{z}),\ \boldsymbol{z} \in \mathbb{D}^n.$$

(In the sequel, we also consider the case of $\lambda > 0$.)

Although we haven't been able to decide if the sub-modules $\mathbb{P}_{\boldsymbol{p}}\mathcal{H}$ and $\mathbb{P}_{\boldsymbol{q}}\mathcal{H}$ are inequivalent when $\chi_{\boldsymbol{p}}(1) = \chi_{\boldsymbol{q}}(1)$, in general, we have obtained the answer in the case of the Bergman module $\mathbb{A}^{(\lambda)}(\mathbb{D}^n)$ for the partitions $\boldsymbol{p}$ of n with $\chi_{\boldsymbol{p}}(1) = 1$. For $n \geq 2$, there are only two such partitions: $\boldsymbol{p} = (n)$ or $(1, \ldots, 1)$. We show that the two sub-modules $\mathbb{P}_{(n)}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$ and $\mathbb{P}_{(1,\ldots,1)}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$ are inequivalent (there is no intertwining module map between them that is unitary) over $\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}$, see Theorem 4.4. Also these summands are locally free of rank 1, therefore they are irreducible and hence minimal. In fact, the reducing submodules $\mathbb{P}_{\boldsymbol{p}}\mathcal{H}$ are minimal whenever $\chi_{\boldsymbol{p}}(1) = 1$ and in case $\chi_{\boldsymbol{p}}(1) > 1$, existence of non-trivial projections $\mathbb{P}_{\boldsymbol{p}}^{ii}$ shows that the submodules are not minimal reducing. Therefore, for $n = 2$, in the decomposition $\mathbb{A}^{(\lambda)}(\mathbb{D}^2) = \mathbb{P}_{(2)}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^2)\big) \oplus \mathbb{P}_{(1,1)}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^2)\big)$, the two summands are minimal and inequivalent and consequently, we have answered the questions (1) - (3). Furthermore, for $n = 3$, it follows that all the submodules in the decomposition $\oplus_{\boldsymbol{p}\vdash 3}\mathbb{P}_{\boldsymbol{p}}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^3)\big)$ are inequivalent, see Corollary 4.17. Along the way, we give an explicit formula, see Theorem 4.10, for the weighted Bergman kernel of the symmetrized polydisc $\mathbb{G}_n$ in the co-ordinates of $\mathbb{G}_n$ rather than that of the polydisc $\mathbb{D}^n$. In an earlier paper [21], the case of $n = 2$ was worked out.

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2. Locally free Hilbert modules

First, we recall several useful definitions following [12], [7] and [6].

Definition 2.1. *A Hilbert space $\mathcal{H}$ is said to be a Hilbert module over the polynomial ring $\mathbb{C}[\boldsymbol{z}]$ in n variables if the map $(p,h) \to p \cdot h$, $p \in \mathbb{C}[\boldsymbol{z}], h \in \mathcal{H}$, defines a homomorphism $p \mapsto T_p$, where T_p is bounded operator defined by $T_p h = p \cdot h$.*

Two Hilbert modules $\mathcal{H}$ and $\tilde{\mathcal{H}}$ are said to be (unitarily) equivalent if there exists a unitary module map $U : \mathcal{H} \to \tilde{\mathcal{H}}$, that is, $UT_p = \tilde{T}_p U$, $p \in \mathbb{C}[\boldsymbol{z}]$.

A Hilbert module $\mathcal{H}$ over $\mathbb{C}[\boldsymbol{z}]$ equipped with the sup norm on some compact set $X \subset \mathbb{C}^n$, is said to be bounded if

$$\|T_p\| \leq C\|p\|_{\infty,X} := C\sup\{|p(\boldsymbol{z})| : \boldsymbol{z} \in X\}$$

for some positive constant C independent of p and it is said to be contractive if C can be chosen to be 1.

In this paper, we study those *analytic Hilbert modules*, where the domain Ω and the kernel K are $\mathfrak{S}_n$-invariant.

Definition 2.2. *A Hilbert module $\mathcal{H}$ is said to be* analytic *over $\mathbb{C}[\boldsymbol{z}]$ if*

1. *$\mathcal{H}$ consists of holomorphic functions on some bounded domain $\Omega \subseteq \mathbb{C}^n$,*
2. *the module action T_p is given by pointwise multiplication, that is, $(\mathfrak{m}_p(h))(\boldsymbol{z}) = p(\boldsymbol{z})h(\boldsymbol{z})$, $\boldsymbol{z} \in \Omega$,*
3. *$\mathcal{H}$ possesses a reproducing kernel on Ω, that is, there exists a function $K : \Omega \times \Omega \to \mathbb{C}$ satisfying the reproducing property with respect to $\mathcal{H}$: $f(\boldsymbol{w}) = \langle f, K(\cdot,\boldsymbol{w})\rangle$, $f \in \mathcal{H}$, $\boldsymbol{w} \in \Omega$,*
4. *$\mathbb{C}[\boldsymbol{z}] \subseteq \mathcal{H}$ is dense in $\mathcal{H}$.*

An analytic Hilbert module $\mathcal{H} \subseteq \mathrm{Hol}(\Omega)$ is said to be contractive if $\|\mathfrak{m}_p\| \leq \|p\|_{\infty,\mathrm{clos}(\Omega)}$. If Ω is assumed to be polynomially convex, contractivity of the module is equivalent to saying that the compact set $\mathrm{clos}(\Omega)$ is a spectral set for the commuting tuple of multiplication operators $(M_{z_1}, \ldots, M_{z_n})$.

Let $\mathbb{C}_{\boldsymbol{w}}$ be the one dimensional module over the polynomial ring $\mathbb{C}[\boldsymbol{z}]$ defined by the evaluation, that is, $(p,c) \to p(\boldsymbol{w})c,\ c \in \mathbb{C}, p \in \mathbb{C}[\boldsymbol{z}]$. Following [12], we define the module tensor product of two Hilbert modules $\mathcal{H}$ and $\mathbb{C}_{\boldsymbol{w}}$ over $\mathbb{C}[\boldsymbol{z}]$ to be the quotient of the Hilbert space tensor product $\mathcal{H} \otimes \mathbb{C}$ by the subspace

$$\begin{aligned} \mathcal{N} &:= \{p \cdot f \otimes 1_{\boldsymbol{w}} - f \otimes p(\boldsymbol{w}) : p \in \mathbb{C}[\boldsymbol{z}], f \in \mathcal{H}\} \\ &= \{\big(p - p(\boldsymbol{w})\big)f : p \in \mathbb{C}[\boldsymbol{z}], f \in \mathcal{H}\}. \end{aligned}$$

Thus

$$\mathcal{H} \otimes_{\mathbb{C}[\boldsymbol{z}]} \mathbb{C}_{\boldsymbol{w}} := (\mathcal{H} \otimes \mathbb{C})/\mathcal{N},$$

where the module action is defined by the compression of the operator $\mathfrak{m}_p \otimes 1_{\boldsymbol{w}},\ p \in \mathbb{C}[\boldsymbol{z}]$, to the subspace $(\mathcal{H} \otimes \mathbb{C})/\mathcal{N}$. We recall the notion of local freeness of a Hilbert module in accordance with [6, Definition 1.4].

Definition 2.3 (Definition 1.4, [6])**.** *Let $\mathcal{H}$ be a Hilbert module over $\mathbb{C}[\boldsymbol{z}]$. Let Ω be a bounded open connected subset of $\mathbb{C}^n$. We say $\mathcal{H}$ is locally free of rank k at $\boldsymbol{w}_0$ in $\Omega^* := \{\boldsymbol{z} \in \mathbb{C}^n : \bar{\boldsymbol{z}} \in \Omega\}$ if there exists a neighbourhood Ω_0^* of $\boldsymbol{w}_0$ and holomorphic functions $\gamma_1, \gamma_2, \ldots, \gamma_k : \Omega_0^* \to \mathcal{H}$ such that the linear* *span of the set of k vectors $\{\gamma_1(\boldsymbol{w}),\ldots,\gamma_k(\boldsymbol{w})\}$ is the module tensor product $\mathcal{H}\otimes_{\mathbb{C}[\boldsymbol{z}]}\mathbb{C}_{\boldsymbol{w}}$. Following the terminology of* [6], *we say that a module $\mathcal{H}$ is locally free on Ω of rank k if it is locally free of rank k at every $\boldsymbol{w}$ in Ω^*.*

Recall that the permutation group $\mathfrak{S}_n$ acts on $\mathbb{C}^n$: σ in $\mathfrak{S}_n$, $(\sigma,\boldsymbol{z})\mapsto\sigma\cdot\boldsymbol{z}:=(z_{\sigma^{-1}(1)},\ldots,z_{\sigma^{-1}(n)})$. For convenience of notation, we sometimes let $\boldsymbol{z}_\sigma$ denote $\sigma\cdot\boldsymbol{z}$. Let $\Omega\subset\mathbb{C}^n$ be a bounded domain invariant under the action of $\mathfrak{S}_n$. Let $\mathcal{H}$ be an analytic Hilbert module on Ω with reproducing kernel K. Let $\boldsymbol{s}:\mathbb{C}^n\to\mathbb{C}^n$ be the symmetrization map $\boldsymbol{s}=(s_1,\ldots,s_n)$, where $s_k(\boldsymbol{z})=\sum_{1\leq i_1,\ldots,i_k\leq n}z_{i_1}\cdots z_{i_k}$, $1\leq k\leq n$. Let $(M_1,\ldots,M_n)$ denote the n-tuple of multiplication operators by the coordinate functions z_i, $1\leq i\leq n$ on $\mathcal{H}$. Clearly, $(M_{s_1},\ldots,M_{s_n})$ defines a commuting tuple of bounded linear operators on $\mathcal{H}$. Define $\Delta(\boldsymbol{z})=\prod_{i<j}(z_i-z_j)$, for $\boldsymbol{z}\in\mathbb{C}^n$. Note that $\Delta(\boldsymbol{z})=J_{\boldsymbol{s}}(\boldsymbol{z})$, the complex jacobian of the symmetrization map $\boldsymbol{s}$. Thus

$$\mathcal{Z}=\{\boldsymbol{z}\in\mathbb{C}^n\mid\Delta(\boldsymbol{z})=0\}=\{\boldsymbol{z}\in\mathbb{C}^n\mid z_i=z_j\text{ for some }i\neq j,1\leq i,j\leq n\}.$$

For every $\boldsymbol{u}\in\boldsymbol{s}(\Omega)\backslash\boldsymbol{s}(\mathcal{Z})$, we note that the set $\boldsymbol{s}^{-1}(\{\boldsymbol{u}\})$ has exactly $n!$ elements. If M_ϕ is a multiplication operator on $\mathcal{H}$ by a holomorphic function ϕ, then $M_\phi^*K_{\boldsymbol{w}}=\overline{\phi(\boldsymbol{w})}K_{\boldsymbol{w}}$ for $\boldsymbol{w}\in\Omega$. Therefore we have the following lemma.

Lemma 2.4. *Let $\mathcal{H}$ be an analytic Hilbert module on an $\mathfrak{S}_n$-invariant domain Ω over $\mathbb{C}[\boldsymbol{z}]$ with reproducing kernel K. For $\sigma\in\mathfrak{S}_n$, $i=1,\ldots,n$, $M_i^*K_{\boldsymbol{w}_\sigma}=\bar{w}_{\sigma^{-1}(i)}K_{\boldsymbol{w}_\sigma}$ and $M_{s_i}^*K_{\boldsymbol{w}_\sigma}=\overline{s_i(\boldsymbol{w})}K_{\boldsymbol{w}_\sigma}$.*

Let $\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}$ be the ring of invariants under the action of $\mathfrak{S}_n$ on $\mathbb{C}[\boldsymbol{z}]$, that is,

$$\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}=\{f\in\mathbb{C}[\boldsymbol{z}]:f(\sigma\cdot\boldsymbol{z})=f(\boldsymbol{z}),\sigma\in\mathfrak{S}_n\}.$$

Furthermore, $\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}=\mathbb{C}[s_1,\ldots,s_n]$, see [23, p. 39]. We now state the main Theorem of this Section.

Theorem 2.5. *If $\mathcal{H}$ is an analytic Hilbert module on an $\mathfrak{S}_n$-invariant domain Ω over $\mathbb{C}[\boldsymbol{z}]$, then $\mathcal{H}$ is a locally free analytic Hilbert module over $\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}$ of rank $n!$ on $\boldsymbol{s}(\Omega)\setminus\boldsymbol{s}(\mathcal{Z})$.*

The proof is facilitated by breaking it up into several pieces. Some of these pieces make essential use of the fact that $\mathbb{C}[\boldsymbol{z}]$ is a finitely generated free module over $\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}$ of rank $n!$ [5, Theorem 1, p. 110]. The motivation for the following lemma and some of the subsequent comments come from [8].

Lemma 2.6. *For any basis $\{p_\sigma\}_{\sigma\in\mathfrak{S}_n}$ of $\mathbb{C}[\boldsymbol{z}]$ over $\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}$, we have*

$$\det\left(\!\!\left(p_\sigma(\boldsymbol{w}_\tau)\right)\!\!\right)_{\sigma,\tau\in\mathfrak{S}_n}\not\equiv 0.$$

Proof. Let $L=\mathbb{C}(\boldsymbol{z})$ denote the field of rational functions and $K=\mathbb{C}(\boldsymbol{z})^{\mathfrak{S}_n}$ be the field of symmetric rational function. From [23, Example 2.22], it is known that L over K is a finite Galois extension with Galois group $\mathrm{Gal}(L/K)=\mathfrak{S}_n$. Let $f\in L$, that is, $f=\frac{p}{q}$ for some polynomials p and q. Pick $\tilde{q}=\prod_{\sigma\in\mathfrak{S}_n}q(\boldsymbol{z}_\sigma)$ and $\tilde{p}=p\prod_{\sigma\in\mathfrak{S}_n,\sigma\neq 1}q(\boldsymbol{z}_\sigma)$. Now, $f=\frac{\tilde{p}}{\tilde{q}}$, where $\tilde{q}$ is symmetric. Again, since $\{p_\sigma\}_{\sigma\in\mathfrak{S}_n}$ is a basis for $\mathbb{C}[\boldsymbol{z}]$ over the ring $\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}$, we have $p=\sum_{\sigma\in\mathfrak{S}_n}p_\sigma h_\sigma$ where h_σ's are symmetric polynomial which in turn shows that $f=\sum_{\sigma\in\mathfrak{S}_n}p_\sigma\frac{h_\sigma}{\tilde{q}}$. Thus $\{p_\sigma\}_{\sigma\in\mathfrak{S}_n}$ forms a basis of L over K. Now we make use of the following basic result from Galois theory [9, Lemma 3.4]:

If N/F is a finite Galois extension with $\mathrm{Gal}(N/F)=\{g_1,\ldots,g_m\}$ and $\{e_1,\ldots,e_m\}$ is a F-basis of N, then $\big(g_1(e_j),\ldots,g_m(e_j)\big)_{j=1}^m$ forms a basis of F^m/F.

Consequently, $\big((p_\sigma\circ\tau^{-1})_{\sigma\in\mathfrak{S}_n}\big)_{\tau\in\mathfrak{S}_n}$ is a basis of $L^{n!}/L$. Hence we have the desired result. ∎

Recall that the length of permutation $\sigma\in\mathfrak{S}_n$ is the number of inversions in σ [17, p. 4]. Here, by an inversion in σ, we mean a pair (i,j) with $1\le i<j\le n$ such that $\sigma(i)>\sigma(j)$. This is the smallest number of transpositions of the form $(i,i+1)$ required to write σ as a product of these transpositions.

Lemma 2.7. *Pick a basis for $\mathbb{C}[\boldsymbol{z}]$ over $\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}$ consisting of homogeneous polynomials p_σ, $\sigma\in\mathfrak{S}_n$, $\deg p_\sigma=\ell(\sigma)$. Then*

(i) *the determinant* $\det\big(\!\!\big(p_\sigma(\boldsymbol{w}_\tau)\big)\!\!\big)_{\sigma,\tau\in\mathfrak{S}_n}$ *is a homogeneous polynomial of degree* $\frac{n!}{2}\binom{n}{2}$,

(ii) $\det\big(\!\!\big(p_\sigma(\boldsymbol{w}_\tau)\big)\!\!\big)_{\sigma,\tau\in\mathfrak{S}_n}$ *is a non-zero constant multiple of* $\Delta(\boldsymbol{w})^{\frac{n!}{2}}$.

Proof. Clearly,

$$\det\big(\!\!\big(p_\sigma(\boldsymbol{w}_\tau)\big)\!\!\big)_{\sigma,\tau\in\mathfrak{S}_n}=\sum_{\nu\in\mathfrak{S}_{n!}}\prod_{\sigma\in\mathfrak{S}_n}p_\sigma(\boldsymbol{w}_{\nu\sigma}).$$

We note that

$$\deg\prod_{\sigma\in\mathfrak{S}_n}p_\sigma(\boldsymbol{w}_{\nu\sigma})=\sum_{\sigma\in\mathfrak{S}_n}\deg p_\sigma(\boldsymbol{w})=\sum_{\sigma\in\mathfrak{S}_n}\deg p_\sigma=\sum_{\sigma\in\mathfrak{S}_n}\ell(\sigma).$$

Let $I_n(k)$ denote the number of k-inversions in $\mathfrak{S}_n$ [20, p. 1]. Alternatively, $I_n(k)=\operatorname{card}\{\sigma\in\mathfrak{S}_n\mid\ell(\sigma)=k\}$. Note that

$$\sum_{\sigma\in\mathfrak{S}_n}\ell(\sigma)=\sum_{k=1}^{\binom{n}{2}}\sum_{\ell(\sigma)=k}\ell(\sigma)=\sum_{k=1}^{\binom{n}{2}}kI_n(k).$$

The generating function formula for $I_n(k)$ is given by [20, Theorem 1]

$$\sum_{k=1}^{\binom{n}{2}}I_n(k)z^k=\prod_{i=1}^{n-1}\sum_{j=0}^{i}z^j.$$

Differentiating with respect to z, we obtain

$$\sum_{k=1}^{\binom{n}{2}}kI_n(k)z^{k-1}=\sum_{i=1}^{n-1}(1+\ldots+iz^{i-1})\prod_{j=1,j\neq i}^{r-1}(1+\ldots+z^j).$$

Putting $z=1$, we have

$$\sum_{k=1}^{\binom{n}{2}}kI_n(k)=\sum_{i=1}^{n-1}\frac{i(i+1)}{2}\prod_{j=1,j\neq i}^{n-1}(j+1)=\frac{n!}{2}\sum_{i=1}^{n-1}i=\frac{n!}{2}\binom{n}{2}.$$

This proves part (i). For part (ii), let us choose i,j with $1\le i<j\le n$. Consider the automorphism of $\mathfrak{S}_n$ given by $\tau\mapsto\tau(i,j)$, where (i,j) is the transposition. This automorphism maps an even permutation to an odd permutation and vice versa. For any polynomial p, clearly, $p(\boldsymbol{z}_\tau)=\sum_{m,n}a_{mn}(\boldsymbol{z}')z_i^mz_j^n\in\mathbb{C}[\boldsymbol{z}]$, where each $a_{mn}(\boldsymbol{z}')$ is a polynomial in the variables $z_1,\ldots,z_{i-1},z_{i+1},\ldots,z_{j-1},z_{j+1},\ldots,z_n$. Thus $p(\boldsymbol{w}_\tau)-p(\boldsymbol{w}_{\tau(i,j)})=\sum_{m,n}a_{mn}(\boldsymbol{w}')(w_i^mw_j^n-w_j^mw_i^n)$ is divisible by w_i-w_j. Thus for each even permutation τ, if we subtract the $\tau(i,j)$-th column $\big(p_\sigma(\boldsymbol{w}_{\tau(i,j)})\big)_{\sigma\in\mathfrak{S}_n}$ from τ-th column $(p_\sigma(\boldsymbol{w}_\tau))_{\sigma\in\mathfrak{S}_n}$, the determinant does not change. Consequently, we see that w_i-w_j is a factor of the determinant. Since we have exactly $\frac{n!}{2}$ even permutations in $\mathfrak{S}_n$, it follows that $(w_i-w_j)^{\frac{n!}{2}}$ must divide the determinant.

