$\widetilde{\mathrm{SL}(2,\mathbb{R})}$-HOMOGENEOUS VECTOR BUNDLES

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ABSTRACT. We describe all the holomorphic Hermitian vector bundles $(E\,,h)$ over the upper half plane in $\mathbb{C}$ with the property that $(f^*E\,,f^*h)$ is holomorphically isometric to $(E\,,h)$ for any holomorphic automorphism f of the upper half plane. We give an explicit construction of all such holomorphic Hermitian vector bundles using some linear algebraic data.

1. INTRODUCTION

Linear spaces with norms defined by a Hermitian form make an appearance in many areas of mathematics and its applications. However, often the bounded linear operators on such spaces or bundle maps of Hermitian vector bundles are the primary object of study. In [Cowen and Douglas 1978], Cowen and Douglas introduce a class of operators possessing an open set of eigenvalues Ω of constant multiplicity, say k. They show that for such an operator T, with the additional assumption that the range $T-w$ is closed for $w\in\Omega$, the map $w\longmapsto \ker(T-w)$ is holomorphic. It then follows that $\pi\,:\,E_T\longrightarrow\Omega$, where

$$E_T\,=\,\{\ker(T-w):w\in\Omega\}\,,\,\pi(\ker(T-w))=w,\,w\in\Omega$$

defines a holomorphic Hermitian vector bundle on Ω. Although, we must emphasize that not all holomorphic Hermitian bundles arise this way. One of the main results of [Cowen and Douglas 1978] is that the unitary equivalence class of T and the equivalence class of the holomorphic Hermitian vector bundle E_T determine each other. For instance, if the multiplicity $k=1$, then the curvature of E_T is a complete invariant of the holomorphic Hermitian line bundle E_T and hence that of the operator T. With this bridge between complex geometry and operator theory at our disposal, we raise the question of homogeneity.

Let T be a bounded linear operator on a Hilbert space H. We will let σ_T denote the spectrum of the operator T. Let Möb be the group of bi-holomorphic automorphisms of the unit disc $\mathbb{D}$. If the function $\varphi\,\in$ Möb is holomorphic on some open set containing σ_T, then $\varphi(T)$ is well-defined via the usual holomorphic functional calculus for T. If $\varphi(T)$ is unitarily equivalent to T for all φ in Möb, then T is said to be *homogeneous*. It turns out that if T is homogeneous then $\sigma_T=\overline{\mathbb{D}}$ and $\varphi(T)$ is unitarily equivalent to T *for all* φ in Möb. The well-known characterization of systems of imprimitivity due to Mackey amounts to classifying normal operators (or commuting tuples of normal operators) in our context (cf. [Bagchi and Misra 1995]). A systematic study of homogeneous operators when the normality assumption is dropped is of more recent origin (cf. [Bagchi and Misra 2001]). A classification of all homogeneous scalar shift operators has been obtained in [Bagchi and Misra 2003].

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Now, assume that T is one of the operators in the Cowen-Douglas class of $\mathbb{D}$ and E_T be the holomorphic Hermitian vector bundle corresponding to T. If T is homogeneous then $\varphi(T)$ is evidently in the Cowen-Douglas class as well for all φ in Möb. The vector bundle $E_{\varphi(T)}$ for the operator $\varphi(T)$ is the pullback of the bundle E_T under φ, that is, $\varphi^* E_T = E_{\varphi(T)}$. It is natural to say that a holomorphic Hermitian vector bundle E defined on the unit disc is homogeneous (Möb - equivariant) if its pullback by any $\varphi \in$ Möb remains (isometrically) isomorphic to E, that is, $\varphi^* E_T \cong E_{\varphi(T)}$ for all φ in Möb. Thus the classification of homogeneous operators in the Cowen-Douglas class of $\mathbb{D}$ is the same as that of the corresponding homogeneous holomorphic Hermitian bundles defined on $\mathbb{D}$. Wilkins [Wilkins 1993] provides a classification of all, not necessarily the ones arising out of Cowen-Douglas operators, homogeneous holomorphic Hermitian bundles. The main goal of this paper is a classification of homogeneous holomorphic Hermitian bundles on $\mathbb{D}$ as well. However, our methods are somewhat different from that of Wilkins and we give a very explicit construction of all homogeneous holomorphic Hermitian bundles on $\mathbb{D}$. The question of which ones of these bundles correspond to operators in the Cowen-Douglas class remains open except when the rank of the bundle is 1. However, a different description of all homogeneous holomorphic Hermitian vector bundles which correspond to homogeneous operators in the Cowen-Douglas class of the unit disc $\mathbb{D}$ is given in [Korányi and Misra 2006, Theorem 4.1].

We point out that the list of $\mathrm{SL}(2,\mathbb{R})$-equivariant vector bundles in [Biswas 2003] is incomplete.

It will be convenient for us to work with the upper half plane in the complex plane rather than the unit disc.

2. Some properties of homogeneous bundles

Let $\mathbb{H} := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ be the upper half plane of the complex plane. The group $\mathrm{PSL}(2,\mathbb{R})$ acts on $\mathbb{H}$ as

$$A(z) = \frac{az+b}{cz+d}, \tag{2.1}$$

where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{R}) \tag{2.2}$$

and $z \in \mathbb{H}$. This action identifies the group of all holomorphic automorphisms of $\mathbb{H}$ with $\mathrm{PSL}(2,\mathbb{R})$. For any A as in Eqn. (2.2), let

$$\phi(A) \in \mathrm{Aut}(\mathbb{H}) \tag{2.3}$$

be the holomorphic automorphism defined in Eqn. (2.1)

The universal cover of the group $\mathrm{PSL}(2,\mathbb{R})$ will be denoted by $\widetilde{\mathrm{SL}(2,\mathbb{R})}$. Using the projection

$$p : \widetilde{\mathrm{SL}(2,\mathbb{R})} \longrightarrow \mathrm{PSL}(2,\mathbb{R}), \tag{2.4}$$

the group $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ acts on $\mathbb{H}$.

Let E be a holomorphic vector bundle over $\mathbb{H}$ equipped with a Hermitian structure h such that the holomorphic Hermitian vector bundle $(\phi(A)^*E, \phi(A)^*h)$ is isomorphic to (E, h) for all $A \in \mathrm{Aut}(\mathbb{H})$. By a *lift* of the action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ to (E, h) as vector bundle automorphisms we mean the following:

- The group $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ acts on the total space of E and the natural projection of E to $\mathbb{H}$ is $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-equivariant;
- for any $A \in \widetilde{\mathrm{SL}(2,\mathbb{R})}$, the action of A on E is a holomorphic isomorphism of the vector bundle $\phi(p(A^{-1}))^*E$ with E, where ϕ and p are defined in Eqn. (2.3) and Eqn. (2.4) respectively; and
- the action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on E preserves the Hermitian structure h.

