JOURNAL OF FUNCTIONAL ANALYSIS **136**, 171–213 (1996)
ARTICLE NO. 0026

Homogeneous Tuples of Multiplication Operators on Twisted Bergman Spaces

BHASKAR BAGCHI AND GADADHAR MISRA

Statistics and Mathematics Unit, Indian Statistical Institute, R. V. College Post, Bangalore 560059, India

Received December 6, 1994

Let B the Bergman kernel on the domain $\Omega_{n,m}$ of $n \times m$ contractive complex matrices ($m \geqslant n \geqslant 1$). Let $\mathscr{W} = \mathscr{W}_{n,m}$ be the associated Wallach set consisting of the $\lambda \geqslant 0$ for which $B^{\lambda/(m+n)}$ is (non-negative definite and hence) the reproducing kernel of a functional Hilbert space $\mathscr{H}^{(\lambda)}$. For $\lambda \in \mathscr{W}$, we examine the mn-tuple $\mathbf{M}^{(\lambda)}$ of operators on $\mathscr{H}^{(\lambda)}$ whose components are multiplications by the mn co-ordinate functions. This tuple is homogeneous with respect to the group action of $PSU(n, m)$ on the matrix ball. Utilising this group action we are able to determine the set of all $\lambda \in \mathscr{W}$ for which (i) $\mathbf{M}^{(\lambda)}$ is bounded, and for which (ii) $\mathbf{M}^{(\lambda)}$ is (bounded and) jointly subnormal. Further, the joint Taylor spectrum of $\mathbf{M}^{(\lambda)}$ is determined for all λ as in (i). The subnormality of $\mathbf{M}^{(\lambda)}$ turns out to be closely tied with the representation theory of $PSU(n, m)$. Namely, $\mathbf{M}^{(\lambda)}$ is subnormal precisely when the natural (projective) representation of $PSU(n, m)$ on the twisted Bergman space $\mathscr{H}^{(\lambda)}$ is a subrepresentation of an induced representation of multiplicity 1. Finally, we examine the values of λ for which $\mathbf{M}^{(\lambda)}$ admits its Taylor spectrum as a k-spectral set, and obtain incomplete results on this question. This question remains open and interesting on $n-1$ gaps, that is, for λ belonging to the union of $n-1$ pairwise disjoint open intervals. Most of the techniques developed in this paper are applicable to all bounded Cartan domains, though we stick to the matrix domains $\Omega_{n,m}$ for concreteness.

1. INTRODUCTION AND MAIN RESULTS

1.1. *The Twisted Bergman Spaces*

Let $m \geqslant n \geqslant 1$ be integers. Throughout this paper $\Omega = \Omega_{n,m}$ will denote the open unit ball of the Banach space $\mathbb{C}^{n \times m}$ of $n \times m$ complex matrices with operator norm. $\mathscr{H}$ will denote the Bergman space on Ω; it is the Hilbert space of analytic functions on Ω which are absolutely square integrable with respect to Lebesgue measure. It is well known that $\mathscr{H}$ is

0022-1236/96 $18.00

a functional Hilbert space with reproducing kernel B, the so-called Bergman kernel, given by:

$$B(z, w) = \det(I_n - zw^*)^{-(m+n)}, \qquad z, w \in \Omega_{n,m}. \tag{1.1}$$

Here I_n is the identity in $\mathbb{C}^{n \times n}$, and $*$ is matrix adjoint.

The Wallach set $\mathscr{W} = \mathscr{W}_{n,m}$ associated with the above set up is the set of all complex numbers λ for which $B^{(\lambda)} \stackrel{\text{def}}{=} B^{\lambda/(m+n)}$ (pointwise power) is a non-negative definite kernel on Ω. The set $\mathscr{W}$ has been determined by several authors (see [5] and [10]). It is:

$$\mathscr{W} = \mathscr{W}_d \cup \mathscr{W}_c, \tag{1.2}$$

where $\mathscr{W}_d$, the discrete part of the Wallach set, and $\mathscr{W}_c$, its continuous part, are given by:

$$\mathscr{W}_d = \{0, 1, ..., n-1\}, \qquad \mathscr{W}_c = \{\lambda: \lambda > n-1\}. \tag{1.3}$$

The standard theory [2] of functional Hilbert spaces implies that for each $\lambda \in \mathscr{W}$, there is a uniquely determined Hilbert space $\mathscr{H}^{(\lambda)} = \mathscr{H}^{(\lambda)}(\Omega_{n,m})$ of analytic functions on $\Omega = \Omega_{n,m}$ whose reproducing kernel is $B^{(\lambda)}$. These spaces $\mathscr{H}^{(\lambda)}$ are the twisted Bergman spaces of the title. (Note that for $\lambda = m+n$ this is the ordinary Bergman space. Also, as is well known, for $\lambda = m$ it is the usual Hardy space on Ω.)

For $\lambda \in \mathscr{W}$, we define the *mn*-tuple $\mathbf{M}^{(\lambda)} = (M_{ij}^{(\lambda)})$ of (a priori densely defined, possibly unbounded) multiplication operators on $\mathscr{H}^{(\lambda)}$ by:

$$(M_{ij}^{(\lambda)} f)(z) = z_{ij} f(z), \qquad z = (z_{ij}) \in \Omega, \quad f \in \mathscr{H}^{(\lambda)}, \quad 1 \leqslant i \leqslant n, \quad 1 \leqslant j \leqslant m. \tag{1.4}$$

This operator tuple $\mathbf{M}^{(\lambda)}$ is the basic object of our study.

1.2. *Main Results*

Our main results are:

THEOREM 1.1. *$\mathbf{M}^{(\lambda)}$ is bounded if and only if $\lambda \in \mathscr{W}_c$.*

THEOREM 1.2. *For $\lambda \in \mathscr{W}_c$, the joint Taylor spectrum of $\mathbf{M}^{(\lambda)}$ is $\bar{\Omega}$.*

THEOREM 1.3. *For $\lambda \in \mathscr{W}_c$, the following are equivalent*:

(i) *$\mathbf{M}^{(\lambda)}$ is jointly subnormal.*

(ii) *$\lambda \in m + \mathscr{W}$.*

(iii) *There is a probability measure μ_λ supported on $\bar{\Omega}$ such that the inner product $\langle \cdot, \cdot \rangle_\lambda$ on $\mathscr{H}^{(\lambda)}$ is given by $\langle f, g \rangle_\lambda = \int f\bar{g}\, d\mu_\lambda$ for polynomials $f, g \in \mathscr{H}^{(\lambda)}$.*

(iv) *The natural projective representation of* $PSU(n, m)$ *on* $\mathscr{H}^{(\lambda)}$ *is a subrepresentation of an induced representation of multiplicity* 1.

Note that (a) $m + \mathscr{W} \subseteq \mathscr{W}_c$; (b) the polynomials belonging to $\mathscr{H}^{(\lambda)}$ are dense in $\mathscr{H}^{(\lambda)}$ (and for $\lambda \in \mathscr{W}_c$, all analytic polynomials belong to $\mathscr{H}^{(\lambda)}$), so that in (iii) above, the probability μ_λ and the inner product $\langle \cdot, \cdot \rangle_\lambda$ determine each other; (c) the natural representation mentioned in Theorem 1.3 (iv) will be discussed in the next section; (d) throughout this paper by "measure" we shall mean regular Borel measure. For the rest of the terminology in this theorem, see [13].

Recall that a d-tuple $\mathbf{T}$ of commuting bounded operators on a Hilbert space is said to admit a compact subset $\mathscr{C}$ of $\mathbb{C}^d$ as a k-spectral set if for all rational functions p with poles off $\mathscr{C}$, we have $\|p(\mathbf{T})\| \leqslant k \sup_{z \in \mathscr{C}} |p(z)|$. The d-tuple $\mathbf{T}$ is said to admit $\mathscr{C}$ as a complete k-spectral set if the same holds for matrix valued p, where $|p(z)|$ is to be interpreted as the operator norm of the matrix $p(z)$. A famous conjecture due to Halmos [6, Problem 6] says that: k-spectral implies complete k'-spectral for some $k' \geqslant k$ [9, Theorem 8.11]. This conjecture, originally stated for a single operator and open even in that case, makes sense and is equally interesting for operator tuples as well. This is the problem that originally motivated our study of the tuple $\mathbf{M}^{(\lambda)}$. In fact, we had hoped for a counterexample to this conjecture (for tuples) among the tuples $\mathbf{M}^{(\lambda)}$. Clearly a jointly subnormal operator tuple admits the joint spectrum of its minimal normal extension as a complete spectral set (i.e., $k=1$). Therefore, the above results imply that $\mathbf{M}^{(\lambda)}$ admits $\bar{\Omega}$ as a complete spectral set if $\lambda \in m + \mathscr{W}$. On the negative side, we find that for $\lambda < m$, $\mathbf{M}^{(\lambda)}$ does not admit $\bar{\Omega}$ as a k-spectral set for any $k < \infty$. Indeed, this is well known for $n = 1$: in this case the monomials are elements of sup norm 1 whose norm in $\mathscr{H}^{(\lambda)}$, $\lambda < m$, goes to infinity as the exponent of the monomial goes to infinity component-wise. The general case is an easy consequence of this since $\Omega_{1,m}$ sits inside $\Omega_{n,m}$ as the set of all $n \times m$ contractive matrices all whose rows, except possibly the first, are zero - and the kernel $B^{(\lambda)}$ on $\Omega_{n,m}$ restricts to the corresponding kernel on $\Omega_{1,m}$.

It turns out that though the tuple $\mathbf{M}^{(\lambda)}$ appears to have a complicated structure, the single operator $\det(\mathbf{M}^{(\lambda)})$ has a simple and tractable structure at least in the case $m = n$. Namely, we find:

THEOREM 1.4. *Let* $m = n$. *Then for* $\lambda \in \mathscr{W}_c$, $\det(\mathbf{M}^{(\lambda)})$ *is a direct sum of weighted forward shifts with explicitly computed weights.*

(i) *If* $\lambda < n$ (*resply if* $\lambda \geqslant n$) *then* $\det(\mathbf{M}^{(\lambda)})$ *does not* (*resply does*) *admit the closed unit disc in* $\mathbb{C}$ *as* k-*spectral set* (*resply complete* k-*spectral set*) *for any* $k < \infty$ (*resply for* $k = 1$).

(ii) *If* $\lambda \geqslant n$ *then* $\det(\mathbf{M}^{(\lambda)})$ *is a subnormal contraction and hence admits the closed unit disc as a complete spectral set.*

In view of the results in this paper, the possibility of $\mathbf{M}^{(\lambda)}$ being a counterexample to the Halmos conjecture remains alive only for λ in one of the $n-1$ "gaps" $(m+i-1, m+i)$, $i = 1, ..., n-1$. This leads to

1.3. *Open Question*

(i) For λ in one of the above mentioned gaps, does $\mathbf{M}^{(\lambda)}$ admit $\bar{\Omega}$ as a k-spectral (or complete k-spectral) set? If yes, what is the best possible value of $k = k(\lambda)$?

(ii) A second question is the extension of Theorem 1.4 to the case $m > n$. On Ω there is a special polynomial (a spherical function) which generalises the usual determinant in the square case $m = n$. This we call the (generalised) determinant on Ω, and $\det(\mathbf{M}^{(\lambda)})$ must be interpreted as multiplication by this generalised determinant. It is not difficult to prove that this operator continues to be a direct sum of weighted shifts in the case $m > n$, but explicit computation of the weights presents unexpected new difficulties.

1.4. *Concluding Remarks*

Let us say that a functional Hilbert space $\mathscr{H}$ of analytic functions on a domain $D \subseteq \mathbb{C}^d$ is a *Hardy like space* if the polynomials are densely contained in $\mathscr{H}$, and there is a (uniquely determined) probability measure μ supported in $\bar{D}$ such that the inner product $\langle f, g \rangle$ is given by $\langle f, g \rangle = \int f\bar{g}\, d\mu$, for analytic polynomials $f, g \in \mathscr{H}$. Thus, Theorem 1.3 says in particular that Hardy likeness of $\mathscr{H}^{(\lambda)}$ is equivalent to joint subnormality of $\mathbf{M}^{(\lambda)}$. Of course, when $\mathscr{H}^{(\lambda)}$ is known to be Hardy-like, boundedness and subnormality of $\mathbf{M}^{(\lambda)}$ are trivial consequences. It is a measure of the success of the techniques evolved in this paper that the main results outlined do go through even when $\mathscr{H}^{(\lambda)}$ is not Hardy like. Indeed, as far as we know, there is no prior instance in the literature where the question of boundedness, subnormality and joint spectrum of a multiplication operator tuple $\mathbf{M}$ on a functional Hilbert space $\mathscr{H}$ has been completely settled even though $\mathbf{M}$ is not a joint weighted shift and $\mathscr{H}$ is not a Hardy like space. (Clearly $\mathbf{M}^{(\lambda)}$ is not a weighted shift for $n \geqslant 2$.)

In this connection, it is perhaps worth pointing out that a d-tuple of multiplication operators on a functional Hilbert space of analytic functions is a joint weighted shift precisely when it is homogeneous in the sense of [8] with respect to the action of the d-dimensional torus group, i.e., the connected component of identity in the full group of linear isometries of the Banach space $l^1(d)$. (Conversely, any joint weighted shift is unitarily equivalent to such a tuple of multiplication operators.) These are well understood classes of operator tuples. A natural generalisation of joint weighted shifts would be the d-tuples of operators homogeneous with respect to the connected component of identity in the full group of linear isometries of some "nice" norm on $\mathbb{C}^d$. A natural choice of "nice" norms are those having the Cartan domains as open unit balls. The tuples $\mathbf{M}^{(\lambda)}$ belong to this class. Indeed, implicit in our discussion of boundedness and subnormality of $\mathbf{M}^{(\lambda)}$ are general criteria for boundedness and subnormality of operator tuples in this general class. For instance, if an mn-tuple of operators has $\bar{\Omega}_{n,m}$ as spectrum, and is homogeneous with respect to the natural action of $PS(U(n) \times U(m))$ on this spectrum, then the arguments leading to Lemma 5.2 below actually yield a subnormality criterion for this tuple, which is very similar to the usual moment-sequence criterion [6, p. 895-896] for the subnormality of joint weighted shifts - with the Schur polynomials taking up the role of monomials.

However, crucial to the techniques used in the determination of the Taylor spectrum, and of course in establishing the connection between subnormality and induced representation, is the fact that $\mathbf{M}^{(\lambda)}$ is homogeneous with respect to the natural action of an even larger (non-linear, non-compact) group of biholomorphic automorphisms, namely $PSU(n, m)$, on its spectrum $\bar{\Omega}_{n,m}$. This fact and other preliminaries are described in Section 2.1. In Section 2.2 we derive an explicit formula for the elementary spherical functions (esf's) in terms of the Schur polynomials. We also obtain a recursion formula (Proposition 2.4) for the Schur polynomials of n variables in terms of those of fewer number of variables. This recursion is used to re-derive Faraut and Koranyi's norm formula for the esf's as elements of the twisted Bergman spaces. While the Faraut-Koranyi proof of their norm formula is computationally simpler, we believe that ours is conceptually simpler. More over, the formula in [5] is not entirely explicit in as much as it involves the dimensions of the $S(U(n) \times U(m))$-irreducible spaces. These dimensions were determined by Upmeier in [12]. On the other hand, we first obtain a completely explicit norm formula and then use it to re-derive Upmeier's dimension formula in an elementary way. However, the results in [5] and [12] are for general Cartan domains, while our proofs apply, as yet, only to domains $\Omega_{n,m}$. Our justification for including the rather lengthy subsection 2.2, devoted mostly to re-deriving known results, (one exception seems to be the recursion formula for Schur polynomials, which we could not locate in the literature) is three-fold: (i) we have tried to make this paper as self-complete and widely accessible as we could, keeping the average operator theorist reader in mind, (ii) the methods and results developed here will be later used to prove the results on boundedness and subnormality, and (iii) we have framed the proofs in such a way that the results here will painlessly generalise to arbitrary Cartan domains as soon as an analogue of our recursion formula

(Proposition 2.4) is available for the Jack polynomials which play the role of the Schur polynomials in the context of arbitrary Cartan domains. Precise conjectures generalising the results of this paper were formulated in [3] where we also announced the results proved here. If Ω is any Cartan domain of genus g and rank r in $\mathbb{C}^d$, one can define analogous operators $M^{(\lambda)}$ for λ in the corresponding Wallach set $\mathscr{W}$. The proofs presented here easily generalise to show that (i) $M^{(\lambda)}$ is unbounded for $\lambda \in \mathscr{W}_d$, (ii) $M^{(\lambda)}$ has Taylor spectrum $\bar{\Omega}$ wherever it is bounded, and (iii) $M^{(\lambda)}$ is subnormal for $\lambda > g-1$ and for at most r values of $\lambda \leqslant g-1$. The recursion formula for Jack polynomials will be needed only to prove its boundedness for $\lambda \in \mathscr{W}_c$ and its subnormality for the r points $\lambda \in d/r + \mathscr{W}_d$.

The proof of Theorem 1.1, as presented in section 3, works by reducing the question of boundedness of the tuple $\mathbf{M}^{(\lambda)}$ to that of a single operator, viz. multiplication by the linear spherical function on Ω. To settle the boundedness of this operator, we reduce it to the question of positivity of an associated kernel and answer it by invoking the explicit formulae for the elementary spherical functions. Theorem 1.2 follows fairly easily from the nature of the action of $PSU(n, m)$ on Ω. The proof given in Section 4 does not involve any explicit calculation of Koszul complexes. The proof of the part (ii) $\Rightarrow$ (i) in Theorem 1.3 depends on the techniques developed in Section 2. Our proof has the advantage of explicitly describing the measures μ_λ, whenever they exist. To establish the part (i) $\Rightarrow$ (ii) of Theorem 1.3 we appeal to the wellknown result that any two quasi-invariant measures on a transitive $\mathbf{G}$-space are equivalent. With the aid of this result, it is not difficult to show that each of the n $\mathbf{G}$-orbits in $\partial\Omega$ supports at most one of the measures μ_λ. The relationship with representation theory is obtained by an appeal to Mackey's theory of *systems of imprimitivity*. Finally in Section 6, we prove Theorem 1.4 on the determinant. This involves an examination of the representation of the maximal compact subgroup of $PSU(n, m)$ on $\mathscr{H}^{(\lambda)}$.

