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The Carathéodory–Fejér interpolation on the polydisc

by

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Abstract. We give an algorithm for finding a solution to the Carathéodory–Fejér interpolation problem on the polydisc, whenever a solution exists. A necessary condition for the existence of a solution becomes apparent from this algorithm. A generalization of the well-known theorem due to Nehari has been obtained. A proof of the Korányi–Pukánszky theorem also follows from these ideas.

1. Introduction. The unit disc $\{z \in \mathbb{C} : |z| < 1\}$ and the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ are denoted by $\mathbb{D}$ and $\mathbb{T}$, respectively. For a Banach space X, let $\mathcal{B}(X)$ denote the set of all bounded linear operators on X. Let $\mathbb{C}[z_1, \ldots, z_n]$ be the set of all complex-valued polynomials in variables $z_1, \ldots, z_n$. For any set S, let S^n denote the n-fold cartesian product of S. For $n \in \mathbb{N} = \{1, 2, \ldots\}$ and $\boldsymbol{z} := (z_1, \ldots, z_n) \in \mathbb{C}^n$, define $\|\boldsymbol{z}\|_\infty = \max\{|z_i| : 1 \le i \le n\}$. Let $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. For every $I = (i_1, \ldots, i_n) \in \mathbb{N}_0^n$, define $|I| := i_1 + \cdots + i_n$. For a holomorphic map $h : \mathbb{D}^n \to \mathbb{C}$, set $h^{(I)}(\boldsymbol{z}) = (\partial_1^{i_1} \cdots \partial_n^{i_n} h)(\boldsymbol{z})$, $\boldsymbol{z} \in \mathbb{D}^n$, $I = (i_1, \ldots, i_n) \in \mathbb{N}_0^n$.

We recall a version of a well-known interpolation problem.

Carathéodory–Fejér interpolation problem (`CF problem` (n, d)). Given a polynomial p in n variables, of degree d, find necessary and sufficient conditions on the coefficients of p to ensure the existence of a holomorphic function h defined on the polydisc $\mathbb{D}^n$ such that $f := p + h$ maps $\mathbb{D}^n$ into $\mathbb{D}$ and $h^{(I)}(\mathbf{0}) = 0$ for any multi-index I with $|I| \le d$.

An explicit solution to this problem for $n = 1$ has been found in [12, p. 179]. More recently, several related results (see [7, 2, 6, 16, 9]) have been obtained for $n > 1$.

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In this article, we present a reformulation of `CF problem` (n,d). It involves associating $d+1$ polynomials $p_0,\dots,p_d$ to the polynomial p appearing in `CF problem` (n,d) according to a well defined and explicit rule. The reformulation asks about the existence of a contractive holomorphic function $f:\mathbb{D}\to\mathcal{B}(L^2(\mathbb{T}^{n-1}))$ extending the polynomial

$$P(z)=M_{p_0}+M_{p_1}z+\cdots+M_{p_d}z^d,$$

where for each $k=0,\dots,d$, M_{p_k} denotes the operator on $L^2(\mathbb{T}^{n-1})$ of multiplication by p_k. Thus the multi-variable interpolation problem is reduced to a problem in one variable, but at the cost of replacing scalar coefficients with operator coefficients. The precise statement follows.

Reformulation of the Carathéodory–Fejér interpolation problem (`CF problem (R)`). Let $P:\mathbb{D}\to\mathcal{B}(L^2(\mathbb{T}^{n-1}))$ be an operator-valued polynomial of the form

$$P(z)=M_{p_0}+M_{p_1}z+\cdots+M_{p_d}z^d,\qquad p_k\in\mathscr{M}_{n-1}^{(k)},$$

where

$$\mathscr{M}_{n-1}^{(k)}:=\operatorname{span}\Big\{z_1^{\alpha_1}\cdots z_{n-1}^{\alpha_{n-1}}:0\le\alpha_1\le\cdots\le\alpha_{n-1},\sum_{j=1}^{n-1}\alpha_j\le(n-1)k\Big\}.$$

Here, for each $k=0,\dots,d$, M_{p_k} is multiplication by p_k on $L^2(\mathbb{T}^{n-1})$. Find necessary and sufficient conditions on $p_0,\dots,p_d$ ensuring the existence of $p_\ell\in\mathscr{M}_{n-1}^{(\ell)}$ for each $\ell>d$ such that $f(z):=\sum_{s=1}^{\infty}M_{p_s}z^s$ maps $\mathbb{D}$ into the unit ball of $\mathcal{B}(L^2(\mathbb{T}^{n-1}))$.

We show that the polynomials $p_k\in\mathscr{M}_{n-1}^{(k)}$, $0\le k\le d$, determine a unique polynomial p in n variables of degree d and vice versa, making `CF problem (R)` a reformulation of `CF problem` (n,d).

In Section 2.2, an algorithm is developed to solve `CF problem` $(2,2)$. It involves finding, inductively, polynomials p_k in $\mathscr{M}_1^{(k)}$ such that a certain block Toeplitz operator made up of multiplication operators corresponding to these polynomials is contractive. A solution to `CF problem` $(2,2)$ exists if and only if this process is completed successfully. Moreover, Theorem 2.5 shows that the algorithm can be executed in certain special circumstances. The conditions in that theorem might appear stringent but we believe that the theorem can be extended to cover many other cases.

In Section 3, it is shown that the algorithm developed for solving `CF problem` $(2,2)$, after obvious necessary modifications, solves the general problem, namely, `CF problem` (n,d).

Our method, in general, gives an (explicit) necessary condition for the existence of a solution to `CF problem` (n,d). The obstruction for it to be sufficient lies in the failure to solve, in general, a matrix completion problem. If the latter admits a solution, then we must find one consisting of multiplication operators alone to ensure the existence of a solution to the original Carathéodory–Fejér interpolation problem on $\mathbb{D}^n$, $n \geq 2$. Specifically, let p be a polynomial in two variables of degree 2 with $p(\mathbf{0}) = 0$. Set

$$\begin{aligned} p_1(z) &= \frac{\partial p}{\partial z_1}(\mathbf{0}) + \frac{\partial p}{\partial z_2}(\mathbf{0})z, \\ p_2(z) &= \frac{1}{2}\frac{\partial^2 p}{\partial z_1^2}(\mathbf{0}) + \frac{\partial^2 p}{\partial z_1 \partial z_2}(\mathbf{0})z + \frac{1}{2}\frac{\partial^2 p}{\partial z_2^2}(\mathbf{0})z^2. \end{aligned} \tag{1.1}$$

In this case, we show that $|p_1(z)|^2 + |p_2(z)| \leq 1$, $z \in \mathbb{D}$, (abbreviated to $|p_1|^2 + |p_2| \leq 1$) is a necessary condition for the existence of a solution to `CF problem` $(2, 2)$ for the polynomial p. By means of an example, we show that this necessary condition is not sufficient. For `CF problem` $(2, 2)$, we isolate a class of polynomials for which our necessary condition is also sufficient. This is verified using a deep theorem of Nehari reproduced below (cf. [17, Theorem 15.14]).

Let $\mathcal{A}(\mathbb{D}^n)$ denote the algebra of functions continuous on the closed polydisc $\overline{\mathbb{D}}^n$ and holomorphic in the interior $\mathbb{D}^n$. The pointwise multiplication $m_h(f) = hf$, $f \in L^2(\mathbb{T}^n)$, defines a bounded operator for each fixed $h \in \mathcal{A}(\mathbb{D}^n)$. The map $m : \mathcal{A}(\mathbb{D}^n) \times L^2(\mathbb{T}^n) \to L^2(\mathbb{T}^n)$ defined by $m(h, f) = m_h(f)$ is called the *module multiplication*. Thus we think of $L^2(\mathbb{T}^n)$ as a module over $\mathcal{A}(\mathbb{D}^n)$. Also, a submodule is a subspace that is invariant under the operators m_h, $h \in \mathcal{A}(\mathbb{D}^n)$. A linear map between two modules over the same algebra is said to be a *module map* if it is bounded between the underlying Hilbert spaces and intertwines the two module multiplications. In Section 4, for a submodule $\mathcal{M} \subseteq L^2(\mathbb{T}^n)$, we let $A : H^2(\mathbb{T}^n) \to \mathcal{M}^\perp \subseteq L^2(\mathbb{T}^n)$ be a module map. For $\varphi \in L^\infty(\mathbb{T}^n)$, the Hankel operator with symbol φ is the module map $H_\phi^{\mathcal{M}} : H^2(\mathbb{T}^n) \to \mathcal{M}^\perp$ defined by setting $H_\varphi^{\mathcal{M}} f = P_{\mathcal{M}^\perp}(\phi f)$. For a specific choice of the submodule $\mathcal{M}$, and Hankel operators of the form $H_\phi^{\mathcal{M}}$ with $\phi \in L^\infty(\mathbb{T}^n)$, we obtain a generalization of Nehari's theorem.

Finally, in Section 5, we give a new proof of the Korányi–Pukánszky theorem using the spectral theorem. In the PhD thesis [8] of the first named author, the proof of Theorem 2.4 below was given using the Korányi–Pukánszky theorem. The proof of Theorem 2.4 in this note does not make use of that theorem. It then appears that the ideas from that proof lead to a different proof of the Korányi–Pukánszky theorem.

Since the bi-holomorphic automorphism group of the polydisc $\mathbb{D}^n$ acts transitively on $\mathbb{D}^n$, the existence of a solution to the CF problem is independent of the constant term in p. Hence throughout this paper we assume, without loss of generality, that $p(\mathbf{0}) = 0$.

Preliminaries. In this subsection, we collect the tools that we use repeatedly in what follows. The first is a variant of the spectral theorem for a pair of commuting normal operators. For $n \in \mathbb{N}$, fix a non-empty subset $\Omega \subseteq \mathbb{C}^n$ and define the supremum norm $\|f\|_{\Omega,\infty}$ of a bounded function f on Ω taking values in a normed linear space $(E, \|\cdot\|)$ to be $\sup_{\boldsymbol{z}\in\Omega} \|f(\boldsymbol{z})\|$. Whenever there is no ambiguity about Ω, we shall write $\|f\|_\infty$ in place of $\|f\|_{\Omega,\infty}$.

Definition 1.1 (Multiplication operator). For $\phi \in L^\infty(\mathbb{T})$, the *multiplication operator* $M_\phi : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ is defined by the pointwise product: $M_\phi(f) = \phi f$, $f \in L^2(\mathbb{T})$.

Since $\phi f \in L^2(\mathbb{T})$ for any $\phi \in L^\infty(\mathbb{T})$ and $f \in L^2(\mathbb{T})$, the operator M_ϕ is well defined for all $\phi \in L^\infty(\mathbb{T})$. Also $\|M_\phi\| = \|\phi\|_\infty$ (see [17, Theorem 13.14]).