This is true for every pair of $i < j$ and $\mathbb{C}[\boldsymbol{z}]$ is a unique factorization domain. Hence $\Delta(\boldsymbol{w})^{\frac{n!}{2}}$ divides the determinant. From part (i) and Lemma 2.6, we see that the degree of the polynomial $\Delta(\boldsymbol{w})^{\frac{n!}{2}}$ is equal to $\frac{n!}{2}\binom{n}{2}$ completing the proof of part (ii). ∎

Remark 2.8. *The degree of the polynomials in a basis consisting of the Descent polynomials* [1, p. 6] *or the Schubert polynomials* [17, Theorem 2.16]*, meet the hypothesis made in Lemma 2.7.*

Lemma 2.9. *Let $\mathcal{H}$ be an analytic Hilbert module on an $\mathfrak{S}_n$-invariant domain Ω over $\mathbb{C}[\boldsymbol{z}]$ with reproducing kernel K. If v is in $\cap_{i=1}^n \ker\big(M_{s_i} - s_i(\boldsymbol{w})\big)^*, \boldsymbol{w} \in \Omega \setminus \mathcal{Z}$, then there exists unique tuple $(c_\sigma)_{\sigma\in\mathfrak{S}_n}$, such that $v = \sum c_\sigma K(\cdot, \boldsymbol{w}_\sigma)$.*

Proof. Clearly, $M_{s_i}^* K(\cdot, \boldsymbol{w}_\sigma) = \overline{s_i(\boldsymbol{w}_\sigma)} K(\cdot, \boldsymbol{w}_\sigma) = \overline{s_i(\boldsymbol{w})} K(\cdot, \boldsymbol{w}_\sigma)$. To complete the proof, given a joint eigenvector v, it is enough to ensure the existence of a unique tuple $(c_\sigma)_{\sigma\in\mathfrak{S}_n}$ of complex numbers such that

$$\langle v, p\rangle = \langle \sum_{\sigma\in\mathfrak{S}_n} c_\sigma K(\cdot, \boldsymbol{w}_\sigma), p\rangle = \sum_{\sigma\in\mathfrak{S}_n} c_\sigma \overline{p(\boldsymbol{w}_\sigma)},$$

for all polynomials p since $\mathbb{C}[\boldsymbol{z}]$ is dense in the Hilbert module $\mathcal{H}$. In particular, if there exists a unique solution for some choice of a basis, say $\{p_\tau\}_{\tau\in\mathfrak{S}_n}$, of $\mathbb{C}[\boldsymbol{z}]$ over the ring $\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}$, then for any $p = \sum_{\tau\in\mathfrak{S}_n} p_\tau h_\tau \in \mathbb{C}[\boldsymbol{z}]$, we have

$$\begin{aligned}
\langle v, p\rangle &= \langle v, \sum_{\tau\in\mathfrak{S}_n} p_\tau h_\tau\rangle = \sum_{\tau\in\mathfrak{S}_n} \langle M_{h_\tau}^* v, p_\tau\rangle = \sum_{\tau\in\mathfrak{S}_n} \overline{h_\tau(\boldsymbol{w})}\langle v, p_\tau\rangle \\
&= \sum_{\tau\in\mathfrak{S}_n} \overline{h_\tau(\boldsymbol{w})} \sum_{\sigma\in\mathfrak{S}_n} c_\sigma \overline{p_\tau(\boldsymbol{w}_\sigma)} = \sum_{\sigma\in\mathfrak{S}_n} c_\sigma \sum_{\tau\in\mathfrak{S}_n} \overline{h_\tau(\boldsymbol{w}_\sigma) p_\tau(\boldsymbol{w}_\sigma)} \\
&= \sum_{\sigma\in\mathfrak{S}_n} c_\sigma \overline{p(\boldsymbol{w}_\sigma)}.
\end{aligned}$$

Thus choosing $\{p_\tau\}_{\tau\in\mathfrak{S}_n}$ as in the hypothesis of Lemma 2.7 and using part (ii) of that Lemma, we have a unique solution $(c_\sigma)_{\sigma\in\mathfrak{S}_n}$ for the system of equations

$$\langle v, p_\tau\rangle = \sum_{\sigma\in\mathfrak{S}_n} c_\sigma \overline{p_\tau(\boldsymbol{w}_\sigma)}$$

as long as $\boldsymbol{w}$ is from $\Omega \setminus \mathcal{Z}$. ∎

As a consequence of the Lemma we have just proved, we see that the set of vectors $\{K_{\boldsymbol{w}_\sigma} \mid \sigma \in \mathfrak{S}_n\}$ is both linearly independent and spanning for the joint kernel $\cap_{i=1}^n \ker\big(M_{s_i} - s_i(\boldsymbol{w})\big)^*, \boldsymbol{w} \in \Omega \setminus \mathcal{Z}$. Therefore, we have the following Corollary.

Corollary 2.10. *Let $\mathcal{H}$ be an analytic Hilbert module on an $\mathfrak{S}_n$-invariant domain Ω over $\mathbb{C}[\boldsymbol{z}]$ with reproducing kernel K. Then $\dim \cap_{i=1}^n \ker\big(M_{s_i} - s_i(\boldsymbol{w})\big)^* = n!$.*

To complete the proof of Theorem 2.5, we need to relate the joint kernel $\cap_{i=1}^n \ker\big(M_{s_i} - s_i(\boldsymbol{w})\big)^*$ to the module tensor product $\mathcal{H} \otimes_{\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}} \mathbb{C}_{\boldsymbol{w}}$. The following Lemma gives an isomorphism between these two. A special case of [12, Lemma 5.11], included in the Lemma below, is used in proving a generalization of Theorem 2.5 to submodules of Hilbert modules over $\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}$.

Recall that $f_1, \ldots, f_k \in \mathcal{H}$ is said to generate the Hilbert module $\mathcal{H}$ if $\{r_1 f_1 + \cdots + r_k f_k : r_1, \ldots, r_k \in \mathcal{R}\}$ is dense in $\mathcal{H}$. The rank of a Hilbert module $\mathcal{H}$ over a ring $\mathcal{R}$ is $\inf|\mathcal{F}|$, where $\mathcal{F} \subseteq \mathcal{H}$ is any subset with the property $\{r_1f_1+\cdots+r_kf_k : f_1,\ldots,f_k\in\mathcal{F}; r_1,\ldots,r_k\in\mathcal{R}\}$ is dense in $\mathcal{H}$ and $|\mathcal{F}|$ denotes the cardinality of $\mathcal{F}$ (cf. [7, Section 2.3]).

Lemma 2.11. *If $\mathcal{H}$ is a Hilbert module over $\mathbb{C}[\boldsymbol{z}]$ consisting of holomorphic functions defined on some bounded domain $\Omega\subseteq\mathbb{C}^n$, then we have*

1. $\mathcal{H}\otimes_{\mathbb{C}[\boldsymbol{z}]}\mathbb{C}_{\boldsymbol{w}}\cong\cap_{p\in\mathbb{C}[\boldsymbol{z}]}\ker M^*_{p-p(\boldsymbol{w})}$;
2. $\mathcal{H}\otimes_{\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}}\mathbb{C}_{\boldsymbol{w}}\cong\cap_{i=1}^n\ker\big(M_{s_i}-s_i(\boldsymbol{w})\big)^*$;
3. *For any set of generators $p_1,\ldots,p_t$ of $\mathcal{H}$ over $\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}$, the vectors*
$$p_1\otimes_{\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}}1_{\boldsymbol{w}},\ldots,p_t\otimes_{\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}}1_{\boldsymbol{w}}$$
span $\mathcal{H}\otimes_{\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}}\mathbb{C}_{\boldsymbol{w}}$.

Proof. We have to show that $\mathcal{H}\otimes_{\mathbb{C}[\boldsymbol{z}]}\mathbb{C}_{\boldsymbol{w}}=\cap_{p\in\mathbb{C}[\boldsymbol{z}]}\ker M^*_{p-p(\boldsymbol{w})}$. Recall that $\mathcal{H}\otimes_{\mathbb{C}[\boldsymbol{z}]}\mathbb{C}_{\boldsymbol{w}}$ is the ortho-complement of the subspace $\mathcal{N}=\{\big(p-p(\boldsymbol{w})\big)f : p\in\mathbb{C}[\boldsymbol{z}], f\in\mathcal{H}\}$ in $\mathcal{H}\otimes\mathbb{C}$. Therefore, we have

$$g\in\mathcal{N}^\perp \iff \langle g,\big(p-p(\boldsymbol{w})\big)f\rangle=0 \text{ for all } p\in\mathbb{C}[\boldsymbol{z}], f\in\mathcal{H} \iff M^*_{\left(p-p(\boldsymbol{w})\right)}g=0,\, p\in\mathbb{C}[\boldsymbol{z}].$$

Similarly, $\cap_{p\in\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}}\ker M^*_{p-p(\boldsymbol{w})}\subseteq\cap_{i=1}^n\ker\big(M_{s_i}-s_i(\boldsymbol{w})\big)^*$. Also, if $f\in\cap_{i=1}^n\ker\big(M_{s_i}-s_i(\boldsymbol{w})\big)^*$, then $M^*_{s_i}f=\overline{s_i(\boldsymbol{w})}f$, $1\le i\le n$. Since $p-p(\boldsymbol{w})$ is a symmetric polynomial, the existence of a polynomial q such that $p-p(\boldsymbol{w})=q\circ\boldsymbol{s}$ follows. Thus

$$M^*_{q\circ\boldsymbol{s}}f=q(M_{s_1},\ldots,M_{s_n})^*f=\overline{q\big(\boldsymbol{s}(\boldsymbol{w})\big)}f=0.$$

To prove the last statement, consider the map $Q:\mathcal{H}\to\mathcal{H}\otimes_{\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}}\mathbb{C}_{\boldsymbol{w}}$ defined by $Qf=f\otimes_{\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}}1_{\boldsymbol{w}}$. Note that Q is the composition of a unitary map from $\mathcal{H}$ to $\mathcal{H}\otimes\mathbb{C}$ followed by the quotient map, hence it is onto and $\|Q\|\le1$. Since $p_1\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}+\cdots+p_t\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}$ is dense in $\mathcal{H}$, it follows that $Q(p_1\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}+\cdots+p_t\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n})$ is dense in $\mathcal{H}\otimes_{\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}}\mathbb{C}_{\boldsymbol{w}}$. Now for any $\sum_{i=1}^t p_if_i\in\mathcal{H}$, where f_i's are in $\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}$, we have

$$Q\big(\sum_{i=1}^t p_if_i\big)=\big(\sum_{i=1}^t p_if_i\big)\otimes_{\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}}1_{\boldsymbol{w}}=\sum_{i=1}^t p_i\otimes_{\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}}f_i\cdot1_{\boldsymbol{w}}=\sum_{i=1}^t f_i(\boldsymbol{w})p_i\otimes_{\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}}1_{\boldsymbol{w}}.$$

Therefore, $Q(p_1\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}+\cdots+p_t\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n})$ is finite dimensional and hence $\mathcal{H}\otimes_{\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}}\mathbb{C}_{\boldsymbol{w}}$ is finite dimensional and is spanned by $p_1\otimes_{\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}}1_{\boldsymbol{w}},\ldots,p_t\otimes_{\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}}1_{\boldsymbol{w}}$. ∎

(Proof of Theorem 2.5). Using Corollary 2.10 and Proposition 2.11, we show that the map $\boldsymbol{t}:\boldsymbol{u}\mapsto\operatorname{span}\{K_{\boldsymbol{w}}\mid\boldsymbol{w}\in\boldsymbol{s}^{-1}(\boldsymbol{u})\}$ taking values in the Grassmannian $\operatorname{Gr}\big(n!,\mathcal{H}\big)$ of the Hilbert space $\mathcal{H}$ of rank $n!$ is anti-holomorphic. Given any $\boldsymbol{u}_0$, fixed but arbitrary, in $\boldsymbol{s}(\Omega)\setminus\boldsymbol{s}(\mathcal{Z})$, there exists a neighborhood of $\boldsymbol{u}_0$, say U, on which $\boldsymbol{s}$ admits $n!$ local inverses. Enumerate them as $\varphi_1,\ldots,\varphi_{n!}$. Then the linearly independent set

$$\big\{\gamma_i:\gamma_i(\boldsymbol{u})=K\big(\cdot,\varphi_i(\boldsymbol{u})\big),\boldsymbol{u}\in U\big\}_{i=1}^{n!}$$

of anti-holomorphic $\mathcal{H}$-valued functions spans the joint kernel $\cap_{i=1}^n\ker\big(M_{s_i}-s_i(\boldsymbol{w})\big)^*$. ∎

Remark 2.12. *An alternative proof of the Corollary 2.10 is possible using Lemma 2.11. For this proof, which is indicated below, it is essential to use a non-trivial result from* [10] *rather than the direct proof that we have presented earlier. From Lemma 2.11, it follows that* $\dim\cap_{i=1}^n\ker\big(M_{s_i}-s_i(\boldsymbol{w})\big)^*\le n!$. *To prove the reverse inequality, we show that for $\boldsymbol{w}\in\Omega\setminus\mathcal{Z}$, the set of vectors $\{K_{\boldsymbol{w}_\sigma}\mid\sigma\in\mathfrak{S}_n\}$ are linearly* *independent. Since the polynomial ring is dense in $\mathcal{H}$, the reproducing kernel K is non-degenerate. From* [10, Lemma 3.6]*, it follows that K is strictly positive, that is, for all $k \geq 1$ the $k \times k$-operator matrix $\left(\!\left(K(\boldsymbol{z}_i, \boldsymbol{z}_j)\right)\!\right)_{1 \leq i,j \leq k}$ is injective for every collection $\{\boldsymbol{z}_1, \ldots, \boldsymbol{z}_k\}$ of distinct points in $\Omega \setminus \mathcal{Z}$. Since the set $\{\boldsymbol{w}_\sigma \mid \sigma \in \mathfrak{S}_n\}$ contains exactly $n!$ distinct points for every $\boldsymbol{w} \in \Omega \setminus \mathcal{Z}$, the matrix $\left(\!\left(\langle K_{\boldsymbol{w}_\sigma}, K_{\boldsymbol{w}_\tau}\rangle\right)\!\right)_{\sigma,\tau \in \mathfrak{S}_n}$ is injective and hence the nonsingularity of the grammian of $\{K_{\boldsymbol{w}_\sigma} \mid \sigma \in \mathfrak{S}_n\}$ gives the linear independence.*

3. Analytic Hilbert module with $\mathfrak{S}_n$-invariant kernel

Let $\mathcal{H}$ be the Hilbert space and $U : \mathfrak{S}_n \to \mathcal{B}(\mathcal{H})$ be a unitary representation. Consider a function $\chi : \mathfrak{S}_n \to \mathbb{C}$ satisfying $\chi(\sigma^{-1}) = \overline{\chi(\sigma)}$. Define an operator on $\mathcal{H}$ by

$$T^\chi = \sum_{\sigma \in \mathfrak{S}_n} \overline{\chi(\sigma)} U(\sigma).$$

Since $U(\sigma)^* = U(\sigma^{-1})$, it follows that

$$(T^\chi)^* = \sum_{\sigma \in \mathfrak{S}_n} \chi(\sigma) U(\sigma)^* = \sum_{\sigma \in \mathfrak{S}_n} \overline{\chi(\sigma^{-1})} U(\sigma^{-1}) = \sum_{\tau \in \mathfrak{S}_n} \overline{\chi(\tau)} U(\tau) = T^\chi.$$

Thus the following Lemma has been proved.

Lemma 3.1. *T^χ is self adjoint on $\mathcal{H}$.*

As before, let $\boldsymbol{\pi_p}$ be a unitary representation of $\mathfrak{S}_n$ in the equivalence class of $\boldsymbol{p} \vdash n$, that is, $\boldsymbol{\pi_p}(\sigma) = \left(\!\left(\boldsymbol{\pi}_{\boldsymbol{p}}^{ij}(\sigma)\right)\!\right)_{i,j=1}^m \in \mathbb{C}^{m \times m}$, $\sigma \in \mathfrak{S}_n$, where $m = \chi_{\boldsymbol{p}}(1)$ and $\chi_{\boldsymbol{p}}$ is the character of the representation $\boldsymbol{\pi_p}$. The following orthogonality relations [19, Proposition 2.9] play a central role in this section.

$$\sum_{\sigma \in \mathfrak{S}_n} \boldsymbol{\pi}_{\boldsymbol{p}}^{ij}(\sigma^{-1}) \boldsymbol{\pi}_{\boldsymbol{q}}^{lm}(\sigma) = \frac{n!}{\chi_{\boldsymbol{p}}(1)} \delta_{\boldsymbol{pq}} \delta_{im} \delta_{jl}, \tag{3.1}$$

where δ is the Kronecker symbol. For any partition $\boldsymbol{p}$ of $\mathbb{N}$ and $1 \leq i, j \leq \chi_{\boldsymbol{p}}(1)$, define the operators $\mathbb{P}_{\boldsymbol{p}}^{ij}, \mathbb{P}_{\boldsymbol{p}} : \mathcal{H} \to \mathcal{H}$ by the formula

$$\mathbb{P}_{\boldsymbol{p}}^{ij} = \frac{\chi_{\boldsymbol{p}}(1)}{n!} \sum_{\sigma \in \mathfrak{S}_n} \boldsymbol{\pi}_{\boldsymbol{p}}^{ji}(\sigma^{-1}) U(\sigma)$$

and

$$\mathbb{P}_{\boldsymbol{p}} = \frac{\chi_{\boldsymbol{p}}(1)}{n!} \sum_{\sigma \in \mathfrak{S}_n} \chi_{\boldsymbol{p}}(\sigma^{-1}) U(\sigma).$$

Clearly,

$$\sum_{i=1}^{\chi_{\boldsymbol{p}}(1)} \mathbb{P}_{\boldsymbol{p}}^{ii} = \mathbb{P}_{\boldsymbol{p}}. \tag{3.2}$$

The following lemma and some of the subsequent discussions are adapted from the properties of projection operators given in [19, p. 162]. We include this for sake of completeness.