2. Group Action and Spherical Functions

2.1. *Group Action*

Let $\mathbf{G}$ denote the connected component of identity in the full group of biholomorphic automorphisms of $\Omega = \Omega_{n,m}$. We have $\mathbf{G} = PSU(n, m)$; abstractly it is the group of linear automorphisms of a non-degenerate unitary form of signature (n, m) on $\mathbb{C}^{n+m}$, modulo scalar matrices. Taking $(I_n) \oplus (-I_m)$ as the matrix of such a form, $\mathbf{G}$ consists of the matrices (modulo scalars) $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $a \in \mathbb{C}^{n\times n}$, $b \in \mathbb{C}^{n\times m}$, $c \in \mathbb{C}^{m\times n}$, $d \in \mathbb{C}^{m\times m}$ satisfying

$$a^*a - c^*c = I_n, \qquad d^*d - b^*b = I_m, \qquad a^*b = c^*d, \qquad \det g = 1.$$

$\mathbf{G}$ acts on $\Omega_{n,m}$ as Möbius transformations:

$$\mathbf{G} \ni g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \to (az+b)(cz+d)^{-1}, \qquad z \in \Omega_{n,m}. \tag{2.1}$$

This action of $\mathbf{G}$ on Ω is transitive. Indeed for each $w \neq 0$ in Ω, there is a unique involution $\phi_w \in \mathbf{G}$ which interchanges 0 and w. It is given by:

$$\phi_w(z) = (1-ww^*)^{-1/2}\,(w-z)(1-w^*z)^{-1}\,(1-w^*w)^{1/2}, \qquad z \in \Omega. \tag{2.2}$$

Thus as homogeneous spaces, we have the identification $\Omega = \mathbf{G}/\mathbf{K}$, where $\mathbf{K}$ is the stabiliser in $\mathbf{G}$ of $0 \in \Omega$. Explicitly, $\mathbf{K} = PS(U(n) \times U(m))$, consisting of pairs (u, v) of unitaries with $\det(u)\det(v) = 1$, modulo scalars. The element $(u, v) \in PS(U(n) \times U(m))$ is identified with the element $u \oplus v$ of $\mathbf{G}$. Specialising (2.1), one sees that $k = (u, v)$ acts on Ω by

$$z \to uzv^*. \tag{2.3}$$

The Shilov boundary S of Ω consists of the maximal partial isometries. The action of $\mathbf{G}$ mentioned above extends naturally to $\bar{\Omega}$, and under this action, $\mathbf{K}$ is transitive on S. We fix a base point $e \in S$. For definiteness, we take $e \in \mathbb{C}^{n \times m}$ given by $e = (I_{n \times n}, 0_{n \times m-n})$. Let $\mathbf{L}$ be the stabiliser of e in $\mathbf{K}$. Thus we have $S = \mathbf{K}/\mathbf{L}$. Explicitly, $\mathbf{L}$ consists of elements $(u, v) \in PS(U(n) \times U(m))$, where $v = u \oplus w$ for some $w \in U(m-n)$. Thus, abstractly, we have $\mathbf{L} = PS(U(n) \times U(m-n))$.

Note that in the special case $n = m$, S is naturally identified with $U(n)$ and the action of $\mathbf{L}$ on S is that of $PSU(n)$ acting on $U(n)$ by conjugation.

For $1 \leqslant j \leqslant n$, let $e_j \in \Omega_{n,m}$ denote the matrix with 1 in the (j, j) position and 0 elsewhere. Also, let Δ_n denote the subset of the n-dimensional box given by

$$\Delta_n = \{(t_1, ..., t_n) : 0 \leqslant t_1 \leqslant \cdots \leqslant t_n \leqslant 1\}. \tag{2.4}$$

Δ_n is embedded in $\bar{\Omega}_{n,m}$ via the identification

$$\Delta_n \ni (t_1, ..., t_n) \leftrightarrow \sum_{1 \leqslant j \leqslant n} t_j e_j \in \bar{\Omega}_{n,m}. \tag{2.5}$$

Using polar decomposition, it is easy to see that each $\mathbf{K}$-orbit in $\bar{\Omega}$ meets Δ_n in a singleton. This gives an identification of the orbit space $\bar{\Omega}/\mathbf{K}$ with Δ_n. Explicitly, the projection $\pi: \bar{\Omega} \to \Delta_n = \bar{\Omega}/\mathbf{K}$ is given by

$$\pi(z) = \text{the } n\text{-tuple of singular values of z arranged in the increasing order.} \tag{2.6}$$

Note that under this identification, $e = \sum_{1 \leqslant j \leqslant n} e_j$ corresponds to the point $\underline{1} = (1, ..., 1) \in \Delta_n$.

For λ in the Wallach set $\mathscr{W}$, $g \in \mathbf{G}$ acts on $\mathscr{H}^{(\lambda)}$ as an unitary operator $U^{(\lambda)}(g)$ by the formula (with Jg = complex Jacobian determinant of g as a function on Ω)

$$U^{(\lambda)}(g)(f) = (Jg)^{\lambda/(m+n)} (f \circ g), \qquad f \in \mathscr{H}^{(\lambda)}. \tag{2.7}$$

The unitarity of $U^{(\lambda)}$ on $\mathscr{H}^{(\lambda)}$ is equivalent to the following transformation rule for the reproducing kernel $B^{(\lambda)}$ (see [1]):

$$J^{\lambda/(m+n)}(g, z)\, \overline{J^{\lambda/(m+n)}(g, w)}\, B^{(\lambda)}(gz, gw) = B^{(\lambda)}(z, w) \tag{2.8}$$

for $g \in \mathbf{G}$, $z, w \in \Omega_{n,m}$.

Excepting when λ is an integer, $g \to U^{(\lambda)}(g)$ is not a "genuine" representation, but is only a projective representation. However the restriction of this action to $\mathbf{K}$ is a genuine representation of $\mathbf{K}$ on $\mathscr{H}^{(\lambda)}$. The decomposition of $\mathscr{H}^{(\lambda)}$ under $\mathbf{K}$-action is described as follows. In the present context, a *signature* is an n-tuple $\underline{s} = (s_1, ..., s_n)$ of integers with $s_1 \geqslant \cdots \geqslant s_n \geqslant 0$. When wishing to emphasise the parameter n, we shall call this a signature of *rank* n. $|\underline{s}| \stackrel{\text{def}}{=} \sum_{j=1}^{n} s_j$ will be called the weight of the signature. The group $\mathbf{K}$ acts by composition on the vector space $\text{Hom}(k)$ of analytic homogeneous polynomials of degree $k \geqslant 0$. Under $\mathbf{K}$-action, $\text{Hom}(k)$ breaks up into inequivalent irreducible components indexed by the signatures of weight k. The component indexed by $\underline{s}$ will be denoted by $\mathscr{P}_{\underline{s}}$. Thus,

$$\text{Hom}(k) = \bigoplus_{\underline{s}:\, |\underline{s}| = k} \mathscr{P}_{\underline{s}}. \tag{2.9}$$

The space $\mathscr{P}_{\underline{s}}$ may be constructed as follows. For $z \in \Omega$, and $1 \leqslant i \leqslant n$, let $z^{(i)}$ denote (temporarily) the top left $i \times i$ submatrix of z. Then the conical polynomial $N_{\underline{s}}$ associated with the signature $\underline{s}$ is defined by

$$N_{\underline{s}}(z) = \det(z^{(1)})^{s_1 - s_2} \cdots \det(z^{(n-1)})^{s_{n-1} - s_n} \det(z^{(n)})^{s_n}. \tag{2.10}$$

Thus $N_{\underline{s}}$ is a homogeneous analytic polynomial of degree $|\underline{s}|$ on Ω. Now, $\mathscr{P}_{\underline{s}}$ is defined to be the minimal $\mathbf{K}$-invariant vector space of polynomials containing $N_{\underline{s}}$.

It is known that $\mathscr{H}^{(\lambda)}$ contains all analytic polynomials if and only if $\lambda \in \mathscr{W}_c$. Indeed for $\lambda \in \mathscr{W}_d$, $\mathscr{H}^{(\lambda)}$ contains precisely those spaces $\mathscr{P}_{\underline{s}}$ for which $\lambda < j \leqslant n \Rightarrow s_j = 0$. Further, the polynomials belonging to $\mathscr{H}^{(\lambda)}$ are dense in

$\mathscr{H}^{(\lambda)}$, whence $\mathscr{H}^{(\lambda)}$ is the direct sum of its subspaces $\mathscr{P}_{\underline{s}}$. In particular, we have

$$\mathscr{H}^{(\lambda)} = \bigoplus_{\underline{s}} \mathscr{P}_{\underline{s}}, \qquad \text{for} \quad \lambda \in \mathscr{W}_c, \tag{2.11}$$

where the direct sum is over all signature $\underline{s}$.

Since the inner product on $\mathscr{H}^{(\lambda)}$ is $\mathbf{K}$-invariant and for distinct signatures $\underline{s}$ the representations of $\mathbf{K}$ on $\mathscr{P}_{\underline{s}}$ are inequivalent irreducible representations, Schur's lemma implies that the above direct sum is an orthogonal direct sum.

Recall from (2.7) the unitary $U^{(\lambda)}(g)$ on $\mathscr{H}^{(\lambda)}$ representing $g \in \mathbf{G}$. The operator tuple $\mathbf{M}^{(\lambda)}$ on $\mathscr{H}^{(\lambda)}$ transforms nicely under the $\mathbf{G}$-action. As g is an analytic function on a neighborhood of $\bar{\Omega}$, there is no ambiguity in the definition of $g(\mathbf{M}^{(\lambda)})$; it is just multiplication by g: $g(\mathbf{M}^{(\lambda)})(f)(z) = g(z) f(z)$, $z \in \Omega$, where defined. And we have

$$g(\mathbf{M}^{(\lambda)}) = U^{(\lambda)}(g)\, \mathbf{M}^{(\lambda)} U^{(\lambda)*}(g), \qquad g \in \mathbf{G}. \tag{2.12}$$

In the language of [8], $\mathbf{M}^{(\lambda)}$ is a $\mathbf{G}$-homogeneous tuple of operators.

For use in Section 5, we recall that under $\mathbf{G}$-action, the boundary $\partial\Omega$ of Ω breaks up into n orbits

$$S_j = \{u \in \partial\Omega : \operatorname{rank}(I_n - uu^*) = j\}, \qquad j = 0, 1, 2, ..., n-1. \tag{2.13}$$

In particular, $S_0 = S$ is the Shilov boundary of Ω. We have

$$\bar{S}_j = S_0 \cup \cdots \cup S_j, \qquad 0 \leqslant j \leqslant n-1. \tag{2.14}$$

Note that the image of S_j under $\pi: \bar{\Omega}_{n,m} \to \Delta_n$ is Δ_j identified as a subset of Δ_n by $\Delta_j = \{(t_1, ..., t_j, 1, ..., 1): 0 \leqslant t_1 \leqslant \ldots t_j \leqslant 1\}$, $0 \leqslant j \leqslant n$.

2.2. *Spherical Functions*

A spherical function on Ω is a bounded analytic function φ fixed by the group $\mathbf{L}$ (acting by composition). An elementary spherical function (esf) is a spherical function φ such that the minimal $\mathbf{K}$-invariant linear space of functions containing φ is $\mathbf{K}$-irreducible.

Each $\mathscr{P}_{\underline{s}}$ contains, upto scalar multiplication, a unique esf $\varphi_{\underline{s}}$, which we normalise by the requirement $\varphi_{\underline{s}}(e) = 1$. (Recall that e is the $\mathbf{L}$-fixed base point in the Shilov boundary S of Ω.) This indexes the esf's by the signatures. Note that we have

$$\varphi_{\underline{s}} = \int_L N_{\underline{s}} \circ \ell \; d\ell,$$

where the integration is with respect to the Haar probability on $\mathbf{L}$, and $N_{\underline{s}}$ is as in formula (2.10). Also,

$$\mathscr{P}_{\underline{s}} = \bigvee \{\varphi_{\underline{s}} \circ k \colon k \in \mathbf{K}\}$$

where $\bigvee$ denotes linear span.

Since $\mathscr{P}_{\underline{s}}$ is $\mathbf{K}$-irreducible, by Schur's Lemma it admits an essentially unique $\mathbf{K}$-invariant inner product $\langle \cdot, \cdot \rangle_{\underline{s}}$. Being finite dimensional, $(\mathscr{P}_{\underline{s}}, \langle \cdot, \cdot \rangle)$ is a functional Hilbert space; say with reproducing kernel $K_{\underline{s}}$. Since the inner product is $\mathbf{K}$-invariant so is the kernel:

$$K_{\underline{s}}(gz, gw) = K_{\underline{s}}(z, w), \quad z, w \in \bar{\Omega}, \qquad g \in \mathbf{K}. \tag{2.15}$$

In particular,

$$K_{\underline{s}}(z, e) = K_{\underline{s}}(\ell z, \ell e) = K_{\underline{s}}(\ell z, e), \qquad \ell \in \mathbf{L}.$$

Thus $K_{\underline{s}}(\cdot, e)$ is an $\mathbf{L}$-fixed element of $\mathscr{P}_{\underline{s}}$, and hence $K_{\underline{s}}(\cdot, e) = \varphi_{\underline{s}}$ after suitable normalisation. Since $K_{\underline{s}}$ is $\mathbf{K}$-invariant and $\mathbf{K}$ acts transitively on the Shilov boundary S, this determines $K_{\underline{s}}(z, w)$ for $w \in S$, and hence (as $K_{\underline{s}}$ is coanalytic in w) for all w. Namely we get:

Proposition 2.1. *Let $\underline{s}$ be a signature. Then, with suitable normalisation, the reproducing kernel of a $\mathbf{K}$-invariant inner product on $\mathscr{P}_{\underline{s}}$ is $K_{\underline{s}}(z, w) = \varphi_{\underline{s}}(zw^*e)$.*

(Note that this implies, in particular, that for any esf $\varphi_{\underline{s}}$, the kernel $(z, w) \to \phi_{\underline{s}}(zw^*e)$ is non-negative definite.)

Proof. By the preceding discussion, upto scalar multiplication there is a unique $\mathbf{K}$-invariant kernel on $\mathscr{P}_{\underline{s}}$. So it suffices to verify that the kernel $(z, w) \to \varphi_{\underline{s}}(zw^*e)$ is $\mathbf{K}$-invariant in the sense of (2.15). In view of the action (2.3) of $\mathbf{K}$, we need to show that for $x = zw^* \in \Omega_{n,m}$ and for $u \in U(n)$, we have $\varphi_{\underline{s}}(uxu^*e) = \varphi_{\underline{s}}(xe)$. Define $v \in U(m)$ by $v = u \oplus 1$. Since $(u^*, v^*) \in \mathbf{L}$ and $\varphi_{\underline{s}}$ is $\mathbf{L}$-fixed, we have $\varphi_{\underline{s}}(uxu^*e) = \phi_{\underline{s}}(xu^*ev)$. Since $u^*ev = e$, this completes the proof. ∎

One interesting consequence of this proposition is:

Corollary 2.1. *For any signature $\underline{s}$, $\|\varphi_{\underline{s}}\|_\infty = 1$.*

(Here, of course, $\|\varphi_{\underline{s}}\|_\infty$ is the supremum over Ω of $|\varphi_{\underline{s}}|$.)

Proof. For $z \in S$,

$$|\varphi_{\underline{s}}(z)|^2 = |K_{\underline{s}}(z, e)|^2 \leqslant K_{\underline{s}}(z, z)\, K_{\underline{s}}(e, e) = \varphi_{\underline{s}}(zz^*e)\, \varphi_{\underline{s}}(ee^*e) = \varphi_{\underline{s}}^2(e) = 1.$$

The inequality is Cauchy-Schwartz applied to the space $\mathscr{P}_{\underline{s}}$ with reproducing kernel $K_{\underline{s}}$. Since $\varphi_{\underline{s}}$ is analytic, S is the Shilov boundary of Ω and $\varphi_{\underline{s}}(e)=1$, this completes the proof. ∎

PROPOSITION 2.2. *For any signature $\underline{s}$, let $\phi_{\underline{s}}$ and $\varphi^*_{\underline{s}}$ denote the corresponding esf's on $\Omega_{n,m}$ and $\Omega_{n,n}$ respectively. Then these two polynomials are related by*

$$\varphi_{\underline{s}}(z_1, z_2) = \varphi^*_{\underline{s}}(z_1), \qquad z_1 \in \mathbb{C}^{n\times n}, \quad z_2 \in \mathbb{C}^{n\times(m-n)}.$$

Proof. In view of the comments preceding Proposition 2.1, the kernel in this proposition is a scalar times $\varphi_{\underline{s}}(z)$ at $w=e$. Equating the value at $z=e$, the scalar must be equal to 1. In other words, we have $\varphi_{\underline{s}}(ze^*e) = \varphi_{\underline{s}}(z)$ for $z \in \mathbb{C}^{n\times m}$. That is, $\varphi_{\underline{s}}(z)$ *depends only on the first n columns of z*.

Embed $\Omega^* = \Omega_{n,n}$ in $\Omega = \Omega_{n,m}$ by $z_1 \mapsto (z_1, 0)$. To complete the proof, it suffices to show that the restriction of $\varphi_{\underline{s}}$ to Ω^* equals $\varphi^*_{\underline{s}}$. The group actions discussed in section 2.1 behave nicely with respect to this embedding; also, clearly, the conical polynomial $N_{\underline{s}}$ restricts to the corresponding conical polynomial $N^*_{\underline{s}}$. It follows that the image of the space $\mathscr{P}_{\underline{s}}$ under restriction contains $\mathscr{P}^*_{\underline{s}}$. Hence there is a $\varphi \in \mathscr{P}_{\underline{s}}$, such that $\varphi|_{\Omega^*} = \varphi^*_{\underline{s}}$. Now, for $\ell \in \mathbf{L}$, φ and $\varphi \circ \ell$ have the same restriction to Ω^*. Therefore, replacing φ by its $\mathbf{L}$-invariantisation $\int \varphi \circ \ell \, d\ell$ (integration with respect to the Haar probability on $\mathbf{L}$) does not change its restriction $\varphi^*_{\underline{s}}$. So we may assume that φ is spherical. Since $\varphi \in \mathscr{P}_{\underline{s}}$, and (as $e \in \Omega^*$) $\varphi(e)=1$, it follows that $\varphi = \varphi_{\underline{s}}$. Hence $\varphi_{\underline{s}}|_{\Omega^*} = \varphi^*_{\underline{s}}$. ∎

Remark (i). The fact that $\varphi_{\underline{s}}(z)$ depends only on the first n columns of z may sound incredible until one notices that in the description of the subgroup $\mathbf{L}$ of $\mathbf{K}$, we have singled out the first n columns by the arbitrary choice of e as base point. A more direct way to establish the same fact is as follows. Put $z=(z_1, z_2)$ with $z_1 \in \mathbb{C}^{n\times n}$, $z_2 \in \mathbb{C}^{n\times(m-n)}$. For $v \in U(m-n)$, $(1_n, 1_n \oplus v) \in \mathbf{L}$, so that $\mathbf{L}$-invariance of $\varphi_{\underline{s}}$ implies $\varphi_{\underline{s}}(z_1, z_2) = \varphi_{\underline{s}}(z_1, z_2 \cdot v)$ for all $v \in U(m-n)$. For each fixed z_1, the polynomial $z_2 \to \varphi_{\underline{s}}(z_1, z_2)$, being an $U(m-n)$-invariant analytic polynomial, is a constant. Thus $\varphi_{\underline{s}}(z)$ depends only on z_1.