Theorem 1.2. *If the power series $\sum_{\boldsymbol{\alpha}\in\mathbb{N}_0^2} a_{\boldsymbol{\alpha}} \boldsymbol{z}^{\boldsymbol{\alpha}}$ represents a holomorphic function f on the bidisc $\mathbb{D}^2$ then $|f(\boldsymbol{z})| \le 1$ for all $\boldsymbol{z} \in \mathbb{D}^2$ if and only if the operator norm of*

$$\begin{pmatrix} & \vdots & \vdots & \vdots & \\ \cdots & M_{p_0} & M_{p_1} & M_{p_2} & \cdots \\ \cdots & 0 & M_{p_0} & M_{p_1} & \cdots \\ \cdots & 0 & 0 & M_{p_0} & \cdots \\ & \vdots & \vdots & \vdots & \end{pmatrix}$$

is at most 1, *where $p_n(z) = \sum_{k=0}^n a_{n-k,k} z^k$ is a polynomial of degree n in one variable for each $n \in \mathbb{N}_0$.*

Proof. Let B^* denote the adjoint of the bilateral shift on $\ell^2(\mathbb{Z})$ and let f be as in the statement. The joint spectrum of $I \otimes B^*$ and $B^* \otimes B^*$ is $\mathbb{T}^2$. By the spectral theorem, the spectrum of $f(I \otimes B^*, B^* \otimes B^*)$ is the same as that of $f(\mathbb{T}^2)$. Therefore, by the maximum modulus principle,

$$\|f\|_{\mathbb{D}^2,\infty} = \|f(I\otimes B^*, B^*\otimes B^*)\| = \|M_{p_0}\otimes I + M_{p_1}\otimes B^* + M_{p_2}\otimes {B^*}^2 + \cdots\|,$$

where $p_n(z) = \sum_{k=0}^n a_{n-k,k} z^k$ for each $n \in \mathbb{N}_0$. For the second equality above, we identify $\ell^2(\mathbb{Z})$ and $L^2(\mathbb{T})$ isometrically. ∎

Remark 1.3. We state separately the special case of Theorem 1.2 in one variable: Suppose $f : \mathbb{D} \to \mathbb{C}$ is a holomorphic function. Then f maps $\mathbb{D}$ into itself if and only if $M_f : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ is a contraction.

Theorem 1.4 (Parrott's theorem [15]). *For $i = 1, 2$, let $\mathbb{H}_i$ and $\mathbb{K}_i$ be Hilbert spaces and set $\mathbb{H} = \mathbb{H}_1 \oplus \mathbb{H}_2$ and $\mathbb{K} = \mathbb{K}_1 \oplus \mathbb{K}_2$. If*

$$\begin{pmatrix} A \\ C \end{pmatrix} : \mathbb{H}_1 \to \mathbb{K} \quad \textit{and} \quad (C\ D) : \mathbb{H} \to \mathbb{K}_2$$

are contractions, then there exists $X \in \mathcal{B}(\mathbb{H}_2, \mathbb{K}_1)$ such that $\left(\begin{smallmatrix} A & X \\ C & D \end{smallmatrix}\right) : \mathbb{H} \to \mathbb{K}$ is a contraction.

In this theorem, all the possible choices for X are of the form

$$(I - ZZ^*)^{1/2} V (I - Y^*Y)^{1/2} - ZC^*Y,$$

where V is an arbitrary contraction and Y, Z are determined from

$$D = (I - CC^*)^{1/2} Y, \quad A = Z(I - C^*C)^{1/2}.$$

We recall a very useful criterion for contractivity, due to Douglas, Muhly and Pearcy [5, Prop. 2.2].

PROPOSITION 1.5 (Douglas–Muhly–Pearcy). *For $i = 1, 2$, let T_i be a contraction on a Hilbert space $\mathcal{H}_i$ and let X be an operator from $\mathcal{H}_2$ into $\mathcal{H}_1$. A necessary and sufficient condition for the operator on $\mathcal{H}_1 \oplus \mathcal{H}_2$ defined by the matrix $\left(\begin{smallmatrix} T_1 & X \\ 0 & T_2 \end{smallmatrix}\right)$ to be a contraction is that there exists a contraction C mapping $\mathcal{H}_2$ into $\mathcal{H}_1$ such that*

$$X = \sqrt{1_{\mathcal{H}_1} - T_1 T_1^*}\, C \, \sqrt{1_{\mathcal{H}_2} - T_2^* T_2}.$$

Let $H^2(\mathbb{T})$ denote the Hardy space, a closed subspace of $L^2(\mathbb{T})$. Let P_- denote the orthogonal projection of $L^2(\mathbb{T})$ onto $L^2(\mathbb{T}) \ominus H^2(\mathbb{T})$.

DEFINITION 1.6 (Hankel operator). A *Hankel operator* $A : H^2(\mathbb{T}) \to H^2(\mathbb{T})^\perp$ is defined to be any operator A such that $P_- M_z A = A M_z|_{H^2(\mathbb{T})}$, where $M_z : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ is the multiplication operator. Clearly, such an operator A has the structure (relative to the standard basis $\{z^n\}_{n \geq 0}$ in $H^2(\mathbb{T})$ and $\{\bar{z}^m\}_{m \geq 1}$ in $H^2(\mathbb{T})^\perp$) of what is known as a Hankel matrix, namely, $\langle A z^n, \bar{z}^m \rangle = \langle A1, \bar{z}^{n+m} \rangle$. Conversely, for such matrices, the defining equation holds.

Finally, we recall the well known theorem due to Nehari relating the quotient norm to that of a Hankel operator.

THEOREM 1.7 (Nehari's theorem [11, 12]). *An operator $A : H^2(\mathbb{T}) \to H^2(\mathbb{T})^\perp$ is a Hankel operator if and only if there exists $\varphi \in L^\infty(\mathbb{T})$ such that $A = H_\varphi$, where $H_\varphi f := P_-(\varphi f)$, $f \in H^2(\mathbb{T})$. Moreover,* $\inf\{\|\phi - g\|_{\mathbb{T},\infty} : g \in H^\infty(\mathbb{T})\} = \|H_\phi\|_{\text{op}}$.

2. `CF problem` $(2,2)$. Several different solutions to the CF problem with $n = 1$ are known (see [12, p. 179]). For $n > 1$, see [2] and [1, Chapter 3] for a comprehensive survey of recent results. In this article, we shall obtain a necessary condition for the existence of a solution to the CF problem and an algorithm to construct a solution if one exists.

2.1. The planar case. Although we state the problem below for polynomials p of degree 2 with $p(0) = 0$, our methods apply to the general case.

$\texttt{CF problem}\ (1,2)$. *Let $p(z) = a_1z + a_2z^2$. Find a necessary and sufficient condition for the existence of a holomorphic function g defined on $\mathbb{D}$ with $g^{(k)}(0) = 0$, $k = 0, 1, 2$, such that $\|p + g\|_{\mathbb{D},\infty} \le 1$.*

Solution. If this problem has a solution, then using Remark 1.3 it can be deduced that

$$A_2 := \begin{pmatrix} a_1 & a_2 \\ 0 & a_1 \end{pmatrix}$$

is a contraction. Thus $\|A_2\| \le 1$ is a necessary condition. On the other hand, if $\|A_2\| \le 1$, then the existence of $a_3 \in \mathbb{C}$ such that

$$A_3 := \begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & a_1 & a_2 \\ 0 & 0 & a_1 \end{pmatrix}$$

has operator norm less than or equal to 1 follows from Parrott's theorem. Repeated use of Parrott's theorem generates a sequence $a_3, a_4, \ldots$ of complex numbers such that $\|M_f\| \le 1$, where $f(z) = a_1z + a_2z^2 + a_3z^3 + \cdots$. Thus $\|A_2\| \le 1$ is a necessary and sufficient condition for the existence of a solution to $\texttt{CF problem}\ (1,2)$.

2.2. Carathéodory–Fejér interpolation in two variables. In [2], the Carathéodory–Fejér interpolation problem for the polydisc is treated. In the case of two variables, a necessary and sufficient condition for the existence of a solution is given in [2, Theorem 5.1]. Also, a slightly different necessary and sufficient condition, again for $\mathbb{D}^2$, appears in [6, Theorem 1]. They discuss separately the case $n = 2$ and state that it is special due to the dilation theorem of Ando for commuting contractions.

Our investigations, giving somewhat different necessary and sufficient conditions, not surprisingly, are also special in the case of $n = 2$. We therefore discuss this case first.

We note in passing that $\texttt{CF problem}\ (2,1)$, where we may assume that $a_{00} = 0$ without loss of generality, as pointed out earlier, is easily settled using the Schwarz lemma for $\mathbb{D}^2$. Hence the first non-trivial instance of the Carathéodory–Fejér interpolation problem is the one we discuss below.

$\texttt{CF problem}\ (2,2)$. *Let p in $\mathbb{C}[z_1, z_2]$ be an arbitrary polynomial, $p(z_1, z_2) = a_{10}z_1 + a_{01}z_2 + a_{20}z_1^2 + a_{11}z_1z_2 + a_{02}z_2^2$. Find necessary and sufficient conditions for the existence of a complex-valued holomorphic function q on $\mathbb{D}^2$ with $(\partial_1^{i_1}\partial_2^{i_2}q)(\mathbf{0}) = 0$, $i_1 + i_2 \le 2$, such that $\|p + q\|_{\mathbb{D}^2,\infty} \le 1$.*

The theorem given below follows from Theorem 1.2 and Proposition 1.5 (compare [5, Proposition 2.2]). As in (1.1), set $p_1(z) = a_{10} + a_{01}z$ and $p_2(z) = a_{20} + a_{11}z + a_{02}z^2$.

THEOREM 2.1. *If p is any complex-valued polynomial in two variables of degree at most 2 with $p(\mathbf{0}) = 0$, then $|p_1|^2 + |p_2| \le 1$ is a necessary condition for the existence of a holomorphic function $q : \mathbb{D}^2 \to \mathbb{C}$ with $q^{(I)}(\mathbf{0}) = 0$, $|I| \le 2$, such that $\|p + q\|_{\mathbb{D}^2,\infty} \le 1$.*

Proof. Suppose p and q are as in the statement. Then from Theorem 1.2, we get

$$\left\| \begin{pmatrix} M_{p_1} & M_{p_2} \\ 0 & M_{p_1} \end{pmatrix} \right\| \le 1.$$

The contractivity criterion of Proposition 1.5 then implies that $|p_1|^2 + |p_2| \le 1$. ■

Combining Theorems 1.2 and 2.1, we obtain the following theorem, which is the reduction to `CF problem (R)` with $n = 2$, $d = 2$.

THEOREM 2.2. *For any polynomial p of the form*

$$p(\boldsymbol{z}) = a_{10}z_1 + a_{01}z_2 + a_{20}z_1^2 + a_{11}z_1z_2 + a_{02}z_2^2,$$

there exists a holomorphic function q on $\mathbb{D}^2$ with $q^{(I)}(\mathbf{0}) = 0$ for $|I| = 0, 1, 2$ such that

$$\|p + q\|_{\mathbb{D}^2,\infty} \le 1$$

if and only if $|p_2| \le 1 - |p_1|^2$ and there exists a polynomial p_k of degree less than or equal to k such that $f : \mathbb{D} \to \mathcal{B}(L^2(\mathbb{T}))$, where

$$\frac{f^{(k)}(0)}{k!} = M_{p_k} \quad \text{for all } k \ge 0, \quad p_0 = 0,$$

defines a holomorphic function with $\sup_{z\in\mathbb{D}} \|f(z)\| \le 1$.

Thus `CF problem` $(2, 2)$ has been reduced to a one-variable problem except it now involves holomorphic functions taking values in $\mathcal{B}(L^2(\mathbb{T}))$. To discuss this variant of the CF problem, we first introduce the following convenient notation.

Let $\mathbb{H}$ be a separable Hilbert space. Given n operators $A_1, \ldots, A_n$ in $\mathcal{B}(\mathbb{H})$, define the operator

$$\mathscr{T}(A_1, \ldots, A_n) := \begin{pmatrix} A_1 & A_2 & A_3 & \cdots & A_n \\ 0 & A_1 & A_2 & \cdots & A_{n-1} \\ 0 & 0 & A_1 & \cdots & A_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_1 \end{pmatrix},$$

which is in $\mathcal{B}(\mathbb{H} \otimes \mathbb{C}^n)$.

DEFINITION 2.3 (Completely polynomially extendible). Suppose k is a natural number and $\{p_j\}_{j=1}^k$ is a sequence of polynomials with $\deg(p_j) \le j$

for all $j = 1, \dots, k$. Then the operator $\mathscr{T}(M_{p_1}, \dots, M_{p_k})$ will be called *m-polynomially e*xtendible if it is a contraction and there exists a sequence $\{p_l\}_{l=k+1}^m$ of polynomials with $\deg(p_l) \le l$ such that $\|\mathscr{T}(M_{p_1}, \dots, M_{p_m})\| \le 1$. Also, $\mathscr{T}(M_{p_1}, \dots, M_{p_k})$ will be called *completely polynomially extendible* if it is m-polynomially extendible for all $m \in \mathbb{N}$.