The following lemma has appeared as Theorem 2.2 in [Korányi and Misra 2006].

Lemma 2.1. *Let $(E\,,h)$ be a holomorphic Hermitian vector bundle over $\mathbb{H}$ such that the holomorphic Hermitian vector bundle $(\phi(A)^*E\,,\phi(A)^*h)$ is isomorphic to $(E\,,h)$ for all $A \in \mathrm{Aut}(\mathbb{H})$. The action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on $\mathbb{H}$ lifts to a holomorphic action on E, as vector bundle automorphisms, preserving the Hermitian structure h.*

Proof. Let G_E denote the group of all pairs of the form $(\tau\,,\sigma)$, where $\tau \in \mathrm{Aut}(\mathbb{H})$ is a biholomorphism of $\mathbb{H}$, and

$$\sigma\,:\,E\,\longrightarrow\,\tau^*E$$

is a holomorphic isomorphism preserving the Hermitian structure h. It is easy to see that G_E is a finite dimensional Lie group. Let $H\,\subset\,G_E$ be the closed subgroup defined by all pairs of the form $(\mathrm{Id}_{\mathbb{H}}\,,\sigma)$; in other words, H is the group of all automorphisms of the holomorphic Hermitian vector bundle $(E\,,h)$. Therefore, we have an exact sequence of groups

$$e\,\longrightarrow\,H\,\longrightarrow\,G_E\,\longrightarrow\,\mathrm{PSL}(2,\mathbb{R})\,\longrightarrow\,e\,. \tag{2.5}$$

We note that the above homomorphism $G_E\,\longrightarrow\,\mathrm{PSL}(2,\mathbb{R})$ is surjective follows from the assumption that the holomorphic Hermitian vector bundle $(\phi(A)^*E\,,\phi(A)^*h)$ is isomorphic to $(E\,,h)$ for all $A \in \mathrm{Aut}(\mathbb{H})$.

Let

$$0\,\longrightarrow\,\mathfrak{h}\,\longrightarrow\,\mathfrak{g}_E\,\longrightarrow\,\mathrm{sl}(2,\mathbb{R})\,\longrightarrow\,0 \tag{2.6}$$

be the exact sequence of Lie algebras obtained from Eqn. (2.5); here $\mathfrak{h}$ and $\mathfrak{g}_E$ are the Lie algebras of H and G_E respectively. Since the Lie algebra $\mathrm{sl}(2,\mathbb{R})$ is semisimple, the exact sequence of Lie algebras in Eqn. (2.6) is right split [Bourbaki 1960, p. 91, Corollaire 3]. In other words, there is a Lie subalgebra $\mathfrak{l}\,\subset\,\mathfrak{g}_E$ that projects isomorphically to $\mathrm{sl}(2,\mathbb{R})$.

Since the group $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ is simply connected, the above isomorphism $\mathrm{sl}(2,\mathbb{R})\,\longrightarrow\,\mathfrak{l}$ gives a homomorphism of groups $\widetilde{\mathrm{SL}(2,\mathbb{R})}\,\longrightarrow\,G_E$. Now $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ acts on E using this homomorphism to G_E (the group G_E has a natural action on E). Therefore, the action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on $\mathbb{H}$ lifts to a holomorphic action on E as vector bundle automorphisms preserving the Hermitian structure h. This completes the proof of the lemma. □

Lemma 2.2. *Let $(E\,,h)$ be as in Lemma 2.1. Assume that the group H in Eqn. (2.5) is a product of copies of the circle group S^1. Then there is a unique lift of the action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ (on $\mathbb{H}$) to a holomorphic action on E as vector bundle automorphisms preserving the Hermitian structure h.*

Proof. Consider the adjoint action of the group G_E on the normal subgroup H in Eqn. (2.5). Since both H and $\mathrm{PSL}(2,\mathbb{R})$ are connected, from Eqn. (2.5) it follows that the group G_E is connected; also the automorphism group of a product of copies of S^1 is discrete. Therefore, the adjoint action of G_E on H is the trivial action. Hence H is contained in the center of G_E.

Since H is contained in the center of G_E, any two right splitting of the exact sequence of Lie algebras in Eqn. (2.6) differ by a Lie algebra homomorphism from $\mathrm{sl}(2,\mathbb{R})$ to $\mathfrak{h}$. As $\mathrm{sl}(2,\mathbb{R})$ is simple and $\mathfrak{h}$ is abelian, there is no nonzero homomorphism from the Lie algebra $\mathrm{sl}(2,\mathbb{R})$ to $\mathfrak{h}$. This means that there is a unique right splitting of the exact sequence of Lie algebras in Eqn. (2.6) (we saw in Lemma 2.1 that there is at least one right splitting). From this it follows immediately that there is a unique lift of the action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on $\mathbb{H}$ to a holomorphic action on E as vector bundle automorphisms preserving the Hermitian structure h. This completes the proof of the lemma. □

Let E be a holomorphic vector bundle over $\mathbb{H}$ equipped with a Hermitian structure h. Then E has a unique Hermitian connection $\nabla = \nabla^{1,0} + \nabla^{0,1}$ such that

$$\nabla^{0,1}s = 0$$

for all locally defined holomorphic sections of E [Kobayashi 1987, p. 11, Proposition (4.9)]. This connection ∇ is called the *Chern connection on E*.

Let $F \subset E$ be a C^∞ subbundle. It is easy to see that the following two statements are equivalent:

(1) For any smooth section s of F defined over some open subset $U \subset \mathbb{H}$, the inclusion

$$\nabla s \in \Omega^1(U, F) = C^\infty(U, F \otimes (T_U^{\mathbb{C}})^*) \tag{2.7}$$

holds, where ∇ is the Chern connection.

(2) Both F and $F^\perp$ are holomorphic subbundles of E.

Assume that the holomorphic Hermitian vector bundle $(\phi(A)^*E\,, \phi(A)^*h)$ is isomorphic to $(E\,, h)$ for all $A \in \mathrm{Aut}(\mathbb{H})$. From Lemma 2.1 we know that the action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on $\mathbb{H}$ lifts to a holomorphic action on E, as vector bundle automorphisms, preserving the Hermitian structure h. Fix such a lift of the action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$. Then we have the following proposition:

Proposition 2.3. *Let $F \subset E$ be a C^∞ subbundle preserved by the Chern connection (i.e., Eqn. (2.7) holds for all smooth sections s of F defined over open subsets of $\mathbb{H}$). Then the action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on E preserves the subbundle F.*

Proof. Let $\mathrm{Aut}^0(E)$ denote the connected component, containing the identity element, of the group of all holomorphic Hermitian automorphisms of the holomorphic Hermitian vector bundle $(E\,, h)$. In other words, $\mathrm{Aut}^0(E)$ is the connected component containing the identity element of the group H in Eqn. (2.5).