Remark (ii). Let Ω and Ω^* be as above, and, for any signature $\underline{s}$, let $\mathscr{P}_{\underline{s}}$ and $\mathscr{P}^*_{\underline{s}}$ be the spaces of polynomials, on these two domains, indexed by $\underline{s}$. Then, by Proposition 2.1 and Proposition 2.2, the reproducing kernel of $\mathscr{P}^*_{\underline{s}}$ is obtained from that of $\mathscr{P}_{\underline{s}}$ by restricting the latter to $\Omega^* \times \Omega^*$. Therefore, the theorem in Aronszajn [2, p. 351] implies that $\mathscr{P}^*_{\underline{s}}$ is the image of $\mathscr{P}_{\underline{s}}$ under the restriction map.

Recall from the representation theory of symmetric groups that the Schur polynomial $Q_n(\cdot \mid \underline{s})$ corresponding to the signature $\underline{s}$ is the polynomial in n variables $\underline{X} = (X_1, ..., X_n)$ given by:

$$Q_n(\underline{X} \mid \underline{s}) = \left(\sum_{\sigma \in \mathrm{Sym}(n)} \operatorname{sgn}(\sigma) \prod_{k=1}^{n} X_{\sigma(k)}^{s_k - k + n} \right) \left(\prod_{1 \leqslant i < j \leqslant n} (X_i - X_j) \right)^{-1} \tag{2.16}$$

(Clearly this is a homogeneous polynomial of degree $|\underline{s}|$. Though named after Schur, these polynomials were first studied by Jacobi and his student Trudi. In the theory of symmetric groups the polynomial $Q_n(\cdot \mid \underline{s})$ is often denoted simply by $\{\underline{s}\}$. Be warned that, at any rate, ours is not the usual notation for Schur polynomials. The standard theory of these polynomials may be found in [7].)

Now we have the following explicit formula for the esf's:

Proposition 2.3. *For complex numbers $t_1, ..., t_n$, and for any signature $\underline{s}$, we have*

$$\varphi_{\underline{s}} \left(\sum_{j=1}^{n} t_j e_j \right) = \frac{Q_n(t_1, ..., t_n \mid \underline{s})}{Q_n(1, ..., 1 \mid \underline{s})}.$$

(Here $e_j \in \bar{\Omega}$ are as in our discussion of $\bar{\Omega}/\mathbf{K}$ in section 2.1. Since each $\mathbf{L}$-orbit inside the Shilov boundary S of Ω intersects the torus $\{\sum_{j=1}^{n} t_j e_j \colon |t_j| = 1\}$, the $\mathbf{L}$-invariant function $\varphi_{\underline{s}}$ is determined on S, and hence by analytic continuation on the whole of Ω, by its restriction to this torus. Thus the formula in Proposition 2.3 determines the esf's uniquely, subject only to $\mathbf{L}$-invariance and analyticity.)

Proof. In view of Proposition 2.2, we may (and do) assume $n = m$.
In this case, the Shilov boundary S of Ω is naturally identified with $U(n)$, and the action of $\mathbf{L} = PSU(n)$ on $S = U(n)$ is by conjugation. Being analytic polynomials, the esf's may be identified with their restriction to $S = U(n)$. Thus viewed, they are class functions on $U(n)$. We claim that upto scaling, the esf's are the irreducible characters of $U(n)$.

Let φ be a spherical function on Ω with $\varphi(e) = 1$. Then φ is an esf if and only if the minimal $\mathbf{K}$-invariant vector space V of polynomials containing ϕ is $\mathbf{K}$-irreducible. Since V is spanned by $\varphi \circ k$, $k \in \mathbf{K}$, this happens if and only if upto a multiplicative constant φ is the only spherical function in V, i.e., if and only if the $\mathbf{L}$-invariantisation of $\varphi \circ k$ is a constant times φ for every $k \in \mathbf{K}$. In view of the specific action of $\mathbf{K} = PS(U(n) \times U(n))$ and $\mathbf{L} = PSU(n)$, this shows that an analytic polynomial φ on Ω with $\varphi(e) = 1$ is an esf if and only if

$$\int_{U(n)} \varphi(uvwv^*)\, dv = \varphi(u)\, \varphi(w) \qquad \text{for all} \quad u, w \in U(n). \tag{2.17}$$

(This is, of course, the usual functional equation for spherical functions as defined in the context of representation theory.)

On the other hand, we claim that a class function χ on $U(n)$ with $\chi(1) \neq 0$ is a scalar times an irreducible character if and only if

$$\int_{U(n)} \chi(uvwv^{-1})\, dv = \chi(u)\, \chi(w)/\chi(1) \qquad \text{for all} \quad u, w \in U(n) \quad (2.18)$$

(More generally, this characterisation is valid for any compact group.)

Indeed, if χ is any irreducible character and π is the matrix representation affording χ, then putting $T = \int \pi(vwv^{-1})\, dv$ for a fixed $w \in U(n)$, we find that T commutes with every $\pi(u)$, $u \in U(n)$, whence $T = cI$ by Schur lemma. Comparing traces, we get $c = \chi(w)/\chi(1)$. Now, trace $(\pi(u)\, T) = c\chi(u)$, which is (2.18).

Conversely, if χ is a class function with $\chi(1) \neq 0$ satisfying (2.18), then we take an irreducible character ψ such that $\langle \chi, \psi \rangle \neq 0$, multiply both sides of 2.18 by $\overline{\psi}(u)$ and integrate with respect to u. Using the fact that ψ also satisfies (2.18), and $\psi(w^{-1}) = \overline{\psi(w)}$, we get $\langle \chi, \psi \rangle (\chi/\chi(1) - \psi/\psi(1)) \equiv 0$. Hence $\chi = c \cdot \psi$, proving the converse.

Comparing the characterisations of esf's and of irreducible characters, we find that esf's are precisely the functions $\chi/\chi(1)$ as χ ranges over the analytic irreducible characters of $U(n)$. Since the irreducible character $\chi_{\underline{s}}$ with highest weight $\underline{s} = (s_1, ..., s_n)$ is analytic if and only if $s_n \geqslant 0$, this proves $\varphi_{\underline{s}} = \chi_{\underline{s}}/\chi_{\underline{s}}(1)$ for a signature $\underline{s}$. (To be precise, a little more work is needed to establish the exact correspondence. We omit this. For our purpose this formula may be taken to define the esf corresponding to the signature $\underline{s}$, in case $n = m$.)

The proposition now follows from Weyl's character formula for $U(n)$ (see [14]): on the torus $\{\sum_{j=1}^{n} t_j e_j : |t_j| = 1\} \subseteq U(n)$, the irreducible character $\chi_{\underline{s}}$ of highest weight $\underline{s}$ is given by

$$\chi_{\underline{s}}\left(\sum_{j=1}^{n} t_j e_j\right) = Q_n(t_1, ..., t_n \mid \underline{s}). \quad \blacksquare$$

Corollary 2.2 *Let ψ and φ be the esfs corresponding to the signatures* (1, ..., 1) *and* (1, 0, ..., 0) *respectively. Then for any signature $\underline{s}$, we have*

(a) $$\varphi\varphi_{\underline{s}} = \frac{1}{n} \sum_{k=1}^{n}{}' \left(\prod_{\substack{i=1 \\ i \neq k}}^{n} \left(1 + \frac{\varepsilon_{i,k}}{|i-k| + |s_i - s_k|} \right) \right) \varphi_{s+\delta_k},$$

(b) $$\psi\varphi_{\underline{s}} = \phi_{\underline{s}+\underline{1}}.$$

Here δ_k is the n-vector with 1 *in the kth slot and* 0 *elsewhere,* $\underline{1}$ *is the n-vector with* 1 *in all the slots,* $\varepsilon_{i,k} = +1$ *if* $i > k$ *and* $= -1$ *if* $i < k$, *and the sum* $\sum'$ *in* (a) *is over those k's* $(1 \leqslant k \leqslant n)$ *for which* $\underline{s} + \delta_k$ *is a signature.*

Proof. By the parenthetical remark following the statement of Proposition 2.3, it suffices to verify these two identities for the variable z ranging over $z = \sum_{j=1}^{n} t_j e_j$ with $t_j \in \mathbb{C}$. In view of Propositions 2.3 and 2.5, these identities follow from

(c) $Q_n(\cdot \mid \underline{s}^0)\, Q_n(\cdot \mid \underline{s}) = \sum' Q_n(\cdot \mid \underline{s} + \delta_k)$,

(d) $Q_n(\cdot \mid \underline{s}^1)\, Q_n(\cdot \mid \underline{s}) = Q_n(\cdot \mid \underline{s} + \underline{1})$,

where $\underline{s}^0 = (1, 0, ..., 0)$, and $\underline{s}^1 = (1, 1, ..., 1)$.

From the defining equation (2.16) one sees that $Q_n(X \mid \underline{s}^1) = \prod_{j=1}^{n} X_j$ and $Q_n(X \mid \underline{s}^0) = \sum_{j=1}^{n} X_j$. One verifies the identities (c) and (d) by substituting these expressions, multiplying both sides by $\prod_{1 \leqslant j < k \leqslant n} (X_j - X_k)$ and equating coefficients of like powers. ∎

Remark. The identities (c) and (d) above are special cases of the Littlewood Richardson rule which expresses the product of any two Schur polynomial as a linear combination of Schur polynomials. (See [7].) In principle, this rule can be used to write the product of any two esf's as a linear combination of esf's. In [16], Zhang has generalised the formulae in Corollary 2.2 to all tube like Cartan domains.

For $n = 1$ the formula (2.16) reduces to $Q_1(X \mid s) = X^s$. This, together with the recursion formula in our next proposition, also determines the Schur polynomials uniquely. To state this formula succinctly, we need some notations.

Notations. For any two finite sets A, B of natural numbers, we put

$$\varepsilon(A, B) = (-1)^{v(A, B)}, \qquad \text{where} \quad v(A, B) = \#\{(x, y) \in A \times B : x > y\}.$$

If k, ℓ are natural numbers, A, B are two (disjoint) sets of size k and ℓ respectively, such that $A \cup B = \{1, 2, ..., k + \ell\}$, then to any signature $\underline{s}$ of rank $k + \ell$ we associate two signatures $\underline{s}^A$, $\underline{s}^B$ of rank k and ℓ respectively, as follows. Say $A = \{a_1, ..., a_k\}$, $B = \{b_1, ..., b_\ell\}$, where $a_1 < \cdots < a_k$, $b_1 < \cdots < b_\ell$. Then,

$$s_i^A = s_{a_i} - a_i + i + \ell, \qquad 1 \leqslant i \leqslant k, \tag{2.19}$$

$$s_j^B = s_{b_j} - b_j + j + k, \qquad 1 \leqslant j \leqslant \ell. \tag{2.20}$$

Note that we then have $|\underline{s}^A| + |\underline{s}^B| = |\underline{s}| + k\ell$.

In terms of these notations, we have:

PROPOSITION 2.4. *Let k, ℓ be natural numbers. Then for any signature $\underline{s}$ of rank $k+\ell$, we have*

$$Q_{k+\ell}(X_1, ..., X_k, Y_1, ..., Y_\ell \mid \underline{s}) = \left(\prod_{\substack{1 \leqslant i \leqslant k \\ 1 \leqslant j \leqslant \ell}} (X_i - Y_j)^{-1} \right) \cdot \sum_{A, B} \varepsilon(A, B)$$
$$\times Q_k(X_1, ..., X_k \mid \underline{s}^A)\, Q_\ell(Y_1, ..., Y_\ell \mid \underline{s}^B),$$

where the sum is over all partitions of $\{1, ..., k+\ell\}$ into two sets A, B of size k and ℓ respectively.

Proof. Let $P_n(\cdot \mid \underline{s})$ denote (for the duration of this proof) the numerator of the formula (2.16): $P_n(Z_1, ..., Z_n \mid \underline{s}) = \sum_{\pi \in \mathrm{Sym}(n)} \prod_{i=1}^n Z_{\pi(i)}^{s_i+n-i}$. Then, with $Z_i = X_i$ for $1 \leqslant i \leqslant k$, $Z_{j+k} = Y_j$ for $1 \leqslant j \leqslant \ell$, $n = k+\ell$, we have to show that

$$P_n(Z_1, ..., Z_n \mid \underline{s}) = \sum_{A, B} \varepsilon(A, B)\, P_k(X_1, ..., X_k \mid \underline{s}^1)\, P_\ell(Y_1, ..., Y_\ell \mid \underline{s}^B).$$

To see this, write the sum over $\pi \in \mathrm{Sym}(k+\ell)$ in the definition of $P_{k+\ell}$ as a double sum $\sum_{A, B} \sum \cdot$ where the outer sum is over all partitions (A, B) as in the statement of this proposition and the inner sum is over all permutations π in $\mathrm{Sym}(k+\ell)$ mapping A and B onto $\{1, ..., k\}$ and $\{k+1, ..., k+\ell\}$ respectively. For a fixed (A, B), any such π may be written as $\pi = (\sigma, \eta) \circ \tau$ for uniquely determined permutations $\sigma \in \mathrm{Sym}(k)$, $\eta \in Sym(\ell)$. Here τ is the element of $\mathrm{Sym}(k+\ell)$ (uniquely determined by (A, B)) mapping A and B onto $\{1, ..., k\}$ and $\{k+1, ..., k+\ell\}$ respectively such that the restrictions of τ to A and B preserve the natural order. Also, for $\sigma \in \mathrm{Sym}(k)$, $\eta \in \mathrm{Sym}(\ell)$, $(\sigma, \eta) \in \mathrm{Sym}(k+\ell)$ is defined by $(\sigma, \eta)(i) = \sigma(i)$ for $1 \leqslant i \leqslant k$, and $(\sigma, \eta)(j+k) = \eta(j)$ for $1 \leqslant j \leqslant \ell$. Now, the inner sum over π may be rewritten as a double sum over $\sigma \in \mathrm{Sym}(k)$, $\eta \in \mathrm{Sym}(\ell)$. This completes the proof since the notations have been so arranged that (for π, σ, η related as above) we have $\mathrm{sgn}(\pi) = \mathrm{sgn}(\tau)\,\mathrm{sgn}(\sigma)\,\mathrm{sgn}(\eta) = \varepsilon(A, B)\ \mathrm{sgn}(\sigma)$ $\mathrm{sgn}(\eta)$ and $\prod_{i=1}^n Z_{\pi(i)}^{s_i+n-i} = \prod_{i=1}^k X_{\sigma(i)}^{s^A{}_i+k-i} \prod_{j=1}^\ell Y_{\eta(j)}^{s^B{}_j+\ell-j}$. ∎

The next proposition is essentially Weyl's dimension formula for $U(n)$. Apart from the fact that our derivation of the formula is much more elementary than the usual one, the identities we come across in the course of this proof will also be useful later on.

PROPOSITION 2.5. *For any signature $\underline{s}$ of rank n we have*

$$Q_n(1, ..., 1 \mid \underline{s}) = \prod_{1 \leqslant i < j \leqslant n} \left(1 + \frac{s_i - s_j}{j - i}\right).$$

Proof. Induction on n. It is trivial for $n=1$. So assume $n \geqslant 2$ and the formula holds for smaller rank. By the case $k=n-1$, $\ell=1$ of Proposition 2.4 and the induction hypothesis, we get, for any $x \neq 0$,

$$Q_n(1, ..., 1, 1-x \mid \underline{s}) = x^{-(n-1)} \sum_{k=1}^{n} (1-x)^{s_k+n-k} \times \prod_{i \leqslant i < j \leqslant n-1} \left(1 + \frac{s_i^{(k)} - s_j^{(k)}}{j-i}\right),$$

where, for $1 \leqslant k \leqslant n$, the signature $\underline{s}^{(k)}$ is defined by

$$s_i^{(k)} = \begin{cases} s_i + 1 & \text{if} \quad 1 \leqslant i < k, \\ s_{i+1} & \text{if} \quad k \leqslant i \leqslant n-1. \end{cases} \tag{2.21}$$

(Thus $\underline{s}^{(i)}$ is nothing but the signature $\underline{s}^A$ defined previously for the special set $A = \{1, ..., i-1, i+1, ..., n\}$.) The limit as $x \to 0$ of the left side of this identity is $Q_n(1, ..., 1 \mid \underline{s})$, while that on the right is the coefficient of x^{n-1} in this polynomial in x. Therefore, the induction hypothesis implies

$$Q_n(1, ..., 1 \mid \underline{s}) = \sum_{k=1}^{n} (-1)^{k-1} \binom{s_k+n-k}{n-1} \prod_{1 \leqslant i < j \leqslant n-1} \left(1 + \frac{s_i^{(k)} - s_j^{(k)}}{j-i}\right).$$

Now, putting $x_i = s_i - i + 1$, $1 \leqslant i \leqslant n$, an elementary calculation yields:

$$\prod_{1 \leqslant i < j \leqslant n-1} \left(1 + \frac{s_i^{(k)} - s_j^{(k)}}{j-i}\right) = (-1)^{k-1} (n-1)! \prod_{\substack{l=1 \\ i \neq k}}^{n} (x_k - x_l)^{-1} \prod_{1 \leqslant i < j \leqslant n} \left(1 + \frac{s_i - s_j}{j-i}\right). \tag{2.22}$$

Letting p denote the polynomial $p(x) = \prod_{l=1}^{n-1} (x+l)$, and substituting the above expression in the previous one, we find that to complete the proof, it suffices to verify the following identity:

$$\sum_{k=1}^{n} p(x_k) \prod_{\substack{i=1 \\ i \neq k}}^{n} (x_k - x_i)^{-1} = 1$$

for distinct $x_1, ..., x_n$ and for any monic polynomial p of degree $n-1$.