For $p_1, p_2 \in \mathbb{C}[z]$ of degree at most 1 and 2 respectively, let $P(z) = M_{p_1}z + M_{p_2}z^2$. We shall say that P is in the *CF class* if there is a holomorphic function $f : \mathbb{D} \to \mathcal{B}(L^2(\mathbb{T}))$ with the properties stated in Theorem 2.2. Such a function f will be called a *CF-extension* of P. It follows that a solution to `CF problem (2,2)` exists if and only if P is in the CF class. We have thus proved the following theorem.

THEOREM 2.4. *A solution to* `CF problem` $(2,2)$ *exists if and only if the corresponding one-variable operator-valued polynomial P is in the CF class, or equivalently the operator $\mathscr{T}(M_{p_1}, M_{p_2})$ is completely polynomially extendible.*

2.3. Algorithm for finding a solution to the CF problem. Now, we have all the tools to produce an algorithm for finding all the polynomials $P(z) = M_{p_1}z + M_{p_2}z^2$ which are in the CF class:

- If $\|\mathscr{T}(M_{p_1}, M_{p_2})\| > 1$, that is, if $|p_2| > 1 - |p_1|^2$, then P is not a CF class polynomial; otherwise go to the next step.
- Parrott's theorem gives all possible operators $T \in \mathcal{B}(L^2(\mathbb{T}))$ such that $\mathscr{T}(M_{p_1}, M_{p_2}, T)$ is a contraction. Let $\mathcal{C}_3$ be the set of all operators T that are multiplication by a polynomial of degree at most 3 and $\mathscr{T}(M_{p_1}, M_{p_2}, T)$ is a contraction. If $\mathcal{C}_3$ is empty then P is not a CF class polynomial.
- For each $k > 3$, using Parrott's theorem we can construct $\mathcal{C}_k$, the set of all operators T that are multiplication by a polynomial of degree at most k and $\mathscr{T}(M_{p_1}, \dots, M_{p_{k-1}}, T)$ is a contraction, where M_{p_j} is an element of $\mathcal{C}_j$ for $j = 3, \dots, k-1$.
- If all of the sets $\mathcal{C}_k$ are non-empty, then (and only then) P is a CF class polynomial.

It is clear, from Theorem 2.2, that $|p_1|^2 + |p_2| \le 1$ is a necessary condition for the existence of a solution to `CF problem` $(2,2)$. This condition, via Parrott's theorem, is also equivalent to $\|\mathscr{T}(M_{p_1}, M_{p_2})\| \le 1$.

We now give some instances where this necessary condition is also sufficient for the existence of a solution to `CF problem` $(2,2)$. This amounts to finding conditions for $\mathscr{T}(M_{p_1}, M_{p_2})$ to be completely polynomially extendible.

THEOREM 2.5. *Let $p_1(z) = \gamma + \delta z$ and $p_2(z) = (\alpha + \beta z)(\gamma + \delta z)$ for some complex numbers α, β, γ and δ. Assume that $|p_1|^2 + |p_2| \le 1$. If either*

$\alpha\beta\gamma\delta = 0$ *or* $\arg(\alpha) - \arg(\beta) = \arg(\gamma) - \arg(\delta)$, *then* $\mathscr{T}(M_{p_1}, M_{p_2})$ *is completely polynomially extendible.*

Proof. Throughout this proof, for brevity, for any holomorphic function $f : \mathbb{D} \to \mathcal{B}(L^2(\mathbb{T}))$, we let $\|f\|$ denote the norm $\sup\{\|f(z)\|_{\mathrm{op}} : z \in \mathbb{D}\}$.

Case 1: $\beta = 0$. Then

$$P(z) = M_{p_1}(z + \alpha z^2).$$

Define a polynomial p in one variable by $p(z) = z + \alpha z^2$. Using Nehari's theorem, we extend p to a holomorphic function $\tilde{p}(z) = z + \alpha z^2 + \alpha_3 z^3 + \cdots$ with $\|\tilde{p}\|_{\mathbb{D},\infty} = \|\mathscr{T}(1, \alpha)\|$. Define $f : \mathbb{D} \to \mathcal{B}(L^2(\mathbb{T}))$ by

$$f(z) = M_{p_1}\tilde{p}(z) = M_{p_1} z + M_{p_2} z^2 + M_{p_3} z^3 + \cdots,$$

where $p_k = \alpha_k p_1$ for $k \in \mathbb{N}$ with $\alpha_1 = 1$ and $\alpha_2 = \alpha$. Also,

$$\|f\| = \sup_{z\in\mathbb{D}} \|M_{p_1}\tilde{p}(z)\| = \|M_{p_1}\| \sup_{z\in\mathbb{D}} |\tilde{p}(z)| = \|M_{p_1}\|\,\|\mathscr{T}(1, \alpha)\|.$$

Thus $\|f\| = \|M_{p_1} \otimes \mathscr{T}(1, \alpha)\| = \|\mathscr{T}(M_{p_1}, M_{p_2})\| \le 1$. Hence, f is a required CF-extension of P.

Case 2: $\alpha = 0$. Then

$$P(z) = M_{p_1}(z + \beta M_z z^2).$$

Define an operator-valued function Q on $\mathbb{D}$ as $Q(z) = z + \beta M_z z^2$ and define a polynomial r on $\mathbb{D}^2$ as $r(z_1, z_2) = z_1(1+\beta z_2)$. Let $s(z_2) = 1+\beta z_2$. Suppose

$$\tilde{s}(z_2) = s(z_2) + \beta_2 z_2^2 + \beta_3 z_2^3 + \cdots$$

is such that $\|\tilde{s}\|_{\mathbb{D},\infty} = \|\mathscr{T}(1, \beta)\|$. If $\tilde{r} := z_1\tilde{s}(z_2)$, then $\|\tilde{r}\| = \|\tilde{s}\| = \|\mathscr{T}(1, \beta)\|$. If

$$\tilde{Q}(z) = z + M_{\beta z} z^2 + M_{\beta_2 z^2} z^2 + \cdots$$

and $f(z) = M_{p_1}\tilde{Q}(z)$, then $\|f\| = \|M_{p_1}\tilde{Q}\| \le \|M_{p_1}\|\,\|\tilde{Q}\|$. Since $\tilde{s}(M_z) = \tilde{Q}(z)$, from the von Neumann inequality it follows that $\|\tilde{Q}\| \le \|\tilde{s}\|$. Therefore, $\|f\| \le \|M_{p_1}\|\,\|\mathscr{T}(1, \beta)\| = \|\mathscr{T}(M_{p_1}, \beta M_{p_1})\|$. Hence,

$$\|f\| \le \left\| \begin{pmatrix} M_z & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} M_{p_1} & \beta M_{p_1} \\ 0 & M_{p_1} \end{pmatrix} \begin{pmatrix} M_z^* & 0 \\ 0 & I \end{pmatrix} \right\| = \|\mathscr{T}(M_{p_1}, M_{p_2})\| \le 1.$$

Consequently, f is a CF-extension of P.

Case 3: $\alpha \ne 0$ and $\beta = 0$. Then $P(z) = M_{p_1}(z + M_{\alpha+\beta z} z^2)$. Let $Q(z) := z + M_{\alpha+\beta z} z^2$. Define $r(z_1, z_2) := z_1 + \alpha z_1^2 + \beta z_1 z_2 = z_1(1 + \alpha z_1 + \beta z_2)$. Let $\lambda := |\alpha|/|\beta|$ and $a := \lambda/(1 + \lambda)$. Define

$$s(z_1, z_2) := 1 + \alpha z_1 + \beta z_2 = (a + \alpha z_1) + (1 - a + \beta z_2).$$

If $h_1(z_1) := a + \alpha z_1$ and $h_2(z_2) := 1 - a + \beta z_2$, then there exist $\tilde{h}_1 = a + \alpha z_1 + \alpha_2 z_1^2 + \cdots$ and $\tilde{h}_2 = 1 - a + \beta z_2 + \beta_2 z_2^2 + \cdots$ with $\|\tilde{h}_1\| = \|\mathscr{T}(a, \alpha)\|$

and $\|\tilde{h}_2\| = \|\mathscr{T}(1-a,\beta)\|$. If

$\tilde{r}(z_1,z_2) := z_1(\tilde{h}_1(z_1)+\tilde{h}_2(z_2)) = z_1 + \alpha z_1^2 + \beta z_1 z_2 + \alpha_2 z_1^3 + \beta_2 z_1 z_2^2 + \cdots,$

then $\|\tilde{r}\| \le \|\tilde{h}_1\| + \|\tilde{h}_2\|$. Define an operator-valued holomorphic map $\tilde{Q}$ as

$$\tilde{Q}(z) = Iz + M_{\alpha+\beta z} z^2 + M_{\alpha_2+\beta_2 z^2} z^3 + \cdots$$

and $f(z) = M_{p_1}\tilde{Q}(z) = \sum_j M_{p_j} z^j$, where $p_{k+1}(z) = (\alpha_k + \beta_k z^k)p_1$ for all $k > 1$. Thus, $\|f\| \le \|M_{p_1}\|\,\|\tilde{Q}\|$. Since $\tilde{Q}(z) = \tilde{r}(z, M_z)$, the von Neumann inequality yields

$$\|f\| \le \|M_{p_1}\|\,\|\tilde{r}\| \le \|M_{p_1}\|(\|\tilde{h}_1\| + \|\tilde{h}_2\|).$$

As $\mathscr{T}(a,|\alpha|) = \lambda\mathscr{T}(1-a,|\beta|)$, we have $\|\tilde{h}_1\| + \|\tilde{h}_2\| = \|\mathscr{T}(1,|\alpha|+|\beta|)\|$ and hence

$$\|f\| \le \|M_{p_1}\|\,\|\mathscr{T}(1,|\alpha|+|\beta|)\| = \big\|\mathscr{T}(\|p_1\|,(|\alpha|+|\beta|)\|p_1\|)\big\|.$$

SUBCASE 1: $\gamma \ne 0$, $\delta \ne 0$ and $\arg(\alpha) - \arg(\beta) = \arg(\gamma) - \arg(\delta)$. Then

$$(|\alpha|+|\beta|)\|p_1\| = \|(\alpha+\beta)p_1\| = \|p_2\|.$$

Our hypothesis clearly implies that $\|p_2\|_\infty + \|p_1\|_\infty^2 \le 1$. Hence the norm of f on $\mathbb{D}$ is at most 1.

SUBCASE 2: $\gamma = \delta = 0$. Then

$$(|\alpha|+|\beta|)\|p_1\| = \|(\alpha+\beta)p_1\| = \|p_2\|.$$

As in Subcase 1, here also $\|f\| \le 1$ can be inferred easily. ∎

REMARK 2.6. In `CF problem` $(2,2)$, if p_1 or p_2 is the zero polynomial and $\|\mathscr{T}(M_{p_1},M_{p_2})\| \le 1$, then $\|P\| \le 1$ and hence f in Theorem 2.2 can be taken to be P itself.

Having verified that the necessary condition $\|\mathscr{T}(M_{p_1},M_{p_2})\| \le 1$ is also sufficient for P to be in the CF class in several cases, we expected it to be sufficient in general. But unfortunately this is not the case. We give an example of a polynomial P for which $\|\mathscr{T}(M_{p_1},M_{p_2})\| \le 1$ but P is not in the CF class.

EXAMPLE. If $p_1(z) = 1/\sqrt{2}$ and $p_2(z) = z^2/2$, then $\mathscr{T}(M_{p_1},M_{p_2})$ is not even 3-polynomially extendible.