Proposition 3.2. *For $1 \leq i, j \leq \chi_{\boldsymbol{p}}(1)$ and $1 \leq l, m \leq \chi_{\boldsymbol{q}}(1)$, $\mathbb{P}_{\boldsymbol{p}}^{ij} \mathbb{P}_{\boldsymbol{q}}^{lm} = \delta_{\boldsymbol{pq}} \delta_{jl} \mathbb{P}_{\boldsymbol{p}}^{im}$.*

Proof. Since $\mathbb{P}_{\boldsymbol{p}}^{ij} = \frac{\chi_{\boldsymbol{p}}(1)}{n!}\sum_{\sigma\in\mathfrak{S}_n}\boldsymbol{\pi}_{\boldsymbol{p}}^{ji}(\sigma^{-1})U(\sigma)$, we have that

$$\begin{aligned}
\mathbb{P}_{\boldsymbol{p}}^{ij}\mathbb{P}_{\boldsymbol{q}}^{lm} &= \frac{\chi_{\boldsymbol{q}}(1)}{n!}\sum_{\sigma\in\mathfrak{S}_n}\boldsymbol{\pi}_{\boldsymbol{q}}^{ml}(\sigma^{-1})\mathbb{P}_{\boldsymbol{p}}^{ij}U(\sigma)\\
&= \frac{\chi_{\boldsymbol{p}}(1)\chi_{\boldsymbol{q}}(1)}{(n!)^2}\sum_{\sigma\in\mathfrak{S}_n}\boldsymbol{\pi}_{\boldsymbol{q}}^{ml}(\sigma^{-1})\{\sum_{\tau\in\mathfrak{S}_n}\boldsymbol{\pi}_{\boldsymbol{p}}^{ji}(\tau^{-1})U(\tau)\}U(\sigma)\\
&= \frac{\chi_{\boldsymbol{p}}(1)\chi_{\boldsymbol{q}}(1)}{(n!)^2}\sum_{\sigma\in\mathfrak{S}_n}\sum_{\tau\in\mathfrak{S}_n}\boldsymbol{\pi}_{\boldsymbol{q}}^{ml}(\sigma^{-1})\boldsymbol{\pi}_{\boldsymbol{p}}^{ji}(\tau^{-1})U(\tau)U(\sigma).
\end{aligned}$$

Let $\eta=\tau\sigma$. Then $\tau^{-1}=\sigma\eta^{-1}$ and

$$\boldsymbol{\pi}_{\boldsymbol{p}}^{ji}(\sigma\eta^{-1}) = (\boldsymbol{\pi}_{\boldsymbol{p}}(\sigma\eta^{-1}))_{ji} = (\boldsymbol{\pi}_{\boldsymbol{p}}(\sigma)\boldsymbol{\pi}_{\boldsymbol{p}}(\eta^{-1}))_{ji} = \sum_{k=1}^{\chi_{\boldsymbol{p}}(1)}\boldsymbol{\pi}_{\boldsymbol{p}}^{jk}(\sigma)\boldsymbol{\pi}_{\boldsymbol{p}}^{ki}(\eta^{-1}).$$

Thus, we also have

$$\begin{aligned}
\mathbb{P}_{\boldsymbol{p}}^{ij}\mathbb{P}_{\boldsymbol{q}}^{lm} &= \frac{\chi_{\boldsymbol{p}}(1)\chi_{\boldsymbol{q}}(1)}{(n!)^2}\sum_{\sigma\in\mathfrak{S}_n}\sum_{\eta\in\mathfrak{S}_n}\boldsymbol{\pi}_{\boldsymbol{q}}^{ml}(\sigma^{-1})\boldsymbol{\pi}_{\boldsymbol{p}}^{ji}(\sigma\eta^{-1})U(\eta)\\
&= \frac{\chi_{\boldsymbol{p}}(1)\chi_{\boldsymbol{q}}(1)}{(n!)^2}\sum_{\sigma\in\mathfrak{S}_n}\sum_{\eta\in\mathfrak{S}_n}\boldsymbol{\pi}_{\boldsymbol{q}}^{ml}(\sigma^{-1})\sum_{k=1}^{\chi_{\boldsymbol{p}}(1)}\boldsymbol{\pi}_{\boldsymbol{p}}^{jk}(\sigma)\boldsymbol{\pi}_{\boldsymbol{p}}^{ki}(\eta^{-1})U(\eta)\\
&= \frac{\chi_{\boldsymbol{p}}(1)\chi_{\boldsymbol{q}}(1)}{(n!)^2}\sum_{\eta\in\mathfrak{S}_n}\sum_{k=1}^{\chi_{\boldsymbol{p}}(1)}\{\sum_{\sigma\in\mathfrak{S}_n}\boldsymbol{\pi}_{\boldsymbol{q}}^{ml}(\sigma^{-1})\boldsymbol{\pi}_{\boldsymbol{p}}^{jk}(\sigma)\}\boldsymbol{\pi}_{\boldsymbol{p}}^{ki}(\eta^{-1})U(\eta)\\
&= \frac{\chi_{\boldsymbol{p}}(1)\chi_{\boldsymbol{q}}(1)}{(n!)^2}\sum_{\eta\in\mathfrak{S}_n}\sum_{k=1}^{\chi_{\boldsymbol{p}}(1)}\{\delta_{\boldsymbol{pq}}\delta_{lj}\delta_{mk}\frac{n!}{\chi_{\boldsymbol{q}}(1)}\}\boldsymbol{\pi}_{\boldsymbol{p}}^{ki}(\eta^{-1})U(\eta),\ \text{(from Equation (3.1))}\\
&= \delta_{\boldsymbol{pq}}\delta_{jl}\frac{\chi_{\boldsymbol{p}}(1)}{n!}\sum_{\eta\in\mathfrak{S}_n}\sum_{k=1}^{\chi_{\boldsymbol{p}}(1)}\delta_{mk}\boldsymbol{\pi}_{\boldsymbol{p}}^{ki}(\eta^{-1})U(\eta)\\
&= \delta_{\boldsymbol{pq}}\delta_{jl}\frac{\chi_{\boldsymbol{p}}(1)}{n!}\sum_{\eta\in\mathfrak{S}_n}\boldsymbol{\pi}_{\boldsymbol{p}}^{mi}(\eta^{-1})U(\eta)\\
&= \delta_{\boldsymbol{pq}}\delta_{jl}\mathbb{P}_{\boldsymbol{p}}^{im}.
\end{aligned}$$

∎

Corollary 3.3. *For each partition $\boldsymbol{p}$ of n and $1\le i\le\chi_{\boldsymbol{p}}(1)$, $\mathbb{P}_{\boldsymbol{p}}^{ii}$ is an orthogonal projection and* $\sum_{\boldsymbol{p}\vdash n}\sum_{i=1}^{\chi_{\boldsymbol{p}}(1)}\mathbb{P}_{\boldsymbol{p}}^{ii}=\mathrm{id}$.

Proof. Since $\boldsymbol{\pi}_{\boldsymbol{p}}$ is a unitary representation, it follows that $\boldsymbol{\pi}_{\boldsymbol{p}}^{ii}(\sigma^{-1})=\overline{\boldsymbol{\pi}_{\boldsymbol{p}}^{ii}(\sigma)}$. Thus from Lemma 3.1, we find that $\mathbb{P}_{\boldsymbol{p}}^{ii}$ is self adjoint. From the Proposition 3.2, it follows that $(\mathbb{P}_{\boldsymbol{p}}^{ii})^2=\mathbb{P}_{\boldsymbol{p}}^{ii}$. Then we see that

$$\sum_{\boldsymbol{p}\vdash n}\sum_{i=1}^{\chi_{\boldsymbol{p}}(1)}\mathbb{P}_{ii}^{\boldsymbol{p}} = \sum_{\boldsymbol{p}\vdash n}\mathbb{P}_{\boldsymbol{p}} = \sum_{\boldsymbol{p}\vdash n}\frac{\chi_{\boldsymbol{p}}(1)}{n!}\sum_{\sigma\in\mathfrak{S}_n}\chi_{\boldsymbol{p}}(\sigma)U(\sigma) = \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n}\Big(\sum_{\boldsymbol{p}\vdash n}\chi_{\boldsymbol{p}}(1)\chi_{\boldsymbol{p}}(\sigma)\Big)U(\sigma) = \mathrm{id},$$

where the last equality follows from the orthogonality relations [19, Proposition 3.8]. This completes the proof. ∎

Let $\Omega \subseteq \mathbb{C}^n$ be a bounded domain invariant under the action of $\mathfrak{S}_n$. Let K be a $\mathfrak{S}_n$-invariant reproducing kernel on Ω, that is,

$$K(\sigma \cdot \boldsymbol{z}, \sigma \cdot \boldsymbol{w}) = K(\boldsymbol{z}, \boldsymbol{w}) \text{ for all } \sigma \in \mathfrak{S}_n.$$

Example 3.4. *Suppose that $\Omega \subseteq \mathbb{C}^n$ is a $\mathfrak{S}_n$-invariant domain and $K : \Omega \times \Omega \to \mathbb{C}$ given by $K(\boldsymbol{z}, \boldsymbol{w}) = \sum_{I,J \in \mathbb{Z}_+^n} a_{IJ} \boldsymbol{z}^I \bar{\boldsymbol{w}}^J$ is a reproducing kernel of some Hilbert space of analytic functions on Ω satisfying $a_{I_\sigma J_\sigma} = a_{IJ}$ for $I, J \in \mathbb{Z}_+^n$ and $\sigma \in \mathfrak{S}_n$. Then K is an $\mathfrak{S}_n$-invariant kernel. Recall that a reproducing kernel K on Ω as above is called a diagonal kernel if $a_{IJ} = 0$ for $I \neq J$. If K is a diagonal reproducing kernel, we write $a_I := a_{II}, I \in \mathbb{Z}_+^n$. Any diagonal reproducing kernel on an $\mathfrak{S}_n$-invariant domain Ω is $\mathfrak{S}_n$-invariant if and only if $a_{I_\sigma} = a_I$ for $I \in \mathbb{Z}_+^n$ and $\sigma \in \mathfrak{S}_n$. Note that if K_1 and K_2 are two $\mathfrak{S}_n$-invariant kernels, then $K_1 + K_2$ and $K_1 K_2$ are also $\mathfrak{S}_n$-invariant.*

The standard inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{C}^n$ is $\mathfrak{S}_n$-invariant, that is, $\langle \sigma \cdot \boldsymbol{z}, \sigma \cdot \boldsymbol{w} \rangle = \langle \boldsymbol{z}, \boldsymbol{w} \rangle$ for $\sigma \in \mathfrak{S}_n$. Let $\mathbb{B}^n$ be the unit ball with respect to the ℓ^2-norm induced by $\langle \cdot, \cdot \rangle$. Suppose that $K : \mathbb{B}_n \times \mathbb{B}_n \to \mathbb{C}$ is given by $K(\boldsymbol{z}, \boldsymbol{w}) = \sum_{k=0}^{\infty} a_k \langle \boldsymbol{z}, \boldsymbol{w} \rangle^k$ with $a_k \geq 0$ for $k \geq 0$. Then K is clearly a positive definite diagonal kernel on $\mathbb{B}_n$ and K is $\mathfrak{S}_n$-invariant. This family includes the important subfamily of kernels on $\mathbb{B}_n$ given by $K_\lambda(\boldsymbol{z}, \boldsymbol{w}) = (1 - \langle \boldsymbol{z}, \boldsymbol{w} \rangle)^{-\lambda}$ for $\lambda > 0$.

Let $\mathbb{D}^n = \{\boldsymbol{z} : |z_1|, \ldots, |z_n| < 1\}$, the unit ball with respect to ℓ^∞-norm, be the polydisc in $\mathbb{C}^n$. Suppose that $K : \mathbb{D}^n \times \mathbb{D}^n \to \mathbb{C}$ is given by $K(\boldsymbol{z}, \boldsymbol{w}) = \prod_{j=1}^n B(z_j, w_j)$, where B is a reproducing kernel on $\mathbb{D}$. Then K is clearly a positive definite diagonal kernel on $\mathbb{D}^n$ and K is $\mathfrak{S}_n$-invariant. This family of $\mathfrak{S}_n$-invariant kernels need not be diagonal, unless B is a diagonal kernel on $\mathbb{D}$. Suppose that $K(\boldsymbol{z}, \boldsymbol{w}) = \sum_{I \in \mathbb{Z}_+^n} a_I \boldsymbol{z}^I \bar{\boldsymbol{w}}^I$ with $a_I \geq 0$ for $I \in \mathbb{Z}_+^n$. Then K is clearly a positive definite diagonal kernel on $\mathbb{D}^n$ and K is $\mathfrak{S}_n$-invariant if and only if $a_{I_\sigma} = a_I$ for $I \in \mathbb{Z}_+^n$ and $\sigma \in \mathfrak{S}_n$. Both of these families of $\mathfrak{S}_n$-invariant kernels on $\mathbb{D}^n$ include the weighted Bergman kernels $K^{(\lambda)}(\boldsymbol{z}, \boldsymbol{w}) = \prod_{j=1}^n (1 - z_j w_j)^{-\lambda}$, $\lambda > 0$, the reproducing kernels of the weighted Bergman modules $\mathbb{A}^{(\lambda)}(\mathbb{D}^n)$. The holomorphic discrete series representations of $\mathrm{Aut}(\mathbb{D}^n)$, the automorphism group of $\mathbb{D}^n$, are realized on this family of Hilbert spaces.

In fact, due to transformation rule of the Bergman kernel under biholomorphic maps, the Bergman kernel B of an $\mathfrak{S}_n$-invariant domain is $\mathfrak{S}_n$-invariant. In particular, the Bergman kernels of the open unit balls in $\mathbb{C}^n$ with respect to the ℓ^p-norm, for $1 \leq p \leq \infty$, are $\mathfrak{S}_n$-invariant. For λ in the Berezin-Wallach set of the $\mathfrak{S}_n$-invariant domain Ω, B^λ is an $\mathfrak{S}_n$-invariant kernel.

Let $\mathcal{H}$ be an analytic Hilbert module with $\mathfrak{S}_n$-invariant kernel K. We claim that the function $f \circ \sigma^{-1}$, $\sigma \in \mathfrak{S}_n$, is in $\mathcal{H}$, whenever f is in $\mathcal{H}$. To see this, recall that f is in $\mathcal{H}$ if and only if there exists a positive real number c such that $K_f(\boldsymbol{z}, \boldsymbol{w}) := \big(c^2 K(\boldsymbol{z}, \boldsymbol{w}) - f(\boldsymbol{z})\overline{f(\boldsymbol{w})}\big)$ is positive definite, see [2, p. 194]. Since

$$\begin{aligned} K_{f\circ\sigma^{-1}}(\boldsymbol{z}, \boldsymbol{w}) &= c^2 K(\boldsymbol{z}, \boldsymbol{w}) - f \circ \sigma^{-1}(\boldsymbol{z})\overline{f \circ \sigma^{-1}(\boldsymbol{w})} \\ &= c^2 K(\sigma \cdot \boldsymbol{u}, \sigma \cdot \boldsymbol{v}) - f(\boldsymbol{u})\overline{f(\boldsymbol{v})} \\ &= c^2 K(\boldsymbol{u}, \boldsymbol{v}) - f(\boldsymbol{u})\overline{f(\boldsymbol{v})} \\ &= K_f(\boldsymbol{u}, \boldsymbol{v}), \end{aligned}$$

where $\sigma \cdot \boldsymbol{u} = \boldsymbol{z}$ and $\sigma \cdot \boldsymbol{v} = \boldsymbol{w}$, it follows that $K_{f\circ\sigma^{-1}}$ is positive definite. Thus the operator $R_\sigma : \mathcal{H} \to \mathcal{H}$, $R_\sigma(f) = f \circ \sigma^{-1}$, is well defined.