But this is equivalent to Corollary 2.3 below. ∎

LEMMA 2.1. *Let k, ℓ be positive integers and let I be an index set of size $k+\ell$. Then, for any $k+\ell$ distinct real numbers x_i, $i \in I$, we have*

$$\sum_{A,\,B} \left(\prod_{h \in B} x_h^p \right) \left(\prod_{i \in A,\, j \in B} (x_i - x_j)^{-1} \right) = \begin{cases} 0 & \text{if} \quad p = 0, 1, ..., k-1 \\ (-1)^{kl} & \text{if} \quad p = k; \end{cases}$$

where the sum is over all ordered partitions (A, B) of I into two sets A, B of size k, ℓ respectively.

Proof. Thought of as a rational function in the complex variable x_i, the left side is everywhere analytic except possibly for simple poles at the points x_j, $j \neq i$. The residue at x_j is a sum over partitions (A, B) as above. Under the involution on the set of these partitions induced by the transposition (i, j), the fixed points contribute 0 to this sum, while the contributions due to the pair of partitions in any non-trivial orbit cancel each other. Thus all the residues are 0, so that the left side is an analytic polynomial in each x_i. But as $x_i \to \infty$, this polynomial clearly goes to a finite limit, and hence it is bounded. By Lieuville, it is independent of each x_i and hence is a constant. Clearly the limit, and hence the constant value, is 0 when $p < k$. The limiting value for $p = k$ may be obtained by induction on ℓ as follows. Clearly it is $= 1$ for $\ell = 0$. So let $\ell > 0$. Then the limit is $(-1)^k$ times a sum as in the Lemma with ℓ replaced by $\ell - 1$ and I replaced by $I - \{i\}$. Hence induction completes the proof. ∎

We shall use the case $\ell = 1$ of this identity more often than the general result, so we record it as

COROLLARY 2.3. *For distinct real numbers $x_1, ..., x_n$, we have*

$$\sum_{i=1}^{n} x_i^p \prod_{\substack{1 \leqslant j \leqslant n \\ j \neq i}} (x_i - x_j)^{-1} = \begin{cases} 0 & \text{if} \quad p = 0, ..., n-2, \\ 1 & \text{if} \quad p = n-1. \end{cases}$$

Remark. The identity in Corollary 2.3 is nothing new. For a purely algebraic proof, and for references to other proofs, see [15].

PROPOSITION 2.6. *For $\lambda \in \mathscr{W}$,*

$$B^{(\lambda)}(z, w) = \sum_{\underline{s}} \varphi_{\underline{s}}(zw^*e)/\|\varphi_{\underline{s}}\|_\lambda^2, \qquad z, w \in \Omega.$$

where the series converges uniformly for (z, w) in compact subsets of $\Omega \times \Omega$.

Here, $\|\varphi_{\underline{s}}\|_\lambda$ is the norm of $\varphi_{\underline{s}}$ as an element of $\mathscr{H}^{(\lambda)}$ and the sum is over all signatures $\underline{s}$, provided we take $\|\varphi_{\underline{s}}\|_\lambda = \infty$ when $\mathscr{P}_{\underline{s}} \not\subseteq \mathscr{H}^{(\lambda)}$.

Proof. Since $\mathscr{P}_{\underline{s}}$ is **K**-irreducible and $\mathscr{H}^{(\lambda)}$ is **K**-invariant, for any signature $\underline{s}$ for which $\varphi_{\underline{s}} \in \mathscr{H}^{(\lambda)}$, i.e. $\|\varphi_{\underline{s}}\| < \infty$, we have $\mathscr{P}_{\underline{s}} \subseteq \mathscr{H}^{(\lambda)}$. For these signatures, let $K_{\underline{s}}^{(\lambda)}(\cdot, w)$ denote the orthogonal projection of $B^{(\lambda)}(\cdot, w)$ to $\mathscr{P}_{\underline{s}}$, and set this $=0$ for the remaining signatures. Then we have $B^{(\lambda)}(\cdot, w) = \sum_{\underline{s}} K_{\underline{s}}^{(\lambda)}(\cdot, w)$ for each fixed $w \in \Omega$, where the convergence is in norm and hence also point-wise. It follows that $\sum \|K_{\underline{s}}^{(\lambda)}(\cdot, w)\|^2 = B^{(\lambda)}(w, w)$ and hence

$$\begin{aligned}\sum |K_{\underline{s}}^{(\lambda)}(z, w)| &= \sum |\langle K_{\underline{s}}^{(\lambda)}(\cdot, z), K_{\underline{s}}^{(\lambda)}(\cdot, w)\rangle| \\ &\leqslant \sqrt{B^{(\lambda)}(z, z)\, B^{(\lambda)}(w, w)}.\end{aligned}$$

Since $z \mapsto B^{(\lambda)}(z, z)$ is (continuous and hence) bounded on compact subsets of Ω, this shows that the series $\sum K_{\underline{s}}^{(\lambda)}(z, w)$ converges uniformly on compact subsets of $\Omega \times \Omega$. We have already seen that the pointwise limit is $B^{(\lambda)}$.

To complete the proof, note that $K_{\underline{s}}^{(\lambda)}$ is the reproducing kernel for the space $\mathscr{P}_{\underline{s}}$ with the inner product inherited from $\mathscr{H}^{(\lambda)}$. Hence by Proposition 2.1 $K_{\underline{s}}^{(\lambda)}(z, w) = c\varphi_{\underline{s}}(zw^*e)$ for some constant $c \geqslant 0$. Now we have

$$c^2 \|\varphi_{\underline{s}}\|_\lambda^2 = \|K_{\underline{s}}^{(\lambda)}(\cdot, e)\|^2 = K_{\underline{s}}^{(\lambda)}(e, e) = c\varphi_{\underline{s}}(e) = c,$$

so that $c = \|\varphi_{\underline{s}}\|_\lambda^{-2}$ and hence $K_{\underline{s}}^{(\lambda)}(z, w) = \varphi_{\underline{s}}(zw^*e)/\|\varphi_{\underline{s}}\|_\lambda^2$. ∎

In [5], Faraut and Koranyi showed that actually the series in the above Proposition converge uniformly on $\Omega \times \bar{\Omega}$. We shall not need this stronger result. These authors explicitly determine the constants $\|\varphi_{\underline{s}}\|_\lambda$. Their formula is for general Cartan domains and involves the dimension of the space $\mathscr{P}_{\underline{s}}$. In view of Upmeier's formula (Lemma 2.6 and 2.7 in [12]) for this dimension, it reduces to the following Proposition in the case of the matrix domains $\Omega_{n,m}$. We include an independent derivation of this formula since it is very crucial to what is to follow.

PROPOSITION 2.7.

$$\|\varphi_{\underline{s}}\|_\lambda^2 = \prod_{1 \leqslant i < j \leqslant n} \left(1 + \frac{s_i - s_j}{j - i}\right)^2 \cdot \prod_{k=1}^{n} \frac{\Gamma(n + s_k - k + 1)\, \Gamma(\lambda - k + 1)}{\Gamma(\lambda + s_k - k + 1)\, \Gamma(n - k + 1)}.$$

Note that this formula is independent of m.

Proof. Apply Proposition 2.6 to "diagonal" elements z, w of $\Omega_{n,m}$ (i.e. elements of the form $\sum_{j=1}^m a_j e_j$). In view of the formula in Proposition 2.3 for the value of esf's on diagonals, this yields

$$\sum_{\underline{s}} \|\varphi_{\underline{s}}\|_\lambda^{-2} Q_n(z_1, ..., z_n \mid \underline{s})/Q_n(1, ..., 1 \mid \underline{s}) = \prod_{j=1}^n (1-z_j)^{-\lambda}$$

where the sum converges uniformly on compact subsets of the polydisc $\{(z_1, ..., z_n): |z_j| < 1\}$. Expand the right side in a power series and for $k = 0, 1, ...,$ equate the homogeneous components of degree k of the two sides. This yields:

$$\sum_{\underline{s}:\, |\underline{s}| = k} \|\varphi_{\underline{s}}\|_\lambda^{-2} Q_n(\cdot \mid \underline{s})/Q_n(1, ..., 1 \mid \underline{s})$$
$$= \sum_{\underline{s}:\, |\underline{s}| = k} \left(\prod_{j=1}^n \left| \binom{-\lambda}{s_j} \right| \right) R_n(\cdot \mid \underline{s}) \tag{2.23}$$

where $R_n(z_1, ..., z_n \mid \underline{s})$ denotes the sum of all the *distinct* monomials of the form $\prod_{j=1}^n z_{\pi(j)}^{s_j}$, as π varies over $\mathrm{Sym}(n)$. That is, letting $\omega(\underline{s})$ denote the order of the isotropy group $\{\pi \in \mathrm{Sym}(n): s_{\pi(j)} = s_j \text{ for } 1 \leqslant j \leqslant n\}$ of the signature $\underline{s}$, we have:

$$R_n(z_1, ..., z_n \mid \underline{s}) = \frac{1}{\omega(\underline{s})} \cdot \sum_{\pi \in \mathrm{Sym}(n)} \prod_{j=1}^n z_{\pi(j)}^{s_j}.$$

Since both sides are polynomials, the identity (2.23) holds throughout $\mathbb{C}^n$, and in particular on the torus $\mathbf{T}^n = \{(z_1, ..., z_n): |z_j| = 1\}$. Let μ be the measure on $\mathbf{T}^n$ defined by

$$d\mu(z_1, ..., z_n) = \frac{1}{n!} \prod_{1 \leqslant i < j \leqslant n} |z_i - z_j|^2 \, dz_1 \ldots dz_n,$$

where $dz = dz_1 \ldots dz_n$ denotes the Haar probability on $\mathbf{T}^n$. From the defining equation (2.16) it is clear that the Schur polynomials form an orthonormal set in $L^2(\mu)$. Therefore, equating the inner products of the two sides of (2.23) with $Q_n(\cdot \mid \underline{s})$ for a fixed signature $\underline{s}$ of weight k, we get

$$\frac{1}{\|\varphi_{\underline{s}}\|_\lambda^2} = Q_n(1, ..., 1 \mid \underline{s}) \sum_{\underline{s}':\, |\underline{s}'| = |\underline{s}|} \left(\prod_{j=1}^n \left| \binom{-\lambda}{s'_j} \right| \right) \langle R_n(\cdot \mid \underline{s}'), Q_n(\cdot \mid \underline{s}) \rangle. \tag{2.24}$$

Here, of course, $\langle \cdot, \cdot \rangle$ is the inner product on $L^2(\mu)$. So, to complete the proof we only need to compute the inner product between $R_n(\cdot \mid \underline{s}')$ and $Q_n(\cdot \mid \underline{s})$ for any two signatures $\underline{s}$ and $\underline{s}'$ of the same weight k. But this is easy. Substituting the defining formula (2.16) for Q_n in the integral representing this inner product, we get

$$\begin{aligned}
&\langle R_n(\cdot \mid \underline{s}'), Q_n(\cdot, \mid \underline{s})\rangle \\
&= \frac{1}{n!} \sum_{\pi \in \mathrm{Sym}(n)} \int \mathrm{sgn}(\pi) \prod_{1 \leqslant i < j \leqslant n} (z_i - z_j) \\
&\quad \times \prod_{i=1}^{n} z_{\pi(i)}^{-s_i - n + i} R_n(z_1, ..., z_n \mid \underline{s}')\, dz_1 \ldots dz_n \\
&= \frac{1}{n!} \sum_{\pi \in \mathrm{Sym}(n)} \int \prod_{1 \leqslant i < j \leqslant n} (z_{\pi(i)} - z_{\pi(j)}) \\
&\quad \times \prod_{i=1}^{n} z_{\pi(i)}^{-s_i - n + i} R_n(z_{\pi(1)}, \ldots, z_{\pi(n)} \mid \underline{s}')\, dz_1 \ldots dz_n \\
&= \int \prod_{1 \leqslant i < j \leqslant n} (z_i - z_j) \prod_{i=1}^{n} z_i^{-s_i - n + i} R_n(z_1, ..., z_n \mid \underline{s}')\, dz_1 \ldots dz_n.
\end{aligned}$$

Now, substituting the defining formula for R_n and the Vandermonde formula

$$\prod_{1 \leqslant i < j \leqslant n} (z_i - z_j) = \sum_{\pi \in \mathrm{Sym}(n)} \mathrm{sgn}(\pi) \prod_{i=1}^{n} z_i^{n - \pi(i)}$$

in the last integral and noting that the monomials are orthonormal in $L^2(dz)$, we get:

$$\langle R_n(\cdot \mid \underline{s}'), Q_n(\cdot \mid \underline{s})\rangle = \frac{1}{\omega(\underline{s}')} \sum_{\substack{\pi,\, \sigma \in \mathrm{Sym}(n) \\ :\, \underline{s}^{\pi} = (\underline{s}')_{\sigma}}} \mathrm{sgn}(\pi).$$

Here $\underline{s}^{\pi}$ denotes the sequence $(s_i + \pi(i) - i$: $1 \leqslant i \leqslant n)$, and $(\underline{s}')_{\sigma}$ denotes the rearrangement of $\underline{s}'$ by the permutation σ. Now, given any π for which the non-increasing rearrangement $(\underline{s}^{\pi})^{\downarrow}$ of $\underline{s}^{\pi}$ equals $\underline{s}'$, the permutations σ satisfying $\underline{s}^{\pi} = (\underline{s}')_{\sigma}$ constitute a coset of the isotropy of $\underline{s}'$ and hence there are $\omega(\underline{s}')$ permutations σ corresponding to each such π. Hence we get

$$\langle R_n(\cdot \mid \underline{s}'), Q_n(\cdot \mid \underline{s})\rangle = \sum_{\substack{\pi \in \mathrm{Sym}(n) \\ :\, (\underline{s}^{\pi})^{\downarrow} = \underline{s}'}} \mathrm{sgn}(\pi).$$

Substitute this formula in (2.24), to find the Laplace expansion of a determinant. Thus,

$$\frac{1}{\|\varphi_{\underline{s}}\|_{\lambda}^{2}} = Q_n(1, ..., 1 \mid \underline{s}) \det(\mathbf{a})$$

where the $n \times n$ matrix $\mathbf{a} = (a_{ij})$ is given by

$$a_{ij} = \left| \binom{-\lambda}{s_i - i + j} \right| = \frac{\Gamma(\lambda + s_i - i + j)}{\Gamma(\lambda)\, \Gamma(s_i - i + j + 1)}.$$

(Here the entry is to be interpreted as 0 when $s_i - i + j < 0$, which is a natural convention since Gamma has poles at non-positive integers. Notice that in view of the functional equation $\Gamma(z+1) = z\Gamma(z)$, the matrix elements are actually polynomials in λ.)

This proves the Proposition for $n = 1$. To compute the determinant for $n > 1$, note that the submatrix of $\mathbf{a}$ obtained by deleting its first column and ith row $(1 \leqslant i \leqslant n)$ has the same form as $\mathbf{a}$ with n replaced by $n-1$, $\underline{s}$ replaced by the signature $\underline{s}^{(i)}$ of rank $n-1$ defined in (2.21). Therefore, expanding det($\mathbf{a}$) along the first column we inductively obtain a formula for this determinant and hence for $\|\varphi_{\underline{s}}\|_\lambda^2$. To show that this formula agrees with the one in the statement of this Proposition, we need to prove an identity which simplifies to:

$$\frac{1}{(n-1)!} \sum_{i=1}^{n} (-1)^{i-1} \frac{Q_{n-1}(1 \mid \underline{s}^{(i)})}{Q_n(1 \mid \underline{s})} \frac{\Gamma(s_i + n - i + 1)}{\Gamma(s_i - i + 2)} \times \left(\prod_{\substack{1 \leqslant k \leqslant n \\ k \neq i}} (\lambda + s_k - k + 1) \right) \left(\prod_{\ell=1}^{n-1} (\lambda - \ell)^{-1} \right) = 1.$$

Now, the left hand side is a rational function of λ, so that to prove this identity it suffices to show that its value at $\lambda = \infty$ is equal to 1 and its apparent poles at the points $\lambda \in \{1, 2, ..., n-1\}$ are not really poles, i.e., the corresponding residues are $= 0$. But, substituting $x_i = s_i - i + 1$, and using Proposition 2.5 and the formula in (2.22), we find that the reside at $\lambda \in \{1, ..., n-1\}$ and the value at infinity are given by(except for a finite multiplicative constant in the first case, but this safely ignored since we only wish to show that these residues are zero):

$$\sum_{i=1}^{n} p(x_i) \prod_{\substack{1 \leqslant \ell \leqslant n \\ \ell \neq i}} (x_i - x_\ell)^{-1},$$

where $p(x) = \prod_{1 \leqslant h \leqslant n-1_{h \neq \lambda}} (x + h)$ in the case of the residue and $p(x) = \prod_{1 \leqslant h \leqslant n-1} (x + h)$ in the case of the value at infinity. Therefore the result follows from Corollary 2.3. ∎

We also have the following formula from Lemma 2.6 and Lemma 2.7 in Upmeier [12].

PROPOSITION 2.8. *For any signature $\underline{s}$, the dimension $d_{\underline{s}}$ of the space $\mathscr{P}_{\underline{s}}$ is*

$$d_{\underline{s}} = \prod_{1 \leqslant i < j \leqslant n} \left(1 + \frac{s_i - s_j}{j-i}\right)^2 \cdot \prod_{k=1}^{n} \frac{\Gamma(m+s_k-k+1)\,\Gamma(n-k+1)}{\Gamma(n+s_k-k+1)\,\Gamma(m-k+1)}.$$

Proof. In view of Proposition 2.7, it suffices to show that $1/d_{\underline{s}}$ is the squared norm $\|\varphi_{\underline{s}}\|_m^2$ of the esf $\varphi_{\underline{s}}$ in the Hardy space $\mathscr{H}^{(m)}$.