It can be seen that $\|\mathscr{T}(M_{p_1},M_{p_2})\| \le 1$. Now suppose there exists a polynomial p_3 of degree at most 3 such that $\|\mathscr{T}(M_{p_1},M_{p_2},M_{p_3})\| \le 1$. Then Parrott's theorem yields a contraction $V \in \mathcal{B}(L^2(\mathbb{T}))$ such that

$$\begin{aligned} M_{p_3} = {} & \big(I - M_{|p_1|^2} - M_{p_2}(I - M_{|p_1|^2})^{-1}M_{p_2}^*\big)V \\ & - M_{p_2}(I - M_{|p_1|^2})^{-1/2}M_{p_1}^*(I - M_{|p_1|^2})^{-1/2}M_{p_2}. \end{aligned}$$

As $(1-|p_1|^2)^2 - |p_2|^2 \equiv 0$,

$$p_3 = \frac{-p_2^2\overline{p}_1}{1-|p_1|^2} = -\frac{z^4}{2\sqrt{2}}.$$

Thus p_3 is of degree more than 3, a contradiction. Hence $\mathscr{T}(M_{p_1}, M_{p_2})$ is not even 3-polynomially extendible.

We close this subsection with an open problem: Find an explicit strengthening of the inequality $\|\mathscr{T}(M_{p_1}, M_{p_2})\| \le 1$ to ensure that P is in the CF class.

3. Carathéodory–Fejér interpolation: the general case. First, we obtain an explicit necessary condition for the existence of a solution to the CF problem in the case of n variables, $n \in \mathbb{N}$. The computations in this case are analogous to those for two variables but they are somewhat cumbersome. Nevertheless, we provide the details. Also, an algorithm to determine the set of all CF class polynomials in n variables analogous to that in the case of two variables is given.

We state below the Carathéodory–Fejér interpolation problem on the polydisc $\mathbb{D}^n$ for a given polynomial in n variables of degree d:

`CF problem` (n, d). *Fix $p \in \mathbb{C}[z_1, \ldots, z_n]$ of degree d of the form*

$$p(z_1, \ldots, z_n) = \sum_{j=1}^{n} a_{e_j} z_j + \cdots + \sum_{i_1,\ldots,i_d=1}^{n} a_{e_{i_1}+\cdots+e_{i_d}} z_{i_1} \cdots z_{i_d}, \tag{3.1}$$

where e_j is the row vector of length n which has 1 at the jth position and 0 elsewhere (e_0 denotes the zero vector). Find necessary and sufficient conditions for the existence of a holomorphic function q defined on $\mathbb{D}^n$ with $q^{(I)}(\mathbf{0}) = 0$ for all $|I| \le d$ such that $\|p+q\|_{\mathbb{D}^n,\infty} \le 1$.

Let f be an analytic function on $\mathbb{D}^n$ represented by the power series

$$f(z_1, \ldots, z_n) = \sum_{k=1}^{\infty} \sum_{i_1,\ldots,i_k=1}^{n} a_{e_{i_1}+\cdots+e_{i_k}} z_{i_1} \cdots z_{i_k}.$$

Replacing z_j by the operator $I^{\otimes(n-j)} \otimes B^{*\otimes j}$ yields

$$f(I^{\otimes(n-1)} \otimes B^*, \ldots, B^{*\otimes n}) = \sum_{k=1}^{\infty} A_k \otimes B^{*\otimes k},$$

where

$$A_k = \sum_{i_1,\ldots,i_k=1}^{n} a_{e_{i_1}+\cdots+e_{i_k}} \prod_{p=1}^{k} (I^{\otimes(n-i_p)} \otimes B^{*\otimes(i_p-1)}). \tag{3.2}$$

In what follows, it will be convenient to replace A_k by the multiplication operator

$$M_{p_k} : L^2(\mathbb{T}^{n-1}) \to L^2(\mathbb{T}^{n-1}), \tag{3.3}$$

where $p_k = \sum_{i_1,\dots,i_k=1}^{n} a_{e_{i_1}+\dots+e_{i_k}} \prod_{p=1}^{k} z_{n-i_p+1} z_{n-i_p+2} \cdots z_{n-1}$, with the understanding that if $i_p = 1$ then the monomial $z_{n-i_p+1} z_{n-i_p+2} \cdots z_{n-1}$ is the constant function 1. Evidently, A_k is unitarily equivalent to M_{p_k}, and therefore this makes no difference.

Note that $p_k \in \mathscr{M}_{n-1}^{(\ell)}$ with $\ell = \sum_{p=1}^{k}(i_p - 1)$ (see Sect. 1). As $\sum_{p=1}^{k}(i_p-1) \le (n-1)k$, the degree of p_k is at most $(n-1)k$.

Proposition 3.1. *The function f maps $\mathbb{D}^n$ into $\mathbb{D}$ if and only if*

$$\mathscr{T}(M_{p_1}, M_{p_2}, \dots) = \begin{pmatrix} & \vdots & \vdots & \vdots & \\ \cdots & M_{p_1} & M_{p_2} & M_{p_3} & \cdots \\ \cdots & 0 & M_{p_1} & M_{p_2} & \cdots \\ \cdots & 0 & 0 & M_{p_1} & \cdots \\ & \vdots & \vdots & \vdots & \end{pmatrix}$$

is a contraction, where the M_{p_k} are as in (3.3).

Proof. Apply the spectral theorem, in the form of functional calculus for a commuting tuple of normal operators, to $(I^{\otimes(n-1)} \otimes B^*, \dots, B^{*\otimes n})$. ∎

First, a necessary condition for the existence of a solution to `CF problem` (n, d) is now evident.

Theorem 3.2. *A solution to* `CF problem` (n, d) *exists only if the operator $\mathscr{T}(M_{p_1}, \dots, M_{p_d})$ is a contraction, where the operators $M_{p_1}, \dots, M_{p_d}$ are as in* (3.3).

Second, we claim that the polynomial p in (3.1) and $p_1, \dots, p_d$ in (3.3) determine each other. Consequently, the reformulation of `CF problem` (n, d) announced in the Introduction follows by using Proposition 3.1.

To prove the claim, first note that the constant $a_{e_{i_1}+\dots+e_{i_k}}$ and the monomial $\prod_{p=1}^{k} z_{n-i_p+1} z_{n-i_p+2} \cdots z_{n-1}$ are invariant under the permutation of $(i_1, \dots, i_k)$. Therefore, to compute the monomial corresponding to $(i_1, \dots, i_k)$ we assume without loss of generality that $1 \le i_1 \le \dots \le i_k \le n$. Now, the monomial corresponding to $(i_1, \dots, i_k)$ is

$$\prod_{l=i_{k-1}}^{i_k-1} z_{n-l} \prod_{l=i_{k-2}}^{i_{k-1}-1} z_{n-l}^2 \cdots \prod_{l=1}^{i_1-1} z_{n-l}^k. \tag{3.4}$$

If all $i_1, \dots, i_k$ are 1, then (3.4) is the constant function 1. Otherwise, there exists $s \ge 1$ such that $i_1 = \dots = i_{s-1} = 1$ and $i_s > 1$. In this case, the exponent of each of the variables $z_{n-i_q+1}, \ldots, z_{n-i_{q-1}}$ in (3.4) is $k+1-q$, $s \leq q \leq k$, where i_0 is assumed to be 1. Thus if $(i_1, \ldots, i_k) \neq (i'_1, \ldots, i'_k)$, $1 \leq i_1 \leq \cdots \leq i_k \leq n$ and $1 \leq i'_1 \leq \cdots \leq i'_k \leq n$, then it is clear from (3.4) that the monomials corresponding to $(i_1, \ldots, i_k)$ and $(i'_1, \ldots, i'_k)$ are distinct.

To give a necessary and sufficient condition for the existence of a solution to `CF problem` (n, d), we need the notion of *complete polynomial extendibility* for the operator $\mathscr{T}(M_{p_1}, \ldots, M_{p_d})$. For each $k \in \{1, \ldots, d\}$, let M_{p_k} be the operator defined in (3.3), where p_k is the homogeneous term of degree k in (3.1).

Definition 3.3 (Completely polynomially extendible). We say that the operator $\mathscr{T}(M_{p_1}, \ldots, M_{p_d})$ with $p_j \in \mathscr{M}_{n-1}^{(j)}$ for $j = 1, \ldots, d$ is *m-polynomially extendible* if there exist $p_j \in \mathscr{M}_{n-1}^{j}$ for $d+1 \leq j \leq m$ such that $\mathscr{T}(M_{p_1}, \ldots, M_{p_m})$ is a contraction. The operator $\mathscr{T}(M_{p_1}, \ldots, M_{p_d})$ is called *completely polynomially extendible* if it is m-polynomially extendible for each m.

It is easy to provide a necessary and sufficient condition for the existence of a solution to `CF problem` (n, d) using the notion of complete polynomially extendibility.

Theorem 3.4. *A solution to* `CF problem` (n, d) *exists if and only if the operator* $\mathscr{T}(M_{p_1}, \ldots, M_{p_d})$ *is completely polynomially extendible.*

We give an algorithm to obtain a solution to `CF problem` (n, d) analogous to the one in the case of two variables. As in that case, for $p_k \in \mathscr{M}_{n-1}^{(k)}$, $1 \leq k \leq d$, let $P(z) = M_{p_0} + M_{p_1} z + \cdots + M_{p_d} z^d$. We shall say that P is in the *CF class* if there is a holomorphic function $f : \mathbb{D} \to \mathcal{B}(L^2(\mathbb{T}^{n-1}))$ with the properties stated in `CF problem (R)`. Such an f will be called a *CF-extension* of P. It follows that a solution to `CF problem` (n, d) exists if and only if P is in the CF class.

3.1. Algorithm for finding a solution to the CF problem. This algorithm identifies all polynomials p such that `CF problem` (n, d) admits a solution.

- If $\|\mathscr{T}(M_{p_1}, \ldots, M_{p_d})\| > 1$, then P is not a CF class polynomial; otherwise move to the next step.
- Parrott's theorem gives all possible operators T for which the operator $\mathscr{T}(M_{p_1}, \ldots, M_{p_d}, T)$ is a contraction. Let $\mathcal{C}_{d+1}$ be the set of all operators T such that $T = M_{p_{d+1}}$ for some $p_{d+1} \in \mathscr{M}_{n-1}^{(d+1)}$ and $\mathscr{T}(M_{p_1}, \ldots, M_{p_d}, T)$ is a contraction. If $\mathcal{C}_{d+1}$ is empty then P is not a CF class polynomial.
- For each $s > d+1$, using Parrott's theorem we can construct $\mathcal{C}_s$, the set of all operators T such that $T = M_{p_s}$ for some $p_s \in \mathscr{M}_{n-1}^{(s)}$ and

$\mathscr{T}(M_{p_1}, \dots, M_{p_{s-1}}, T)$ is a contraction, where $M_{p_j} \in \mathcal{C}_j$ for $j = d+1, \dots, s-1$.
- If all of the sets $\mathcal{C}_s$ are non-empty, then (and only then) P is a CF class polynomial.

4. A generalization of Nehari's theorem. Let M be a closed subspace of a Hilbert space $\mathbb{H}$. For any vector h in $\mathbb{H}$, the distance from h to M is attained at $P(h)$, where P is the orthogonal projection from $\mathbb{H}$ onto M. A deep result due to Nehari, recalled in Theorem 1.7, shows that the distance from a vector ϕ in $L^\infty(\mathbb{T})$ to the closed subspace $H^\infty(\mathbb{T})$ is the norm of the Hankel operator H_ϕ with symbol ϕ.