Lemma 3.5. *The map $R : \sigma \mapsto R_\sigma$ is a unitary representation of $\mathfrak{S}_n$ on $\mathcal{H}$.*

Proof. Note that $R_{\sigma\tau}f(\boldsymbol{z}) = f\big((\sigma\tau)^{-1}\cdot\boldsymbol{z})\big) = f(\tau^{-1}\sigma^{-1}\cdot\boldsymbol{z}) = (R_\tau f)(\sigma^{-1}\cdot\boldsymbol{z}) = R_\sigma(R_\tau f)(\boldsymbol{z})$. Thus $R_{\sigma\tau} = R_\sigma R_\tau$. Since the set $\{K_{\boldsymbol{w}} \mid \boldsymbol{w}\in\Omega\}$ is total in $\mathcal{H}$, it is enough to check R_σ is unitary on $\{K_{\boldsymbol{w}} \mid \boldsymbol{w}\in\Omega\}$. Also,

$$R_\sigma K_{\boldsymbol{w}}(\boldsymbol{z}) = K_{\boldsymbol{w}}(\sigma^{-1}\cdot\boldsymbol{z}) = K(\sigma^{-1}\cdot\boldsymbol{z},\boldsymbol{w}) = K(\boldsymbol{z},\sigma\cdot\boldsymbol{w}) = K_{\sigma\cdot\boldsymbol{w}}(\boldsymbol{z}),$$

that is, $R_\sigma K_{\boldsymbol{w}} = K_{\sigma\cdot\boldsymbol{w}}$. Thus

$$\langle R_\sigma K_{\boldsymbol{w}}, R_\sigma K_{\boldsymbol{w}'}\rangle = \langle K_{\sigma\cdot\boldsymbol{w}}, K_{\sigma\cdot\boldsymbol{w}'}\rangle = K(\sigma\cdot\boldsymbol{w}',\sigma\cdot\boldsymbol{w}) = K(\boldsymbol{w}',\boldsymbol{w}) = \langle K_{\boldsymbol{w}}, K_{\boldsymbol{w}'}\rangle.$$

This completes the proof. ∎

In the remaining portion of this section, we will specialize to the representation R. Now, the formulae for $\mathbb{P}_{\boldsymbol{p}}^{ij}$ and $\mathbb{P}_{\boldsymbol{p}}$ simplify to

$$\mathbb{P}_{\boldsymbol{p}}^{ij}f(\boldsymbol{z}) = \frac{\chi_{\boldsymbol{p}}(1)}{n!}\sum_{\sigma\in\mathfrak{S}_n}\boldsymbol{\pi}_{\boldsymbol{p}}^{ji}(\sigma^{-1})(R_\sigma f)(\boldsymbol{z}) = \frac{\chi_{\boldsymbol{p}}(1)}{n!}\sum_{\sigma\in\mathfrak{S}_n}\boldsymbol{\pi}_{\boldsymbol{p}}^{ji}(\sigma^{-1})f(\sigma^{-1}\cdot\boldsymbol{z}) \tag{3.3}$$

and

$$\mathbb{P}_{\boldsymbol{p}}f(\boldsymbol{z}) = \frac{\chi_{\boldsymbol{p}}(1)}{n!}\sum_{\sigma\in\mathfrak{S}_n}\overline{\chi_{\boldsymbol{p}}(\sigma)}R_\sigma f(\boldsymbol{z}) = \frac{\chi_{\boldsymbol{p}}(1)}{n!}\sum_{\sigma\in\mathfrak{S}_n}\overline{\chi_{\boldsymbol{p}}(\sigma)}f(\sigma^{-1}\cdot\boldsymbol{z}). \tag{3.4}$$

In view of Corollary 3.3, we obtain that any analytic Hilbert module $\mathcal{H}$ with $\mathfrak{S}_n$-invariant kernel K can be decomposed as follows:

$$\mathcal{H} = \bigoplus_{\boldsymbol{p}\vdash n}\mathbb{P}_{\boldsymbol{p}}\mathcal{H} = \bigoplus_{\boldsymbol{p}\vdash n}\bigoplus_{i=1}^{\chi_{\boldsymbol{p}}(1)}\mathbb{P}_{\boldsymbol{p}}^{ii}\mathcal{H}. \tag{3.5}$$

It is natural to ask whether each of the projections $\mathbb{P}_{\boldsymbol{p}}$ and $\mathbb{P}_{\boldsymbol{p}}^{ii}$ is nontrivial. To see $\mathbb{P}_{\boldsymbol{p}}$ is non-trivial, we are going to use the following well known result which is analogous to the fact that the polynomial ring $\mathbb{C}[\boldsymbol{z}]$ is a finitely generated free module over $\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}$ of rank $n!$.

Theorem 3.6. *The module $\mathbb{P}_{\boldsymbol{p}}\mathbb{C}[\boldsymbol{z}]$ is a finitely generated free module over $\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}$ of rank $\chi_{\boldsymbol{p}}(1)^2$.*

We are unable to locate a proof of this Theorem and therefore indicate a proof using results from [25].

Proof. Set $\mathbb{C}[\boldsymbol{z}]_{\boldsymbol{p}} := \mathbb{P}_{\boldsymbol{p}}\mathbb{C}[\boldsymbol{z}]$. There exist homogeneous polynomials in $\mathbb{C}[\boldsymbol{z}]_{\boldsymbol{p}}$, whose images in the quotient module $\mathcal{S}_{\boldsymbol{p}} = \mathbb{C}[\boldsymbol{z}]_{\boldsymbol{p}}/\{s_1\mathbb{C}[\boldsymbol{z}]_{\boldsymbol{p}}+\cdots+s_n\mathbb{C}[\boldsymbol{z}]_{\boldsymbol{p}}\}$ forms a $\mathbb{C}$-basis for $\mathcal{S}_{\boldsymbol{p}}$, see [25, Theorem 1.3]). Also, from [25, Theorem 3.10], it follows that $p_1,\ldots,p_\mu$ is a basis for the free module $\mathbb{C}[\boldsymbol{z}]_{\boldsymbol{p}}$ over $\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}$. Now to see that $\mu = \chi_{\boldsymbol{p}}(1)^2$, we make use of [25, Theorem 4.9] and its proof along with [25, Corollary 4.9]. It says that the action of $\mathfrak{S}_n$ on the quotient ring $\mathbb{C}[\boldsymbol{z}]/\{s_1\mathbb{C}[\boldsymbol{z}]+\cdots+s_n\mathbb{C}[\boldsymbol{z}]\} \cong \oplus_{\boldsymbol{p}\vdash n}\mathcal{S}_{\boldsymbol{p}}$ is isomorphic to the regular representation of $\mathfrak{S}_n$, where the action on $\mathcal{S}_{\boldsymbol{p}}$ is isomorphic to the representation $\pi_{\boldsymbol{p}}$ corresponding to $\boldsymbol{p}\vdash n$ with multiplicity $\chi_{\boldsymbol{p}}(1)$. ∎

The proof of the following Corollary is immediate from Theorem 3.6 and Lemma 2.11.

Corollary 3.7. *If $\mathcal{H}$ is an analytic Hilbert module with an $\mathfrak{S}_n$-invariant kernel, then the Hilbert module $\mathbb{P}_{\boldsymbol{p}}\mathcal{H}$ over $\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}$ is non-trivial and is of rank at most $\chi_{\boldsymbol{p}}(1)^2$.*

We record the non-triviality of the projections $\mathbb{P}_{\boldsymbol{p}}^{ii}$ as a separate Lemma. The main ingredient of the proof is borrowed from [19, p. - 166].

Lemma 3.8. *For each $\boldsymbol{p} \vdash n$ and $1 \le i \le \chi_{\boldsymbol{p}}(1)$, $\mathbb{P}_{\boldsymbol{p}}^{ii} \neq 0$.*

Proof. From Proposition 3.2, we have

$$\mathbb{P}_{\boldsymbol{p}}^{ij}\mathbb{P}_{\boldsymbol{p}}^{jj} = \mathbb{P}_{\boldsymbol{p}}^{ij} = \mathbb{P}_{\boldsymbol{p}}^{ii}\mathbb{P}_{\boldsymbol{p}}^{ij},$$

and it then follows that $\mathbb{P}_{\boldsymbol{p}}^{ij}\mathbb{P}_{\boldsymbol{p}}^{jj}\mathcal{H} \subseteq \mathbb{P}_{\boldsymbol{p}}^{ii}\mathcal{H}$. Also for $f \in \mathcal{H}$,

$$\mathbb{P}_{\boldsymbol{p}}^{ii}f = \mathbb{P}_{\boldsymbol{p}}^{ij}\mathbb{P}_{\boldsymbol{p}}^{ji}f = \mathbb{P}_{\boldsymbol{p}}^{ij}\mathbb{P}_{\boldsymbol{p}}^{jj}\mathbb{P}_{\boldsymbol{p}}^{ji}f$$

and thus $\mathbb{P}_{\boldsymbol{p}}^{ii}\mathcal{H} \subseteq \mathbb{P}_{\boldsymbol{p}}^{ij}\mathbb{P}_{\boldsymbol{p}}^{jj}\mathcal{H}$. Consequently, $\mathbb{P}_{\boldsymbol{p}}^{ij}$ is a surjective map from $\mathbb{P}_{\boldsymbol{p}}^{jj}\mathcal{H}$ onto $\mathbb{P}_{\boldsymbol{p}}^{ii}\mathcal{H}$. Now $\mathbb{P}_{\boldsymbol{p}}^{ij}\mathbb{P}_{\boldsymbol{p}}^{jj}f = 0$ implies that $\mathbb{P}_{\boldsymbol{p}}^{ji}\mathbb{P}_{\boldsymbol{p}}^{ij}\mathbb{P}_{\boldsymbol{p}}^{jj}f = 0$ and hence $\mathbb{P}_{\boldsymbol{p}}^{jj}f = (\mathbb{P}_{\boldsymbol{p}}^{jj})^2 f = 0$. This shows that $\mathbb{P}_{\boldsymbol{p}}^{ij}$ is injective on $\mathbb{P}_{\boldsymbol{p}}^{jj}\mathcal{H}$. The operator $\mathbb{P}_{\boldsymbol{p}}^{ij}$, being a finite linear combination of unitaries, is bounded and hence an invertible map (by the open mapping theorem) from $\mathbb{P}_{\boldsymbol{p}}^{jj}\mathcal{H}$ onto $\mathbb{P}_{\boldsymbol{p}}^{ii}\mathcal{H}$. Since each $\mathbb{P}_{\boldsymbol{p}}$ is non-trivial, from Equation (3.2), it follows that each $\mathbb{P}_{\boldsymbol{p}}^{ii}$ is non-trivial. ∎

Proposition 3.9. *For each $\boldsymbol{p} \vdash n$, $1 \le i, j \le \chi_{\boldsymbol{p}}(1)$ and $k = 1, \ldots, n$, $M_{s_k}\mathbb{P}_{\boldsymbol{p}}^{ij} = \mathbb{P}_{\boldsymbol{p}}^{ij}M_{s_k}$.*

Proof. For $f \in \mathcal{H}$, from the Equation (3.3) we have

$$\begin{aligned}
\big(M_{s_k}\mathbb{P}_{\boldsymbol{p}}^{ij}f\big)(\boldsymbol{z}) &= \frac{\chi_{\boldsymbol{p}}(1)}{n!}\sum_{\sigma\in\mathfrak{S}_n}\boldsymbol{\pi}_{\boldsymbol{p}}^{ji}(\sigma^{-1})M_{s_k}f(\sigma^{-1}\cdot\boldsymbol{z})\\
&= \frac{\chi_{\boldsymbol{p}}(1)}{n!}\sum_{\sigma\in\mathfrak{S}_n}\boldsymbol{\pi}_{\boldsymbol{p}}^{ji}(\sigma^{-1})s_k(\sigma^{-1}\cdot\boldsymbol{z})f(\sigma^{-1}\cdot\boldsymbol{z})\\
&= \frac{\chi_{\boldsymbol{p}}(1)}{n!}\sum_{\sigma\in\mathfrak{S}_n}\boldsymbol{\pi}_{\boldsymbol{p}}^{ji}(\sigma^{-1})(s_kf)(\sigma^{-1}\cdot\boldsymbol{z})\\
&= \big(\mathbb{P}_{\boldsymbol{p}}^{ij}M_{s_k}f\big)(\boldsymbol{z}).
\end{aligned}$$

This completes the proof. ∎

In particular for each $\boldsymbol{p} \vdash n$ and i, $1 \le i \le \chi_{\boldsymbol{p}}(1)$, the projections $\mathbb{P}_{\boldsymbol{p}}^{ii}$ commute with M_{s_k} for each $k, 1 \le k \le n$ and we have the following corollary.

Corollary 3.10. *Let $\mathcal{H}$ be an analytic Hilbert module with an $\mathfrak{S}_n$-invariant kernel. Then $\mathbb{P}_{\boldsymbol{p}}^{ii}\mathcal{H}$ is a joint reducing subspace for $M_{s_k}, k = 1, \ldots, n$, for every partition $\boldsymbol{p}$ of n and for each i, $1 \le i \le \chi_{\boldsymbol{p}}(1)$.*

3.1. **Inequivalence.** Having obtained the decomposition (3.5) and having shown that each $\mathbb{P}_{\boldsymbol{p}}\mathcal{H}$ and $\mathbb{P}_{\boldsymbol{p}}^{ii}\mathcal{H}$ is a reducing sub-module (Corollary 3.10 and Equation 3.2) over the ring of symmetric polynomials $\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}$ of the Hilbert module $\mathcal{H}$, it is natural to ask whether these sub-modules are inequivalent for distinct pairs $\boldsymbol{p}$ or $(\boldsymbol{p}, i)$ of a partition $\boldsymbol{p}$ of n and i, $1 \le i \le \chi_{\boldsymbol{p}}(1)$. We first prove few results which will be relevant for this discussion.

Set $M_{s_k}^{(\boldsymbol{p})} = M_{s_k}|_{\mathbb{P}_{\boldsymbol{p}}\mathcal{H}}$. Since each $\mathbb{P}_{\boldsymbol{p}}\mathcal{H}$ is a reducing subspace of M_{s_k} for each $k, 1 \le k \le n$. Therefore, $M_{s_k}^* = \oplus_{\boldsymbol{p}\vdash n}\big(M_{s_k}^{(p)}\big)^*$ and we have

$$\cap_{k=1}^n \ker M_{s_k - s_k(\boldsymbol{w})}^* = \oplus_{\boldsymbol{p}\vdash n} \cap_{k=1}^n \ker \big(M_{s_k - s_k(\boldsymbol{w})}^{(\boldsymbol{p})}\big)^*.$$

Proposition 3.11. *Let $\mathcal{H}$ be an analytic Hilbert module with an $\mathfrak{S}_n$-invariant kernel.*

$$\dim \cap_{k=1}^n \ker \left(M_{s_k - s_k(\boldsymbol{w})}^{(\boldsymbol{p})}\right)^* = \chi_{\boldsymbol{p}}(1)^2,\ \boldsymbol{w} \in \Omega \setminus \mathcal{Z}.$$

Proof. From Corollary 3.7 and Lemma 2.11, it follows that $\dim \cap_{k=1}^n \ker \left(M_{s_k - s_k(\boldsymbol{w})}^{(\boldsymbol{p})}\right)^* \leq \chi_{\boldsymbol{p}}(1)^2$. However, if it is strictly less for some $\boldsymbol{p} \vdash n$ we have the following contradiction:

$$n! = \dim \cap_{k=1}^n \ker M_{s_k - s_k(\boldsymbol{w})}^* = \sum_{\boldsymbol{p} \vdash n} \dim \cap_{k=1}^n \ker \left(M_{s_k - s_k(\boldsymbol{w})}^{(\boldsymbol{p})}\right)^* < \sum_{\boldsymbol{p} \vdash n} \chi_{\boldsymbol{p}}(1)^2 = n!.$$

For the last equality, see [19, Theorem 3.4]. ∎

From the Proposition given above and the proof of Theorem 2.5, the following generalization to $\mathbb{P}_{\boldsymbol{p}}\mathcal{H}$ is evident.

Corollary 3.12. *Let $\mathcal{H}$ be an analytic Hilbert module with an $\mathfrak{S}_n$-invariant kernel. The Hilbert module $\mathbb{P}_{\boldsymbol{p}}\mathcal{H}$ over $\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}$ is locally free of rank $\chi_{\boldsymbol{p}}(1)^2$ on $\boldsymbol{s}(\Omega) \setminus \boldsymbol{s}(\mathcal{Z})$.*

Remark 3.13. *Since $\mathbb{P}_{\boldsymbol{p}}\mathcal{H}$ is assumed to be locally free at $\boldsymbol{w} \in \boldsymbol{s}(\Omega) \setminus \boldsymbol{s}(\mathcal{Z})$, it follows that $E_{\boldsymbol{p}} = \{(\boldsymbol{u}, x) \in U \times \mathbb{P}_{\boldsymbol{p}}\mathcal{H} \mid x \in \cap_{k=1}^n \ker \left(M_{s_k - u_k}^{(\boldsymbol{p})}\right)^*\}$ and $\pi(\boldsymbol{u}, x) = \boldsymbol{u}$ defines a rank $\chi_{\boldsymbol{p}}(1)^2$ hermitian anti-holomorphic vector bundle on some open neighbourhood W of $\boldsymbol{w}$. The equivalence class of this vector bundle $E_{\boldsymbol{p}}$ determines the isomorphism class of the module $\mathbb{P}_{\boldsymbol{p}}\mathcal{H}$ and conversely. The vector bundle E corresponding to the module $\mathcal{H}$ is therefore the direct sum $\oplus_{\boldsymbol{p} \vdash n} E_{\boldsymbol{p}}$.*

We now state the main theorem of this subsection.

Theorem 3.14. *Let $\mathcal{H}$ be an analytic Hilbert module with an $\mathfrak{S}_n$-invariant kernel. If $\boldsymbol{p}$ and $\boldsymbol{q}$ are two partitions of n such that $\chi_{\boldsymbol{p}}(1) \neq \chi_{\boldsymbol{q}}(1)$, then*

(a) *the sub-modules $\mathbb{P}_{\boldsymbol{p}}^{ii}\mathcal{H}$ and $\mathbb{P}_{\boldsymbol{q}}^{jj}\mathcal{H}$ are not unitarily equivalent for any i, j, $1 \leq i \leq \chi_{\boldsymbol{p}}(1)$ and $1 \leq j \leq \chi_{\boldsymbol{q}}(1)$.*

(b) *the sub-modules $\mathbb{P}_{\boldsymbol{p}}\mathcal{H}$ and $\mathbb{P}_{\boldsymbol{q}}\mathcal{H}$ are not unitarily equivalent.*

Proof. Set $M_{s_k}^{(\boldsymbol{p},i)} := M_{s_k}|_{\mathbb{P}_{\boldsymbol{p}}^{ii}\mathcal{H}}$, $1 \leq k \leq n, 1 \leq i \leq \chi_{\boldsymbol{p}}(1)$. From Corollary 3.10, it follows that

$$\bigcap_{k=1}^n \ker \left(M_{s_k}^{(\boldsymbol{p})} - s_k(\boldsymbol{w})\right)^* = \bigoplus_{i=1}^{\chi_{\boldsymbol{p}}(1)} \bigcap_{k=1}^n \ker \left(M_{s_k}^{(\boldsymbol{p},i)} - s_k(\boldsymbol{w})\right)^*.$$

Arguments similar to the ones given in the proof of Lemma 3.8 applied to the sub-modules $\mathbb{P}_{\boldsymbol{p}}^{ii}\mathcal{H}$ show that $\cap_{k=1}^n \ker \left(M_{s_k}^{(\boldsymbol{p},i)} - s_k(\boldsymbol{w})\right)^*$ are isomorphic for all i, $1 \leq i \leq \chi_{\boldsymbol{p}}(1)$. Thus from Proposition 3.11, it follows that $\dim \cap_{k=1}^n \ker \left(M_{s_k}^{(\boldsymbol{p},i)} - s_k(\boldsymbol{w})\right)^* = \chi_{\boldsymbol{p}}(1)$ for all i. From the proof of Theorem 2.5, it follows that each of the sub-modules $\mathbb{P}_{\boldsymbol{p}}^{ii}\mathcal{H}$ is locally free of rank $\chi_{\boldsymbol{p}}(1)$ on $\boldsymbol{s}(\Omega) \setminus \boldsymbol{s}(\mathcal{Z})$. The rank being an invariant for locally free Hilbert modules, the proof of (a) is complete. The proof of (b) follows from Corollary 3.12. ∎

The theorem above leaves open the question of equivalence when $\chi_{\boldsymbol{p}}(1) = \chi_{\boldsymbol{q}}(1)$. While we are not able to settle this question in its entirety, we answer this question in the important example of weighted Bergman module in the next section for $\chi_{\boldsymbol{p}}(1) = 1 = \chi_{\boldsymbol{q}}(1)$, or equivalently, $\boldsymbol{p} = (n)$ and $\boldsymbol{q} = (1, \ldots, 1)$ since one dimensional representations of $\mathfrak{S}_n$ are the trivial and the sign representations.