Recall that the inner product on the Hardy space $\mathscr{H}^{(m)}$ is given by

$$\langle f, g\rangle_m = \int_K (f \circ k^{-1})(e)\overline{(g \circ k^{-1})(e)}\, dk.$$

Also, if the space $\mathscr{P}_{\underline{s}}$ is equipped with the inner product it inherits as a subspace of $\mathscr{H}^{(m)}$, then its reproducing kernel is

$$K(z, w) = \frac{\varphi_{\underline{s}}(zw^*e)}{\|\varphi_{\underline{s}}\|_m^2}.$$

On the other hand, if $\{f_j\colon 1 \leqslant j \leqslant d_{\underline{s}}\}$ is any orthonormal basis for $\mathscr{P}_{\underline{s}}$ then from the general theory of reproducing kernels we get

$$K(z, w) = \sum_{j=1}^{d_{\underline{s}}} f_j(z)\overline{f_j(w)}.$$

Now we have

$$\frac{1}{\|\varphi_{\underline{s}}\|_m^2} = K(e, e) = K(k^{-1}e, k^{-1}e) = \sum_{1 \leqslant j \leqslant d_{\underline{s}}} f_j(k^{-1}e)\overline{f_j(k^{-1}e)}.$$

Integrating both sides with respect to dk, we get

$$\frac{1}{\|\varphi_{\underline{s}}\|_m^2} = \sum_{1 \leqslant j \leqslant d_{\underline{s}}} \langle f_j, f_j\rangle_m = \sum_{1 \leqslant j \leqslant d_{\underline{s}}} 1 = d_{\underline{s}}. \quad \blacksquare$$

We shall also need the following formula for the invariantisation of $|\varphi_{\underline{s}}|^2$ by the group **K**. It is Lemma 3.3 in Faraut and Koranyi [5]. Though in the same spirit, our proof is technically simpler than the one in [5] in as much as it appeals to the general theory of reproducing kernels instead of using Schur's orthogonality relations.

PROPOSITION 2.9. *For any signature $\underline{s}$, we have*

$$\int_K |(\varphi_{\underline{s}} \circ k)(z)|^2\, dk = \frac{1}{d_{\underline{s}}}\, \varphi_{\underline{s}}(zz^*e), \qquad z \in \bar{\Omega}.$$

Proof. Continuing with the notations in the previous proof, one obtains

$$\begin{aligned}(\varphi_{\underline{s}} \circ k)(z) &= \|\varphi_{\underline{s}}\|_m^2 K(kz, e) \\ &= \|\varphi_{\underline{s}}\|_m^2 K(z, k^{-1}e) \\ &= \|\varphi_{\underline{s}}\|_m^2 \sum_{j=1}^{d_{\underline{s}}} f_j(z) \overline{f_j(k^{-1}e)}.\end{aligned}$$

Hence,

$$|(\varphi_{\underline{s}} \circ k)(z)|^2 = \|\varphi_{\underline{s}}\|_m^4 \sum_{1 \leqslant j, l \leqslant d_{\underline{s}}} f_j(z) \overline{f_j(k^{-1}e) f_\ell(z)} f_\ell(k^{-1}e).$$

Integrating both sides with respect to dk, it follows that

$$\begin{aligned}\int_K |(\varphi_{\underline{s}} \circ k)(z)|^2 \, dk &= \|\varphi_{\underline{s}}\|_m^4 \sum_{1 \leqslant j, l \leqslant d_s} f_j(z) \overline{f_l(z)} \langle f_l, j_j \rangle_m \\ &= \|\varphi_{\underline{s}}\|_m^4 \sum_{1 \leqslant j \leqslant d_{\underline{s}}} f_j(s) \overline{f_j(z)} \\ &= \|\varphi_{\underline{s}}\|_m^4 K(z, z) \\ &= \|\varphi_{\underline{s}}\|_m^2 \phi_{\underline{s}}(zz^*e).\end{aligned}$$

Therefore, an appeal to the previous Proposition completes the proof. ∎

Remark. From the above, it is easy to deduce a formula for the **K**-invariantisation $(f\bar{g})^K$ of $f\bar{g}$ for any two elements f, g of $\mathscr{H}^{(\lambda)}$. Namely, if $f = \sum_{\underline{s}} f_{\underline{s}}$, $g = \sum_{\underline{s}} g_{\underline{s}}$ are the break-ups of f, g along the orthogonal decomposition (2.11) then

$$(f\bar{g})^K(w) = \sum_{\underline{s}} \frac{\langle f_{\underline{s}}, g_{\underline{s}} \rangle_\lambda}{d_{\underline{s}}} \frac{\varphi_{\underline{s}}(ww^*e)}{\|\varphi_{\underline{s}}\|_\lambda^2}.$$

To prove this, note that for any fixed w in Ω, $\langle f, g \rangle \stackrel{\text{def}}{=} (f\bar{g})^K(w)$ defines a **K**-invariant inner product on $\mathscr{H}^{(\lambda)}$ which is continuous with respect to the norm on the latter. Since the same is true of the right hand side of the above formula, to prove it, it suffices to verify it for $f = g = \varphi_{\underline{s}}$; but in this case the formula reduces to Proposition 2.9.

3. Boundedness

3.1. *Some General Facts*

We begin with some generalities on reproducing kernels. Recall that if $K: X \times X \to \mathbb{C}$ is non-negative definite (*nnd*) in the sense that for any

$x_1, ..., x_n \in X$ the matrix $((K(x_i, x_j))$ is *nnd*, then there is a uniquely determined Hilbert space $\mathscr{H}(K)$ of functions on X such that K is the reproducing kernel of $\mathscr{H}(K)$, in the sense that $K(\cdot, x) \in \mathscr{H}(K)$ for all $x \in X$, and we have $\langle f, K(\cdot, x)\rangle = f(x)$ for all $f \in \mathscr{H}(K)$. The usual construction of $\mathscr{H}(K)$ is as follows. Take $\mathscr{F}$ to be the linear span of $K(\cdot, x)$, $x \in X$, and define a sesqui-linear form $\langle \cdot, \cdot \rangle$ on $\mathscr{F}$ by $\langle K(\cdot, y), K(\cdot, x)\rangle = K(x, y)$. Non-negative definiteness of K implies this form is *nnd*, whence Cauchy–Schwarz yields $|f(x)|^2 = |\langle f, K(\cdot, x)\rangle|^2 \leqslant \langle f, f\rangle\, K(x, x)$, $x \in X$. Hence $\langle f, f\rangle = 0$ implies $f(x) = 0$ for all x, that is, $f = 0$. Thus, $\langle \cdot, \cdot \rangle$ is an inner product on $\mathscr{F}$. (Thus, the usual requirement that K be positive definite is unnecessary. It is enough to have K *nnd.*) This inequality also implies that $\langle \cdot, \cdot \rangle$-Cauchy sequences are pointwise Cauchy, and the completion $\mathscr{H}(K)$ of $(\mathscr{F}, \langle \cdot, \cdot \rangle)$ is naturally identified with a Hilbert space of functions on X, with K as its reproducing kernel.

An alternative and more direct description of $\mathscr{H}(K)$ is as follows. For two kernels K_1, K_2 on X, let's write $K_1 \leqslant K_2$ to mean $K_2 - K_1$ is *nnd.* For any complex valued function f on X, define $\|f\|$ by

$$\|f\| = \inf\{c > 0 : f\bar{f} \leqslant c^2 K\}. \tag{3.1}$$

(Explicitly, the condition within braces means that the kernel on X given by $(x, y) \to (c^2 K(x, y) - f(x)\overline{f(y)})$ is nnd.) Then

$$\mathscr{H}(K) = \{f : \|f\| < \infty\}. \tag{3.2}$$

To see the equivalence of the two definitions, let $\widehat{\|\cdot\|}$ denote the norm on the Hilbert space $\mathscr{H}(K)$ in the first construction. For any orthonormal basis $\{f_n\}$ of $\mathscr{H}(K)$, K is recovered by the formula $K = \sum_{n \geqslant 1} f_n \bar{f}_n$. Taking an orthonormal basis with $f_1 = f/c$, $c = \widehat{\|f\|}$, we find $K - (1/c)^2 f\bar{f} = \sum_{n \geqslant 2} f_n \bar{f}_n \geqslant 0$, whence $f\bar{f} \leqslant c^2 K$, so that $\|f\| \leqslant \widehat{\|f\|}$. On the other hand, if $\|f\| = c$, define the kernel K_1 on X by $K_1 = f\bar{f}/c^2$. Then both K and K_1 are nnd kernels and $K_1 \leqslant K$. By [2, Theorem I, p. 354], this implies that the Hilbert space $\mathscr{H}(K_1)$ is a subset of the Hilbert space $\mathscr{H}(K)$, and the norm on $\mathscr{H}(K_1)$ is point-wise greater than or equal to the norm on $\mathscr{H}(K)$. Since $\mathscr{H}(K_1)$ is clearly the one dimensional space spanned by f and the norm of f in $\mathscr{H}(K_1)$ equals $c = \|f\|$, this means that $\|f\| \geqslant \widehat{\|f\|}$. Thus we have $\|f\| = \widehat{\|f\|}$ for all f, proving equivalence of the two definitions of $\mathscr{H}(K)$. This also shows that for any *nnd* kernel K, (3.1) and (3.2) define a functional Hilbert space. It should be amusing to construct an ab initio proof of this fact.

Now, let K be an *nnd* kernel on X, $\mathscr{H}(K)$ the associated Hilbert space, and f any function on X. Let M_f denote multiplication by f. When is M_f a bounded operator on $\mathscr{H}(K)$? If there is a finite c such that $(c^2 - f\bar{f})\, K \geqslant 0$

then $f\bar{f}K \leqslant c^2 K$ and $g\bar{g} \leqslant c_0^2 K$ imply that $f\bar{f}g\bar{g} \leqslant c_0^2 f\bar{f}K \leqslant (c_0 c)^2 K$, so that M_f is bounded and $\|M_f\| \leqslant c$ for any such c. (Here we have used the wellknown fact that the pointwise product of two *nnd* kernels is *nnd*.) Thus we have shown

Lemma 3.1. *Suppose there is a* $c \in [0, \infty)$ *such that* $(c^2 - f\bar{f})\, K \geqslant 0$. *Then* M_f *is bounded on* $\mathscr{H}(K)$.

(In fact, it can be shown that $\|M_f\|$ is the infimum of all $c \geqslant 0$ for which $(c^2 - f\bar{f})\, K \geqslant 0$, and the condition in the Lemma is necessary and sufficient.)

3.2. *A Reduction*

Now we come to the proof of Theorem 1.1. If $\lambda \in \mathscr{W}_d$, then there is a polynomial ψ such that $\psi \notin \mathscr{H}^{(\lambda)}$. (For instance, ψ can be taken to be the generalised determinant on Ω, i.e., the esf corresponding to the signature $(1, 1, ..., 1)$.) But the constant function 1 is in $\mathscr{H}^{(\lambda)}$. Since $1 \in \mathscr{H}^{(\lambda)}$, $\psi \notin \mathscr{H}^{(\lambda)}$, $\psi(M^{(\lambda)}) =$ multiplication by ψ is not bounded. A fortiori, $M^{(\lambda)}$ is not bounded. So, from now on, we assume $\lambda \in \mathscr{W}_c$, i.e., $\lambda > n-1$.

Till the end of this section, let φ be the generalised trace on Ω, i.e., the unique esf of degree 1 corresponding to the signature $(1, 0, ..., 0)$. Suppose we can show that $M_\varphi^{(\lambda)} \stackrel{\text{def}}{=} \varphi(M^{(\lambda)})$ is bounded on $\mathscr{H}^{(\lambda)}$, $\lambda > n-1$. Since the action of $\mathbf{K}$ by composition is unitary on $\mathscr{H}^{(\lambda)}$, it will follow that multiplication by $\phi \circ k$ is bounded on $\mathscr{H}^{(\lambda)}$ for all $k \in \mathbf{K}$. But $\varphi \circ k$, $k \in \mathbf{K}$ spans the space $\mathscr{P}_{\underline{s}}$ for $\underline{s} = (1, 0, ..., 0)$, so that multiplication by each $f \in \mathscr{P}_{\underline{s}}$ is bounded. But as $\underline{s}$ is the only signature of weight 1, $\mathscr{P}_{\underline{s}} = \mathrm{Hom}(1)$, the space of all linear homogeneous polynomials; in particular, all the *mn* coordinate functions belong here. Hence the components of $M^{(\lambda)}$ are bounded. Thus, it suffices to show that $M_\varphi^{(\lambda)}$ is bounded on $\mathscr{H}^{(\lambda)}$ for $\lambda > n-1$.

3.3 *Multiplication by Trace*

Fix $\lambda > n-1$. In view of Lemma 3.1, we only have to show that there is a finite constant c such that $(c^2 - \varphi\bar{\varphi})\, B^{(\lambda)} \geqslant 0$. Recall that $\mathrm{Hom}(1)$ is a functional Hilbert space with reproducing kernel $K(z, w) = \varphi(zw^*e)$. Since $\varphi = K(\cdot, e)$, φ has norm $K(e, e) = \varphi(e) = 1$ as an element of this space. Hence by (3.1), $\varphi\bar{\varphi} \leqslant K$. Hence, writing

$$(c^2 - \varphi(z)\,\overline{\varphi(w)})\, B^{(\lambda)}(z, w) = (c^2 - \varphi(zw^*e))\, B^{(\lambda)}(z, w) + (\varphi(zw^*e) - \varphi(z)\,\overline{\varphi(w)})\, B^{(\lambda)}(z, w), \qquad (3.3)$$

we see that the second term is an *nnd* kernel. Hence, for our purpose, it suffices to exhibit a finite c for which $(c^2-\varphi(zw^*e))\, B^{(\lambda)}(z,w)$ is an *nnd* kernel. Now using the expansion in Proposition 2.6, we get

$$(c^2-\varphi(zw^*e))\, B^{(\lambda)}(z,w)=\sum_{\underline{s}} \frac{c^2}{\|\varphi_{\underline{s}}\|_\lambda^2}\, \varphi_{\underline{s}}(zw^*e)-\sum_{\underline{s}} \frac{1}{\|\varphi_{\underline{s}}\|_\lambda^2}\,(\varphi\varphi_{\underline{s}})(zw^*e).$$

Now use the formula (a) from Corollary 2.2 to get

$$(c^2-\varphi(zw^*e))\, B^{(\lambda)}(z,w)=\sum_{\underline{s}} \left(\frac{c^2}{\|\varphi_{\underline{s}}\|_\lambda^2}-\beta_{\underline{s}}(\lambda)\right) \varphi_{\underline{s}}(zw^*e),$$

where

$$\beta_{\underline{s}}(\lambda)=\sum_{\underline{\tilde{s}}}{}' \frac{a(\underline{\tilde{s}},\underline{s})}{\|\varphi_{\underline{\tilde{s}}}\|_\lambda^2},$$

the sum $\sum'$ is over all signatures $\tilde{s}$ such that $|\underline{\tilde{s}}|=|\underline{s}|-1$, and $\underline{\tilde{s}}\leqslant \underline{s}$ component wise; the coefficients $a(\underline{\tilde{s}},\underline{s})$ are given, for such pairs $\underline{s}$, $\underline{\tilde{s}}$ of signatures, by the formula

$$a(\underline{\tilde{s}},\underline{s})=\frac{1}{n}\prod_{\substack{i=1\\ i\neq k}}^{n}\left(1+\frac{\varepsilon_{i,k}}{|i-k|+|s_i-s_k+1|}\right),$$

where k is the unique coordinate position for which $\tilde{s}_k<s_k$, and $\varepsilon_{i,k}=1$ if $i>k$ and $=-1$ if $i<k$.

Recall from Proposition 2.1 that $(z,w)\to\phi_{\underline{s}}(zw^*e)$ is an nnd kernel for each signature $\underline{s}$. Hence the nonnegative definiteness of $(c^2-\phi(zw^*e))\, B^{(\lambda)}(z,w)$ follows if c can be chosen so that each coefficient $c^2/\|\varphi_{\underline{s}}\|_\lambda^2-\beta_{\underline{s}}(\lambda)$ is nonnegative. Note that, as $\lambda>n-1$, each $\|\varphi_{\underline{s}}\|_\lambda<\infty$. Therefore, this argument gives the estimate

$$\|M_\varphi^{(\lambda)}\|^2\leqslant \sup_{\underline{s}} \beta_{\underline{s}}(\lambda)\, \|\varphi_{\underline{s}}\|_\lambda^2,$$

where the supremum is over all signatures. Thus to conclude the proof of Theorem 1.1, we need only show that this supremum is finite for $\lambda>n-1$. But, using the explicit norm formula from Proposition 2.7, we get:

$$\beta_{\underline{s}}(\lambda)\, \|\varphi_{\underline{s}}\|_\lambda^2=\frac{1}{n}\sum_{\substack{1\leqslant k\leqslant n\\ : s_{k+1}<s_k}} \frac{(n+s_k-k)}{(\lambda+s_k-k)} \prod_{\substack{1\leqslant i\leqslant n\\ i\neq k}}\left(1+\frac{\varepsilon_{ik}}{|i-k|+|s_i-s_k+1|}\right)^{-1},$$

with ε_{ik} as above. (We adopt the convention that $s_{n+1}=0$.) For $1 \leqslant k \leqslant n$ the kth term in the above formula is clearly a bounded function of $\underline{s}$, so that this supremum is finite. This completes the proof of Theorem 1.1.

4. Joint Spectrum

Through out the rest of this paper, we assume $\mathbf{M}^{(\lambda)}$ is bounded, i.e., $\lambda > n-1$. In this section, we will prove Theorem 1.2: the joint Taylor spectrum $\mathbf{M}^{(\lambda)}$ is $\bar{\Omega}$. Actually our proof goes through for any notion of joint spectrum of a commuting d-tuple $\mathbf{T}$ of bounded operators provided this notion satisfies

(i) it is a unitary invariant,

(ii) the joint spectrum contains the eigenvalues of $\mathbf{T}$,

(iii) the associated functional calculus has the correct mapping property: if f: $U \to V$ is an analytic or co-analytic function between complex domains such that U contains the spectrum of $\mathbf{T}$ then f maps the spectrum of $\mathbf{T}$ into the spectrum of $f(\mathbf{T})$, and

(iv) if further, f is component wise rational then $f(\mathbf{T})$ is obtained by "plugging in" $\mathbf{T}$ into this rational expression.

Note that the Taylor spectrum has all these properties (cf. [4] and [11, Theorem 1.5]). We have stated Theorem 1.2 for the Taylor spectrum because there is an agreement among experts that Taylor's notions is the minimal (and hence best) among all reasonable notions of joint spectrum.