To obtain a multi-variable generalization, one must first consider what operators might be designated to be Hankel operators. One possibility is to look for operators $A : H^2(\mathbb{T}^n) \to \mathcal{M}^\perp$, where $\mathcal{M} \subseteq L^2(\mathbb{T}^n)$ is any invariant subspace for the multiplication operators $M_{z_i} : L^2(\mathbb{T}^n) \to L^2(\mathbb{T}^n)$, $i = 1, \dots, n$. As in Definition 1.6 and following [4, Section 4], we say that $A : H^2(\mathbb{T}^n) \to \mathcal{M}^\perp$ is a *Hankel operator* if $PM_{z_i}A = AM_{z_i}|_{H^2(\mathbb{T}^n)}$, $i = 1, \dots, n$, where P is the projection onto $\mathcal{M}^\perp$. An alternative and probably more pleasing way to state this requirement is to simply say that A is a module map from $H^2(\mathbb{T}^n)$ to $\mathcal{M}^\perp$ (see [4, Section 4] for the details). The operators $H_\varphi^{\mathcal{M}} : H^2(\mathbb{T}^n) \to \mathcal{M}^\perp$ with symbol $\varphi \in L^\infty(\mathbb{T}^n)$, where $H_\varphi^{\mathcal{M}} f = P(\varphi f)$, $f \in H^2(\mathbb{T}^n)$, are bounded and define a module map. In consequence, they are Hankel operators. However, unlike the case of one variable, here many "natural" choices for the subspace $\mathcal{M}$ exist. In particular, the "big" and the "small" Hankel operators have been frequently discussed in the literature [1, Section 4.4]. We choose, in the case of two variables, the subspace $\mathcal{M} \subseteq L^2(\mathbb{T}^2)$ to be $L^2(\mathbb{T}) \otimes \ell^2(\mathbb{N}_0) \simeq L^2(\mathbb{T}) \otimes H^2(\mathbb{T})$. Having made this choice, we prove a Nehari type theorem for these operators. Of course, it is not clear what happens if one makes a different choice of $\mathcal{M}$. Our choice was dictated by the form of the matrix representation which agrees with an existing class of Hankel operators possessing operator symbols discussed in [3, p. 34].

Remark 4.1. We are grateful to one of the reviewers for pointing out that Theorem 4.6 below follows immediately from [13, Theorem 3.1] (see also [14, Theorem 4]) by applying it to the case where the coefficient space C is equal to $\ell^2(\mathbb{Z})$. However, Lemmas 4.4–4.6 below are possibly of independent interest. In particular, these ideas are used to give the matrix-theoretic proof of the Korányi–Pukánsky theorem. The reviewer also points out that our approach fails for the case of the "small" and "big" Hankel operators, making the theory for these operators much different.

4.1. Nehari's theorem for $L^\infty(\mathbb{T}^2)$. In this subsection, we present a possible multivariate generalization of Nehari's theorem for $L^\infty(\mathbb{T}^2)$. This generalization is most conveniently stated in terms of the D-slice ordering on $\mathbb{Z}^2$.

For $k \in \mathbb{Z}$, define $P_k := \{(x,y) : x+y = k\}$. The subsets P_k of $\mathbb{Z}^2$ are disjoint and $\bigsqcup_{k\in\mathbb{Z}} P_k = \mathbb{Z}^2$. An order on $\mathbb{Z}^2$, which we call the *D-slice ordering*, is defined below. It is obtained from the usual co-lexicographic ordering by rotating it through an angle of $\pi/4$.

DEFINITION 4.2 (D-slice ordering). For $(x_1,y_1) \in P_l$ and $(x_2,y_2) \in P_m$,

- if $l = m$, then the order between (x_1,y_1) and (x_2,y_2) is determined by the lexicographic ordering on $P_l \subseteq \mathbb{Z}^2$;
- if $l < m$ (resp., if $l > m$), then $(x_1,y_1) < (x_2,y_2)$ (resp., $(x_1,y_1) > (x_2,y_2)$).

Define

$$H_1 := \Big\{ f := \sum_{(m,n)\in A_1} a_{m,n} z_1^m z_2^n : f \in L^\infty(\mathbb{T}^2) \Big\},$$
$$H_2 := \Big\{ f := \sum_{(m,n)\in A_2} a_{m,n} z_1^m z_2^n : f \in L^\infty(\mathbb{T}^2) \Big\},$$

where $A_1 := \{(m,n) \in \mathbb{Z}^2 : m+n \geq 0\}$ and $A_2 := \{(m,n) \in \mathbb{Z}^2 : m+n < 0\}$. Then H_1 and H_2 are disjoint closed subspaces of $L^\infty(\mathbb{T}^2)$ satisfying $L^\infty(\mathbb{T}^2) = H_1 + H_2$. The answer to the following question would be one possible generalization of Nehari's theorem. Let $\operatorname{dist}_\infty(\phi, H_1)$ denote the distance from ϕ to the subspace H_1.

QUESTION 4.3. *For any ϕ in $L^\infty(\mathbb{T}^2)$, what is* $\operatorname{dist}_\infty(\phi, H_1)$?

To answer this question, it will be convenient to introduce the notion of a Hankel operator with symbol $\phi \in L^\infty(\mathbb{T}^2)$.

4.2. The Hankel matrix corresponding to ϕ. Any $f \in L^2(\mathbb{T}^2)$ can be represented as a power series

$$f(z_1,z_2) = \sum_{m,n\in\mathbb{Z}} a_{m,n} z_1^m z_2^n = \sum_{m,n\in A_1} a_{m,n} z_1^m z_2^n + \sum_{m,n\in A_2} a_{m,n} z_1^m z_2^n.$$

Suppose $z_2 = \lambda z_1$. Then

$$f(z_1, \lambda z_1) = \sum_{k\geq 0}\Big(\sum_{m+n=k} a_{m,n}\lambda^n\Big) z_1^k + \sum_{k<0}\Big(\sum_{m+n=k} a_{m,n}\lambda^n\Big) z_1^k.$$

Setting $f_k(\lambda) := \sum_{m+n=k} a_{m,n}\lambda^n$, we have

$$f(z_1, \lambda z_1) = \sum_{k\in\mathbb{Z}} f_k(\lambda) z_1^k. \tag{4.1}$$

In this way, $L^2(\mathbb{T}^2)$ is first identified with $L^2(\mathbb{T}) \otimes L^2(\mathbb{T})$ and then with $L^2(\mathbb{T}) \otimes \ell^2(\mathbb{Z})$, the identifications in both cases being isometric. For any $\phi \in L^\infty(\mathbb{T}^2)$, define the multiplication operator $M_\phi : L^2(\mathbb{T}) \otimes \ell^2(\mathbb{Z}) \to L^2(\mathbb{T}) \otimes \ell^2(\mathbb{Z})$ as follows:

$$M_\phi\Big(\sum_{j\in\mathbb{Z}} g_j \otimes e_j\Big) := \sum_{k\in\mathbb{Z}}\Big(\sum_{q\in\mathbb{Z}} g_q\phi_{q+k}\Big) \otimes e_k,$$

where ϕ_j satisfies $\phi(z_1, \lambda z_1) = \sum_{j\in\mathbb{Z}} \phi_j(\lambda) z_1^k$.

LEMMA 4.4. *For any $\phi \in L^\infty(\mathbb{T}^2)$, we have $\|M_\phi\| = \|\phi\|_{\mathbb{T}^2,\infty}$.*

Proof. Let $\phi \in L^\infty(\mathbb{T}^2)$. From (4.1), it follows that

$$\phi(z, \lambda z) = \sum_{k\in\mathbb{Z}} \phi_k(\lambda) z^k$$

for some ϕ_k in $L^\infty(\mathbb{T})$. Note that $\{z^i \otimes e_j : (i,j) \in \mathbb{Z}^2\}$ is an orthonormal basis in $L^2(\mathbb{T}) \otimes \ell^2(\mathbb{Z})$. The matrix of M_ϕ with respect to this basis and the D-slice ordering on its index set is of the form

$$\begin{pmatrix} & \vdots & \vdots & \vdots & \\ \cdots & M_{\phi_{-1}} & M_{\phi_0} & M_{\phi_1} & \cdots \\ \cdots & M_{\phi_{-2}} & M_{\phi_{-1}} & M_{\phi_0} & \cdots \\ \cdots & M_{\phi_{-3}} & M_{\phi_{-2}} & M_{\phi_{-1}} & \cdots \\ & \vdots & \vdots & \vdots & \end{pmatrix}.$$

We know that $\|\phi\|_{\mathbb{T}^2,\infty} = \sup_{\lambda\in\mathbb{T}} \sup_{z\in\mathbb{T}} |\sum_{k\in\mathbb{Z}} \phi_k(\lambda) z^k|$. Thus

$$\|\phi\|_{\mathbb{T}^2,\infty} = \sup_{\lambda\in\mathbb{T}} \left\| \begin{pmatrix} & \vdots & \vdots & \vdots & \\ \cdots & \phi_{-1}(\lambda) & \phi_0(\lambda) & \phi_1(\lambda) & \cdots \\ \cdots & \phi_{-2}(\lambda) & \phi_{-1}(\lambda) & \phi_0(\lambda) & \cdots \\ \cdots & \phi_{-3}(\lambda) & \phi_{-2}(\lambda) & \phi_{-1}(\lambda) & \cdots \\ & \vdots & \vdots & \vdots & \end{pmatrix} \right\|$$

$$= \left\| \begin{pmatrix} & \vdots & \vdots & \vdots & \\ \cdots & M_{\phi_{-1}} & M_{\phi_0} & M_{\phi_1} & \cdots \\ \cdots & M_{\phi_{-2}} & M_{\phi_{-1}} & M_{\phi_0} & \cdots \\ \cdots & M_{\phi_{-3}} & M_{\phi_{-2}} & M_{\phi_{-1}} & \cdots \\ & \vdots & \vdots & \vdots & \end{pmatrix} \right\|.$$

Hence $\|\phi\|_{\mathbb{T}^2,\infty} = \|M_\phi\|$. ∎

The Hilbert space $\ell^2(\mathbb{N}_0)$ and the normed linear subspace

$$\Big\{(\ldots,0,x_0,x_1,\ldots):\sum_{i\geq 0}|x_i|^2<\infty \text{ with } x_0 \text{ at the 0th position}\Big\}$$

of $\ell^2(\mathbb{Z})$ are naturally isometrically isomorphic. Let $H:=L^2(\mathbb{T})\otimes\ell^2(\mathbb{N}_0)$. Then H is a closed subspace of $L^2(\mathbb{T})\otimes\ell^2(\mathbb{Z})$. We define the Hankel operator H_ϕ with symbol $\phi\in L^\infty(\mathbb{T}^2)$ to be $P_{H^\perp}\circ M_{\phi_{|H}}$. Writing down the matrix for H_ϕ with respect to the bases $\{z^i\otimes e_j : i\in\mathbb{Z},\ j=0,1,2,\ldots\}$ and $\{z^i\otimes e_{-j}: i\in\mathbb{Z},\ j=1,2,\ldots\}$ in H and $H^\perp$ respectively, we get

$$H_\phi=\begin{pmatrix} M_{\phi_{-1}} & M_{\phi_{-2}} & M_{\phi_{-3}} & \cdots\\ M_{\phi_{-2}} & M_{\phi_{-3}} & M_{\phi_{-4}} & \cdots\\ M_{\phi_{-3}} & M_{\phi_{-4}} & M_{\phi_{-5}} & \cdots\\ \vdots & \vdots & \vdots & \end{pmatrix},\qquad \phi\in L^\infty(\mathbb{T}^2).$$

We note that H_ϕ is the Hankel operator with symbol ϕ modulo the signs of the indices [3, p. 34]. However, it is different from the usual definition of either the big or small Hankel operator in two variables as defined in [1, Section 4.4].