Using the action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on E, the group $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ acts as automorphisms of $\mathrm{Aut}^0(E)$ as follows: The action of any $g \in \widetilde{\mathrm{SL}(2,\mathbb{R})}$ sends any $T \in \mathrm{Aut}^0(E)$ to the automorphism $g^{-1} \circ T \circ g$ of $(E\,, h)$. Let

$$\rho_E : \widetilde{\mathrm{SL}(2,\mathbb{R})} \longrightarrow \mathrm{Aut}(\mathrm{Aut}^0(E)) \tag{2.8}$$

be the resulting homomorphism, where $\mathrm{Aut}(\mathrm{Aut}^0(E))$ is the group of automorphism of the group $\mathrm{Aut}^0(E)$.

Since $\mathrm{Aut}^0(E)$ is a compact Lie group, any element in the connected component of $\mathrm{Aut}(\mathrm{Aut}^0(E))$ containing the identity automorphism must be an inner conjugation. As $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ is connected, and the connected component of $\mathrm{Aut}(\mathrm{Aut}^0(E))$ containing the identity automorphism is $\mathrm{Aut}^0(E)/Z$, where $Z \subset \mathrm{Aut}^0(E)$ is the center, we conclude that the homomorphism ρ_E in Eqn. (2.8) gives a homomorphism

$$\overline{\rho}_E \,:\, \widetilde{\mathrm{SL}(2,\mathbb{R})} \longrightarrow \mathrm{Aut}^0(E)/Z\,. \tag{2.9}$$

Since $\mathrm{Aut}^0(E)/Z$ is compact, the image of any unipotent element in $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ under the homomorphism $\overline{\rho}_E$ in Eqn. (2.9) must be the identity element. The group $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ being simple, this implies that $\overline{\rho}_E$ is the trivial homomorphism. Consequently, the homomorphism ρ_E in Eqn. (2.8) is the trivial homomorphism.

For any $\theta \in [0\,,\pi/2] \subset \mathbb{R}$, consider the automorphism T_θ of E defined by

$$T_\theta \,=\, \exp(\pi\sqrt{-1}\theta)\mathrm{Id}_F + \exp(-\pi\sqrt{-1}\theta)\mathrm{Id}_{F^\perp}\,,$$

where $F^\perp \subset E$ is the orthogonal complement of F. We noted earlier that the given condition that Eqn. (2.7) holds for all smooth sections s of F defined over open subsets of $\mathbb{H}$ is equivalent to the condition that both F and $F^\perp$ are holomorphic subbundles of E. Hence we conclude that $T_\theta \in \mathrm{Aut}^0(E)$.

Since the homomorphism ρ_E in Eqn. (2.8) is the trivial homomorphism, the above automorphism T_θ of E commutes with the action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on E. This immediately implies that the action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on E preserves the eigenbundles of T_θ. Therefore, the action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on E preserves the subbundle F. This completes the proof of the proposition. □

In the next section we will investigate the holomorphic Hermitian line bundles over $\mathbb{H}$ equipped with an action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$.

3. $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-homogeneous line bundles

We start with a simple lemma.

Lemma 3.1. *Let $(L_1\,,h_1)$ and $(L_2\,,h_2)$ be two holomorphic Hermitian line bundles over $\mathbb{H}$. Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be the curvatures of the Chern connections on L_1 and L_2 respectively (so $\mathcal{K}_1$ and $\mathcal{K}_2$ are smooth $(1\,,1)$-form on $\mathbb{H}$). The two holomorphic Hermitian line bundles $(L_1\,,h_1)$ and $(L_2\,,h_2)$ are isomorphic if and only if $\mathcal{K}_1 \,=\, \mathcal{K}_2$.*

Proof. If L_1 is holomorphically isometric to L_2, then obviously $\mathcal{K}_1 \,=\, \mathcal{K}_2$. To prove the converse, assume that $\mathcal{K}_1 \,=\, \mathcal{K}_2$. The Chern connection on L_1 and L_2 will be denoted by ∇_1 and ∇_2 respectively.

Consider the holomorphic line bundle $L_1^*\bigotimes L_2$ over $\mathbb{H}$ equipped with the Hermitian structure induced by the pair h_1 and h_2. The Chern connection on $L_1^*\bigotimes L_2$ is $\nabla \,=\, \nabla_1^*\bigotimes \mathrm{Id}_{L_2} + \mathrm{Id}_{L_1}\bigotimes \nabla_2$, where ∇_1^* is the connection on L^* induced by ∇_1. The curvature of ∇ is $\mathcal{K}_2 - \mathcal{K}_1 \,=\, 0$.

Since $\mathbb{H}$ is simply connected, and the curvature of ∇ vanishes, the flat line bundle $L_1^*\bigotimes L_2$ is isomorphic to the trivial line bundle equipped with the constant metric. Fixing a flat section of $L_1^*\bigotimes L_2$ of (pointwise) norm one, we get a holomorphic isomorphism of L_1 with L_2 that takes the Hermitian structure h_1 to h_2. This completes the proof of the lemma. □

We will now construct a family of holomorphic Hermitian line bundle parametrized by the real numbers.

Take any $\lambda \in \mathbb{R}$. Let ξ denote the trivial holomorphic line bundle $\mathbb{H} \times \mathbb{C}$ over $\mathbb{H}$. Consider the Hermitian structure h^{ξ}_{λ} on ξ defined as follows: For any $z \in \mathbb{H}$ and any $c \in \xi_z = \mathbb{C}$,

$$h^{\xi}_{\lambda}(c\,,c) := \langle c\,,c\rangle_{\lambda} := \frac{|c|^2}{\mathrm{Im}(z)^{\lambda}}\,. \tag{3.1}$$

Let ξ_λ denote the holomorphic line bundle ξ over $\mathbb{H}$ equipped with the Hermitian structure h^{ξ}_{λ} defined above.

It is easy to check that the curvature $\mathcal{K}_\lambda$ of the Chern connection on ξ_λ is

$$\mathcal{K}_\lambda = -\lambda\overline{\partial}\partial(\mathrm{Im}(z)) \in \Omega^{1,1}_{\mathbb{H}}\,,$$

which is a smooth $(1\,,1)$-form on $\mathbb{H}$. It is straight-forward to check that the form $\mathcal{K}_\lambda$ is left invariant by the action of $\mathrm{PSL}(2,\mathbb{R})$ on $\mathbb{H}$. Therefore, using Lemma 2.2 and Lemma 3.1 we have the following corollary:

Corollary 3.2. *There is a unique lift of the action of* $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ *(on* $\mathbb{H}$*) to a holomorphic action on* ξ_λ *as vector bundle automorphisms preserving its Hermitian structure* h^{ξ}_{λ}.