Lemma 4.1. *Let $z_0 \in \mathbb{C}^{n\times m}$ with $\|z_0\| = t > 1$. Then there is a $g \in \mathbf{G}$ such that*

(i) *g is analytic in a neighborhood of $t\bar{\Omega}$, and*

(ii) $\|g(z_0)\| > \|z_0\|$.

(Here $\|\cdot\|$ is the usual operator norm on $\mathbb{C}^{n\times m}$. Recall that Ω is the open unit ball with respect to this norm.)

Proof. Let ε be a small positive number and put $a = z_0/t^2 + \varepsilon \in \Omega$. We claim that $g = \phi_a \in \mathbf{G}$ works provided ε is sufficiently small. Recall from (2.2) that ϕ_a is the unique involution in $\mathbf{G}$ interchanging 0 and a.

$$\phi_a(z_0) = (1-aa^*)^{-1/2}\,(a-z_0)(1-a^*z_0)^{-1}\,(1-a^*a)^{1/2}.$$

Since $z_0^* z_0$ is *nnd*, $t^2 = \|z_0^* z_0\|$ is an eigenvalue of $z_0^* z_0$. Let $u \in \mathbb{C}^m$ be a corresponding eigenvector of norm 1. Put $v = z_0 u \in \mathbb{C}^n$. Since u is an eigenvector of $z_0^* z_0$ corresponding to a nonzero eigenvalue, we have $v \neq 0$. An easy computation shows

$$(1-aa^*)^{1/2}\,\phi_a(z_0)\,u = \left(1-\frac{t^2}{(t^2+\varepsilon)^2}\right)^{1/2}\left(1-\frac{t^2}{t^2+\varepsilon}\right)^{-1}\left(\frac{1}{t^2+\varepsilon}-1\right)v.$$

Hence $\|(1-aa^*)^{1/2}\,\phi_a(z_0)\,u\| \sim (c_1/\varepsilon)$ as $\varepsilon \to 0$, where $c_1>0$ is independent of ε. Therefore,

$$\|\phi_a(z_0)\,u\| \geqslant \|(1-aa^*)^{1/2}\|^{-1}\,\|(1-aa^*)^{1/2}\,\phi_a(z_0)\,u\| \sim \frac{c_2}{\varepsilon} \qquad \text{as} \quad \varepsilon \to 0,$$

where $c_2>0$ is independent of ε.

Hence $\|\phi_a(z_0)\| \geqslant \|\phi_a(z_0)\,u\| \to \infty$ as $\varepsilon \to 0$. This completes the proof. ∎

Now the proof of Theorem 1.2 is surprisingly easy. Suppose, if possible, the spectrum is not contained in $\bar{\Omega}$. Choose a z_0 in the spectrum which maximises $\|z_0\|$, say $\|z_0\| = t > 1$. Then the spectrum is contained in $t\bar{\Omega}$, and if g is as guaranteed in Lemma 4.1, then $g(z_0)$ is in the spectrum of $g(\mathbf{M}^{(\lambda)})$. But by homogeneity, $g(\mathbf{M}^{(\lambda)})$ is unitarily equivalent to $\mathbf{M}^{(\lambda)}$. Hence $g(z_0)$ is in the spectrum of $\mathbf{M}^{(\lambda)}$. But this contradicts the maximality of z_0 since $\|g(z_0)\| > \|z_0\|$. Thus the spectrum is contained in $\bar{\Omega}$. Next, take any $z \in \Omega$. A straightforward computation shows that $\bar{z}$ is a joint eigenvalue for the adjoint of $\mathbf{M}^{(\lambda)}$ with eigenvector $B^{(\lambda)}(\cdot, z) \in \mathscr{H}^{(\lambda)}$. Therefore, z is in the spectrum of $\mathbf{M}^{(\lambda)}$, whence the spectrum contains Ω and therefore $\bar{\Omega}$. This completes the proof of Theorem 1.2.

5. Joint Subnormality

Recall that a commuting tuple of bounded operators on a Hilbert space is called jointly subnormal if it is the restriction of a commuting tuple of normal operators to a common invariant subspace. In this section, we prove Theorem 1.3 and, in particular, determine the range of λ for which $\mathbf{M}^{(\lambda)}$ is jointly subnormal.

5.1. *A Question of Measure*

We begin by proving a general theorem which implies, in particular, the equivalence of (i) and (iii) in the statement of Theorem 1.3.

Theorem 5.1. *Let $X \subseteq \mathbb{C}^d$ be a bounded domain and let $\mathscr{H}$ be a Hilbert space of analytic functions on X such that the set of analytic polynomials is densely contained in $\mathscr{H}$. Let* $\mathbf{M}$ *be the* (*densely defined*) *d-tuple of multiplication by coordinate functions on $\mathscr{H}$. Suppose the Taylor spectrum of* $\mathbf{M}$ *is $\bar{X}$. Then the following are equivalent*:

(i) **M** *is a subnormal tuple of bounded operators,*

(ii) *There is a uniquely determined finite measure* μ *supported inside* $\bar{X}$ *such that the inner product* $\langle \cdot, \cdot \rangle$ *on* $\mathscr{H}$ *is given by* $\langle f, g \rangle = \int f\bar{g}\, d\mu$ *for all analytic polynomials* f, g.

Proof. (ii) clearly implies (i), since, under (ii), $\mathscr{H}$ is naturally embedded as a closed subspace in $L^2(\mu)$ and the natural extension of **M** to $L^2(\mu)$ is normal.

So, assume (i). If **T** is any commuting tuple of bounded subnormal operators on $\mathscr{H}$, then letting **S** denote a normal extension of **T** and letting E be the spectral measure of **S**, we have for all $x \in \mathscr{H}$ and all multi-indices I, J,

$$\langle \mathbf{T}^I x, \mathbf{T}^J x \rangle = \langle \mathbf{S}^I x, \mathbf{S}^J x \rangle = \int z^I \bar{z}^J \, d\langle Ex, x \rangle.$$

Taking x to be the constant function $\underline{1} \in \mathscr{H}$, $\mathbf{T} = \mathbf{M}$, and $\mu = d\langle E\underline{1}, \underline{1} \rangle$, where E is the spectral measure of the minimal normal extension of **M**, this formula specialises to

$$\langle z^I, z^J \rangle = \int z^I \bar{z}^J \, d\mu(z),$$

for all analytic monomials z^I, z^J. Hence the integral representation in (ii) follows. The support of μ is contained in the spectrum of the minimal normal extension, which in turn is contained in $\bar{X}$. Also, since the linear span of $\{f\bar{g}: f, g \text{ analytic polynomials}\}$ is dense in $C(X)$ by Stone-Weirstrass, μ is uniquely determined by the integral $\int f\bar{g}\, d\mu$. This completes the proof. ∎

LEMMA 5.1. *Let* $\lambda \in \mathscr{W}_c$. *Then* $\mathbf{M}^{(\lambda)}$ *is subnormal if and only if there is a probability measure* μ_λ *supported inside* $\bar{\Omega}$ *such that* μ_λ *is quasi-invariant under* **G**-*action, with*

$$\left(\frac{d\mu_\lambda \circ g}{d\mu_\lambda}\right)(z) = |Jg(z)|^{2\lambda/(m+n)}, \qquad \forall g \in \mathbf{G}, \qquad \forall z \in \bar{\Omega}. \tag{5.1}$$

In this case, μ_λ *is uniquely determined by this condition. Moreover, either* $\mu_\lambda(\Omega) = 1$ *or* $\mu_\lambda(S_i) = 1$ *for a uniquely determined value of* i, $0 \leqslant i \leqslant n-1$.

(Here S_i is the ith boundary component as defined in (2.13).)

Proof. Let $M^{(\lambda)}$ be jointly subnormal. By Theorems 1.2 and 5.1, there is a finite measure μ_λ supported in $\bar{\Omega}$ such that $\langle f_1, f_2 \rangle_\lambda = \int f_1 \bar{f}_2 \, d\mu_\lambda$ for analytic polynomials f_1, f_2. Since the constant function $\underline{1} = B^{(\lambda)}(\cdot, 0) \in \mathscr{H}^{(\lambda)}$

has norm $B^{(\lambda)}(0,0)=1$, taking $f_1=f_2=\underline{1}$ we find $\mu_\lambda(\bar{\Omega})=\int \underline{1}\, d\mu_\lambda = \|\underline{1}\|^2=1$, so that μ_λ is a probability measure. Fix $g\in\mathbf{G}$. Since the operator $U^{(\lambda)}(g^{-1})$ defined by (2.7) is unitary on $\mathscr{H}^{(\lambda)}$, it follows that for any two analytic polynomials f_1, f_2, we have

$$\begin{aligned}
\int f_1\bar{f}_2\, d\mu_\lambda &= \langle f_1, f_2\rangle_\lambda \\
&= \langle U^{(\lambda)}(g^{-1})(f_1),\, U^{(\lambda)}(g^{-1})(f_2)\rangle_\lambda \\
&= \int U^{(\lambda)}(g^{-1})(f_1)\, \overline{U^{(\lambda)}(g^{-1})(f_2)}\, d\mu_\lambda \\
&= \int |Jg^{-1}(z)|^{2\lambda/(m+n)}\, (f_1\circ g^{-1}(z))\, \overline{(f_2\circ g^{-1}(z))}\, d\mu_\lambda(z) \\
&= \int |Jg(w)|^{-2\lambda/(m+n)}\, f_1(w)\, \overline{f_2(w)}\, d\mu_\lambda\circ g(w).
\end{aligned}$$

Since the finite linear combinations of the function $f_1\bar{f}_2$ form a dense set in $C(\bar{\Omega})$, it follows that μ_λ is quasi-invariant (i.e., $\mu_\lambda\circ g$ and μ_λ are equivalent measures for all $g\in\mathbf{G}$) and the density $d\mu_\lambda\circ g/d\mu_\lambda$ is given by (5.1).

Now assume the probability μ_λ satisfies (5.1). In particular, μ_λ is **K**-invariant. Thus, μ_λ is invariant under $z\to e^{i\theta}z$, for each $\theta\in[-\pi,\pi]$. Also, if f is analytic in a neighbourhood of $\bar{\Omega}$ then for each fixed $z\in\bar{\Omega}$, we have $f(0)=1/2\pi\int_{-\pi}^{\pi} f(e^{i\theta}z)\, d\theta$.

Hence, for any such f,

$$\begin{aligned}
|f(0)|^2 &= \int_{\bar{\Omega}} |f(0)|^2\, d\mu_\lambda(z) \\
&= \int_{\bar{\Omega}} \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta}z)\, d\theta\right|^2 d\mu_\lambda(z). \\
&\leqslant \int_{\bar{\Omega}} \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(e^{i\theta}z)|^2\, d\theta\, d\mu_\lambda(z). \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{\bar{\Omega}} |f(e^{i\theta}z)|^2\, d\mu_\lambda(z)\, d\theta \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{\bar{\Omega}} |f(z)|^2\, d\mu_\lambda(z)\, d\theta \\
&= \int_{\bar{\Omega}} |f(z)|^2\, d\mu_\lambda(z) \\
&= \|f\|^2.
\end{aligned}$$

Thus, if $f \in L^2(\mu_\lambda)$ is analytic in a neighbourhood of $\bar{\Omega}$, then we have $|f(0)| \leqslant \|f\|$. That is $f \to f(0)$ is bounded on the subspace P of $L^2(\mu_\lambda)$ consisting of such functions. Since $\mathbf{G}$ is transitive on Ω and μ_λ is quasi-invariant with respect to $\mathbf{G}$-action, it follows that for each fixed $z \in \Omega$, $z \to f(z)$ is bounded on P, the bound being uniform for z in compact subsets of Ω. Putting $H = \bar{P}$, we see that H is a closed subspace of $L^2(\mu_\lambda)$, and indeed, it is a functional Hilbert space of analytic functions on Ω. Let K be the reproducing kernel of H. Retracing the computations in the beginning of this proof, with $f_j = K(\cdot, w_j)$, $j = 1, 2$, $w_j \in \Omega$, we find that K transforms under $\mathbf{G}$ action exactly like $B^{(\lambda)}$. Also, $K(\cdot, 0)$, being $\mathbf{K}$-invariant, is a constant function. Hence $K = cB^{(\lambda)}$ for this constant c. But, $c = K(0, 0) = \int 1 \, d\mu_\lambda = 1$. Thus, $K = B^{(\lambda)}$, and hence $H = \mathscr{H}^{(\lambda)}$. Thus, the inner product on $\mathscr{H}^{(\lambda)}$ is "given by" the probability measure μ_λ, so that $\mathbf{M}^{(\lambda)}$ is subnormal on $\mathscr{H}^{(\lambda)}$.

The uniqueness of μ_λ now follows from the uniqueness statement in Theorem 5.1. Since $\bar{\Omega}$ is the union of the $n+1$ $\mathbf{G}$-orbits Ω and S_i, $0 \leqslant i \leqslant n-1$, there is at least one of these orbits, say Θ, for which $\mu_\lambda(\Theta) > 0$. Define the probability μ by $\mu(A) = \mu_\lambda(A \cap \Theta)/\mu_\lambda(\Theta)$. Then μ also satisfies (5.1), so that by the uniqueness $\mu = \mu_\lambda$. Hence $\mu_\lambda(\Theta) = \mu(\Theta) = 1$, which proves the last statement in the Lemma. ▮

5.2. *Subnormality versus Induced Representations*

Given Lemma 5.1, the equivalence of (i) and (iv) of Theorem 1.3 is very easy. Namely, if $\mathbf{M}^{(\lambda)}$ is subnormal on $\mathscr{H}^{(\lambda)}$, then let μ_λ be the probability measure guaranteed by this Lemma. Define the $*$-algebra homomorphism $\psi_\lambda \colon C(\bar{\Omega}) \to \mathscr{L}(L^2(\mu_\lambda))$ by $\psi_\lambda(p)$ equal to multiplication by p. The projective representation $U^{(\lambda)}$ of $\mathbf{G} = PSU(n, m)$ on $\mathscr{H}^{(\lambda)}$ extends naturally to a representation $\tilde{U}^{(\lambda)}$ on $L^2(\mu_\lambda)$ having the property:

$$\tilde{U}_g^{(\lambda)} \psi_\lambda(p) \, \tilde{U}_g^{(\lambda)*} = \psi_\lambda(g \cdot p), \qquad p \in C(\bar{\Omega}), \qquad g \in \mathbf{G}, \qquad (g \cdot p \stackrel{\text{def}}{=} p \circ g^{-1}).$$

Thus $(\bar{\Omega}, \psi_\lambda, U^{(\lambda)})$ is a system of imprimitivity of multiplicity one, which is transitive since by Lemma 5.1, μ_λ sits on a single $\mathbf{G}$-orbit in $\bar{\Omega}$. Hence by Mackey's imprimitivity theorem [13, Theorem 6.12, p. 223], the projective representation $\tilde{U}^{(\lambda)}$ of $\mathbf{G}$ on $L^2(\mu_\lambda)$ is induced from a one dimensional representation (of the isotropy subgroup of any point in the orbit on which μ_λ sits), and $U^{(\lambda)}$ on $\mathscr{H}^{(\lambda)}$ is a subpresentation of this induced representation. Clearly this argument goes backwards to prove the converse as well.

5.3. *Measure for Measure*

In this subsection, we prove the implication (ii) $\Rightarrow$ (iii) of Theorem 1.3. That is, $\mathbf{M}^{(\lambda)}$ is subnormal if $\lambda \in m + \mathscr{W}$. We begin by stating our final criterion for subnormality of $\mathbf{M}^{(\lambda)}$:

LEMMA 5.2. *$\mathbf{M}^{(\lambda)}$ is subnormal if and only if there is a probability measure $\mathbf{m}_\lambda = \mathbf{m}_\lambda^{n\times m}$ on Δ_n satisfying*

$$\int_{\Delta_n} Q_n(\underset{\sim}{x} \mid \underset{\sim}{s})\, d\mathbf{m}_\lambda(\mathbf{x})$$
$$= \prod_{1 \leqslant i < j \leqslant n} \left(1 + \frac{s_i - s_j}{j-i}\right) \prod_{k=1}^{n} \frac{\Gamma(m + s_k - k + 1)\, \Gamma(\lambda - k + 1)}{\Gamma(\lambda + s_k - k + 1)\, \Gamma(m - k + 1)}$$

for all signatures $\underset{\sim}{s}$ of rank n.

(Note that, since $Q_n(\cdot \mid \underset{\sim}{0})$ is the constant function $\underset{\sim}{1}$, the measure $\mathbf{m}_\lambda^{n\times m}$, when it exists, is necessarily a probability measure.)

Proof. By Theorem 5.1., $\mathbf{M}^{(\lambda)}$ is subnormal if and only if there is a probability measure μ on $\bar{\Omega}$ such that $\|f\|_\lambda^2 = \int |f|^2\, d\mu$ for all $f \in \mathscr{H}^{(\lambda)}$. Since the norm on $\mathscr{H}^{(\lambda)}$ is $\mathbf{K}$-invariant, such a probability μ, when it exists, satisfies:

$$\mu \circ k = \mu \qquad \text{for all} \quad k \in \mathbf{K}, \qquad \|\varphi_{\underset{\sim}{s}}\|_\lambda^2 = \int |\varphi_{\underset{\sim}{s}}|^2\, d\mu \qquad \text{for all signatures } \underset{\sim}{s}. \tag{5.2}$$

Conversely, if a probability μ satisfies the conditions in (5.2), then it defines a $\mathbf{K}$-invariant inner product on the space of polynomials on $\bar{\Omega}$. By Schur Lemma, the $\mathbf{K}$-irreducible subspaces $\mathscr{P}_{\underset{\sim}{s}}$ are mutually orthogonal with respect to this inner product. Now, by assumption the norm defined by μ agrees with $\|\cdot\|_\lambda$ on at least one element (viz. the esf) in each of these subspaces. Since by Schur Lemma the $\mathbf{K}$-invariant inner product on each irreducible subspace is unique upto a scalar multiple, and since $\mathscr{H}^{(\lambda)}$ is the orthogonal direct sum of these subspaces, it follows that the norm defined by μ is precisely $\|\cdot\|_\lambda$ under the hypotheses in (5.2). This shows that $\mathbf{M}^{(\lambda)}$ is subnormal if and only if there is a probability measure μ on $\bar{\Omega}$ satisfying (5.2).