LEMMA 4.5. *For any ϕ in $L^\infty(\mathbb{T}^2)$, we have $\|H_\phi\|\leq \operatorname{dist}_\infty(\phi,H_1)$.*

Proof. From the definition of H_ϕ and Lemma 4.4, it can be seen that

$$\|H_\phi\|=\|P_{H^\perp}\circ M_{\phi_{|H}}\|\leq\|M_\phi\|=\|\phi\|_{\mathbb{T}^2,\infty}.$$

From the matrix representation of H_ϕ, it is clear that for any g in H_1, $H_{\phi-g}=H_\phi$. Hence $\|H_\phi\|=\|H_{\phi-g}\|\leq\|\phi-g\|_{\mathbb{T}^2,\infty}$. ∎

For $n\in\mathbb{N}$, $a_0,a_1,\ldots,a_{n-1}\in\mathbb{C}$ and $(b_m)_{m\in\mathbb{N}}$, $b_m\in\mathbb{C}$, define

$$T_n((b_m),a_0,a_1,\ldots,a_{n-1}):=\begin{pmatrix} a_0 & a_1 & \cdots & a_{n-1}\\ b_1 & a_0 & \cdots & a_{n-2}\\ \vdots & \vdots & \ddots & \vdots\\ b_{n-1} & b_{n-2} & \cdots & a_0\\ \vdots & \vdots & & \vdots \end{pmatrix}.$$

LEMMA 4.6. *Suppose $f_0,f_1,\ldots,f_{n-1}\in L^\infty(\mathbb{T})$ and $(g_m)_{m\in\mathbb{N}}$, $g_m\in L^\infty(\mathbb{T})$, are such that*

$$\sup_{\lambda\in\mathbb{T}}\|T_n((g_m(\lambda)),f_0(\lambda),\ldots,f_{n-1}(\lambda))\|\leq 1.$$

Then there exists $f_n\in L^\infty(\mathbb{T})$ satisfying

$$\sup_{\lambda\in\mathbb{T}}\|T_{n+1}((g_m(\lambda)),f_0(\lambda),\ldots,f_n(\lambda))\|\leq 1.$$

Proof. Let

$$Q(\lambda) = (f_0(\lambda) \ \cdots \ f_{n-1}(\lambda)), \qquad R(\lambda) = (f_{n-1}(\lambda) \ \cdots \ f_0(\lambda)\ g_1(\lambda) \ \cdots)^{\mathrm{t}},$$

$$S(\lambda) = \begin{pmatrix} g_1(\lambda) & f_0(\lambda) & \cdots & f_{n-3}(\lambda) \\ g_2(\lambda) & g_1(\lambda) & \cdots & f_{n-4}(\lambda) \\ \vdots & \vdots & \ddots & \vdots \\ g_{n-1}(\lambda) & g_{n-2}(\lambda) & \cdots & g_1(\lambda) \\ \vdots & \vdots & & \vdots \end{pmatrix}.$$

All possible choices of $f_n(\lambda)$ for which $T_{n+1}((g_m(\lambda)), f_0(\lambda), \ldots, f_n(\lambda))$ is a contraction are given, via Parrott's theorem (cf. [17, Chapter 12, p. 152]), by the formula

$$f_n(\lambda) = (I - ZZ^*)^{1/2} V (I - Y^*Y)^{1/2} - ZS(\lambda)^* Y, \tag{4.2}$$

where V is an arbitrary contraction and the operators Y, Z are obtained from the formulae $R(\lambda) = (I - S(\lambda)S(\lambda)^*)^{1/2} Y$, $Q(\lambda) = Z(I - S(\lambda)^* S(\lambda))^{1/2}$.

We note that every entry of $I - S(\lambda)^* S(\lambda)$ is in L^∞ as a function of λ. Thus all entries in $(I - S(\lambda)^* S(\lambda))^{1/2}$ are measurable functions which are essentially bounded. Consequently, so are all entries of Z. A similar assertion can be made for Y. Therefore, choosing $V = 0$ in (4.2), we get f_n with the required property. In fact, one can choose V to be any contraction whose entries are L^∞ functions. ∎

Let $\mathbb{H}$ be a Hilbert space. For any $(T_n)_{n \in \mathbb{N}}$, $T_n \in \mathcal{B}(\mathbb{H})$, define an operator $H(T_1, T_2, \ldots)$ as follows:

$$H(T_1, T_2, \ldots) = \begin{pmatrix} T_1 & T_2 & T_3 & \cdots \\ T_2 & T_3 & T_4 & \cdots \\ T_3 & T_4 & T_5 & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}.$$

Theorem 4.7 (Nehari's theorem for $L^\infty(\mathbb{T}^2)$). *If $\phi \in L^\infty(\mathbb{T}^2)$, then $\|H_\phi\| = \mathrm{dist}_\infty(\phi, H_1)$.*

Proof. From Lemma 4.5, we know that $\|H_\phi\| \le \mathrm{dist}_\infty(\phi, H_1)$. Without loss of generality we assume that $\|H_\phi\| = 1$. Using Lemma 4.6, we find $\phi_0 \in L^\infty(\mathbb{T})$ such that the norm of the operator $H(M_{\phi_0}, M_{\phi_{-1}}, \ldots)$ is at most 1. Now, one proves the desired conclusion by repeated use of Lemma 4.6. ∎

4.3. Nehari's theorem in n variables. The generalization of Nehari's theorem to n variables is very similar. Therefore we will be brief. The key is the *D-slice ordering* on $\mathbb{Z}^n$, defined below.

For $k \in \mathbb{Z}$, define $P_k := \{(x_1, \dots, x_n) \in \mathbb{Z}^n : x_1 + \cdots + x_n = k\}$. The subsets P_k of $\mathbb{Z}^n$ are disjoint and $\bigsqcup_{k\in\mathbb{Z}} P_k = \mathbb{Z}^n$.

Definition 4.8 (D-slice ordering for $\mathbb{Z}^n$). For $l, m \in \mathbb{Z}$, and $(x_1, \dots, x_n) \in P_l$ and $(y_1, \dots, y_n) \in P_m$,

- if $l = m$, then the order between $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$ is determined by the lexicographic ordering on $P_l \subseteq \mathbb{Z}^n$,
- if $l < m$ (respectively if $l > m$), then $(x_1, \dots, x_n) < (y_1, \dots, y_n)$ (respectively $(x_1, \dots, x_n) > (y_1, \dots, y_n)$).

Define

$$H_1 := \Big\{f := \sum_{(m_1,\dots,m_n)\in A_1} a_{m_1,\dots,m_n} z_1^{m_1} \cdots z_n^{m_n} : f \in L^\infty(\mathbb{T}^2)\Big\},$$

$$H_2 := \Big\{f := \sum_{(m_1,\dots,m_n)\in A_2} a_{m_1,\dots,m_n} z_1^{m_1} \cdots z_n^{m_n} : f \in L^\infty(\mathbb{T}^2)\Big\},$$

where $A_1 := \{(m_1, \dots, m_n) \in \mathbb{Z}^n : m_1 + \cdots + m_n \geq 0\}$ and $A_2 := \{(m_1, \dots, m_n) \in \mathbb{Z}^n : m_1 + \cdots + m_n < 0\}$. The subspaces H_1 and H_2 of $L^\infty(\mathbb{T}^n)$ are closed disjoint, and $L^\infty(\mathbb{T}^n) = H_1 + H_2$. Let $f \in L^2(\mathbb{T}^2)$ have power series expansion

$$\begin{aligned} f(z_1, \dots, z_n) &= \sum_{m_1,\dots,m_n\in\mathbb{Z}} a_{m_1,\dots,m_n} z_1^{m_1} \cdots z_n^{m_n} \\ &= \sum_{(m_1,\dots,m_n)\in A_1} a_{m_1,\dots,m_n} z_1^{m_1} \cdots z_n^{m_n} \\ &\quad + \sum_{(m_1,\dots,m_n)\in A_2} a_{m_1,\dots,m_n} z_1^{m_1} \cdots z_n^{m_n}. \end{aligned}$$

Suppose $z_j = \lambda_{j-1} z_1$ with $\lambda_{j-1} \in \mathbb{D}$ for $j = 2, \dots, n$. Then

$$\begin{aligned} f(z_1, \lambda_1 z_1, \dots, \lambda_{n-1} z_1) &= \sum_{k\geq 0} \Big(\sum_{m_1+\cdots+m_n=k} a_{m_1,\dots,m_n} \lambda_1^{m_2} \cdots \lambda_{n-1}^{m_n} \Big) z_1^k \\ &\quad + \sum_{k<0} \Big(\sum_{m_1+\cdots+m_n=k} a_{m_1,\dots,m_n} \lambda_1^{m_2} \cdots \lambda_{n-1}^{m_n} \Big) z_1^k. \end{aligned}$$

For each $k \in \mathbb{Z}$, we set

$$f_k(\lambda_1, \dots, \lambda_{n-1}) := \sum_{m_1+\cdots+m_n=k} a_{m_1,\dots,m_n} \lambda_1^{m_2} \cdots \lambda_{n-1}^{m_n}.$$

For any $\phi \in L^\infty(\mathbb{T}^n)$, define $M_\phi : L^2(\mathbb{T}^{n-1}) \otimes \ell^2(\mathbb{Z}) \to L^2(\mathbb{T}^{n-1}) \otimes \ell^2(\mathbb{Z})$ by

$$M_\phi\Big(\sum_{j\in\mathbb{Z}} g_j \otimes e_j\Big) := \sum_{k\in\mathbb{Z}} \Big(\sum_{q\in\mathbb{Z}} g_q \phi_{q+k}\Big) \otimes e_k,$$

where $\phi(z_1, \lambda_1 z_1, \ldots, \lambda_{n-1} z_1) = \sum_{j\in\mathbb{Z}} \phi_j(\lambda_1, \ldots, \lambda_{n-1}) z_1^k$. Now, we define the Hankel operator H_ϕ corresponding to ϕ by

$$H_\phi = \begin{pmatrix} M_{\phi_{-1}} & M_{\phi_{-2}} & M_{\phi_{-3}} & \cdots \\ M_{\phi_{-2}} & M_{\phi_{-3}} & M_{\phi_{-4}} & \cdots \\ M_{\phi_{-3}} & M_{\phi_{-4}} & M_{\phi_{-5}} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}.$$

The proof of the following theorem is very similar to that of Theorem 4.7; we omit the details.

THEOREM 4.9 (Nehari's theorem for $L^\infty(\mathbb{T}^n)$). *If $\phi \in L^\infty(\mathbb{T}^n)$, then $\|H_\phi\| = \operatorname{dist}_\infty(\phi, H_1)$.*

4.4. CF problem in $\mathbb{D}^2$ and Nehari's theorem for $L^\infty(\mathbb{T}^2)$. Fix $p \in \mathbb{C}[z_1, z_2]$ with

$$p(z_1, z_2) = a_{10}z_1 + a_{01}z_2 + a_{20}z_1^2 + a_{11}z_1z_2 + a_{02}z_2^2.$$

Denote

$$\phi(z_1, z_2) := \overline{z}_1^3 p(z_1, z_2) = a_{10}\overline{z}_1^2 + a_{01}\overline{z}_1^3 z_2 + a_{20}\overline{z}_1 + a_{11}\overline{z}_1^2 z_2 + a_{02}\overline{z}_1^3 z_2^2.$$

Suppose $p_1(\lambda) = a_{10} + a_{01}\lambda$ and $p_2(\lambda) = a_{20} + a_{11}\lambda + a_{02}\lambda^2$. Then $\|H_\phi\| = \operatorname{dist}_\infty(\phi, H_1)$, where

$$H_\phi = \begin{pmatrix} M_{p_2} & M_{p_1} & 0 & \cdots \\ M_{p_1} & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

Thus, if there exists a holomorphic function $q : \mathbb{D}^2 \to \mathbb{C}$ with $q^{(I)}(0) = 0$ for $|I| \le 2$ such that $\|p + q\|_{\mathbb{D}^2,\infty} \le 1$, then $\|H_\phi\| \le \|p + q\|_{\mathbb{D}^2,\infty}$. Hence $\|H_\phi\| \le 1$ is a necessary condition for such a q to exist. As we have seen before, this necessary condition, however, is not sufficient.