In cases where $\chi_{\boldsymbol{p}}(1) > 1$, we believe, the work of [29] and [13], may be useful in answering the question of mutual equivalence of the sub-modules $\mathbb{P}^{ii}_{\boldsymbol{p}}\mathcal{H}$. We intend to explore this possibility in our future work.

Let $\mathcal{H}$ be a locally free Hilbert module over $\Omega \subseteq \mathbb{C}^n$. Following [29] and [13], we define a holomorphic section $\gamma : \Omega \to \mathcal{H}$ to be a *spanning holomorphic cross-section* for $\mathcal{H}$ if

$$\bigvee\{\gamma(\boldsymbol{z}) : \boldsymbol{z} \in \Omega\} = \mathcal{H}.$$

Building on the work in [29], the existence of a spanning holomorphic cross-section for a large class of Hilbert modules over an admissible set was proved in [13]. However, in the case of the sub-modules $\mathbb{P}^{ii}_{\boldsymbol{p}}\mathcal{H}$, the existence of a spanning holomorphic cross-section is easily established by exhibiting such a section. Indeed, we give an explicit realization of the spanning holomorphic cross-section for these sub-modules.

Since $\mathbb{P}^{ii}_{\boldsymbol{p}}K(\cdot, \boldsymbol{w})$ is the reproducing kernel for $\mathbb{P}^{ii}_{\boldsymbol{p}}\mathcal{H}$, it can vanish only on a set $F \subseteq \Omega$ such that the real dimension of F is at most $2n - 2$. Also note from Lemma 2.4 and Proposition 3.9 that

$$M^{(\boldsymbol{p},i)}_{s_k}\mathbb{P}^{ii}_{\boldsymbol{p}}K(\cdot, \boldsymbol{w}) = \overline{s_k(\boldsymbol{w})}\mathbb{P}^{ii}_{\boldsymbol{p}}K(\cdot, \boldsymbol{w}). \tag{3.6}$$

Let U be an open neighbourhood of $\boldsymbol{u}_0$ in $\big(\boldsymbol{s}(\Omega) \setminus \boldsymbol{s}(\mathcal{Z})\big) \cap \boldsymbol{s}\big(\Omega \setminus F\big)$. The function $\boldsymbol{s}$ admits $n!$ local inverses on the open set U. Fix one such, say ϕ. Define $\gamma(\boldsymbol{u}) = \mathbb{P}^{ii}_{\boldsymbol{p}}K\big(\cdot, \phi(\bar{\boldsymbol{u}})\big)$, $\boldsymbol{u} \in U^*$. From Equation (3.6), it follows that γ is a spanning holomorphic cross-section for $\mathbb{P}^{ii}_{\boldsymbol{p}}\mathcal{H}$ on U^*. Let

$$E^{(i)}_{\boldsymbol{p}} = \{(\boldsymbol{u}, x) \in U^* \times \mathbb{P}^{ii}_{\boldsymbol{p}}\mathcal{H} \mid x = c\gamma(\boldsymbol{u}) \text{ for some } c \in \mathbb{C}\}$$

denote the corresponding holomorphic hermitian line bundle and

$$\mathscr{K}^{(i)}_{\boldsymbol{p}}(\boldsymbol{u}) = -\sum_{j,k=1}^{n} \partial_j\bar{\partial}_k \log \|\gamma(\boldsymbol{u})\|^2 du_j \wedge d\bar{u}_k,$$

be the curvature of $E^{(i)}_{\boldsymbol{p}}$. Now, we restate Theorem 5.2 of [13] using the spanning cross-sections we have found here.

Theorem 3.15. *Let $\mathcal{H}$ be an analytic Hilbert module with an $\mathfrak{S}_n$-invariant kernel. Let $\boldsymbol{p}$ and $\boldsymbol{q}$ be any two partitions of n. The sub-modules $\mathbb{P}^{ii}_{\boldsymbol{p}}\mathcal{H}$ and $\mathbb{P}^{jj}_{\boldsymbol{q}}\mathcal{H}$ are equivalent if and only if $\mathscr{K}^{(i)}_{\boldsymbol{p}} = \mathscr{K}^{(j)}_{\boldsymbol{q}}$, $1 \leq i \leq \chi_{\boldsymbol{p}}(1)$, $1 \leq j \leq \chi_{\boldsymbol{q}}(1)$.*

3.2. **Spectrum and spectral set.** Recall $M^{(\boldsymbol{p},i)}_{s_k} := M_{s_k}|_{\mathbb{P}^{ii}_{\boldsymbol{p}}\mathcal{H}}$, $1 \leq i \leq \chi_{\boldsymbol{p}}(1)$. To find the spectrum of the commuting n-tuple $(M^{(\boldsymbol{p},i)}_{s_1}, \ldots, M^{(\boldsymbol{p},i)}_{s_n})$, we first establish, following [27, Lemma 1.2], the spectral inclusion for the direct sum of two commuting n-tuples.

Proposition 3.16. *Let $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ be two commuting n-tuples of bounded linear operators acting on the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Then the Taylor joint spectrum $\sigma(\boldsymbol{S}_1)$ and $\sigma(\boldsymbol{S}_2)$ are contained in the Taylor joint spectrum $\sigma(\boldsymbol{S}_1 \oplus \boldsymbol{S}_2)$.*

Proof. Let $\iota : \mathcal{H}_1 \oplus \{0\} \to \mathcal{H}_1 \oplus \mathcal{H}_2$ be the inclusion map, $(f, 0) \mapsto (f, 0)$ and $P : \mathcal{H}_1 \oplus \mathcal{H}_2 \to \{0\} \oplus \mathcal{H}_2$ be the projection, $(f, g) \mapsto (0, g)$. Apply Lemma 1.2 of [27] to the short exact sequence

$$0 \to \mathcal{H}_1 \oplus \{0\} \xrightarrow{\iota} \mathcal{H}_1 \oplus \mathcal{H}_2 \xrightarrow{P} \{0\} \oplus \mathcal{H}_2 \to 0$$

and the direct sum $\boldsymbol{S}_1 \oplus \boldsymbol{S}_2$ to complete the proof. ∎

Theorem 3.17. *If $\mathcal{H}$ be a contractive analytic Hilbert module on an $\mathfrak{S}_n$-invariant domain Ω over $\mathbb{C}[\boldsymbol{z}]$ with an $\mathfrak{S}_n$-invariant kernel, then the Taylor joint spectrum of the n-tuple $(M_{s_1}^{(\boldsymbol{p},i)}, \ldots, M_{s_n}^{(\boldsymbol{p},i)})$ is* $\mathrm{clos}\big(\boldsymbol{s}(\Omega)\big)$.

Proof. From Proposition 3.16, it follows that $\sigma\big(M_{s_1}^{(\boldsymbol{p},i)}, \ldots, M_{s_n}^{(\boldsymbol{p},i)}\big) \subseteq \sigma(M_{s_1}, \ldots, M_{s_n})$. The Taylor functional calculus shows that $\sigma\big(M_{s_1}, \ldots, M_{s_n}\big) = \boldsymbol{s}\big(\sigma(M_1, \ldots, M_n)\big) = \boldsymbol{s}\big(\mathrm{clos}(\Omega)\big)$. Thus, from Equation (3.6), we have

$$\boldsymbol{s}(\Omega) \setminus \boldsymbol{s}(F) \subseteq \boldsymbol{s}(\Omega \setminus F) \subseteq \sigma\big(M_{s_1}^{(\boldsymbol{p},i)}, \ldots, M_{s_n}^{(\boldsymbol{p},i)}\big) \subseteq \mathrm{clos}\big(\boldsymbol{s}(\Omega)\big).$$

Since $\mathrm{clos}\big(\boldsymbol{s}(\Omega) \setminus \boldsymbol{s}(F)\big) = \mathrm{clos}\big(\boldsymbol{s}(\Omega)\big)$ and the spectrum is compact, the proof is complete. ∎

Following the usual convention, set $\mathbb{G}_n = \boldsymbol{s}(\mathbb{D}^n), \Gamma_n = \mathrm{clos}(\mathbb{G}_n)$ and note that $\Gamma_n = \boldsymbol{s}\big(\mathrm{clos}(\mathbb{D}^n)\big)$. Specializing to $\Omega = \mathbb{D}^n$, the following corollary is immediate.

Corollary 3.18. *If $\mathcal{H}$ be a contractive analytic Hilbert module on $\mathbb{D}^n$ over $\mathbb{C}[\boldsymbol{z}]$ with an $\mathfrak{S}_n$-invariant kernel, then the Taylor joint spectrum of the n-tuple $\big(M_{s_1}^{(\boldsymbol{p},i)}, \ldots, M_{s_n}^{(\boldsymbol{p},i)}\big)$ is Γ_n.*

The computation of the Taylor joint spectrum has some immediate applications. Commuting n-tuples of joint weighted shifts are discussed in [16]. It is shown (see [16, Corollary 3]), among other things, that the spectrum of a joint weighted shift must be Reinhardt (invariant under the action of the torus group). It is easy to see that Γ_n is not Reinhardt. Indeed, $(1, \frac{1}{2}, \ldots, 0)$ is in Γ_n while $(1, -\frac{1}{2}, 0, \ldots, 0)$ is not in Γ_n. This follows from the observation that $(\mu_1, \ldots, \mu_k, 0, \ldots, 0)$ is in Γ_n if and only if $(\mu_1, \ldots, \mu_k)$ is in Γ_k. Therefore we have proved the following corollary.

Corollary 3.19. *The n-tuple $\big(M_{s_1}^{(\boldsymbol{p},i)}, \ldots, M_{s_n}^{(\boldsymbol{p},i)}\big)$ is not unitarily equivalent to any joint weighted shift.*

Let $X \subseteq \mathbb{C}^n$ be a polynomially convex set. A commuting n-tuple $\boldsymbol{T}$ of operators is said to admit X as a spectral set if $\|p(\boldsymbol{T})\| \leq \|p\|_{\infty,X}$. If Ω is a bounded domain in $\mathbb{C}^n$ such that $\mathrm{clos}(\Omega)$ is polynomially convex, then since the symmetrization map $\boldsymbol{s}$ is a proper holomorphic map and $\mathrm{clos}(\Omega) = \boldsymbol{s}^{-1}\big(\boldsymbol{s}(\mathrm{clos}(\Omega)\big)$. Therefore, $\boldsymbol{s}\big(\mathrm{clos}(\Omega)\big)$ is polynomially convex by [26, Theorem 1.6.24]. Since $\boldsymbol{s}$ is a proper map, it is closed [24, p. 301] and therefore $\boldsymbol{s}\big(\mathrm{clos}(\Omega)\big) = \mathrm{clos}\big(\boldsymbol{s}(\Omega)\big)$. In the particular case of $X = \mathrm{clos}(\Omega)$ with an $\mathfrak{S}_n$-invariant domain Ω, the following theorem is immediate, generalizing [4, Theorem 3.12].

Theorem 3.20. *If $\mathcal{H}$ be a contractive analytic Hilbert module on an $\mathfrak{S}_n$-invariant domain Ω over $\mathbb{C}[\boldsymbol{z}]$ with an $\mathfrak{S}_n$-invariant kernel, then commuting n-tuple $(M_{s_1}, \ldots, M_{s_n})$ acting on the Hilbert space $\mathbb{P}_{\boldsymbol{p}}^{ii}\mathcal{H}$ admits $\mathrm{clos}\big(\boldsymbol{s}(\Omega)\big)$ as a spectral set for every partition $\boldsymbol{p}$ of n, $1 \leq i \leq \chi_{\boldsymbol{p}}(1)$ and all $\lambda \geq 1$.*

Remark 3.21. *Using the polynomial convexity of $\mathrm{clos}\big(\boldsymbol{s}(\Omega)\big)$, it is easy to see that the Taylor joint spectrum of the commuting n-tuple $\big(M_{s_1}^{(\boldsymbol{p},i)}, \ldots, M_{s_n}^{(\boldsymbol{p},i)}\big)$ on a contractive analytic Hilbert module $\mathcal{H}$ on an $\mathfrak{S}_n$-invariant domain Ω over $\mathbb{C}[\boldsymbol{z}]$ with an $\mathfrak{S}_n$-invariant kernel is contained in $\mathrm{clos}\big(\boldsymbol{s}(\Omega)\big)$. Here we emphasize that the Taylor joint spectrum $\mathrm{clos}\big(\boldsymbol{s}(\Omega)\big)$ of the n-tuple $\big(M_{s_1}^{(\boldsymbol{p},i)}, \ldots, M_{s_n}^{(\boldsymbol{p},i)}\big)$ is a spectral set.*

In the particular case of $X = \Gamma_n$, such a commuting n-tuple $\boldsymbol{T}$ is said to be a Γ_n-contraction. Since the restriction of a Γ_n-contraction to a reducing subspace is again a Γ_n-contraction, the proof of the following theorem is evident from [4, Proposition 2.13 and Corollary 3.11].

Theorem 3.22. *If $\mathcal{H}$ be a contractive analytic Hilbert module on $\mathbb{D}^n$ over $\mathbb{C}[\boldsymbol{z}]$ with an $\mathfrak{S}_n$-invariant kernel, then the commuting n-tuple $(M_{s_1}, \ldots, M_{s_n})$ acting on the Hilbert space $\mathbb{P}^{ii}_{\boldsymbol{p}}\mathcal{H}$ is a Γ_n-contraction for every partition $\boldsymbol{p}$ of n, $1 \leq i \leq \chi_{\boldsymbol{p}}(1)$ and all $\lambda \geq 1$.*

Consider the diagonal kernel $K(\boldsymbol{z}, \boldsymbol{w}) = \sum_{I \in \mathbb{Z}^n_+} a_I \boldsymbol{z}^I \bar{\boldsymbol{w}}^I$ on $\mathbb{D}^n$ where $a_{I_\sigma} = a_I > 0$, for $I \in \mathbb{Z}^n_+$ and $\sigma \in \mathfrak{S}_n$. As mentioned in Example 3.4, these kernels are $\mathfrak{S}_n$-invariant kernel. If M_i denotes the multiplication operator by the co-ordinate function z_i and M_i is a contraction for $i = 1, \ldots, n$, then it follows from [15] that the corresponding Hilbert module is a contractive analytic Hilbert module on $\mathbb{D}^n$. This leads to a large class of examples where the results of this section applies.

4. Weighted Bergman modules $\mathbb{A}^{(\lambda)}(\mathbb{D}^n)$

Recall that the weighted Bergman module $\mathbb{A}^{(\lambda)}(\mathbb{D}^n)$, consisting of holomorphic functions on $\mathbb{D}^n$, is determined by the reproducing kernel $K^{(\lambda)} : \mathbb{D}^n \times \mathbb{D}^n \to \mathbb{C}$ given by the formula

$$K^{(\lambda)}(\boldsymbol{z}, \boldsymbol{w}) = \prod_{j=1}^{n} (1 - z_j \bar{w}_j)^{-\lambda}, \;\; \boldsymbol{z}, \boldsymbol{w} \in \mathbb{D}^n.$$

For $\lambda > 1$, this coincides with the usual notion of the weighted Bergman spaces $\mathbb{A}^{(\lambda)}(\mathbb{D}^n)$ defined as the Hilbert space of square integrable holomorphic functions on $\mathbb{D}^n$ with respect to the measure $dV^{(\lambda)} = \big(\frac{\lambda-1}{\pi}\big)^n \Big(\prod_{i=1}^n (1 - r_i^2)^{\lambda-2} r_i dr_i d\theta_i \Big)$. The limiting case of $\lambda = 1$ is the Hardy space $H^2(\mathbb{D}^n)$. Throughout the rest of this paper, we will assume that $\lambda > 0$. In this section, we show that the two submodules $\mathbb{P}_{(n)}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$ and $\mathbb{P}_{(1,\ldots,1)}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$ are inequivalent over $\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}$. Along the way, we realize each of these submodules as analytic Hilbert module on $\mathbb{G}_n$ and consequently, they become locally free on all of $\mathbb{G}_n$. As a by product, we obtain an explicit formula in Theorem 4.10, for the weighted Bergman kernel of the symmetrized polydisc $\mathbb{G}_n$ in the co-ordinates of $\mathbb{G}_n$.

We begin by setting up some notation which will be useful in the discussion to follow. The length $\ell(\boldsymbol{p})$ of a partition $\boldsymbol{p}$ of n is the number of positive summands of $\boldsymbol{p}$. For a positive integer n, we define the following two subsets of $\mathbb{Z}^n_+ := \{(m_1, \ldots, m_n) \in \mathbb{Z}^n : m_1, \ldots, m_n \geq 0\}$:

$$[n] = \{\boldsymbol{m} \in \mathbb{Z}^n_+ : m_i \geq m_j \text{ for } i < j\} \text{ and } [\![n]\!] = \{\boldsymbol{m} \in \mathbb{Z}^n_+ : m_i > m_j \text{ for } i < j\}.$$

If $\boldsymbol{p} \in [\![n]\!]$, then we can write $\boldsymbol{p} = \boldsymbol{m} + \boldsymbol{\delta}$, where $\boldsymbol{m} \in [n]$ and $\boldsymbol{\delta} = (n-1, n-2, \ldots, 1, 0)$. So,

$$[\![n]\!] = \{\boldsymbol{m} + \boldsymbol{\delta} : \boldsymbol{m} \in [n]\}.$$

Recall from equation (3.4) that for a partition $\boldsymbol{p}$ of n, the linear map $\mathbb{P}_{\boldsymbol{p}} : \mathbb{A}^{(\lambda)}(\mathbb{D}^n) \to \mathbb{A}^{(\lambda)}(\mathbb{D}^n)$ by

$$\mathbb{P}_{\boldsymbol{p}} f = \frac{\chi_{\boldsymbol{p}}(1)}{n!} \sum_{\sigma \in \mathfrak{S}_n} \overline{\chi_{\boldsymbol{p}}(\sigma)} f \circ \sigma^{-1}, \tag{4.1}$$

where $\chi_{\boldsymbol{p}}$ is the character of the representation corresponding to the partition $\boldsymbol{p}$ of n. Choosing the partition $\boldsymbol{p}$ of n to be $(n) := (n, 0, \ldots, 0)$ in Equation (4.1), it is easy to see that

$$\mathbb{P}_{(n)}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big) = \{f \in \mathbb{A}^{(\lambda)}(\mathbb{D}^n) : f \circ \sigma^{-1} = f \text{ for } \sigma \in \mathfrak{S}_n\},$$

that is, $\mathbb{P}_{(n)}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$ consists of symmetric functions in $\mathbb{A}^{(\lambda)}(\mathbb{D}^n)$. Thus $\mathbb{A}^{(\lambda)}_{\text{sym}}(\mathbb{D}^n) = \mathbb{P}_{(n)}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$. In view of [4, Equation (3.1)], the following proposition is a particular case of [4, Proposition 3.6] for $\boldsymbol{p} = (n)$.