Proof. Using Lemma 3.1 it follows that for any $A \in \mathrm{PSL}(2,\mathbb{R})$, the holomorphic Hermitian line bundle $\phi(A)^*\xi_\lambda$ is holomorphically isometric to ξ_λ, where ϕ is defined in Eqn. (2.3). Since the group H (see Lemma 2.2) in this case is S^1, the proof is completed using Lemma 2.2. □

Lemma 3.3. *Let* $(L\,,h)$ *be a holomorphic Hermitian line bundle over* $\mathbb{H}$ *equipped with a lift of the action of* $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ *(on* $\mathbb{H}$*) to a holomorphic action on* L *preserving the Hermitian structure* h*. Then there is a unique real number* λ *such that* ξ_λ *is holomorphically isometric to* $(L\,,h)$.

Proof. Since the curvature of ξ_λ is $-\lambda\overline{\partial}\partial(\mathrm{Im}(z))$, and $\overline{\partial}\partial(\mathrm{Im}(z)) \neq 0$, there can be at most one λ with the above property. To prove that there is actually one λ, consider the curvature $\mathcal{K}_h$ of the Chern connection on $(L\,,h)$. As $(L\,,h)$ is equipped with a lift of the action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$, we conclude that

$$\phi(A)^*\mathcal{K}_h = \mathcal{K}_h \tag{3.2}$$

for all $A \in \mathrm{PSL}(2,\mathbb{R})$, where ϕ is defined in Eqn. (2.3).

From the identity in Eqn. (3.2) it follows that $\mathcal{K}_h = \mu\overline{\partial}\partial(\mathrm{Im}(z))$ for some $\mu \in \mathbb{R}$ (we also use the fact that the form $\mathcal{K}_h$ is real). Now the proof of the lemma is completed using Lemma 3.1. □

Proposition 3.4. *If the line bundle* ξ_λ *admits a smooth nonzero section invariant under the action of* $\widetilde{\mathrm{SL}(2,\mathbb{R})}$*, then* $\lambda = 0$.

Proof. Let s be a smooth section of ξ_λ satisfying the following two conditions:

- the action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on ξ_λ leaves s invariant, and
- the section s is not identically equal to zero.

Note that these two conditions imply that s is nowhere vanishing. Let $\overline{\partial}_{\xi_\lambda}$ be the Dolbeault operator defining the holomorphic structure of ξ_λ. So $\overline{\partial}_{\xi_\lambda}s$ is a smooth section of $\xi_\lambda \bigotimes (T^{0,1}_{\mathbb{H}})^*$ which is left invariant by the action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$.

Since s is nowhere vanishing, we have

$$\overline{\partial}_{\xi_\lambda}s = s \otimes \omega\,,$$

where ω is a $(0\,,1)$-form on $\mathbb{H}$ which is left invariant by the action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on $\mathbb{H}$. But there is no nonzero $(0\,,1)$-form on $\mathbb{H}$ which is left invariant by the action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$. Indeed, this follows immediately from the fact that the isotropy subgroup of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ for any point $z \in \mathbb{H}$ acts nontrivially on the line $(T^{0,1}_z)^*$ (the fiber of $(T^{0,1}_{\mathbb{H}})^*$ over z).

Therefore, the $(0\,,1)$-form ω vanishes identically. Hence $\overline{\partial}_{\xi_\lambda} s \,=\, 0$. In other words, the section s is holomorphic. Since s is left invariant by the action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$, the inner product $\langle s(z)\,, s(z)\rangle_\lambda$ is independent of $z \in \mathbb{H}$ (see Eqn. (3.1)). As s is a holomorphic section of ξ_λ whose norm is a constant function on $\mathbb{H}$, the Chern connection on the holomorphic Hermitian line bundle ξ_λ is flat. Therefore, $\lambda \,=\, 0$. This completes the proof of the proposition. □

Proposition 3.4 has the following corollary:

Corollary 3.5. *Let $f \,:\, \xi_\lambda \,\longrightarrow\, \xi_{\lambda'}$ be a C^∞ homomorphism of line bundles that intertwines the actions of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$. If f is not identically zero, then $\lambda \,=\, \lambda'$. In that case, f is a constant scalar multiplication.*

Proof. Consider the Hermitian structure on the holomorphic line bundle $\xi^*_\lambda \bigotimes \xi_{\lambda'}$ induced by the Hermitian structures on ξ_λ and $\xi_{\lambda'}$. The curvature of the Chern connection on $\xi^*_\lambda \bigotimes \xi_{\lambda'}$ is $-(\lambda' - \lambda)\overline{\partial}\partial(\mathrm{Im}(z))$. Using Lemma 3.3 it follows that the Holomorphic Hermitian line bundle $\xi^*_\lambda \bigotimes \xi_{\lambda'}$ holomorphically isometric to $\xi_{\lambda'-\lambda}$. Now the proof is completed by applying Proposition 3.4 to $\xi_{\lambda'-\lambda}$. □

4. Data for higher rank vector bundles

For any point $z \in \mathbb{H}$, let $\Gamma_z \subset \widetilde{\mathrm{SL}(2,\mathbb{R})}$ be the subgroup that fixes z. It is easy to see that Γ_z is isomorphic to the additive group $\mathbb{R}$. Fix a point $z_0 \in \mathbb{H}$, and fix an isomorphism

$$\alpha \,:\, \Gamma_{z_0} \longrightarrow \mathbb{R}\,. \tag{4.1}$$

Let $(E\,,h)$ be a holomorphic Hermitian vector bundle over $\mathbb{H}$ equipped with an action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ compatible with both holomorphic and Hermitian structures. The fiber of the vector bundle E over a point $z \in \mathbb{H}$ will be denoted by E_z. The isotropy subgroup $\Gamma_z \subset \widetilde{\mathrm{SL}(2,\mathbb{R})}$ acts on E_z preserving the Hermitian structure on E_z. Since Γ_{z_0} preserves the Hermitian structure of E_{z_0}, using the isomorphism α in Eqn. (4.1) we have a positive integer ℓ, real number $r_1, \cdots, r_\ell$ and positive integers $m_1, \cdots, m_\ell$ satisfying the following conditions:

- $\sum_{i=1}^{\ell} m_i \,=\, \mathrm{rank}(E)$, and
- for each $i \in [1\,,\ell]$, there is a subspace $V_{z_0,r_i} \subset E_{z_0}$ of dimension m_i such that $\alpha^{-1}(t)$ acts on V_{z_0,r_i} as multiplication by $\exp(\sqrt{-1}r_i t)$ for all $t \in \mathbb{R}$.