5. Alternative proof of the Korányi–Pukánszky theorem.

We recall the following theorem of Korányi and Pukánszky [10, Corollary, p. 452]. It gives a necessary and sufficient condition for the range of a holomorphic function on $\mathbb{D}^n$ to be in the right half-plane H_+.

THEOREM 5.1 (Korányi–Pukánszky theorem). *Suppose $\sum_{\boldsymbol{\alpha}\in\mathbb{N}_0^n} a_{\boldsymbol{\alpha}} \boldsymbol{z}^{\boldsymbol{\alpha}}$ represents a holomorphic function f on $\mathbb{D}^n$. Then $\Re(f(\boldsymbol{z})) \ge 0$ for all $\boldsymbol{z} \in \mathbb{D}^n$ if and only if the map $\phi : \mathbb{Z}^n \to \mathbb{C}$ defined by*

$$\phi(\boldsymbol{\alpha}) = \begin{cases} 2\Re a_{\boldsymbol{\alpha}} & \textit{if } \boldsymbol{\alpha} = 0, \\ a_{\boldsymbol{\alpha}} & \textit{if } \boldsymbol{\alpha} > 0, \\ a_{-\boldsymbol{\alpha}} & \textit{if } \boldsymbol{\alpha} < 0, \\ 0 & \textit{otherwise}, \end{cases}$$

is positive, that is, the $k \times k$ *matrix* $(\phi(m_i - m_j))_{i,j}$ *is non-negative definite for any* $k \in \mathbb{N}$ *and* $m_1, \ldots, m_k \in \mathbb{Z}^n$.

We call ϕ the *Korányi–Pukánszky function* corresponding to the coefficients $(a_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha} \in \mathbb{N}_0^n}$.

5.1. The planar case. Suppose $f : \mathbb{D} \to H_+$ is holomorphic. Without loss of generality we can assume $f(0) = 1$. Consider the Cayley map $\chi : H_+ \to \mathbb{D}$ defined by

$$\chi(z) = \frac{1-z}{1+z},$$

which is a bi-holomorphism. Suppose $\chi \circ f$ mapping $\mathbb{D}$ into $\mathbb{D}$ has a power series expansion $\sum_{n=1}^{\infty} a_n z^n$. Then

$$f(z) = \frac{1 + \chi \circ f(z)}{1 - \chi \circ f(z)} = 2\Big(c_0 + \sum_{n=1}^{\infty} c_n z^n\Big), \tag{5.1}$$

where $2c_n = f^{(n)}(0)/n!$ for all $n \in \mathbb{N}_0$. The exact relationship between the coefficients c_n and a_n is obtained in the lemma below.

Lemma 5.2. *For each* $n \in \mathbb{N}$,

$$c_n = a_n + \sum_{j=1}^{n-1} a_j c_{n-j}.$$

Proof. Consider the expression

$$\begin{aligned} f(z) &= 2\Big(c_0 + \sum_{n=1}^{\infty} c_n z^n\Big) \\ &= 2\Big(\frac{1}{2} + (\chi \circ f)(z) + (\chi \circ f)(z)^2 + (\chi \circ f)(z)^3 + \cdots\Big). \end{aligned}$$

Rewriting, we get

$$\frac{1}{1 - (\chi \circ f)(z)} = 1 + \sum_{n=1}^{\infty} c_n z^n.$$

Hence,

$$\Big(1 + \sum_{n=1}^{\infty} c_n z^n\Big)\Big(1 - \sum_{n=1}^{\infty} a_n z^n\Big) = 1.$$

A comparison of the coefficients completes the verification. ∎

Let ϕ denote the Korányi–Pukánszky function corresponding to $(c_n)_{n=0}^{\infty}$. The matrix $(\phi(j-k))_{j,k}$ is given by

$$\begin{array}{c c} & \begin{array}{ccccc} \cdots & -1 & 0 & 1 & \cdots \end{array} \\ \begin{array}{c} \vdots \\ -1 \\ 0 \\ 1 \\ \vdots \end{array} & \begin{pmatrix} & \vdots & \vdots & \vdots & \\ \cdots & 1 & \overline{c}_1 & \overline{c}_2 & \cdots \\ \cdots & c_1 & 1 & \overline{c}_1 & \cdots \\ \cdots & c_2 & c_1 & 1 & \cdots \\ & \vdots & \vdots & \vdots & \end{pmatrix} \end{array}. \tag{5.2}$$

For each $n \in \mathbb{N}$, let

$$C_n := \begin{pmatrix} 1 & \overline{c}_1 & \overline{c}_2 & \cdots & \overline{c}_n \\ c_1 & 1 & \overline{c}_1 & \cdots & \overline{c}_{n-1} \\ c_2 & c_1 & 1 & \cdots & \overline{c}_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_n & c_{n-1} & c_{n-2} & \cdots & 1 \end{pmatrix}, \quad A_n := \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ 0 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & 0 & a_1 & \cdots & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_1 \end{pmatrix},$$

$$P_n := \begin{pmatrix} 1 & -a_1 & -a_2 & \cdots & -a_n \\ 0 & 1 & -a_1 & \cdots & -a_{n-1} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -a_1 \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Lemma 5.3. *For all $n \in \mathbb{N}$, $P_n C_n^{\mathrm{t}} P_n^* = (I - A_n A_n^*) \oplus 1$.*

Proof. We use induction on n. The case $n = 1$ is trivial. Assume the result is valid for $n-1$, $n > 1$. For each $n \in \mathbb{N}$, let

$$\tilde{P}_n := (-a_n, -a_{n-1}, \dots, -a_1)^{\mathrm{t}} \quad \text{and} \quad \tilde{C}_n := (c_n, c_{n-1}, \dots, c_1)^{\mathrm{t}}.$$

The verification of the identity

$$P_n C_n^{\mathrm{t}} P_n^* = \begin{pmatrix} P_{n-1} & \tilde{P}_n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} C_{n-1}^{\mathrm{t}} & \tilde{C}_n \\ \tilde{C}_n^* & 1 \end{pmatrix} \begin{pmatrix} P_{n-1}^* & 0 \\ \tilde{P}_n^* & 1 \end{pmatrix}$$

is easy. Hence $P_n C_n^{\mathrm{t}} P_n^*$ takes the form

$$\begin{pmatrix} P_{n-1} C_{n-1}^{\mathrm{t}} P_{n-1}^* + \tilde{P}_n \tilde{C}_n^* P_{n-1}^* + \tilde{P}_n^* (P_{n-1} \tilde{C}_n + \tilde{P}_n) & P_{n-1} \tilde{C}_n + \tilde{P}_n \\ (P_{n-1} \tilde{C}_n + \tilde{P}_n)^* & 1 \end{pmatrix}.$$

From Lemma 5.2 we have $P_{n-1} \tilde{C}_n + \tilde{P}_n = 0$, and therefore

$$P_n C_n^{\mathrm{t}} P_n^* = \begin{pmatrix} P_{n-1} C_{n-1}^{\mathrm{t}} P_{n-1}^* + \tilde{P}_n \tilde{C}_n^* P_{n-1}^* & 0 \\ 0 & 1 \end{pmatrix}.$$

Now,

$$\tilde{P}_n \tilde{C}_n^* P_{n-1}^* = \begin{pmatrix} -a_n \\ \vdots \\ -a_1 \end{pmatrix} \Big(\overline{c}_n - \sum_{i=1}^{n-1} a_i c_{n-i} \;\; \overline{c}_{n-1} - \sum_{i=1}^{n-2} a_i c_{n-i} \;\; \cdots \;\; \overline{c}_1 \Big).$$

From Lemma 5.2, we get

$$\tilde{P}_n \tilde{C}_n^* P_{n-1}^* = \begin{pmatrix} -a_n \\ \vdots \\ -a_1 \end{pmatrix} (\overline{a}_n \;\; \cdots \;\; \overline{a}_1) = (-a_{n-i}\overline{a}_{n-j})_{i,j=0}^{n-1}.$$

Since

$$I - A_k A_k^* = \begin{pmatrix} 1 - \sum_{j=1}^{k} |a_j|^2 & -\sum_{j=2}^{k} a_j \overline{a}_{j-1} & \cdots & -a_k \overline{a}_1 \\ -\sum_{j=2}^{k} \overline{a}_j a_{j-1} & 1 - \sum_{j=1}^{k-1} |a_j|^2 & \cdots & -a_{k-1}\overline{a}_1 \\ \vdots & \vdots & \ddots & \vdots \\ -a_1 \overline{a}_k & -a_1 \overline{a}_{k-1} & \cdots & 1 - |a_1|^2 \end{pmatrix},$$

we obtain

$$I - A_n A_n^* = ((I - A_{n-1}A_{n-1}^*) \oplus 1) + (-a_{n-j}\overline{a}_{n-l})_{1 \le j,l \le k-1}.$$

Thus $I - A_n A_n^* = P_{n-1} C_{n-1}^{\mathrm{t}} P_{n-1}^* + \tilde{P}_n \tilde{C}_n P_{n-1}^*$, and the proof is complete. ■

An immediate corollary to Lemma 5.3 is the following proposition.

PROPOSITION 5.4. *The matrix C_n is non-negative definite if and only if $\|A_n\| \le 1$.*

Since $\chi \circ f$ $(= g$, say) is a holomorphic map from $\mathbb{D}$ to $\mathbb{D}$, the multiplication operator M_g on $L^2(\mathbb{T})$ has the property that $\|M_g\| = \|g\|_{\mathbb{D},\infty}$ (see [17, Theorem 13.14]). Writing the matrix for M_g with respect to the basis $\{\ldots, z^{-2}, z^{-1}, 1, z^1, z^2, \ldots\}$, we conclude that M_g is a contraction if and only if A_n is a contraction for each $n \in \mathbb{N}$. Using Proposition 5.4 together with the equality $\|M_g\| = \|g\|_{\mathbb{D},\infty}$, we see that f maps $\mathbb{D}$ into H_+ if and only if C_n is non-negative definite for each $n \in \mathbb{N}$. Thus we recover the solution of the Korányi–Pukánszky problem (solvability criterion in terms of the positivity of the associated Herglotz matrix) for the single-variable case.

5.2. The case of several variables. In this subsection, all the computations are given for $n = 2$ only. These computations are easily seen to work equally well, using the D-slice ordering on $\mathbb{Z}^n$, for any $n \in \mathbb{N}$. The details are briefly indicated in Subsection 5.3.

Suppose $f : \mathbb{D} \to H_+$ is holomorphic. Without loss of generality, we assume that $f(\mathbf{0}) = 1$. As before, let $\chi : \mathbb{H} \to \mathbb{D}$ be the Cayley map and

$$\chi \circ f(\boldsymbol{z}) = \sum_{m,n=0}^{\infty} a_{mn} z_1^m z_2^n.$$

Thus $\chi \circ f$ maps $\mathbb{D}^2$ into $\mathbb{D}$ and $a_{00} = 0$. With the understanding that $c_0 = 1/2$, we have

$$f(\boldsymbol{z}) = \frac{1 + \chi \circ f(\boldsymbol{z})}{1 - \chi \circ f(\boldsymbol{z})} = 2\Big(c_{00} + \sum_{m,n=1}^{\infty} c_{mn} z_1^m z_2^n\Big).$$

Let ϕ be the Korányi–Pukánszky function corresponding to (c_{mn}). The following theorem describes ϕ with respect to the D-slice ordering on $\mathbb{Z}^2$.