Now, there is a natural bijection between the set of $\mathbf{K}$-invariant measures μ on $\bar{\Omega}$ and the set of all measures ν on Δ_n, given by $\nu = \mu \circ \pi^{-1}$. Here $\pi\colon \bar{\Omega} \to \Delta_n = \bar{\Omega}/\mathbf{K}$ is the quotient map given by (2.6). For μ and ν thus related, we have

$$\int_{\bar{\Omega}} f\, d\mu = \int_{\Delta_n} f^K\, d\nu \qquad \text{for all} \quad f \in L^1(\mu),$$

where f^K is the $\mathbf{K}$-invariantisation of f: $f^K = \int_K f \circ k\, dk$. In particular, by Propositions 2.3 and 2.9, the $\mathbf{K}$-invariantisation of $|\varphi_{\underset{\sim}{s}}|^2$ at $z = \sum_{k=1}^n x_k e_k$, is $d_{\underline{s}}^{-1} Q_n(x_1^2, ..., x_n^2 \mid \underline{s})/Q_n(\underline{1} \mid \underline{s})$. Therefore, a measure μ on $\bar{\Omega}$ satisfies the two conditions in (5.2) iff the corresponding measure $\nu = \mu \circ \pi^{-1}$ satisfies

$$\int_{\Delta_n} Q_n(x_1^2, ..., x_n^2 \mid \underline{s}) \, d\nu = d_{\underline{s}} Q_n(\underline{1} \mid \underline{s}) \, \|\phi_{\underline{s}}\|_\lambda^2. \tag{5.3}$$

Now, letting $\sigma: \Delta_n \to \Delta_n$ denote the squaring map $(x_1, ..., x_n) \mapsto (x_1^2, ..., x_n^2)$, and taking $\mathbf{m}$ to be the measure on Δ_n given by $\mathbf{m} = \nu \circ \sigma^{-1}$, we see that the above holds iff $\int_{\Delta_n} Q_n(\cdot \mid \underline{s}) \, d\mathbf{m}$ equals the righthand side of (5.3). Since by the formulae in Propositions 2.3, 2.7 and 2.8, the right hand side of (5.3) equals that of the equation in the statement of Lemma 5.2, this completes the proof. ▮

Next we prove a Lemma which shows that for the proof of the equivalence (ii) $\Leftrightarrow$ (iii) in Theorem 1.3, one loses no generality in assuming $m = n$.

LEMMA 5.3. *For any λ, $\mathbf{M}^{(\lambda)}$ is subnormal on $\mathscr{H}^{(\lambda)}(\Omega_{n,n})$ if and only if $\mathbf{M}^{(\lambda+m-n)}$ is subnormal on $\mathscr{H}^{(\lambda+m-n)}(\Omega_{n,m})$ for all $m \geqslant n$.*

Proof. First suppose that λ is such that $\mathbf{M}^{(\lambda)}$ is subnormal on $\mathscr{H}^{(\lambda)}(\Omega_{n,n})$. Then by Lemma 5.2, there is a measure $\mathbf{m}$ on Δ_n for which

$$\int_{\Delta_n} Q_n(\mathbf{x} \mid \underline{s}) \, d\mathbf{m}(\mathbf{x}) = \prod_{k=1}^{n} \frac{\Gamma(n+s_k-k+1) \, \Gamma(\lambda-k+1)}{\Gamma(\lambda+s_k-k+1) \, \Gamma(n-k+1)}.$$

By repeated application of the formula (d) in the proof of Corollary 2.2, we have

$$\prod_{i=1}^{n} x_i^{m-n} \cdot Q_n(x_1, ..., x_n \mid \underline{s}) = Q_n(x_1, ..., x_n \mid \underline{s} + (m-n) \, \underline{1}).$$

Therefore, letting $\tilde{\mathbf{m}}$ denote the measure on Δ_n defined by

$$d\tilde{\mathbf{m}}(\mathbf{x}) = c \cdot \prod_{i=1}^{n} x_i^{m-n} \, d\mathbf{m}(\mathbf{x}), \tag{5.4}$$

(where c is a suitable constant to make this a probability) we get

$$\int_{\Delta_n} Q_n(\mathbf{x} \mid \underline{s}) \, d\tilde{\mathbf{m}}(\mathbf{x}) = \int_{\Delta_n} Q_n(\mathbf{x} \mid \underline{s} + (m-n) \, \underline{1}) \, d\mathbf{m}(\mathbf{x}).$$

From the assumption on the measure on the right, we find that this integral is

$$c \cdot \prod_{k=1}^{n} \frac{\Gamma(m+s_k-k+1)\,\Gamma(\lambda-k+1)}{\Gamma(\lambda+m-n+s_k-k+1)\,\Gamma(n-k+1)}$$
$$= \prod_{k=1}^{n} \frac{\Gamma(m+s_k-k+1)\,\Gamma(\lambda+m-n-k+1)}{\Gamma(\lambda+m-n+s_k-k+1)\,\Gamma(m-k+1)},$$

so that $\tilde{\mathbf{m}}$ satisfies the requirement of Lemma 5.2 with λ replaced by $\lambda+m-n$. Hence $\mathbf{M}^{(\lambda+m-n)}$ is subnormal on $\mathscr{H}^{(\lambda+m-n)}(\Omega_{n,m})$.

For the converse, assume that the measure $\tilde{\mathbf{m}}$ satisfies the requirement of Lemma 5.2 with λ replaced by $\lambda+m-n$. By the note following this Lemma, $\tilde{\mathbf{m}}$ is a probability measure; also, Propositions 2.1, 2.3 and Corollary 2.1 imply that

$$0 \leqslant \frac{Q_n(\mathbf{x} \mid \underline{s})}{Q_n(\underline{1})} \leqslant 1 \qquad \text{for all} \quad \mathbf{x} \in \varDelta_n.$$

Therefore, we get

$$0 \leqslant \int_{\varDelta_n} \frac{Q_n(\cdot \mid \underline{s})}{Q_n(\underline{1})}\, d\tilde{\mathbf{m}} \leqslant 1.$$

But if $\lambda < n$, then the assumption on $\tilde{\mathbf{m}}$ implies that the integral goes to infinity as the signature $\underline{s}$ goes to infinity coordinate wise. Thus we must have $\lambda \geqslant n$. The Dirac delta measure at $\underline{1} \in \varDelta_n$ satisfies the requirement of Lemma 5.2 with $\lambda = n$, $m = n$. Therefore there is nothing to prove in case $\lambda = n$, and we may assume $\lambda > n$.

As above, repeated application of the formula (d) in the proof of Corollary 2.2 shows that for each nonnegative integer h, $\tilde{\mathbf{m}}$ satisfies

$$\int_{\varDelta_n} \prod_{k=1}^{n} x_k^h \cdot Q_n(\mathbf{x} \mid \underline{s})\, d\tilde{\mathbf{m}}(\mathbf{x})$$
$$= \prod_{k=1}^{n} \frac{\Gamma(h+m+s_k-k+1)\,\Gamma(\lambda+m-n-k+1)}{\Gamma(h+\lambda+m-n+s_k-k+1)\,\Gamma(m-k+1)}. \tag{5.5}$$

That is, if $f\colon \varDelta_n \to [0, 1]$ denotes the function $f(x) = \prod_{k=1}^{n} x_k$, then the hth moment of the probability $((Q_n(\mathbf{x} \mid \underline{s})\, d\tilde{\mathbf{m}}(\mathbf{x})) \circ f^{-1}$ on $[0,1]$ is given by the right hand side of (5.5). But, Euler's identity relating the Beta and the Gamma integral shows that this is also the hth moment of the probability $\mathbf{n} \circ f^{-1}$ on $[0,1]$, where the measure $\mathbf{n}$ on $\varDelta_n$ is defined by

$$d\mathbf{n}(\mathbf{x}) = \prod_{k=1}^{n} \frac{x_k^{m+s_k-k}(1-x_k)^{\lambda-n-1}}{\beta(m-k+1, \lambda-n)}\, dx_k.$$

Therefore, by Weirstrass' approximation theorem, these two probabilities on $[0, 1]$ are equal, and hence have the same hth moment not only for $h \geqslant 0$, but for all h for which the second probability has finite hth moment, viz. for $h \geqslant n-m$. (Here we have made use of the assumption $\lambda > n$.) Hence, in particular, the equation (5.5) holds for $h = n-m$. That is, the measure $\mathbf{m}$ defined by the equation (5.4) satisfies the requirement of Lemma 5.2 with $m = n$. ∎

In view of this Lemma, we assume $m = n$ and break up the proof of the implication (ii) ⇒ (iii) (in Theorem 1.3) in this case into two parts:

Claim 1. If $m = n$, and $\lambda > 2n-1$, then the measure $\mathbf{m}_\lambda^{n \times m}$ on Δ_n given by

$$d\mathbf{m}_\lambda^{n \times n}(x_1, ..., x_n) = c_\lambda(n) \prod_{1 \leqslant i < j \leqslant n} (x_i - x_j)^2 \cdot \prod_{k=1}^{n} (1-x_k)^{\lambda - 2n} \, d\mathbf{x} \tag{5.6}$$

with

$$c_\lambda(n) = \prod_{h=1}^{n} \frac{\Gamma(\lambda - n + h)}{\Gamma(h)^2 \, \Gamma(\lambda - 2n + h)} \tag{5.7}$$

satisfies the requirement of Lemma 5.2.

Claim 2. If $m = n$ and $\lambda = n + k$ for some $k \in \{0, 1, ..., n-1\}$, then the measure $\mathbf{m}_\lambda^{n \times n}$ given on Δ_n by

$$\mathbf{m}_\lambda^{n \times n} = \mathbf{m}_\lambda^{k \times k} \circ \pi_{k,n}^{-1} \tag{5.8}$$

satisfies the requirement of Lemma 5.2. Here, of course, $\mathbf{m}_\lambda^{k \times k}$ is the measure given in *Claim* 1 (with the same λ and with n replaced by k) and $\pi_{k,n}\colon \Delta_k \to \Delta_n$ is the embedding $(x_1, ..., x_k) \mapsto (x_1, ..., x_k, 1, ..., 1)$.

(In particular, for $\lambda = n$, i.e., $k = 0$, the measure $\mathbf{m}_\lambda^{n \times n}$ given above is to be interpreted as the Dirac delta measure (what else?) on the singleton set $\Delta_0 = \{\underline{1}\}$.)

Proof. In case $\lambda > 2n-1$, we have,

$$\begin{aligned}
&\int_{\Delta_n} Q_n(\mathbf{x} \mid \underline{s}) \prod_{1 \leqslant i < j \leqslant n} (x_i - x_j)^2 \prod_{k=1}^{n} (1-x_k)^{\lambda - 2n} \, d\mathbf{x} \\
&\quad = \frac{1}{n!} \sum_{\pi \in \mathrm{Sym}(n)} \int_{[0,1]^n} \mathrm{sgn}(\pi) \prod_{1 \leqslant i < j \leqslant n} (x_i - x_j) \\
&\qquad \cdot \prod_{k=1}^{n} x_{\pi(k)}^{s_k + n - k} \cdot \prod_{k=1}^{n} (1-x_k)^{\lambda - 2n} \, d\mathbf{x}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{n!} \sum_{\pi \in \mathrm{Sym}(n)} \int_{[0,1]^n} \prod_{1 \leqslant i<j \leqslant n} (x_{\pi(i)} - x_{\pi(j)}) \\
&\quad \cdot \prod_{k=1}^{n} (x_{\pi(k)}^{s_k+n-k} (1 - x_{\pi(k)})^{\lambda-2n})\, d\mathbf{x} \\
&= \int_{[0,1]^n} \prod_{1 \leqslant i<j \leqslant n} (y_i - y_j) \cdot \prod_{k=1}^{n} y_k^{s_k+n-k} (1-y_k)^{\lambda-2n}\, d\mathbf{y} \\
&= \sum_{\sigma \in \mathrm{Sym}(n)} \mathrm{sgn}(\sigma) \int_{[0,1]^n} \prod_{k=1}^{n} y_k^{s_k+2n-k-\sigma(k)} (1-y_k)^{\lambda-2n}\, d\mathbf{y} \\
&= \sum_{\sigma \in \mathrm{Sym}(n)} \mathrm{sgn}(\sigma) \prod_{k=1}^{n} \int_{[0,1]} y^{s_k+2n-k-\sigma(k)} (1-y)^{\lambda-2n}\, dy \\
&= \sum_{\sigma \in \mathrm{Sym}(n)} \mathrm{sgn}(\sigma) \prod_{k=1}^{n} \frac{\Gamma(s_k+2n-k-\sigma(k)+1)\, \Gamma(\lambda-2n+1)}{\Gamma(s_k+\lambda-k-\sigma(k)+2)}.
\end{aligned}
$$

Here we have used the symmetry of the integrand in the variables x_k and the fact that Δ_n is a fundamental domain for the action of $\mathrm{Sym}(n)$ on $[0, 1]^n$ and the latter is the union of $n!$ essentially disjoint copies of the fundamental domain. Also, at one point we have used the expression for $\prod_{1 \leqslant i<j \leqslant n} (y_i - y_j)$ as the Laplace expansion of a Vandermonde determinant. But the last expression obtained is the Laplace expansion of yet another determinant. Thus we have:

$$
\int_{\Delta_n} Q_n(\mathbf{x} \mid \underline{s})\, d\mathbf{m}_\lambda^{n \times n}(\mathbf{x}) = \prod_{h=1}^{n} \frac{\Gamma(\lambda-n+h)}{\Gamma(h)^2\, \Gamma(\lambda-2n+h)} \cdot \det(\mathbf{b}),
$$

where the $n \times n$ matrix $\mathbf{b} = (b_{ij})$, $1 \leqslant i, j \leqslant n$, is given by

$$
b_{ij} = \frac{\Gamma(\lambda-2n+1)\, \Gamma(2n+1+s_i-i-j)}{\Gamma(\lambda+2+s_i-i-j)}.
$$

Therefore, to establish *Claim* 1, we have to show that

$$
\begin{aligned}
\det(\mathbf{b}) = &\prod_{1 \leqslant i<j \leqslant n} \left(1 + \frac{s_i - s_j}{j-i}\right) \\
&\cdot \prod_{k=1}^{n} \frac{\Gamma(k)\, \Gamma(\lambda-2n+k)\, \Gamma(s_k+n-k+1)}{\Gamma(s_k+\lambda-k+1)}.
\end{aligned}
$$

This is trivial for $n=1$. To do a proof by induction, note that the submatrix of $\mathbf{b}$ obtained by deleting the first column and ith row has the same form as $\mathbf{b}$ with n, λ, $\underline{s}$ replaced by $n-1$, $\lambda-2$, $\underline{s}^{(i)}$, respectively. (Here the signature $\underline{s}^{(i)}$ is as in the formula (2.21).) Therefore to complete the inductive calculation of $\det(\mathbf{b})$, we need to prove an identity which simplifies (with $x_i = s_i - i + 1$) to

$$\sum_{i=1}^{n} \left(\prod_{\substack{k=1 \\ k \neq i}}^{n} \frac{\lambda + x_k - 1}{x_i - x_k} \right) \left(\prod_{\ell=1}^{n-1} \frac{x_i + \ell + n - 1}{\lambda - \ell - n} \right) = 1.$$

But this may be proved exactly as we proved the analogous identity which came up in the proof of Proposition 2.7.

Next, let $\lambda = n + k$, $k = 0, 1, ..., n-1$. With $\mathbf{m}_\lambda^{n \times n}$, defined as in *Claim* 2, we have, with $\ell = n - k$,

$$\begin{aligned}
&\int_{\Delta_n} Q_n(x_1, ..., x_n \mid \underline{s})\, d\mathbf{m}_\lambda^{n\times n}(\mathbf{x}) \\
&\quad = \int_{\Delta_k} Q_n(x_1, ..., x_k, 1, ..., 1 \mid \underline{s})\, d\mathbf{m}_\lambda^{k\times k}(\mathbf{x}) \\
&\quad = (-1)^{k\ell}\, c_k(n+k) \sum_{(A,\, B)} \varepsilon(A, B)\, Q_\ell(\underline{1} \mid \underline{s}^B) \\
&\qquad \times \int_{\Delta_k} Q_k(x_1, ..., x_k \mid \underline{s}^A) \prod_{1 \leqslant i < j \leqslant k} (x_i - x_j)^2\, d\mathbf{x} \\
&\quad = (-1)^{k\ell} \frac{c_k(n+k)}{c_k(2k)} \sum_{(A,\, B)} \varepsilon(A, B)\, Q_\ell(\underline{1} \mid \underline{s}^B) \\
&\qquad \times \int_{\Delta_k} Q_k(x_1, ..., x_k \mid \underline{s}^A)\, d\mathbf{m}_{2k}^{k\times k}(\mathbf{x}) \\
&\quad = (-1)^{k\ell} \sum_{(A,\, B)} \varepsilon(A, B)\, Q_k(\underline{1} \mid \underline{s}^A)\, Q_\ell(\underline{1} \mid s^B) \\
&\qquad \times \left(\prod_{p=1}^{k} \frac{(k + s_p^A - p)!\, (2k-p)!}{(2k + s_p^A - p)!\, (k-p)!} \right).
\end{aligned}$$

Here the sum $\sum_{(A,\, B)}$ is over all partitions of $\{1, ..., k+\ell\}$ into two sets of size k and ℓ respectively. In this computation, we have applied Proposition 2.4 once and *Claim* 1 above twice (with n replaced by k and λ replaced once by $k+n$ and once by $2k$). Now, to prove *Claim* 2, we have to show that the last expression above equals

$$\prod_{1 \leqslant i < j \leqslant k+\ell} \left(1 + \frac{s_i - s_j}{j - i} \right) \cdot \prod_{p=1}^{k+\ell} \frac{(k + \ell + s_p - p)!\, (2k + \ell - p)!}{(2k + \ell + s_p - p)!\, (k + \ell - p)!}.$$

Substituting $x_i = s_i - i + 1$, $1 \leqslant i \leqslant k + \ell$, and using the definition of $\underline{s}^A$ and $\underline{s}^B$ from (2.19) and (2.20), this reduces to

$$\sum_{(A,\,B)} \prod_{\substack{i \in A \\ j \in B}} (x_i - x_j)^{-1} \prod_{h \in B} f(x_h) = (-1)^{k\ell}$$

where f is the monic polynomial of degree k given by $f(x) = \prod_{p=1}^{k} (x + k + \ell + p - 1)$. But this is immediate from Lemma 2.1. This proves *Claim* 2.

With both claims thus established, the subnormality of $\mathbf{M}^{(\lambda)}$ for $\lambda \in m + \mathscr{W}$ follows from Lemma 5.2 and Lemma 5.3.