Proposition 4.1. *The reproducing kernel $K^{(\lambda)}_{\rm sym}$ of $\mathbb{A}^{(\lambda)}_{\rm sym}(\mathbb{D}^n)$ is given explicitly by the formula:*

$$K^{(\lambda)}_{\rm sym}(\boldsymbol{z},\boldsymbol{w}) = \frac{1}{n!}\mathrm{per}\Big(\big(\!\!\big((1-z_j\bar{w}_k)^{-\lambda}\big)\!\!\big)_{j,k=1}^n\Big),\ \ \boldsymbol{z},\boldsymbol{w}\in\mathbb{D}^n,$$

where $\mathrm{per}\Big(\big(\!\!\big(a_{ij}\big)\!\!\big)_{i,j=1}^n\Big) = \sum_{\sigma\in\mathfrak{S}_n}\prod_{k=1}^n a_{k\sigma(k)}$.

The Hilbert space $\mathbb{A}^{(\lambda)}_{\rm sym}(\mathbb{D}^n)$ can be thought of as a space of functions defined on the symmetrized polydisc $\mathbb{G}_n$ as follows. Recall that $\boldsymbol{s}$ is the symmetrization map and note that

$$\mathbb{A}^{(\lambda)}_{\rm sym}(\mathbb{D}^n) = \{f\in\mathbb{A}^{(\lambda)}(\mathbb{D}^n) : f = g\circ\boldsymbol{s} \text{ for some } g:\mathbb{G}_n\longrightarrow\mathbb{C} \text{ holomorphic }\}.$$

Let

$$\mathcal{H}^{(\lambda)}(\mathbb{G}_n) := \{g:\mathbb{G}_n\longrightarrow\mathbb{C} \text{ is holomorphic } : g\circ\boldsymbol{s}\in\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\}.$$

The inner product on $\mathcal{H}^{(\lambda)}(\mathbb{G}_n)$ is given by $\langle f_1,f_2\rangle_{\mathcal{H}^{(\lambda)}(\mathbb{G}_n)} := \langle f_1\circ s, f_2\circ s\rangle_{\mathbb{A}^{(\lambda)}(\mathbb{D}^n)}$. Now, the following corollary is immediate from Proposition 4.1.

Corollary 4.2. *The reproducing kernel $K^{(\lambda)}_{\mathbb{G}_n}$ of $\mathcal{H}^{(\lambda)}(\mathbb{G}_n)$ is given explicitly by the formula:*

$$K^{(\lambda)}_{\mathbb{G}_n}\big(\boldsymbol{s}(\boldsymbol{z}),\boldsymbol{s}(\boldsymbol{w})\big) = \frac{1}{n!}\mathrm{per}\Big(\big(\!\!\big((1-z_j\bar{w}_k)^{-\lambda}\big)\!\!\big)_{j,k=1}^n\Big),\ \ \boldsymbol{z},\boldsymbol{w}\in\mathbb{D}^n.$$

Choosing the partition $\boldsymbol{p}$ of n to be $(1^n) := (1,\ldots,1)\in[n]$, we see that

$$\mathbb{P}_{(1^n)}\big(\mathbb{A}^{(\mu)}(\mathbb{D}^n)\big) = \{f\in\mathbb{A}^{(\mu)}(\mathbb{D}^n) : f\circ\sigma^{-1} = \mathrm{sgn}(\sigma)f \text{ for } \sigma\in\mathfrak{S}_n\}.$$

Since $\mathbb{P}_{(1^n)}\big(\mathbb{A}^{(\mu)}(\mathbb{D}^n)\big)$ consists of anti-symmetric functions, therefore

$$\mathbb{A}^{(\mu)}_{\rm anti}(\mathbb{D}^n) = \mathbb{P}_{(1^n)}\big(\mathbb{A}^{(\mu)}(\mathbb{D}^n)\big).$$

Appealing to [4, Proposition 3.8] for $\boldsymbol{p}=(n)$ and $\boldsymbol{p}=(1^n)$, we have a particular case of [4, Proposition 3.8], which we record below for future reference.

Lemma 4.3. *The Hilbert spaces $\mathbb{A}^{(\lambda)}_{\rm sym}(\mathbb{D}^n)$ and $\mathbb{A}^{(\mu)}_{\rm anti}(\mathbb{D}^n)$ are Hilbert modules over $\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}$, under its natural action for $\lambda,\mu>0$ and $n\geq 2$.*

The theorem below provides an affirmative answer to the question we raised in the beginning of this section.

Theorem 4.4. *The Hilbert modules $\mathbb{A}^{(\lambda)}_{\rm sym}(\mathbb{D}^n)$ and $\mathbb{A}^{(\lambda)}_{\rm anti}(\mathbb{D}^n)$ over $\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}$ are not equivalent for any $\lambda>0$ and $n\geq 2$.*

We recall that $\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n} = \mathbb{C}[s_1,\ldots,s_n]$. In view of this fact $\mathcal{H}^{(\lambda)}(\mathbb{G}_n)$ is a Hilbert module over $\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}$, under the natural action of $\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}$. Consider the map from $\mathcal{H}^{(\lambda)}(\mathbb{G}_n)$ to $\mathbb{A}^{(\lambda)}_{\rm sym}(\mathbb{D}^n)$ defined by $f\mapsto f\circ\boldsymbol{s}$ and note that it is a unitary map which intertwines the n-tuple $(M_{s_1},M_{s_2},\ldots,M_{s_n})$ of multiplication operators by the coordinate functions $s_1,\ldots,s_n$ and the tuple $(M_{s_1(\boldsymbol{z})},M_{s_2(\boldsymbol{z})},\ldots,M_{s_n(\boldsymbol{z})})$, where $s_i(\boldsymbol{z})$ is the i-th elementary symmetric function in $z_1,\ldots,z_n$ for $i=1,\ldots,n$. Therefore, there is a unitary module map between the Hilbert modules $\mathcal{H}^{(\lambda)}(\mathbb{G}_n)$ and $\mathbb{A}^{(\lambda)}_{\rm sym}(\mathbb{D}^n)$ over $\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}$. We record this observation in the form of a lemma.

Lemma 4.5. *For $\lambda > 0$, the Hilbert modules $\mathcal{H}^{(\lambda)}(\mathbb{G}_n)$ and $\mathbb{A}^{(\lambda)}_{\text{sym}}(\mathbb{D}^n)$ are equivalent as modules over $\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}$.*

Now we describe the weighted Bergman space on the symmetrized polydisc $\mathbb{G}_n$ as a module over $\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}$. For $\mu > 1$, let $dV^{(\mu)}$ be the probability measure $\left(\frac{\mu-1}{\pi}\right)^n\Big(\prod_{i=1}^n(1-r_i^2)^{\mu-2}r_idr_id\theta_i\Big)$ on the polydisc $\mathbb{D}^n$. Let $dV_{\boldsymbol{s}}^{(\mu)}$ be the measure on the symmetrized polydisc $\mathbb{G}_n$ obtained by the change of variables formula [3, p. 106]:

$$\int_{\mathbb{G}_n} f dV_{\boldsymbol{s}}^{(\mu)} = \frac{1}{n!}\int_{\mathbb{D}^n}(f\circ\boldsymbol{s})|J_{\boldsymbol{s}}|^2 dV^{(\mu)},\ \mu > 1, \tag{4.2}$$

where $J_{\boldsymbol{s}}(\boldsymbol{z}) = \Delta(\boldsymbol{z})$ is the complex jacobian of the symmetrization map $\boldsymbol{s}$. The weighted Bergman space $\mathbb{A}^{(\mu)}(\mathbb{G}_n)$, $\mu > 1$, on the symmetrized polydisc $\mathbb{G}_n$ is the subspace of $L^2(\mathbb{G}_n, dV_{\boldsymbol{s}}^{(\mu)})$ consisting of holomorphic functions. For $\mu > 1$, consider the map $\Gamma : \mathbb{A}^{(\mu)}(\mathbb{G}_n) \to \mathbb{A}^{(\mu)}(\mathbb{D}^n)$ defined by

$$\Gamma f = \frac{1}{\sqrt{n!}} J_{\boldsymbol{s}}(f\circ\boldsymbol{s}),\ f \in \mathbb{A}^{(\mu)}(\mathbb{G}_n). \tag{4.3}$$

It follows from Equation (4.2) that Γ is an isometry onto $\mathbb{A}^{(\mu)}_{\text{anti}}(\mathbb{D}^n)$ [21, p. 2363]. One can easily check that $\|\boldsymbol{z}^{\boldsymbol{m}}\|^2_{\mathbb{A}^{(\mu)}(\mathbb{D}^n)} = \big\|z_1^{m_1}\ldots z_n^{m_n}\big\|^2_{\mathbb{A}^{(\mu)}(\mathbb{D}^n)} = \frac{m_1!\ldots m_n!}{(\mu)_{m_1}\cdots(\mu)_{m_n}}$. For a partition $\boldsymbol{m} = (m_1,\ldots,m_n) \in [\![n]\!]$, put $a_{\boldsymbol{m}}(\boldsymbol{z}) = a_{\boldsymbol{p}+\boldsymbol{\delta}}(\boldsymbol{z}) = \det\Big(\big(\!\big(z_i^{m_j}\big)\!\big)_{i,j=1}^n\Big)$, where $\boldsymbol{p} \in [n]$ and $\boldsymbol{m} = \boldsymbol{p}+\boldsymbol{\delta}$. The norm of $a_{\boldsymbol{m}}$ in $\mathbb{A}^{(\mu)}(\mathbb{D}^n)$ is easily calculated using orthogonality of distinct monomials in $\mathbb{A}^{(\mu)}(\mathbb{D}^n)$:

$$\|a_{\boldsymbol{m}}\|^2_{\mathbb{A}^{(\mu)}(\mathbb{D}^n)} = \Big\|\sum_{\sigma\in\mathfrak{S}_n}\operatorname{sgn}(\sigma)\prod_{k=1}^n z_k^{m_{\sigma(k)}}\Big\|^2_{\mathbb{A}^{(\mu)}(\mathbb{D}^n)} = \sum_{\sigma\in\mathfrak{S}_n}\|\prod_{k=1}^n z_k^{m_{\sigma(k)}}\|^2_{\mathbb{A}^{(\mu)}(\mathbb{D}^n)} = \frac{n!\boldsymbol{m}!}{(\mu)_{\boldsymbol{m}}},$$

where $\boldsymbol{m}! = \prod_{j=1}^n m_j!$ and $(\mu)_{\boldsymbol{m}} = \prod_{j=1}^n(\mu)_{m_j}$. Here $(\mu)_{m_j}$ is the Pochhammer symbol $(\mu)_{m_j} = \mu(\mu+1)\ldots(\mu+m_j-1)$. Putting $c_{\boldsymbol{m}} = \sqrt{\frac{(\mu)_{\boldsymbol{m}}}{n!\boldsymbol{m}!}}$, it follows from [21, p. 2364] that

$$\{e_{\boldsymbol{m}} = c_{\boldsymbol{m}}a_{\boldsymbol{m}} : \boldsymbol{m} \in [\![n]\!]\}$$

is an orthonormal basis of $\mathbb{A}^{(\mu)}_{\text{anti}}(\mathbb{D}^n)$.

The determinant function $a_{\boldsymbol{p}+\boldsymbol{\delta}}$ is a polynomial and is divisible by each of the differences $z_i - z_j, 1 \le i < j \le n$ and hence by the product

$$\prod_{1\le i<j\le n}^n (z_i - z_j) = \det\Big(\big(\!\big(z_i^{n-j}\big)\!\big)_{i,j=1}^n\Big) = a_{\boldsymbol{\delta}}(\boldsymbol{z}) = \Delta(\boldsymbol{z}).$$

For $\boldsymbol{p} \in [n]$, the quotient $S_{\boldsymbol{p}} := \frac{a_{\boldsymbol{p}+\boldsymbol{\delta}}}{a_{\boldsymbol{\delta}}}$, is therefore well-defined and is called the Schur polynomial [14, p. 454]. For $\boldsymbol{p} \in [n]$ and $\boldsymbol{m} = \boldsymbol{p}+\boldsymbol{\delta}$, recall that $c_{\boldsymbol{m}} = c_{\boldsymbol{p}+\boldsymbol{\delta}} = \sqrt{\frac{(\mu)_{\boldsymbol{p}+\boldsymbol{\delta}}}{n!(\boldsymbol{p}+\boldsymbol{\delta})!}}$, now it follows from Equation (4.3) that

$$\Gamma\left(\sqrt{\frac{(\mu)_{\boldsymbol{p}+\boldsymbol{\delta}}}{(\boldsymbol{p}+\boldsymbol{\delta})!}}S_{\boldsymbol{p}}\right) = \Gamma\Big(\sqrt{n!}c_{\boldsymbol{p}+\boldsymbol{\delta}}S_{\boldsymbol{p}}\Big) = c_{\boldsymbol{p}+\boldsymbol{\delta}}a_{\boldsymbol{p}+\boldsymbol{\delta}} = c_{\boldsymbol{m}}a_{\boldsymbol{m}},\ \boldsymbol{m} \in [\![n]\!].$$

Since the map $\Gamma : \mathbb{A}^{(\mu)}(\mathbb{G}_n) \to \mathbb{A}^{(\mu)}_{\text{anti}}(\mathbb{D}^n)$ defined by Equation (4.3) is a unitary [21, p. 2363], the set

$$\{\gamma_{\boldsymbol{p}}S_{\boldsymbol{p}} : \boldsymbol{p} \in [n]\},\ \text{where } \gamma_{\boldsymbol{p}} = \sqrt{\frac{(\mu)_{\boldsymbol{p}+\boldsymbol{\delta}}}{(\boldsymbol{p}+\boldsymbol{\delta})!}},$$

is an orthonormal basis for $\mathbb{A}^{(\mu)}(\mathbb{G}_n)$. Hence we have the following proposition,

Proposition 4.6. *The reproducing kernel $\mathbf{B}^{(\mu)}_{\mathbb{G}_n}$ for $\mathbb{A}^{(\mu)}(\mathbb{G}_n)$ is given by*

$$\mathbf{B}^{(\mu)}_{\mathbb{G}_n}\big(\boldsymbol{s}(\boldsymbol{z}),\boldsymbol{s}(\boldsymbol{w})\big)=\sum_{\boldsymbol{p}\in[n]}\gamma^2_{\boldsymbol{p}}S_{\boldsymbol{p}}(\boldsymbol{z})\overline{S_{\boldsymbol{p}}(\boldsymbol{w})},\ \boldsymbol{z},\boldsymbol{w}\in\mathbb{D}^n,\mu>1. \tag{4.4}$$

From [21, p. 2363], it follows that $\mathbb{A}^{(\mu)}_{\text{anti}}(\mathbb{D}^n)$ and the weighted Bergman module $\mathbb{A}^{(\mu)}(\mathbb{G}_n)$ are unitarily equivalent as modules over $\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}$ for $\mu>1$. The limiting case $\mu=1$, is discussed in [21, p. 2367]. It is not difficult to show that the function $\mathbf{B}^{(\mu)}_{\mathbb{G}_n}:\mathbb{G}_n\times\mathbb{G}_n\to\mathbb{C}$, defined by the Equation (4.4), is positive definite for $\mu>0$. For $0<\mu<1$, let $\mathbb{A}^{(\mu)}(\mathbb{G}_n)$ be the Hilbert space of holomorphic functions having $\mathbf{B}^{(\mu)}_{\mathbb{G}_n}$ as its reproducing kernel. If we assume that the set $\{S_{\boldsymbol{p}}\}_{\boldsymbol{p}\in[n]}$ is orthogonal in $\mathbb{A}^{(\mu)}(\mathbb{G}_n)$ and $\|S_{\boldsymbol{p}}\|^2=\frac{(\boldsymbol{p}+\boldsymbol{\delta})!}{(\mu)_{\boldsymbol{p}+\boldsymbol{\delta}}}$, then it is easy to verify that the injective linear map $\Gamma:\mathbb{A}^{(\mu)}(\mathbb{G}_n)\to\mathbb{A}^{(\mu)}(\mathbb{D}^n)$ defined in Equation (4.3) is an isometry. By similar arguments as in the case $\mu>1$, we reach the desired conclusion for $0<\mu<1$ as well. This observation is recorded in the following Lemma.

Lemma 4.7. *For $\mu>0$, the Hilbert modules $\mathbb{A}^{(\mu)}(\mathbb{G}_n)$ and $\mathbb{A}^{(\mu)}_{\text{anti}}(\mathbb{D}^n)$ are equivalent, as modules over $\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}$.*

In view of Lemma 4.5 and Lemma 4.7, proving Theorem 4.4 amounts to proving the following theorem.

Theorem 4.8. *The Hilbert modules $\mathbb{A}^{(\lambda)}(\mathbb{G}_n)$ and $\mathcal{H}^{(\lambda)}(\mathbb{G}_n)$ over $\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}$ are not equivalent for any $\lambda>0$ and $n\geq 2$.*

To prove this theorem, we recall the notion of a normalized kernel from [10]. Let $\Omega\subseteq\mathbb{C}^n$ be domain. A kernel function $K:\Omega\times\Omega\to\mathbb{C}$ is said to be normalized at $\boldsymbol{w}_0\in\Omega$ if $K(\boldsymbol{z},\boldsymbol{w}_0)=1$ for $\boldsymbol{z}\in\Omega_0$, where $\Omega_0\subseteq\Omega$, is a neighborhood of $\boldsymbol{w}_0$. We note that $S_{\boldsymbol{p}}$ is a homogeneous symmetric polynomial of degree $|\boldsymbol{p}|:=\sum_{i=1}^n p_i$, so $S_{\boldsymbol{0}}\equiv 1$ and $S_{\boldsymbol{p}}(\boldsymbol{0})=0$ for $\boldsymbol{p}\neq\boldsymbol{0}$, where $\boldsymbol{0}\in[n]$ with all components equal to 0. From Equation (4.4) and the discussion following Proposition 4.6, we see that $\mathbf{B}^{(\mu)}_{\mathbb{G}_n}\big(\boldsymbol{s}(\boldsymbol{z}),\boldsymbol{0}\big)=\gamma^2_{\boldsymbol{0}}=\frac{(\mu)_{\boldsymbol{\delta}}}{\boldsymbol{\delta}!}$ for $\boldsymbol{z}\in\mathbb{D}^n$ and $\mu>0$. We record the following obvious corollary of Proposition 4.6 for future reference.