THEOREM 5.5. *Let $(c_{mn})_{m,n\in\mathbb{N}_0}$ be an infinite array of complex numbers. The matrix of the Korányi–Pukánszky function ϕ, in the D-slice ordering, corresponding to this array is*

$$\begin{array}{c} \\ \vdots \\ P_{-1} \\ P_0 \\ P_1 \\ \vdots \end{array}
\begin{array}{c}
\begin{array}{ccccc} \cdots & P_{-1} & P_0 & P_1 & \cdots \end{array} \\
\begin{pmatrix}
 & \vdots & \vdots & \vdots & \\
\cdots & I & C_1^* & C_2^* & \cdots \\
\cdots & C_1 & I & C_1^* & \cdots \\
\cdots & C_2 & C_1 & I & \cdots \\
 & \vdots & \vdots & \vdots &
\end{pmatrix}
\end{array},$$

where $C_n := c_{n0}I + c_{n-1,1}B^* + \cdots + c_{0n}B^{*n}$, $n \in \mathbb{N}$, and B is the bilateral shift on $\ell^2(\mathbb{Z})$.

Proof. With respect to the D-slice ordering on $\mathbb{Z}^2$, the matrix corresponding to ϕ is a doubly infinite block matrix. The (k, n) element $\phi((k, -k+l) - (n, -n+m))$, in the (l, m) block in this matrix, is computed below separately:

$k - n < 0$: The quantity $\phi((k, -k + l) - (n, -n + m))$ is non-zero only if $k - n \geq l - m$. Hence if $l \geq m$, then $\phi((k, -k + l) - (n, -n + m)) = 0$. Now, assume $l < m$. In this case, if $k - n \notin \{l - m, l - m + 1, \ldots, -1\}$ then $\phi((k, -k + l) - (n, -n + m)) = 0$. For $p \in \{0, 1, \ldots, -l + m - 1\}$ and $k - n = l - m + p$, we have

$$\phi((k, -k + l) - (n, -n + m)) = \bar{c}_{m-l-p,p}.$$

$k - n = 0$:

$$\phi(0, l - m) = \begin{cases} c_{0,l-m} & \text{if } l \geq m, \\ \bar{c}_{0,m-l} & \text{if } l < m. \end{cases}$$

$k - n > 0$: The quantity $\phi((k, -k + l) - (n, -n + m))$ is non-zero only if $k - n \leq l - m$. Hence if $l \leq m$, then $\phi((k, -k + l) - (n, -n + m)) = 0$. Now, assume $l > m$. In this case, if $k - n \notin \{l - m, l - m - 1, \ldots, 1\}$

then $\phi((k,-k+l)-(n,-n+m))=0$. For $p\in\{0,1,\dots,l-m-1\}$ and $k-n=l-m-p$, we have

$$\phi((k,-k+l)-(n,-n+m))=c_{l-m-p,p}.$$

Therefore, the (l,m) block $\phi(l,m)$ in the matrix of ϕ is of the form

$$\phi(l,m)=\begin{cases} C^*_{m-l} & \text{if } l<m,\\ I & \text{if } m=l,\\ C_{l-m} & \text{if } l>m.\end{cases}$$

Hence the block matrix of the Korányi–Pukánszky function ϕ, in the D-slice ordering, corresponding to the array (c_{mn}) takes the form

$$\begin{array}{c} \\ \vdots \\ P_{-1} \\ P_0 \\ P_1 \\ \vdots \end{array}
\begin{array}{c} \begin{array}{ccccc} \cdots & P_{-1} & P_0 & P_1 & \cdots \end{array} \\
\begin{pmatrix} & \vdots & \vdots & \vdots & \\ \cdots & I & C_1^* & C_2^* & \cdots \\ \cdots & C_1 & I & C_1^* & \cdots \\ \cdots & C_2 & C_1 & I & \cdots \\ & \vdots & \vdots & \vdots & \end{pmatrix} \end{array}. \blacksquare$$

LEMMA 5.6. *For all $n\in\mathbb{N}$, setting $A_n:=a_{n0}I+a_{n-1,1}B^*+\dots+a_{0n}B^{*n}$ and $C_n=c_{n0}I+c_{n-1,1}B^*+\dots+c_{0n}B^{*n}$, we have*

$$C_n=A_n+\sum_{j=1}^{n-1}A_jC_{n-j}.$$

Proof. Let $C(z_1,z_2):=\sum_{i,j=0}^{\infty}c_{ij}z_1^iz_2^j$. We have

$$1+\chi\circ f(z_1,z_2)+\chi\circ f(z_1,z_2)^2+\dots=\frac{f(z_1,z_2)}{2}+c_{00}=C(z_1,z_2).$$

Thus $C(z_1,z_2)(1-\chi\circ f(z_1,z_2))=1$, which is the same as

$$\begin{aligned}(1&+c_{10}z_1+c_{01}z_2+c_{20}z_1^2+c_{11}z_1z_2+c_{02}z_2^2+\cdots)\\ &\times(1-a_{10}z_1-a_{01}z_2-a_{20}z_1^2-a_{11}z_1z_2-a_{02}z_2^2+\cdots)=1.\end{aligned}$$

For each $k\in\mathbb{N}$, comparing the coefficient of the monomial $z_1^{n-k}z_2^k$, we have

$$c_{n-k,k}=\sum_{p=0}^{k}\sum_{j=k}^{n}a_{n-j,p}c_{j-k,k-p},$$

where $a_{00}=0$. The coefficient of B^{*k} in $A_n+\sum_{i=1}^{n-1}A_iC_{n-i}$ is

$$\begin{aligned}
a_{n-k,k}c_{00} + a_{n-k,k-1}c_{01} + a_{n-k-1,k}c_{10} + a_{n-k,k-2}c_{02}& \\
+ a_{n-k-1,k-1}c_{11} + a_{n-k-2,k}c_{20} + \cdots& \\
= (a_{n-k,k}c_{00} + a_{n-k,k-1}c_{01} + \cdots + a_{n-k,0}c_{0k})& \\
+ (a_{n-k-1,k}c_{10} + a_{n-k-1,k-1}c_{11} + \cdots + a_{n-k-1,0}c_{1,k})& \\
+ \cdots + (a_{0k}c_{n-k,0} + a_{0,k-1}c_{n-k,1} + \cdots + a_{00}c_{n-k,k})& \\
= \sum_{p=0}^{k}\sum_{j=k}^{n} a_{n-j,p}c_{j-k,k-p},&
\end{aligned}$$

completing the proof of the lemma. ∎

The relationship between A_n and C_n is given by the following lemma.

LEMMA 5.7. *If A_n and C_n are defined as above, then*

$$\begin{pmatrix} I & C_1^* & C_2^* & \cdots & C_n^* \\ C_1 & I & C_1^* & \cdots & C_{n-1}^* \\ C_2 & C_1 & I & \cdots & C_{n-2}^* \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_n & C_{n-1} & C_{n-2} & \cdots & I \end{pmatrix}$$

is non-negative definite if and only if

$$\left\| \begin{pmatrix} A_1 & A_2 & A_3 & \cdots & A_n \\ 0 & A_1 & A_2 & \cdots & A_{n-1} \\ 0 & 0 & A_1 & \cdots & A_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_1 \end{pmatrix} \right\| \le 1.$$

Proof. For each $n \in \mathbb{N}$, C_n commutes with C_m and A_m for all $m \in \mathbb{N}$, and hence we can adapt the proof of Lemma 5.3 to this case. ∎

An application of the spectral theorem along with Lemma 5.7 gives an alternative proof of the Korányi–Pukánszky theorem, as shown below.

Proof of the Korányi–Pukánszky theorem. The operators $I \otimes B^*$ and $B^* \otimes B^*$ are commuting unitaries and they have $\mathbb{T}^2$ as their joint spectrum. Applying the spectral theorem and the maximum modulus principle, we get

$$\|\chi \circ f(I \otimes B^*, B^* \otimes B^*)\| = \|\chi \circ f\|_{\mathbb{D}^2,\infty}. \tag{5.3}$$

Note that $\chi \circ f(I \otimes B^*, B^* \otimes B^*) = A_1 \otimes B^* + A_2 \otimes B^{*2} + \cdots$, where $A_n := a_{n0}I + a_{n-1,1}B^* + \cdots + a_{0n}B^{*n}$ as in Lemma 5.7. Since $\|\chi \circ f\|_{\mathbb{D}^2,\infty} \le 1$, it follows from (5.3) that $\|\mathscr{T}(A_1,\dots,A_n)\| \le 1$ for all $n \in \mathbb{N}$. From Lemma 5.7, we conclude that

$$(5.4)\qquad \begin{pmatrix} I & C_1^* & C_2^* & \cdots & C_n^* \\ C_1 & I & C_1^* & \cdots & C_{n-1}^* \\ C_2 & C_1 & I & \cdots & C_{n-2}^* \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_n & C_{n-1} & C_{n-2} & \cdots & I \end{pmatrix}$$

is non-negative definite for all $n \in \mathbb{N}$, where $C_n := c_{n0}I + c_{n-1,1}B^* + \cdots + c_{0n}B^{*n}$. Hence from Theorem 5.5, we see that the Korányi–Pukánszky function ϕ corresponding to the array (c_{jk}) is positive.

Conversely, suppose the Korányi–Pukánszky function ϕ corresponding to (c_{jk}) is positive, where c_{00} is assumed to be $1/2$. Then from Theorem 5.5 it follows that the operator in (5.4) is non-negative definite for all $n \in \mathbb{N}$. From Lemma 5.7 and (5.3), we conclude that $\|\chi^{-1} \circ g\|_{\mathbb{D}^2,\infty} \le 1$, where $g(z_1,z_2) = 2\sum_{m,n=0}^{\infty} c_{mn} z_1^m z_2^n$. This is so if and only if g maps $\mathbb{D}^2$ into H_+. Hence the theorem is proved. ∎

5.3. The case of n variables. Suppose $f : \mathbb{D}^n \to H_+$ is holomorphic. Without loss of generality, we assume that $f(\mathbf{0}) = 1$. Let

$$\chi \circ f(\boldsymbol{z}) = \sum_{k=0}^{\infty} \sum_{i_1,\dots,i_k=1}^{n} a_{e_{i_1}+\cdots+e_{i_k}} z_{i_1}\cdots z_{i_k}.$$

As before, $\chi \circ f$ maps $\mathbb{D}^n$ into $\mathbb{D}$ and $a_{\mathbf{0}} = 0$. Then with the understanding that $c_{\mathbf{0}} = 1/2$, we have

$$f(\boldsymbol{z}) = \frac{1+\chi\circ f(\boldsymbol{z})}{1-\chi\circ f(\boldsymbol{z})} = 2\Big(c_{\mathbf{0}} + \sum_{k=1}^{\infty} \sum_{i_1,\dots,i_k=1}^{n} c_{e_{i_1}+\cdots+e_{i_k}} z_{i_1}\cdots z_{i_k}\Big).$$

For $k \in \mathbb{N}$, let

$$C_k := \sum_{i_1,\dots,i_k=1}^{n} c_{e_{i_1}+\cdots+e_{i_k}} \prod_{p=1}^{k} (I^{\otimes(n-i_p)} \otimes B^{*\otimes(i_p-1)}),$$
$$A_k := \sum_{i_1,\dots,i_k=1}^{n} a_{e_{i_1}+\cdots+e_{i_k}} \prod_{p=1}^{k} (I^{\otimes(n-i_p)} \otimes B^{*\otimes(i_p-1)}).$$

Computations similar to the case of $n = 2$, using A_k and C_k, $k \in \mathbb{N}$, prove a result analogous to Lemma 5.7. Hence, as before, using the spectral theorem for the operators $I^{\otimes(n-j)} \otimes B^{*\otimes j}$, $j = 1,\dots,n$, we deduce the Korányi–Pukánszky theorem for the polydisc $\mathbb{D}^n$.

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