Using the action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on E, the above decomposition

$$E_{z_0} \,=\, \bigoplus_{i=1}^{\ell} V_{z_0,r_i}$$

induces a C^∞ decomposition of the vector bundle E

$$E \,=\, \bigoplus_{i=1}^{\ell} V_{r_i}\,. \tag{4.2}$$

In other words, V_{r_i} is the orbit of V_{z_0,r_i} for the action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on E. It is easy to see that for any $z \in \mathbb{H}$, the decomposition

$$E_z = \bigoplus_{i=1}^{\ell} (V_{r_i})_z$$

obtained from Eqn. (4.2) is the isotypical decomposition of the Γ_z-module E_z.

We will interpret the decomposition in Eqn. (4.2) using the line bundles ξ_λ constructed in Section 3.

In Corollary 3.2 we noted that there is a unique lift of the action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on $\mathbb{H}$ to a holomorphic action on ξ_λ preserving its Hermitian structure. For any $\lambda \in \mathbb{R}$, let

$$\mathcal{W}_{E,\lambda} := C^\infty(\mathbb{H},\, \xi_\lambda^* \otimes E)^{\widetilde{\mathrm{SL}(2,\mathbb{R})}} \tag{4.3}$$

be the space of all smooth homomorphisms from ξ_λ to E that intertwine the actions of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on ξ_λ and E (these homomorphisms need not be holomorphic). Since $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ acts transitively on $\mathbb{H}$, for any point $z \in \mathbb{H}$, the evaluation at z

$$\mathcal{W}_{E,\lambda} \longrightarrow (\xi_\lambda^* \otimes E)_z \tag{4.4}$$

is an injective homomorphism. In particular, $\mathcal{W}_{E,\lambda}$ is a finite dimensional vector space.

We will construct an inner product on the vector space $\mathcal{W}_{E,\lambda}$. Consider the inner product on $\mathcal{W}_{E,\lambda}$ given by the Hermitian structure on the fiber $(\xi_\lambda^* \otimes E)_z$ using the injective homomorphism in Eqn. (4.4). It is now an easy exercise to check that this inner product on $\mathcal{W}_{E,\lambda}$ does not depend on the choice of the point z. Therefore, $\mathcal{W}_{E,\lambda}$ is canonically a finite dimensional Hilbert space.

The Hermitian structure of ξ_λ and the inner product on $\mathcal{W}_{E,\lambda}$ together induce a Hermitian structure on the holomorphic vector bundle

$$\mathcal{E}_\lambda := \xi_\lambda \otimes_{\mathbb{C}} \mathcal{W}_{E,\lambda}\,. \tag{4.5}$$

The actions of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on ξ_λ and the trivial action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on $\mathcal{W}_{E,\lambda}$ together induce an action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on the vector bundle $\mathcal{E}_\lambda$ constructed in Eqn. (4.5). This action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on $\mathcal{E}_\lambda$ clearly preserves the holomorphic structure as well as the Hermitian structure.

Associate to $(E\,,h)$ the holomorphic Hermitian vector bundle

$$\mathcal{E} := \bigoplus_{\lambda\in\mathbb{R}} \mathcal{E}_\lambda = \bigoplus_{\lambda\in\mathbb{R}} \xi_\lambda \otimes_{\mathbb{C}} \mathcal{W}_{E,\lambda}\,. \tag{4.6}$$

Note that $\mathcal{W}_{E,\lambda} = 0$ for all but finitely many λs in $\mathbb{R}$. The Hermitian structure on $\mathcal{E}$, which we will denote by $\mathcal{H}$, is the above mentioned Hermitian structure on each $\mathcal{E}_\lambda$, and the direct sum in Eqn. (4.6) is an orthogonal direct sum.

The actions of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on $\mathcal{E}_\lambda$, $\lambda \in \mathbb{R}$, together induce an action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on the holomorphic Hermitian vector bundle $(\mathcal{E}\,,\mathcal{H})$. This action clearly preserves the holomorphic structure as well as the the Hermitian structure.

The evaluation homomorphisms in Eqn. (4.4) for different values of λ together define a C^∞ isomorphism

$$\gamma_E : \mathcal{E} \longrightarrow E \tag{4.7}$$

which takes the metric $\mathcal{H}$ to h. This homomorphism γ_E clearly intertwines the actions of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on $\mathcal{E}$ and E. However, the homomorphism γ_E need not be holomorphic.

Let

$$\overline{\partial}_E \,:\, E \longrightarrow E \otimes \Omega^{0,1}_{\mathbb{H}}$$

be the Dolbeault operator defining the holomorphic structure of the holomorphic vector bundle E. Let

$$\overline{\partial}^{\phi}_E \,:=\, (\gamma_E^{-1} \otimes \mathrm{Id}_{\Omega^{0,1}_{\mathbb{H}}}) \circ \overline{\partial}_E \circ \gamma_E \,:\, \mathcal{E} \longrightarrow \mathcal{E} \otimes \Omega^{0,1}_{\mathbb{H}} \tag{4.8}$$

be the Dolbeault operator obtained by transporting $\overline{\partial}_E$ to $\mathcal{E}$ using the isomorphism γ_E in Eqn. (4.7).

Let $\overline{\partial}_{\mathcal{E}} \,:\, \mathcal{E} \longrightarrow \mathcal{E} \otimes \Omega^{0,1}_{\mathbb{H}}$ be the Dolbeault operator defining the holomorphic structure of the holomorphic vector bundle $\mathcal{E}$. Therefore,

$$\Theta \,:=\, \overline{\partial}^{\phi}_E - \overline{\partial}_{\mathcal{E}} \,\in\, C^\infty(\mathbb{H},\, \mathrm{End} \otimes (T^{0,1}_{\mathbb{H}})^*) \,=\, \Omega^{0,1}_{\mathbb{H}}(\mathrm{End}(\mathcal{E})) \tag{4.9}$$

is a $(0\,,1)$-form on $\mathbb{H}$ with values in the vector bundle $\mathrm{End}(\mathcal{E})$, where $\overline{\partial}^{\phi}_E$ is defined in Eqn. (4.8).

Since the C^∞ isomorphism γ_E (constructed in Eqn. (4.7)) intertwines the actions of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$, the section Θ in Eqn. (4.9) is preserved by the action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ (the action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on $\mathcal{E}$ induces an action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on $\mathrm{End}(\mathcal{E})$).

Proposition 4.1. *For any $\lambda \in \mathbb{R}$, the homomorphism Θ in Eqn.* (4.9) *sends the direct summand $\mathcal{E}_\lambda \subset \mathcal{E}$ (see Eqn.* (4.6)*) to the direct summand $\mathcal{E}_{\lambda-2} \bigotimes (T^{0,1}_{\mathbb{H}})^*$.*

Proof. We recall that the holomorphic tangent bundle $T^{1,0}_{\mathbb{H}}$ has the Poincaré metric defined by $\|\frac{\partial}{\partial z}\| = 1/\mathrm{Im}(z)$. Hence the holomorphic Hermitian line bundle $T^{1,0}_{\mathbb{H}}$ is holomorphically isometric to ξ_2.