5.4. *The Points of Subnormality*

To conclude the proof of Theorem 1.3, it remains to show that the only values of $\lambda \in \mathscr{W}$ for which $\mathbf{M}^{(\lambda)}$ is subnormal are those for which subnormality has been already established. In view of Lemma 5.3 we may (and do) assume that $m = n$. (This is only a simplifying assumption that could be avoided.) For the proof we need:

LEMMA 5.4. *Let λ, λ' in $\mathscr{W}$ be such that both $\mathbf{M}^{(\lambda)}$ and $\mathbf{M}^{(\lambda')}$ are subnormal. Assume that the corresponding probability measures μ_λ, $\mu_{\lambda'}$ (as guaranteed by Lemma* 5.1) *have the same support. Then either*

(a) *the common support is contained in $\partial\Omega$ and $\lambda = \lambda'$, or*

(b) *the common support is $\bar{\Omega}$ and $(d\mu_{\lambda'}/d\mu_\lambda)(w) = \alpha \det(1_n - ww^*)^{\lambda - \lambda'}$ for $w \in \Omega$, where $\alpha > 0$ is a suitable normalising constant.*

Proof. By Lemma 5.1., $\mu_\lambda(\Theta) = 1 = \mu_{\lambda'}(\Theta)$, where $\Theta = S_j$, $0 \leqslant j \leqslant n-1$ or $\Theta = \Omega$. Since Θ is transitive $\mathbf{G}$-space and these are quasi invariant measures on Θ, [13, Theorem 5.19] implies that these two measures are equivalent. Define F on Θ by $F = d\mu_{\lambda'}/d\mu_\lambda$. The transformation rules (as in Lemma 5.1) for these two measures, along with chain rule, imply that

$$(F \circ g)(z) = |J(g, z)|^{(2(\lambda' - \lambda))/(m+n)} F(z), \qquad z \in \Theta, \quad g \in \mathbf{G}. \tag{5.9}$$

In particular, if $z \in \Theta$ is such that $F(z) \neq 0$ (almost all (μ_λ) z in Θ satisfies this) then (5.9), applied to g in the isotropy subgroup $\mathbf{G}_z$ of z in $\mathbf{G}$, simplifies to

$$|J(g, z)| = 1 \qquad \text{for} \quad g \in \mathbf{G}_z, \qquad z \in \Theta, \tag{5.10}$$

provided $\lambda \neq \lambda'$. (Since $\mathbf{G}$ is transitive on Θ, chain rule implies that (5.10) holds for all $z \in \Theta$ once it holds for some z.)

It is easy to see that (5.10) is a true statement when $\Theta = \Omega$. We prove part (a) of this lemma by showing that (5.10) is false for $z \in S_j \subseteq \partial\Omega$. To see this, choose $z=\sum_{i=j+1}^{n} e_i$, where, as before, e_i is the element of $\partial\Omega$ with 1 in the (i,i) slot and 0 elsewhere. Take $w=rz\in\Omega$, where $0\leqslant r<1$. Then the formula (2.2) simplifies to $\phi_w(z)=(r-1)\,z(1-rz^2)^{-1}$, so that z and $\phi_w(z)$ have the same singular values, counting multiplicity. Hence there is a $k\in\mathbf{K}$ such that $g=k\phi_w\in\mathbf{G}_z$. Since elements of $\mathbf{K}$ have unimodular Jacobian determinants, (5.10) applied to our choice of g and z implies that $|J(\phi_w, z)|=1$. But this is certainly false at least for sufficiently large r, since as $r\to 1-$, $w\to z$ and $J(\phi_w, z)\to\infty$. This proves (a). To Prove (b), assume $\Theta=\Omega$, and apply (5.9) to $g=\phi_w$, $z=0$. This yields the required formula for $F(x)$ with $\alpha=F(0)$. ∎

Now let $\lambda\in\mathscr{W}$ be such that $\mathbf{M}^{(\lambda)}$ is subnormal. By Lemma 5.1, either $\mu_\lambda(S_j)=1$ for some j, $0\leqslant j\leqslant n-1$, or $\mu_\lambda(\Omega)=1$. Take $\lambda'=m+j$ in the first case and $\lambda'=m+n$ in the second. In view of the explicit determination of the measure $\mu_{\lambda'}$ in the previous subsection, we have $\mu_{\lambda'}(S_j)=1$ in the first case and $\mu_{\lambda'}$ is the normalised Lebesgue measure on Ω in the second case. So Lemma 5.4 yields $\lambda=\lambda'=m+j$ in the first case and $d\mu_\lambda(w)=c\det(1-ww^*)^{\lambda-m-n}\,dw$ in the second. But this last is not a finite measure unless $\lambda>m+n-1$, so we have $\lambda\in m+\mathscr{W}$ in either case. This completes the proof of the implication (i) ⇒ (ii) in Theorem 1.3.

6. Multiplication by Determinant

Throughout this section, we assume $m=n$. The esf $\varphi_{\underline{s}}$ corresponding to the signature $\underline{1}=(1, ..., 1)$ will be denoted simply by ψ. It is the determinant function on $\Omega_{n,n}$. We shall now investigate the operator

$$M_\psi^{(\lambda)} \stackrel{\text{def}}{=} \psi(\mathbf{M}^{(\lambda)})$$

of multiplication by ψ on $\mathscr{H}^{(\lambda)}$. Since there is nothing to prove for $\lambda\leqslant n-1$, we assume $\lambda>n-1$.

Note that ψ is almost $\mathbf{K}$-invariant. More precisely, there is a one dimensional character χ on $\mathbf{K}$, given by $\chi(k)=\det(uv^*)$ for $k=(u,v)\in\mathbf{K}$, for which

$$\psi\circ k=\chi(k)\psi, \qquad k\in\mathbf{K}. \tag{6.1}$$

By part (b) of Corollary 2.2 we get

$$M_\psi^{(\lambda)}(\varphi_{\underline{s}})=\varphi_{\underline{s}+\underline{1}}, \tag{6.2}$$

where addition of signatures is component wise. From (6.1) and (6.2) we get $M_\psi^{(\lambda)}(\varphi_{\underline{s}}\circ k)=\overline{\chi(k)}\,\varphi_{\underline{s}+\underline{1}}\circ k$. Since $\varphi_{\underline{s}}\circ k$ (respectively $\varphi_{\underline{s}+\underline{1}}\circ k$), $k\in\mathbf{K}$, span $\mathscr{P}_{\underline{s}}$ (respectively $\mathscr{P}_{\underline{s}+\underline{1}}$), this shows that $M_\psi^{(\lambda)}$ maps $\mathscr{P}_{\underline{s}}$ onto $\mathscr{P}_{\underline{s}+\underline{1}}$. Also,

$M_{\psi}^{(\lambda)}$ is clearly one-one. Hence $M_{\psi}^{(\lambda)}$ pushes the inner product on $\mathscr{P}_{\underline{s}}$ (inherited from $\mathscr{H}^{(\lambda)}$) down to a **K**-invariant inner product on $\mathscr{P}_{\underline{s}+\underline{1}}$, and the latter must be a positive scalar times the inner product $\mathscr{P}_{\underline{s}+\underline{1}}$ inherits from $\mathscr{H}^{(\lambda)}$. Comparing the norms of $\varphi_{\underline{s}}$ and $\varphi_{\underline{s}+\underline{1}} = M_{\psi}^{(\lambda)}(\varphi_{\underline{s}}$, one sees that the scalar must be $\|\varphi_{\underline{s}+\underline{1}}\|_{\lambda}/\|\varphi_{\underline{s}}\|_{\lambda}$. But by Proposition 2.7, we get

$$\frac{\|\varphi_{\underline{s}+\underline{1}}\|_{\lambda}}{\|\varphi_{\underline{s}}\|_{\lambda}} = \prod_{j=1}^{n} \left(\frac{n+s_j-j+1}{\lambda+s_j-j+1}\right)^{1/2}.$$

Thus we have proved:

Lemma 6.1. *Let $f \in \mathscr{H}^{(\lambda)}$ and let $\underline{s}$ be any signature. Then $f \in \mathscr{P}_{\underline{s}}$ if and only if $M_{\psi}^{(\lambda)}(f) \in \mathscr{P}_{\underline{s}+\underline{1}}$. Also, if this holds then*

$$\frac{\|M_{\psi}^{(\lambda)} f\|_{\lambda}}{\|f\|_{\lambda}} = \prod_{j=1}^{n} \left(\frac{n+s_j-j+1}{\lambda+s_j-j+1}\right)^{1/2}.$$

Let us temporarily put $T = M_{\psi}^{(\lambda)}$. Then Lemma 6.1 implies the estimate, for $h = 0, 1, 2, ...,$

$$\|T^h\| \geqslant \frac{\|T^h \varphi_{\underline{s}}\|}{\|\varphi_{\underline{s}}\|_{\lambda}} = \prod_{j=1}^{n} \prod_{\ell=s_j}^{s_j+h-1} \left(\frac{n+\ell-j+1}{\lambda+\ell-j+1}\right)^{1/2}.$$

But if $\lambda < n$, then Sterling's approximation for the factorial shows that the right hand side above goes to infinity with h, so that the powers of $M_{\psi}^{(\lambda)}$ go to infinity in norm. Hence for $\lambda < n$, $M_{\psi}^{(\lambda)}$ does not admit its spectrum (which, by Theorem 1.2 and the mapping property of the spectrum, is the closed unit disc) as a k-spectral set for any k. A fortiori, $M_{\psi}^{(\lambda)}$ is not subnormal in this case.

Let $\mathscr{S}_0$ be the set of all signatures $\underline{s}$ with $s_n = 0$. For any $\underline{s} \in \mathscr{S}_0$, let $\mathscr{H}_{\underline{s}}^{(\lambda)}$ be the closed subspace of $\mathscr{H}^{(\lambda)}$ defined by

$$\mathscr{H}_{\underline{s}}^{(\lambda)} = \bigoplus_{h=0}^{\infty} \mathscr{P}_{\underline{s}+h\cdot\underline{1}}.$$

Then by (2.11) we have

$$\mathscr{H}^{(\lambda)} = \bigoplus_{\underline{s}\in\mathscr{S}_0} \mathscr{H}_{\underline{s}}^{(\lambda)}.$$

Also Lemma 6.1 implies

Lemma 6.2. *For each $\underline{s} \in \mathscr{S}_0$, $\mathscr{H}_{\underline{s}}^{(\lambda)}$ is a reducing subspace for $M_{\psi}^{(\lambda)}$, and the restriction of $M_{\psi}^{(\lambda)}$ to $\mathscr{H}_{\underline{s}}^{(\lambda)}$ is the direct sum of $d_{\underline{s}}$ copies of a weighted* *shift operator* $T_{\underline{s}}^{(\lambda)}$ *with weight sequence* $\{a_h = a_h(\underline{s}, \lambda): h = 0, 1, 2, \ldots\}$ *given by*

$$a_h = \prod_{j=1}^{n} \left(\frac{h+n+s_j-j+1}{h+\lambda+s_j-j+1} \right)^{1/2}. \tag{6.3.}$$

(Recall that this means that $T_{\underline{s}}^{(\lambda)}$ is an operator on a Hilbert space with an orthonormal basis $\{x_h: h = 0, 1, 2, \ldots\}$ such that $T_{\underline{s}}^{(\lambda)} x_h = a_h x_{h+1}$ for $h \geqslant 0$.)

Note that for $\lambda \geqslant n$ (6.3) implies $\sup_{h \geqslant 0} a_h = 1$, so that $\|T_{\underline{s}}^{(\lambda)}\| = 1$ $\forall \underline{s} \in \mathscr{S}_0$, whence $\|M_\psi^{(\lambda)}\| = 1$ for $\lambda \geqslant n$. This is in sharp contrast with the case $\lambda < n$, where $M_\psi^{(\lambda)}$ is not power bounded.

From Lemma 6.2 it is clear that $M_\psi^{(\lambda)}$ is subnormal if and only if $T_{\underline{s}}^{(\lambda)}$ is subnormal for all $\underline{s} \in \mathscr{S}_0$. Now, there is a necessary and sufficient condition in [6, p. 895–896] for the subnormality of the weighted shift operators of norm 1 with weight sequence $\{a_h: h \geqslant 0\}$. Namely, the sequence $\{b_k\}$ of partial products $b_k = \prod_{h=0}^{k} a_h^2$ must be the moment sequence of a probability on [0,1] with 1 in its support. For $T_{\underline{s}}^{(\lambda)}$, this sequence is given by

$$b_k = \prod_{j=1}^{n} \frac{\Gamma(\lambda+s_j-j+1)\, \Gamma(k+n+s_j-j+1)}{\Gamma(n+s_j-j+1)\, \Gamma(k+\lambda+s_j-j+1)}. \tag{6.4}$$

Hence we have:

Lemma 6.3. *$M_\psi^{(\lambda)}$ is subnormal if and only if for each $\underline{s}$ in $\mathscr{S}_0$ there is a probability $\sigma_{\underline{s}}$ supported in* [0,1] *such that* 1 *is in the support of $\sigma_{\underline{s}}$ and such that for $k = 0, 1, 2, \ldots$*

$$\int_0^1 x^k \, d\sigma_{\underline{s}}(x) = b_k.$$

(Here b_k is as in (6.4).)

Now note that the moment sequence of the product of finitely many stochastically independent random variables is the term wise product of the moment sequences of the factors. Further the support of the product is the element-wise product of the supports of the factors. Therefore, Lemma 6.3 implies:

Lemma 6.4. *For $M_\psi^{(\lambda)}$ to be subnormal it is sufficient to have, for each j with $1 \leqslant j \leqslant n$ and for each signature $\underline{s}$ in $\mathscr{S}_0$, a probability $\sigma_{j,\underline{s}}$ supported in* [0, 1] *such that* 1 *belongs to the support of $\sigma_{j,\underline{s}}$, and such that for $k = 0, 1, 2, \ldots$*

$$\int_0^1 x^k \, d\sigma_{j,\underline{s}}(x) = \frac{\Gamma(\lambda+s_j-j+1)\, \Gamma(k+n+s_j-j+1)}{\Gamma(n+s_j-j+1)\, \Gamma(k+\lambda+s_j-j+1)}.$$

Finally we observe:

LEMMA 6.5. *For $b > a > 0$ there is a probability σ such that the support of σ is $[0,1]$ and for $k = 0, 1, 2, \ldots,$*

$$\int_0^1 x^k \, d\sigma(x) = \frac{\Gamma(b)\,\Gamma(a+k)}{\Gamma(a)\,\Gamma(b+k)}.$$

Proof. By Euler's identity connecting his Beta and Gamma integrals, the measure

$$d\sigma(x) = \frac{1}{\beta(a, b-a)} x^{a-1}(1-x)^{b-a-1}\, dx, \qquad 0 \leqslant x \leqslant 1,$$

satisfies the requirement. ∎

Now taking $a = n + s_j - j + 1$, $b = \lambda + s_j - j + 1$ in Lemma 6.5, we get a probability $\sigma_{j, \underline{s}}$ satisfying the requirement of Lemma 6.4, provided $\lambda > n$. If $\lambda = n$ then this argument fails since the Beta integral $\beta(x, y)$ diverges for $x = 0$ or $y = 0$. However, in case $\lambda = n$, (6.4) reduces to $b_k = 1$ for all k, so that the Dirac delta measure at 1 works as σ in this case. Thus for $\lambda \geqslant n$, $M_\psi^{(\lambda)}$ is subnormal. This completes the proof of Theorem 1.4.

ACKNOWLEDGMENT

We are thankful to V. Pati and V. S. Sunder for many helpful conversations on the subject matter of this paper. We thank J. Arazy for having written the beautiful survey article [1] on bounded symmetric domains. Much of what we know about this subject, we learned from this survey. We have freely drawn on this survey throughout this paper, particularly in the second section.

REFERENCES

1. J. ARAZY, A survey of invariant Hilbert spaces of analytic functions on bounded symmetric domains, *Contemporary Math.* **185** (1995).
2. N. ARONSZAJN, Theory of reproducing kernels, *Trans. Amer. Math. Soc.* **68** (1950), 337–404.
3. B. BAGCHI AND G. MISRA, Homogeneous operators and systems of imprimitivity, *in* "Proceedings of the Seattle Conference on Multivariable Operator Theory, 1994", *Contemporary Math.* **185** (1995), 67–76.
4. R. CURTO, Application of several complex variables to multiparameter spectral theory, *in* "Surveys of Some Recent Results in Operator Theory, II" (J. Conway and B. Morrel, Eds.), Pitman Res. Notes in Mathematics, Vol. 192, pp. 25–90.
5. J. FARAUT AND A. KORANYI, Function spaces and reproducing kernels on bounded symmetric domains, *J. Funct. Anal.* **88** (1990), 64–89.

6. P. Halmos, Ten problems in Hilbert space, *Bull. Amer. Math. Soc.* **76** (1970), 887–993.
7. I. G. Macdonald, "Symmetric Functions and Hall Polynomials," Clarendon Press, Oxford, 1979.
8. G. Misra and N. S. N. Sastry, Homogeneous tuples of operators and holomorphic discrete series representation of some classical groups, *J. Operator Theory*, **24** (1990), 23–32.
9. V. I. Paulsen, "Completely Bounded Maps and Dilations," Pitman Research Notes in Mathematics, Vol. 146, Pitman, London, 1986.
10. H. Rossi and M. Vergne, Analytic continuation of the holomorphic discrete series of a semisimple Lie group, *Acta. Math.* **136** (1976), 1–59.
11. N. Salinas, Products of kernel functions and module tensor product, *in* "Operator Theory: Advances and Applications," Birkhäuser, Basel, 1988.
12. H. Upmeier, Jordan algebras and harmonic analysis on symmetric spaces, *Amer. J. Math.* **108** (1986), 1–25.
13. V. S. Varadarajan, "Geometry of Quantum Theory," Springer-Verlag, New York, 1985.
14. V. S. Varadarajan, "An Introduction to Harmonic Analysis on Semisimple Lie Groups," Cambridge Studies in Advanced Mathematics, Vol. 16, Cambridge Univ. Press, Cambridge, 1989.
15. W. C. Waterhouse, letter to the editor, *Amer. Math. Monthly* **100**, No. 8 (1993), 789.
16. G. Zhang, Some recurrence formulas for spherical polynomials on tube domains, *Trans. Amer. Math. Soc.* **347** (1995), 1725–1734.