Corollary 4.9. *The normalized reproducing kernel $\widetilde{\mathbf{B}}^{(\mu)}_{\mathbb{G}_n}$ for $\mathbb{A}^{(\mu)}(\mathbb{G}_n)$ is given by*

$$\widetilde{\mathbf{B}}^{(\mu)}_{\mathbb{G}_n}\big(\boldsymbol{s}(\boldsymbol{z}),\boldsymbol{s}(\boldsymbol{w})\big)=\frac{\boldsymbol{\delta}!}{(\mu)_{\boldsymbol{\delta}}}\sum_{\boldsymbol{p}\in[n]}\gamma^2_{\boldsymbol{p}}S_{\boldsymbol{p}}(\boldsymbol{z})\overline{S_{\boldsymbol{p}}(\boldsymbol{w})},\ \boldsymbol{z},\boldsymbol{w}\in\mathbb{D}^n,\mu>0. \tag{4.5}$$

It is of independent interest to express the reproducing kernel $\mathbf{B}^{(\mu)}_{\mathbb{G}_n}$ in terms of coordinates of $\mathbb{G}_n$, that is, in terms of elementary symmetric polynomials. In order to do that, we need to introduce some terminologies. To a partition $\boldsymbol{p}=(p_1,\ldots,p_n)\in[n]$ is associated a *Young diagram* [14, Section 4.1] with p_i boxes in the i-th row, the rows of boxes lined up on the left. The *conjugate partition* $\boldsymbol{p}'=(p'_1,\ldots,p'_r)$ to the partition $\boldsymbol{p}$ is defined by interchanging rows and columns in the Young diagram, that is, reflecting the diagram in the 45° line. For example, the conjugate partition to the partition $(3,3,2,1,1)$ is $(5,3,2)$. For the conjugate partition $\boldsymbol{p}'=(p'_1,\ldots,p'_r)$ to $\boldsymbol{p}$, let us require that $p'_r>0$ and call r the length of $\boldsymbol{p}'$. Let us agree to call s_k the k-th elementary symmetric polynomial in n variables for $k=0,1,\ldots,$ with the convention that $s_k\equiv 0$ if $k>n$. We are now ready to state the second of *Giambelli's formulas* expressing the Schur polynomials as functions of elementary symmetric polynomials. Here is Giambelli's second formula [14, p. 455]:

$$S_{\boldsymbol{p}} = \det\Big((\!(s_{p'_i+j-i})\!)_{i,j=1}^{r} \Big),\ \boldsymbol{p} \in [n], \tag{4.6}$$

where $\boldsymbol{p}' = (p'_1, \ldots, p'_r)$ is the conjugate partition to $\boldsymbol{p}$.

Combining Corollary 4.9 with the Equation (4.6), we obtain the following theorem.

Theorem 4.10. *The normalized reproducing kernel $\widetilde{\mathbf{B}}^{(\mu)}_{\mathbb{G}_n}$ for $\mathbb{A}^{(\mu)}(\mathbb{G}_n)$ is given by*

$$\widetilde{\mathbf{B}}^{(\mu)}_{\mathbb{G}_n}(\boldsymbol{s}, \boldsymbol{t}) = \frac{\boldsymbol{\delta}!}{(\mu)_{\boldsymbol{\delta}}} \sum_{\boldsymbol{p} \in [n]} \gamma_{\boldsymbol{p}}^2 \det\Big((\!(s_{p'_i+j-i})\!)_{i,j=1}^{r} \Big) \overline{\det\Big((\!(t_{p'_i+j-i})\!)_{i,j=1}^{r} \Big)},$$

for $\boldsymbol{s} = (s_1, \ldots, s_n), \boldsymbol{t} = (t_1, \ldots, t_n) \in \mathbb{G}_n, \mu > 0$ and $\boldsymbol{p}' = (p'_1, \ldots, p'_r)$ is the conjugate partition to $\boldsymbol{p} \in [n]$.

The following lemma will be useful in proving Theorem 4.8.

Lemma 4.11. *Let $\widetilde{\mathbf{B}}^{(\mu)}_{\mathbb{G}_n}$ be the normalized reproducing kernel for $\mathbb{A}^{(\mu)}(\mathbb{G}_n)$. Then*

(i) *the coefficient of $s_1(\boldsymbol{z})\overline{s_1(\boldsymbol{w})}$ in $\widetilde{\mathbf{B}}^{(\mu)}_{\mathbb{G}_n}\big(\boldsymbol{s}(\boldsymbol{z}), \boldsymbol{s}(\boldsymbol{w})\big)$ is $\frac{\mu+n-1}{n}$,*

(ii) *the coefficient of $s_1(\boldsymbol{z})^2\overline{s_1(\boldsymbol{w})^2}$ in $\widetilde{\mathbf{B}}^{(\mu)}_{\mathbb{G}_n}\big(\boldsymbol{s}(\boldsymbol{z}), \boldsymbol{s}(\boldsymbol{w})\big)$ is $\frac{(\mu+n-1)(\mu+n)}{n(n+1)}$.*

Proof. Since the Schur polynomial $S_{\boldsymbol{p}}$ is a homogeneous symmetric polynomial of degree $|\boldsymbol{p}| := \sum_{i=1}^n p_i$, therefore, it is a polynomial in the elementary symmetric polynomials $s_i(\boldsymbol{z})$ for $i = 1, \ldots, n$. For a fixed $k, q \in \mathbb{Z}_+$, the term $s_k(\boldsymbol{z})^q\overline{s_k(\boldsymbol{w})^q}$ in $\widetilde{\mathbf{B}}^{(\mu)}_{\mathbb{G}_n}\big(\boldsymbol{s}(\boldsymbol{z}), \boldsymbol{s}(\boldsymbol{w})\big)$ comes only from the terms which involve $S_{\boldsymbol{p}}(\boldsymbol{z})\overline{S_{\boldsymbol{p}}(\boldsymbol{w})}$ in the series for $\widetilde{\mathbf{B}}^{(\mu)}_{\mathbb{G}_n}\big(\boldsymbol{s}(\boldsymbol{z}), \boldsymbol{s}(\boldsymbol{w})\big)$ in Equation (4.5), where $\boldsymbol{p} = (p_1, \ldots, p_n) \in [n]$ such that $\sum_{i=1}^n p_i = kq$.

To get the coefficient of $s_1(\boldsymbol{z})\overline{s_1(\boldsymbol{w})}$ in $\widetilde{\mathbf{B}}^{(\mu)}_{\mathbb{G}_n}\big(\boldsymbol{s}(\boldsymbol{z}), \boldsymbol{s}(\boldsymbol{w})\big)$, take $\boldsymbol{p} = (1, 0, ..., 0)$. From Equation (4.5), the coefficient of $s_1(\boldsymbol{z})\overline{s_1(\boldsymbol{w})}$ in $\widetilde{\mathbf{B}}^{(\mu)}_{\mathbb{G}_n}\big(\boldsymbol{s}(\boldsymbol{z}), \boldsymbol{s}(\boldsymbol{w})\big)$ is

$$\frac{\boldsymbol{\delta}!}{(\mu)_{\boldsymbol{\delta}}}\gamma_{\boldsymbol{p}}^2 = \frac{\boldsymbol{\delta}!}{(\mu)_{\boldsymbol{\delta}}} \cdot \frac{(\mu)_{\boldsymbol{p}+\boldsymbol{\delta}}}{(\boldsymbol{p}+\boldsymbol{\delta})!} = \frac{\mu+n-1}{n},$$

where $\boldsymbol{p} = (1, 0, \ldots, 0)$. This proves (i).

Similarly, to obtain the coefficient of $s_1(\boldsymbol{z})^2\overline{s_1(\boldsymbol{w})^2}$ in $\widetilde{\mathbf{B}}^{(\mu)}_{\mathbf{G}_n}\big(\boldsymbol{s}(\boldsymbol{z}), \boldsymbol{s}(\boldsymbol{w})\big)$, we need to consider terms corresponding to $\boldsymbol{p} = (2, 0, \ldots, 0)$ and $\boldsymbol{p} = (1, 1, 0, \ldots, 0)$. From the Giambelli's formula (4.6), we get $S_{(2,0,\ldots,0,0)}(\boldsymbol{z}) = ({s_1}^2 - s_2)(\boldsymbol{z})$ and $S_{(1,1,\ldots,0,0)}(\boldsymbol{z}) = s_2(\boldsymbol{z})$.

Since s_1^2 appears only in $S_{(2,0,\ldots,0)}$, from Equation (4.5), it follows that the coefficient of $s_1(\boldsymbol{z})^2\overline{s_1(\boldsymbol{w})^2}$ in $\widetilde{\mathbf{B}}^{(\mu)}_{\mathbb{G}_n}\big(\boldsymbol{s}(\boldsymbol{z}), \boldsymbol{s}(\boldsymbol{w})\big)$ is

$$\frac{\boldsymbol{\delta}!}{(\mu)_{\boldsymbol{\delta}}}\gamma_{\boldsymbol{p}}^2 = \frac{\boldsymbol{\delta}!}{(\mu)_{\boldsymbol{\delta}}} \cdot \frac{(\mu)_{\boldsymbol{p}+\boldsymbol{\delta}}}{(\boldsymbol{p}+\boldsymbol{\delta})!} = \frac{(\mu+n-1)(\mu+n)}{n(n+1)},$$

where $\boldsymbol{p} = (2, 0, \ldots, 0)$. This proves (ii). ∎

Consider the restriction of the action of $\mathfrak{S}_n$ to $\mathbb{Z}_+^n$. Let $\mathfrak{S}_n\boldsymbol{m}$ denote the orbit of $\boldsymbol{m} \in \mathbb{Z}_+^n$. If $\boldsymbol{m} \in [n]$ has $k(\leq n)$ distinct components, that is, there are k distinct non-negative integers $m_1 > \ldots > m_k$ such that

$$\boldsymbol{m} = (m_1, \ldots, m_1, m_2, \ldots, m_2, \ldots, m_k, \ldots, m_k),$$

where each m_i is repeated α_i times, for $i = 1, \ldots, k$, then $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_k)$ is said to be the *multiplicity* of $\boldsymbol{m} \in [n]$. For any $\boldsymbol{m} \in \mathbb{Z}_+^n$ the components of $\boldsymbol{m}$ can be arranged in the decreasing order to obtain, say, $\widetilde{\boldsymbol{m}} \in [n]$. We say that $\boldsymbol{m} \in \mathbb{Z}_+^n$ is of *multiplicity* $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_k)$ if $\widetilde{\boldsymbol{m}}$ has multiplicity $\boldsymbol{\alpha}$. In particular, the elements of $[\![n]\!]$ are of multiplicity (1^n), that is, 1 occurs n-times.

We recall that the number of distinct n-letter words with k distinct letters is $\frac{n!}{\boldsymbol{\alpha}!} = \frac{n!}{\alpha_1! \ldots \alpha_k!}$, where the k distinct letters $a_1, \ldots, a_k$ are repeated $\alpha_1, \ldots, \alpha_k$ times, respectively $(\alpha_1 + \ldots + \alpha_k = n)$. In other words, for a fixed $\boldsymbol{m} \in \mathbb{Z}_+^n$, we have $|\mathfrak{S}_n \boldsymbol{m}| = \frac{n!}{\boldsymbol{\alpha}!}$, where $|X|$ denotes the cardinality of a set X. Let $\mathbb{Z}_+^n / \mathfrak{S}_n$ denote the set of all orbits of $\mathbb{Z}_+^n$ under the action of $\mathfrak{S}_n$. We record the following as a lemma for later use.

Lemma 4.12. *The set $\mathbb{Z}_+^n / \mathfrak{S}_n$ is in one-one correspondence with the set $[n]$.*

Proof. First, we prove that each $\mathfrak{S}_n$ orbit of $\mathbb{Z}_+^n$ has exactly one n-tuple in decreasing order. To see this, observe that each orbit contains an n-tuple in decreasing order, and hence enough to prove it is unique. Suppose there are two n-tuples in decreasing order, say $\boldsymbol{m}, \boldsymbol{m}'$, in the same orbit. Since a permutation only changes the position of a component, it follows that all n-tuples in an orbit have the same multiplicity. Therefore the multiplicity of $\boldsymbol{m}$ and $\boldsymbol{m}'$ is the same and hence $\boldsymbol{m} = \boldsymbol{m}'$. Note that each element in $[n]$ is in some orbit and hence the proof is complete. ∎

Consider the monomial symmetric polynomials [14, p. 454]

$$\mathbb{M}_{\boldsymbol{m}}(\boldsymbol{z}) = \sum_{\boldsymbol{\beta}} \boldsymbol{z}^{\boldsymbol{\beta}},$$

where the sum is over all distinct permutations $\boldsymbol{\beta} = (\beta_1, \beta_2, ..., \beta_n)$ of $\boldsymbol{m} \in [n]$ and $\boldsymbol{z}^{\boldsymbol{\beta}} = z_1^{\beta_1} z_2^{\beta_2} ... z_n^{\beta_n}$. This definition of $\mathbb{M}_{\boldsymbol{m}}$ makes sense for $\boldsymbol{m} \in \mathbb{Z}_+^n$ as well and we use it in the sequel. Observe that $\mathfrak{S}_n \boldsymbol{m}$ is the set of all distinct permutations of $\boldsymbol{m}$, so,

$$\mathbb{M}_{\boldsymbol{m}}(\boldsymbol{z}) = \sum_{\boldsymbol{\beta} \in \mathfrak{S}_n \boldsymbol{m}} \boldsymbol{z}^{\boldsymbol{\beta}} = \mathbb{M}_{\boldsymbol{m}'}(\boldsymbol{z}) \text{ for } \boldsymbol{m}, \boldsymbol{m}' \in \mathfrak{S}_n \boldsymbol{m}. \tag{4.7}$$

The following lemma that gives us an expression for the reproducing kernel $K_{\mathbb{G}_n}^{(\lambda)}$ for $\mathcal{H}^{(\lambda)}(\mathbb{G}_n)$ will play a significant role in the sequel.

Lemma 4.13. *The reproducing kernel $K_{\mathbb{G}_n}^{(\lambda)}$ for $\mathcal{H}^{(\lambda)}(\mathbb{G}_n)$ is given by the formula:*

$$K_{\mathbb{G}_n}^{(\lambda)}\big(\boldsymbol{s}(\boldsymbol{z}), \boldsymbol{s}(\boldsymbol{w})\big) = \frac{1}{n!} \sum_{\boldsymbol{m} \in [n]} \frac{\boldsymbol{\alpha}! (\lambda)_{\boldsymbol{m}}}{\boldsymbol{m}!} \mathbb{M}_{\boldsymbol{m}}(\boldsymbol{z}) \overline{\mathbb{M}_{\boldsymbol{m}}(\boldsymbol{w})},\ \boldsymbol{z}, \boldsymbol{w} \in \mathbb{D}^n,$$

where $\boldsymbol{m}$ is of multiplicity $\boldsymbol{\alpha}$.

Proof. If $\boldsymbol{m} \in [n]$ is of multiplicity $\boldsymbol{\alpha}$, then $\mathbb{M}_{\boldsymbol{m}}(\boldsymbol{z})$ is the sum of $|\mathfrak{S}_n \boldsymbol{m}| = \frac{n!}{\boldsymbol{\alpha}!}$ distinct monomials. We then observe that

$$\operatorname{per}\Big((\!(z_i^{m_j})\!)_{i,j=1}^n \Big) = \sum_{\sigma \in \mathfrak{S}_n} \prod_{i=1}^n z_i^{m_{\sigma(i)}} = \sum_{\sigma \in \mathfrak{S}_n} \prod_{i=1}^n z_i^{m_{\sigma^{-1}(i)}} = \sum_{\sigma \in \mathfrak{S}_n} \boldsymbol{z}^{\sigma \cdot \boldsymbol{m}}, \tag{4.8}$$

is the sum of $n!$ monomials, from which exactly $|\mathfrak{S}_n \boldsymbol{m}| = \frac{n!}{\boldsymbol{\alpha}!}$ are distinct (since there can be only $\frac{n!}{\boldsymbol{\alpha}!}$ distinct permutations of a $\boldsymbol{m} \in [n]$ with multiplicity $\boldsymbol{\alpha}$). So, each distinct term must be repeated $\boldsymbol{\alpha}!$ times. Thus, from equation (4.7) we conclude that

$$\mathrm{per}\Big(\!\big(\!\big(z_i^{m_j}\big)\!\big)_{i,j=1}^n\Big) = \boldsymbol{\alpha}!\mathbb{M}_{\boldsymbol{m}'}(\boldsymbol{z}), \quad \text{for any } \boldsymbol{m}' \in \mathfrak{S}_n\boldsymbol{m}. \tag{4.9}$$

Since $\mathbb{Z}_+^n$ is the disjoint union of its $\mathfrak{S}_n$-orbits, from Lemma 4.12 we have

$$\mathbb{Z}_+^n = \cup_{\boldsymbol{m}\in[n]}\mathfrak{S}_n\boldsymbol{m}. \tag{4.10}$$