The $(1\,,1)$-form on $\mathbb{H}$ given by the Poincaré metric, which is a $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-invariant Kähler form, gives a smooth $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-equivariant isomorphism of $T^{1,0}_{\mathbb{H}}$ with $(T^{0,1}_{\mathbb{H}})^*$.

Therefore, from Corollary 3.5 it follows that if there is a nonzero $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-equivariant homomorphism from the C^∞ line bundle ξ_λ to the C^∞ line bundle $\xi_{\lambda'} \bigotimes (T^{0,1}_{\mathbb{H}})^*$, then $\lambda' \,=\, \lambda - 2$.

Since the action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on $\mathcal{E}_\lambda$ is defined using the trivial action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on $\mathcal{W}_{E,\lambda}$ (see Eqn. (4.5)), applying the above assertion to the $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-invariant section Θ, the proof of the proposition is complete. □

In the proof of Proposition 4.1 we saw that the holomorphic tangent bundle $T^{1,0}_{\mathbb{H}}$ equipped with the Poincaré metric is holomorphically isometric to ξ_2. On the other hand, $\xi_{\lambda_1+\lambda_2}$ is holomorphically isometric to $\xi_{\lambda_1} \bigotimes \xi_{\lambda_2}$ equipped with the Hermitian structure induced by the Hermitian structures on ξ_{λ_1} and ξ_{λ_2}. Therefore, $\xi_\lambda \bigotimes T^{1,0}_{\mathbb{H}}$ is holomorphically isometric to $\xi_{\lambda+2}$ for all $\lambda \in \mathbb{R}$.

Fix once and for all a holomorphic bundle map

$$f_\lambda \,:\, \xi_\lambda \longrightarrow \xi_{\lambda-2} \otimes T^{1,0}_{\mathbb{H}},\ \lambda \in \mathbb{R} \tag{4.10}$$

which is isometric. The isomorphism f_λ clearly intertwines the actions of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$.

We also noted in the proof of Proposition 4.1 that the Kähler form on $\mathbb{H}$ for the Poincaré metric gives a $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-equivariant smooth isomorphism of $T^{1,0}_{\mathbb{H}}$ with $(T^{0,1}_{\mathbb{H}})^*$. Combining this with the isomorphism

f_λ in Eqn. (4.10) we get a $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-equivariant smooth isomorphism

$$f'_\lambda : \xi_\lambda \longrightarrow \xi_{\lambda-2} \otimes (T^{0,1}_{\mathbb{H}})^* . \tag{4.11}$$

Let

$$\Theta_\lambda : \mathcal{E}_\lambda \longrightarrow \mathcal{E}_{\lambda-2} \otimes (T^{0,1}_{\mathbb{H}})^*$$

be the homomorphism obtained by restricting the smooth homomorphism Θ constructed in Eqn. (4.9) (see Proposition 4.1). This homomorphism Θ_λ and the smooth isomorphism f'_λ in Eqn. (4.11) together give a $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-equivariant smooth homomorphism of vector bundles

$$\Theta'_\lambda : \mathcal{E}_\lambda := \xi_\lambda \otimes_{\mathbb{C}} \mathcal{W}_{E,\lambda} \longrightarrow \xi_\lambda \otimes_{\mathbb{C}} \mathcal{W}_{E,\lambda-2} \tag{4.12}$$

(see Eqn. (4.5)).

Lemma 4.2. *The homomorphism Θ'_λ in Eqn.* (4.12) *is of the form $\mathrm{Id}_{\xi_\lambda} \bigotimes T$, for some linear transformation $T \in \mathrm{Hom}(\mathcal{W}_{E,\lambda}, \mathcal{W}_{E,\lambda-2})$.*

Proof. Recall that the action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on $\mathcal{E}_\lambda$ was constructed using the trivial action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on the vector space $\mathcal{W}_{E,\lambda}$. Therefore, the lemma follows immediately from the fact that Θ'_λ is $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-equivariant. □

Thus, given a holomorphic Hermitian vector bundle (E, h) over $\mathbb{H}$ equipped with an action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ compatible with both holomorphic and Hermitian structures, we have constructed the following data:

(1) A direct sum of finite dimensional Hilbert spaces indexed by real numbers

$$H(E) := \bigoplus_{\lambda\in\mathbb{R}} \mathcal{W}_{E,\lambda} \tag{4.13}$$

such that $\mathcal{W}_{E,\lambda} = 0$ for all but finitely many real numbers, and

(2) a linear map

$$\Theta' : H(E) \longrightarrow H(E) \tag{4.14}$$

such that $\Theta'(\mathcal{W}_{E,\lambda}) \subset \mathcal{W}_{E,\lambda-2}$ for all λ.

By making Eqn. (4.13) an orthogonal direct sum we get an inner product on the vector space $H(E)$.

In the next section we will invert the above construction in the sense that starting from a data of the above type, we will construct a holomorphic Hermitian vector bundle over $\mathbb{H}$ equipped with a compatible action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$.

5. Higher rank equivariant bundles on $\mathbb{H}$

Suppose we are given the following data:

(1) An orthogonal direct sum of finite dimensional complex Hilbert spaces indexed by real numbers

$$H := \bigoplus_{\lambda\in\mathbb{R}} H_\lambda$$

such that $H_\lambda = 0$ for all but finitely many real numbers, and

(2) a linear map

$$\rho\,:\,H\,\longrightarrow\,H \tag{5.1}$$

such that $\rho(H_\lambda)\,\subset\,H_{\lambda-2}$ for all $\lambda\,\in\,\mathbb{R}$.

Let F denote the smooth vector bundle

$$F\,:=\,\bigoplus_{\lambda\in\mathbb{R}}\xi_\lambda\otimes_{\mathbb{C}}H_\lambda \tag{5.2}$$

(recall that H_λ is nonzero for only finitely many λ). The Hermitian structures on the line bundles ξ_λ and the inner product on the vector spaces H_λ together give a Hermitian structure on the vector bundle F defined in Eqn. (5.2). This Hermitian structure on F will be denoted by h_F.

Equip each H_λ with the trivial action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$. Now using the action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on the line bundles ξ_λ we get an action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on F that preserves the above defined Hermitian structure h_F.

The holomorphic structures on the line bundles ξ_λ together induce a holomorphic structure on F. This holomorphic structure on F is clearly preserved by the above action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$.

Using ρ (in Eqn. (5.1)) we will construct a new holomorphic structure on F.

Let $\overline{\partial}_F$ denote the Dolbeault operator on F giving the holomorphic structure of F obtained using the holomorphic structures on the line bundles ξ_λ.

For any $\lambda\,\in\,\mathbb{R}$, let $\rho(\lambda)\,:\,H_\lambda\,\longrightarrow\,H_{\lambda-2}$ be the restriction of the homomorphism ρ in Eqn. (5.1). Using the isomorphism f'_λ in Eqn. (4.11), this homomorphism $\rho(\lambda)$ gives a smooth homomorphism of vector bundles

$$S_\lambda\,=\,f'_\lambda\otimes\rho(\lambda)\,:\,\xi_\lambda\otimes_{\mathbb{C}}H_\lambda\,\longrightarrow\,(\xi_{\lambda-2}\otimes(T^{0,1}_{\mathbb{H}})^*)\otimes_{\mathbb{C}}H_{\lambda-2}\,=\,(\xi_{\lambda-2}\otimes_{\mathbb{C}}H_{\lambda-2})\otimes(T^{0,1}_{\mathbb{H}})^*\,. \tag{5.3}$$

It is easy to see that S_λ intertwines the actions of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$.

Let

$$\overline{\partial}'_F\,:=\,\overline{\partial}_F+\left(\bigoplus_{\lambda\in\mathbb{R}}S_\lambda\right) \tag{5.4}$$

be the Dolbeault operator on F, where $\overline{\partial}_F$ is the earlier defined Dolbeault operator on F, and S_λ is defined in Eqn. (5.3). Since both $\overline{\partial}_F$ and S_λ are preserved by the action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on F, the holomorphic structure defined by $\overline{\partial}'_F$ (see Eqn. (5.4)) is also preserved by the action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$.

Therefore, $((F\,,\overline{\partial}_F)\,,h_F)$ is a holomorphic Hermitian vector bundle over $\mathbb{H}$ equipped with an action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ which is compatible with both the holomorphic and the Hermitian structures. It is straightforward to check the following: If we substitute $((F\,,\overline{\partial}_F)\,,h_F)$ for $(E\,,h)$ in Section 4, then the data we get from it (see the end of Section 4) coincide with the above data $(H\,,\rho)$.

Conversely, if we perform the above construction using the data $(H(E)\,,\Theta')$ in Eqn. (4.13) and Eqn. (4.14), then we get back the holomorphic Hermitian vector bundle $(E\,,h)$ from which the data $(H(E)\,,\Theta')$ were constructed.

Therefore, we have proved the following theorem.

Theorem 5.1. *Once the isomorphisms f'_λ in Eqn. (4.11) are fixed, there is a natural bijective correspondence between the following two collections:*

(1) *holomorphic Hermitian vector bundle over* $\mathbb{H}$ *equipped with an action of* $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ *which is compatible with both the holomorphic and the Hermitian structures, and*
(2) *pairs of the form* $(H\,,\rho)$*, where*

$$H \,:=\, \bigoplus_{\lambda\in\mathbb{R}} H_\lambda$$

is an orthogonal direct sum of finite dimensional Hilbert spaces with $H_\lambda \,=\, 0$ *for all but finitely many real numbers, and*

$$\rho \,:\, H \,\longrightarrow\, H$$

is a linear map with $\rho(H_\lambda) \,\subset\, H_{\lambda-2}$ *for all* $\lambda \,\in\, \mathbb{R}$.

6. Classification of isomorphism classes

Let $(E_1\,,h_1)$ and $(E_2\,,h_2)$ be two holomorphic Hermitian vector bundles over $\mathbb{H}$ equipped with actions of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ which are compatible with both the holomorphic and the Hermitian structures. They will be called *equivalent* if there is a holomorphic isometry $E_1 \,\longrightarrow\, E_2$ that intertwines the actions of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$.

Our aim in this section is to investigate the conditions on the data of the type $(H\,,\rho)$ in Theorem 5.1 that translate into the above equivalence condition on equivariant holomorphic Hermitian vector bundles.

Take a pair $(H\,,\rho)$, where

$$H \,:=\, \bigoplus_{\lambda\in\mathbb{R}} H_\lambda$$

is an orthogonal direct sum of finite dimensional Hilbert spaces with $H_\lambda \,=\, 0$ for all but finitely many real numbers, and

$$\rho \,:\, H \,\longrightarrow\, H$$

is a linear map with $\rho(H_\lambda) \,\subset\, H_{\lambda-2}$ for all $\lambda \,\in\, \mathbb{R}$. Let

$$H' \,:=\, \bigoplus_{\lambda\in\mathbb{R}} H'_\lambda$$

be another finite dimensional Hilbert space of above type, and let

$$\rho' \,:\, H' \,\longrightarrow\, H'$$

be a linear map such that $\rho'(H'_\lambda) \,\subset\, H'_{\lambda-2}$ for all $\lambda \,\in\, \mathbb{R}$. Assume that there is a linear isometry

$$\psi \,:\, H \,\longrightarrow\, H' \tag{6.1}$$

such that $\psi(H_\lambda) \,=\, H'_\lambda$ for all $\lambda \,\in\, \mathbb{R}$, and furthermore,

$$\rho'\circ\psi \,=\, \psi\circ\rho\,.$$

Let $((F\,,\overline{\partial}_F)\,,h_F)$ (respectively, $((F'\,,\overline{\partial}_{F'})\,,h_{F'})$) be the vector bundle corresponding to $(H\,,\rho)$ (respectively, $(H'\,,\rho')$) given by Theorem 5.1. Therefore,

$$F \,:=\, \bigoplus_{\lambda\in\mathbb{R}} \xi_\lambda\otimes_{\mathbb{C}} H_\lambda\,,$$

and

$$F' \,:=\, \bigoplus_{\lambda\in\mathbb{R}} \xi_\lambda\otimes_{\mathbb{C}} H'_\lambda$$

(see eqn. (5.2)).

Consider the smooth isomorphism between vector bundles

$$\widetilde{\psi} := \bigoplus_{\lambda\in\mathbb{R}} (\mathrm{Id}_{\xi_\lambda}\otimes\psi_\lambda) \,:\, F \,=\, \bigoplus_{\lambda\in\mathbb{R}} \xi_\lambda\otimes_{\mathbb{C}} H_\lambda \,\longrightarrow\, \bigoplus_{\lambda\in\mathbb{R}} \xi_\lambda\otimes_{\mathbb{C}} H'_\lambda \,=\, F'\,,$$

where ψ_λ is the restriction to H_λ of the isomorphism ψ in Eqn. (6.1). It is straight-forward to check that this isomorphism $\widetilde{\psi}$ takes the holomorphic structure $\overline{\partial}_F$ to $\overline{\partial}_{F'}$, and it takes the Hermitian structure h_F to $h_{F'}$. Furthermore, $\widetilde{\psi}$ intertwines the actions of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$.

Therefore, $((F\,,\overline{\partial}_F)\,,h_F)$ and $((F'\,,\overline{\partial}_{F'})\,,h_{F'})$ are equivalent.