Therefore, from Corollary 4.2, we have

$$\begin{aligned}
K_{\mathbb{G}_n}^{(\lambda)}\big(\boldsymbol{s}(\boldsymbol{z}),\boldsymbol{s}(\boldsymbol{w})\big) &= \frac{1}{n!}\mathrm{per}\Big(\!\big(\!\big((1-z_j\bar{w}_k)^{-\lambda}\big)\!\big)_{j,k=1}^n\Big)\\
&= \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n}\prod_{i=1}^n(1-z_i\bar{w}_{\sigma(i)})^{-\lambda}\\
&= \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n}\sum_{\boldsymbol{m}\in\mathbb{Z}_+^n}\frac{(\lambda)_{\boldsymbol{m}}}{\boldsymbol{m}!}\prod_{i=1}^n z_i^{m_i}\prod_{i=1}^n\bar{w}_{\sigma(i)}^{m_i}\\
&= \frac{1}{n!}\sum_{\boldsymbol{m}\in\mathbb{Z}_+^n}\frac{(\lambda)_{\boldsymbol{m}}}{\boldsymbol{m}!}\prod_{i=1}^n z_i^{m_i}\sum_{\sigma\in\mathfrak{S}_n}\prod_{i=1}^n\bar{w}_{\sigma(i)}^{m_i}\\
&= \frac{1}{n!}\sum_{\boldsymbol{m}\in\mathbb{Z}_+^n}\frac{(\lambda)_{\boldsymbol{m}}}{\boldsymbol{m}!}\prod_{i=1}^n z_i^{m_i}\mathrm{per}\Big(\!\big(\!\big(\bar{w}_i^{m_j}\big)\!\big)_{i,j=1}^n\Big)\\
&= \frac{1}{n!}\sum_{\boldsymbol{m}\in[n]}\sum_{\boldsymbol{m}'\in\mathfrak{S}_n\boldsymbol{m}}\frac{(\lambda)_{\boldsymbol{m}'}}{\boldsymbol{m}'!}\prod_{i=1}^n z_i^{m_i'}\mathrm{per}\Big(\!\big(\!\big(\bar{w}_i^{m_j'}\big)\!\big)_{i,j=1}^n\Big) \text{ (using (4.10))}\\
&= \frac{1}{n!}\sum_{\boldsymbol{m}\in[n]}\frac{(\lambda)_{\boldsymbol{m}}}{\boldsymbol{m}!}\mathrm{per}\Big(\!\big(\!\big(\bar{w}_i^{m_j}\big)\!\big)_{i,j=1}^n\Big)\sum_{\boldsymbol{m}'\in\mathfrak{S}_n\boldsymbol{m}}\prod_{i=1}^n z_i^{m_i'}\\
&= \frac{1}{n!}\sum_{\boldsymbol{m}\in[n]}\frac{(\lambda)_{\boldsymbol{m}}}{\boldsymbol{m}!}\boldsymbol{\alpha}!\overline{\mathbb{M}_{\boldsymbol{m}}(\boldsymbol{w})}\sum_{\boldsymbol{m}'\in\mathfrak{S}_n\boldsymbol{m}}\boldsymbol{z}^{\boldsymbol{m}'} \text{ (using (4.9))}\\
&= \frac{1}{n!}\sum_{\boldsymbol{m}\in[n]}\frac{\boldsymbol{\alpha}!(\lambda)_{\boldsymbol{m}}}{\boldsymbol{m}!}\mathbb{M}_{\boldsymbol{m}}(\boldsymbol{z})\overline{\mathbb{M}_{\boldsymbol{m}}(\boldsymbol{w})},
\end{aligned}$$

where the last equality follows from Equation (4.7). ∎

Remark 4.14. *One could also write the reproducing kernel in terms of permanent using the equations* (4.9) *and* (4.10) *and the equality* $|\mathfrak{S}_n\boldsymbol{m}| = \frac{n!}{\boldsymbol{\alpha}!}$*, as follows:*

$$\begin{aligned}
K_{\mathbb{G}_n}^{(\lambda)}\big(\boldsymbol{s}(\boldsymbol{z}),\boldsymbol{s}(\boldsymbol{w})\big) &= \frac{1}{n!}\sum_{\boldsymbol{m}\in[n]}\sum_{\boldsymbol{m}'\in\mathfrak{S}_n\boldsymbol{m}}\big(\frac{n!}{\boldsymbol{\alpha}!}\big)^{-1}\frac{\boldsymbol{\alpha}!(\lambda)_{\boldsymbol{m}'}}{\boldsymbol{m}'!}\frac{1}{\boldsymbol{\alpha}!}\mathrm{per}\Big(\!\big(\!\big(z_i^{m_j'}\big)\!\big)_{i,j=1}^n\Big)\frac{1}{\boldsymbol{\alpha}!}\mathrm{per}\Big(\!\big(\!\big(\bar{w}_i^{m_j'}\big)\!\big)_{i,j=1}^n\Big)\\
&= \frac{1}{(n!)^2}\sum_{\boldsymbol{m}\in\mathbb{Z}_+^n}\frac{(\lambda)_{\boldsymbol{m}}}{\boldsymbol{m}!}\mathrm{per}\Big(\!\big(\!\big(z_i^{m_j}\big)\!\big)_{i,j=1}^n\Big)\mathrm{per}\Big(\!\big(\!\big(\bar{w}_i^{m_j}\big)\!\big)_{i,j=1}^n\Big),
\end{aligned}$$

for $\boldsymbol{z},\boldsymbol{w}\in\mathbb{D}^n$.

We note that the kernels $B_{\mathbb{G}_n}^{(\lambda)}$ and $K_{\mathbb{G}_n}^{(\lambda)}$ are defined on all of $\mathbb{G}_n$. Hence the Hilbert modules $\mathbb{P}_{(n)}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$ and $\mathbb{P}_{(1,\ldots,1)}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$ are locally free on all of $\mathbb{G}_n$ strengthening our earlier assertion (Corollary 3.12) that they are locally free only on $\mathbb{G}_n\setminus\boldsymbol{s}(\mathcal{Z})$. Thus we have proved the following corollary.

Corollary 4.15. *The Hilbert modules* $\mathbb{P}_{(n)}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$ *and* $\mathbb{P}_{(1,\ldots,1)}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$ *are locally free of rank* 1 *on* $\mathbb{G}_n$.

Like Lemma 4.11, the following lemma will be useful in proving Theorem 4.8.

Lemma 4.16. *Let* $K^{(\lambda)}_{\mathbb{G}_n}$ *be the normalized reproducing kernel for* $\mathcal{H}^{(\lambda)}(\mathbb{G}_n)$*. Then*

(i) *the coefficient of* $s_1(\boldsymbol{z})\overline{s_1(\boldsymbol{w})}$ *in* $K^{(\lambda)}_{\mathbb{G}_n}\big(\boldsymbol{s}(\boldsymbol{z}),\boldsymbol{s}(\boldsymbol{w})\big)$ *is* $\frac{\lambda}{n}$,

(ii) *the coefficient of* $s_1(\boldsymbol{z})^2\overline{s_1(\boldsymbol{w})^2}$ *in* $K^{(\lambda)}_{\mathbb{G}_n}\big(\boldsymbol{s}(\boldsymbol{z}),\boldsymbol{s}(\boldsymbol{w})\big)$ *is* $\frac{\lambda(\lambda+1)}{2n}$.

Proof. Since the monomial symmetric polynomial $\mathbb{M}_{\boldsymbol{m}}$ is a homogeneous symmetric polynomial of degree $|\boldsymbol{m}| := \sum_{i=1}^n m_i$, therefore, it is a polynomial in the elementary symmetric polynomials $s_i(\boldsymbol{z})$ for $i = 1,\ldots,n$. For a fixed $k, q \in \mathbb{Z}_+$, the term $s_k(\boldsymbol{z})^q\overline{s_k(\boldsymbol{w})^q}$ in $K^{(\lambda)}_{\mathbb{G}_n}\big(\boldsymbol{s}(\boldsymbol{z}),\boldsymbol{s}(\boldsymbol{w})\big)$ comes only from the terms involving $\mathbb{M}_{\boldsymbol{m}}(\boldsymbol{z})\overline{\mathbb{M}_{\boldsymbol{m}}(\boldsymbol{w})}$ in the series for $K^{(\lambda)}_{\mathbb{G}_n}\big(\boldsymbol{s}(\boldsymbol{z}),\boldsymbol{s}(\boldsymbol{w})\big)$ in Lemma 4.13, where $\boldsymbol{m} = (m_1,\ldots,m_n) \in [n]$ such that $\sum_{i=1}^n m_i = kq$.

To obtain the coefficient of $s_1(\boldsymbol{z})\overline{s_1(\boldsymbol{w})}$ in $K^{(\lambda)}_{\mathbb{G}_n}\big(\boldsymbol{s}(\boldsymbol{z}),\boldsymbol{s}(\boldsymbol{w})\big)$, we only need to consider the term $\mathbb{M}_{\boldsymbol{m}}(\boldsymbol{z})\overline{\mathbb{M}_{\boldsymbol{m}}(\boldsymbol{w})}$, for $\boldsymbol{m} = (1,0,...,0)$. Note that $\mathbb{M}_{\boldsymbol{m}}(\boldsymbol{z}) = s_1(\boldsymbol{z})$. Since $\boldsymbol{m} = (1,0,\ldots,0)$ has multiplicity $\alpha = (1,(n-1))$, it follows that the coefficient of $s_1(\boldsymbol{z})\overline{s_1(\boldsymbol{w})}$ in $K^{(\lambda)}_{\mathbb{G}_n}\big(\boldsymbol{s}(\boldsymbol{z}),\boldsymbol{s}(\boldsymbol{w})\big)$ is

$$\frac{1}{n!}\cdot\frac{\boldsymbol{\alpha}!(\lambda)_{\boldsymbol{m}}}{\boldsymbol{m}!} = \frac{(n-1)!1!(\lambda)_1}{1!n!} = \frac{\lambda}{n}.$$

This proves (i).

Analogously, to find the coefficient of $s_1(\boldsymbol{z})^2\overline{s_1(\boldsymbol{w})^2}$ in $K^{(\lambda)}_{\mathbb{G}_n}\big(\boldsymbol{s}(\boldsymbol{z}),\boldsymbol{s}(\boldsymbol{w})\big)$, we need to consider terms corresponding to $\boldsymbol{m} = (2,0,...,0)$ and $\boldsymbol{m} = (1,1,0,...,0)$. Note that $\mathbb{M}_{\boldsymbol{m}}(\boldsymbol{z}) = s_2(\boldsymbol{z})$ for $\boldsymbol{m} = (1,1,0,\ldots,0)$, so the coefficient of the term $\mathbb{M}_{\boldsymbol{m}}(\boldsymbol{z})\overline{\mathbb{M}_{\boldsymbol{m}}(\boldsymbol{w})}$ for $\boldsymbol{m} = (1,1,0,\ldots,0)$, will not contribute here. Now $\mathbb{M}_{\boldsymbol{m}}(\boldsymbol{z}) = s_1(\boldsymbol{z})^2 - 2s_2(\boldsymbol{z})$ for $\boldsymbol{m} = (2,0,\ldots,0)$. Since $\boldsymbol{m} = (2,0,\ldots,0)$ has multiplicity $\boldsymbol{\alpha} = (1,n-1)$, it follows that the coefficient of $s_1(\boldsymbol{z})^2\overline{s_1(\boldsymbol{w})^2}$ in $K^{(\lambda)}_{\mathbb{G}_n}\big(\boldsymbol{s}(\boldsymbol{z}),\boldsymbol{s}(\boldsymbol{w})\big)$ is

$$\frac{1}{n!}\cdot\frac{\boldsymbol{\alpha}!(\lambda)_{\boldsymbol{m}}}{\boldsymbol{m}!} = \frac{(n-1)!(\lambda)_2}{2!n!} = \frac{\lambda(\lambda+1)}{2n}.$$

This proves (ii). ∎

Proof of Theorem 4.8. If possible, let these two modules be unitarily equivalent. Recall that the reproducing kernels $\widetilde{\mathbf{B}}^{(\lambda)}_{\mathbb{G}_n}$ and $K^{(\lambda)}_{\mathbb{G}_n}$ have the property that

$$\widetilde{\mathbf{B}}^{(\lambda)}_{\mathbb{G}_n}\big(\boldsymbol{s}(\boldsymbol{z}),0\big) = K^{(\lambda)}_{\mathbb{G}_n}\big(\boldsymbol{s}(\boldsymbol{z}),0\big) = 1 \text{ for } \boldsymbol{s}(\boldsymbol{z}) \in \mathbb{G}_n,$$

that is, these are the normalized reproducing kernels at 0 of the respective Hilbert spaces. Since by construction, the polynomial ring $\mathbb{C}[s_1,\ldots,s_n] = \mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}$ in n variables is dense in both $\mathcal{H}^{(\lambda)}(\mathbb{G}_n)$ and $\mathbb{A}^{(\lambda)}(\mathbb{G}_n)$, it follows (cf. [11, Remark, p. 285]) that the dimension of the joint kernel is 1 on $\mathbb{G}_n$. Therefore, by [10, Lemma 4.8(c)], we infer that

$$\widetilde{\mathbf{B}}^{(\lambda)}_{\mathbb{G}_n}\big(\boldsymbol{s}(\boldsymbol{z}),\boldsymbol{s}(\boldsymbol{w})\big) = K^{(\lambda)}_{\mathbb{G}_n}\big(\boldsymbol{s}(\boldsymbol{z}),\boldsymbol{s}(\boldsymbol{w})\big) \text{ for } \boldsymbol{s}(\boldsymbol{z}),\boldsymbol{s}(\boldsymbol{w}) \in \mathbb{G}_n.$$

Equating the coefficients of $s_1(\boldsymbol{z})\overline{s_1(\boldsymbol{z})}$ from Lemma 4.11 we see that $\lambda = \lambda + n - 1$. Thus we must have $n = 1$ completing the proof of the Theorem. ∎

Corollary 4.17. *In the decomposition of the Hilbert module* $\mathbb{A}^{(\lambda)}(\mathbb{D}^3)$:

$$\mathbb{A}^{(\lambda)}(\mathbb{D}^3) = \mathbb{P}_{(3)}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^3)\big) \oplus \mathbb{P}_{(2,1)}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^3)\big) \oplus \mathbb{P}_{(1,1,1)}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^3)\big),$$

all the sub-modules on the right hand side of the equality are inequivalent.

Proof. We have just proved that $\mathbb{P}_{(3)}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^3)\big)$ cannot be equivalent to $\mathbb{P}_{(1,1,1)}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^3)\big)$, in general. Since the rank of the sub-module $\mathbb{P}_{(2,1)}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^3)\big)$ is $\chi_{(2,1)}(1)^2 = 4$, [14, Example 2.6], it cannot be equivalent to either of these. ∎

Remark 4.18. *The proof of Theorem 4.4 shows that we have proved a little more than what is claimed in the Theorem, namely: The Hilbert modules* $\mathbb{A}^{(\lambda)}_{\text{sym}}(\mathbb{D}^n)$ *and* $\mathbb{A}^{(\mu)}_{\text{anti}}(\mathbb{D}^n)$ *over* $\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}$ *are not equivalent for any* $\lambda, \mu > 0$ *and* $n \geq 2$. *To prove this more general claim, we merely note, as before, that equating the coefficients of* $s_1(\boldsymbol{z})\overline{s_1(\boldsymbol{z})}$ *and* $s_1(\boldsymbol{z})^2\overline{s_1(\boldsymbol{z})^2}$ *from Lemma 4.11 and Lemma 4.16, we obtain*

$$\lambda = \mu + n - 1 \quad \text{and} \quad \frac{\lambda(\lambda+1)}{2n} = \frac{(\mu+n-1)(\mu+n)}{n(n+1)}.$$

Combining these equations, we have that $n = 1$, *which proves our claim. Indeed, the two modules* $\mathbb{P}^{ii}_{\boldsymbol{p}}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$ *and* $\mathbb{P}^{jj}_{\boldsymbol{q}}\big(\mathbb{A}^{(\mu)}(\mathbb{D}^n)\big)$ *are not equivalent either for any* $1 \leq i \leq \chi_{\boldsymbol{p}}(1)$ *and* $1 \leq j \leq \chi_{\boldsymbol{q}}(1)$ *for which* $\chi_{\boldsymbol{p}}(1) \neq \chi_{\boldsymbol{q}}(1)$.

Appendix A. The bi-holomorphic automorphism group of $\mathbb{D}^n$ and the weigeted Bergman modules

The bi-holomorphic automorphism group $\operatorname{Aut}(\mathbb{D}^n)$ is the semi-direct product $\operatorname{Aut}(\mathbb{D})^n \rtimes \mathfrak{S}_n$, where $\operatorname{Aut}(\mathbb{D})$ is the bi-holomorphic automorphism group of $\mathbb{D}$. For $\Phi \in \operatorname{Aut}(\mathbb{D}^n)$, define $U : \operatorname{Aut}(\mathbb{D}^n) \to \mathcal{L}\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n)\big)$ by the formula:

$$U(\Phi^{-1})h = \big(\det(D\Phi)\big)^{\lambda/2} h \circ \Phi,\ h \in \mathbb{A}^{(\lambda)}(\mathbb{D}^n).$$

Since the map $(\Phi, \boldsymbol{z}) \mapsto \big(\det(D\Phi)\big)^{\lambda/2}(\boldsymbol{z})$ from $\operatorname{Aut}(\mathbb{D}^n) \times \mathbb{D}^n$ to $\mathbb{C}$ is a (projective) cocycle, it follows that the map U defines a (projective) unitary representation. The Hilbert space $\mathbb{A}^{(\lambda)}(\mathbb{D}^n)$ is also a module over the polynomial ring $\mathbb{C}[\boldsymbol{z}]$, namely,

$$\mathfrak{m}_p(h) = p \cdot h,\ p \in \mathbb{C}[\boldsymbol{z}],\ h \in \mathbb{A}^{(\lambda)}(\mathbb{D}^n),$$

where $p \cdot h$ is the point-wise multiplication. Setting $(\Phi \cdot f)(\boldsymbol{z}) = f\big(\Phi^{-1}(\boldsymbol{z})\big)$, we have the relationship $\mathfrak{m}_{\Phi\cdot p} = U(\Phi)^* \mathfrak{m}_p U(\Phi)$, $\Phi \in \operatorname{Aut}(\mathbb{D}^n), p \in \mathbb{C}[\boldsymbol{z}]$, which is analogous to the imprimitivity introduced by Mackey (cf. [28, Chapter 6]). The imprimitivities of Mackey have been studied extensively and are related to induced representations, representations of the semi-direct product and homogeneous vector bundles, see Theorems 6.12, 6.20 and 6.24 in [28], respectively. However, the situation we have described is different in that the module action is defined over the ring of analytic polynomials rather than the algebra of continuous functions. This, we believe, merits a detailed investigation and the outcome, see [18], [22], so far is very encouraging. Also, the restriction of the representation U to the subgroup $\triangle := \{(\varphi, \ldots, \varphi) : \varphi \in \operatorname{Aut}(\mathbb{D})\}$ of $\operatorname{Aut}(\mathbb{D}^n)$ has a decomposition into irreducible components known as the Clebsch-Gordan decomposition. On the other hand, the symmetric group acts on $\mathbb{A}^{(\lambda)}(\mathbb{D}^n)$ via the unitary map $R_{\sigma^{-1}} : h \to h \circ \sigma$, $\sigma \in \mathfrak{S}_n$. The Hilbert space $\mathbb{A}^{(\lambda)}(\mathbb{D}^n)$ is also a module over the ring of the symmetric polynomials $\mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}$, where the module map is given by the formula: $\mathfrak{m}_p(h) = p \cdot h,\ p \in \mathbb{C}[\boldsymbol{z}]^{\mathfrak{S}_n}$. In this paper, we have studied the imprimitivity $\big(\mathbb{A}^{(\lambda)}(\mathbb{D}^n), \mathfrak{m}_p, R_\sigma\big)$ and an orthogonal decomposition into sub-modules like in the more familiar Clebsch-Gordan decomposition mentioned above. The question of finding a decomposition where each component is minimal and any two of them are inequivalent remains open in general.

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