Conversely, let $(E\,,h)$ and $(E'\,,h')$ be two holomorphic Hermitian vector bundles over $\mathbb{H}$ equipped with actions of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ that are compatible with the holomorphic and Hermitian structures. Let

$$\delta\,:\,E\,\longrightarrow\,E' \tag{6.2}$$

be a holomorphic isometry intertwining the actions of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$. In other words, δ gives an equivalence.

For any $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-equivariant smooth homomorphism of vector bundles

$$\alpha\,:\,\xi_\lambda\,\longrightarrow\,E\,,$$

the composition $\delta\circ\alpha$ is a $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-equivariant smooth homomorphism from ξ_λ to E', where δ is the homomorphism in Eqn. (6.2). The map defined by $\alpha\,\longmapsto\,\delta\circ\alpha$ gives a linear isometry of the Hilbert space $\mathcal{W}_{E,\lambda}$ (defined in Eqn. (4.3)) with the Hilbert space $\mathcal{W}_{E',\lambda}$, where $\mathcal{W}_{E',\lambda}$ denotes the Hilbert space obtained by substituting $(E'\,,h')$ for $(E\,,h)$ in the construction of $\mathcal{W}_{E,\lambda}$; see Eqn. (4.3). Let

$$\psi_\lambda\,:\,\mathcal{W}_{E,\lambda}\,\longrightarrow\,\mathcal{W}_{E',\lambda} \tag{6.3}$$

be the isometry obtained this way.

Let

$$\rho\,:\,\bigoplus_{\lambda\in\mathbb{R}}\mathcal{W}_{E,\lambda}\,\longrightarrow\,\bigoplus_{\lambda\in\mathbb{R}}\mathcal{W}_{E,\lambda}$$

and

$$\rho'\,:\,\bigoplus_{\lambda\in\mathbb{R}}\mathcal{W}_{E',\lambda}\,\longrightarrow\,\bigoplus_{\lambda\in\mathbb{R}}\mathcal{W}_{E,\lambda}$$

be the homomorphisms constructed as in Eqn. (4.14) from $(E\,,h)$ and $(E'\,,h')$ respectively. The restriction of ρ (respectively, ρ') to $\mathcal{W}_{E,\lambda}$ (respectively, $\mathcal{W}_{E',\lambda}$) will be denoted by ρ_λ (respectively, ρ'_λ). Now, it is easy to check that ψ_λ constructed in Eqn. (6.3) has the property that

$$\rho'_\lambda\circ\psi_\lambda\,=\,\psi_{\lambda-2}\circ\rho_\lambda\,,$$

Therefore, we have proved the following:

Theorem 6.1. *Let $(E\,,h)$ and $(E'\,,h')$ be two holomorphic Hermitian vector bundles over $\mathbb{H}$ equipped with actions of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ which are compatible with both the holomorphic and the Hermitian structures. Let $(H=\bigoplus_{\lambda\in\mathbb{R}}H_\lambda\,,\rho)$ and $(H'=\bigoplus_{\lambda\in\mathbb{R}}H'_\lambda\,,\rho')$ be the data for $(E\,,h)$ and $(E'\,,h')$ respectively given by Theorem 5.1. Then the following two statements are equivalent:*

1. *There is a holomorphic isometry $E\,\longrightarrow\,E'$ intertwining the actions of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$.*
2. *There is a linear isometry $H\,\longrightarrow\,H'$ that takes H_λ to H'_λ for all λ, and it intertwines ρ and ρ'.*

Let $(E\,,h)$ be a holomorphic Hermitian vector bundle over $\mathbb{H}$ equipped with an action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ which is compatible with both the holomorphic and the Hermitian structures. We will say that $(E\,,h)$ is *decomposable* if there is a smooth nonzero subbundle $F\,\subset\,E$ such that

- $\mathrm{rank}(F)\,<\,\mathrm{rank}(E)$, and
- the Chern connection on E leaves F invariant.

These two conditions are equivalent to the condition that the orthogonal complement $F^\perp$ is nonzero with both F and $F^\perp$ being left invariant by the Chern connection on E. From the equivalence of statements prior to Proposition 2.3 it follows that the above two conditions are equivalent to the condition that both F and $F^\perp$ are holomorphic nonzero subbundles. If a smooth subbundle $F\,\subset\,E$ is left invariant by the Chern connection on E, then Proposition 2.3 says that F is also left invariant by the action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on E.

We will say that $(E\,,h)$ is *indecomposable* if it is not decomposable.

Theorem 5.1 associates to $(E\,,h)$ data of following type:

- A Hilbert space H and an orthogonal decomposition
$$H\,:=\,\bigoplus_{\lambda\in\mathbb{R}} H_\lambda$$
with $H_\lambda\,=\,0$ for all but finitely many real numbers, and
- a linear map
$$\rho\,:\,H\,\longrightarrow\,H$$
with $\rho(H_\lambda)\,\subset\,H_{\lambda-2}$ for all $\lambda\,\in\,\mathbb{R}$.

We will call $(H=\bigoplus_{\lambda\in\mathbb{R}} H_\lambda\,,\rho)$ to be *decomposable* if the following conditions hold:

(1) for each $\lambda\,\in\,\mathbb{R}$, there is a linear subspace $H'_\lambda\,\subset\,H_\lambda$ such that
$$0\,\neq\,\bigoplus_{\lambda\in\mathbb{R}} H'_\lambda\,=:\,H'\,\neq\,H$$
and
(2) the subspaces $H'\,\subset\,H$ and its orthogonal complement $H'^\perp$ are both preserved by the homomorphism ρ.

We will say that $(H=\bigoplus_{\lambda\in\mathbb{R}} H_\lambda\,,\rho)$ is *indecomposable* if it is not decomposable.

From the construction of the bijective correspondence in Theorem 5.1, we have the following corollary:

Corollary 6.2. *Let $(E\,,h)$ be a holomorphic Hermitian vector bundle over $\mathbb{H}$ equipped with an action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ which is compatible with both the holomorphic and the Hermitian structures. Let $(H=\bigoplus_{\lambda\in\mathbb{R}} H_\lambda\,,\rho)$ be the corresponding data given by Theorem 5.1. Then the following two statements are equivalent:*

(1) *The holomorphic Hermitian vector bundle $(E\,,h)$ is indecomposable.*
(2) *The data $(H=\bigoplus_{\lambda\in\mathbb{R}} H_\lambda\,,\rho)$ is indecomposable.*

7. Some numerical invariants

Let $(E\,,h)$ be a holomorphic Hermitian vector bundle over $\mathbb{H}$ equipped with an action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ which is compatible with both the holomorphic and the Hermitian structures. Let ∇ denote the Chern connection on E. The curvature of ∇ will be denoted by $\mathcal{K}(\nabla)$. Therefore, $\mathcal{K}(\nabla)$ is a smooth $(1\,,1)$-form on $\mathbb{H}$, with values in $\mathrm{End}(E)$, which is left invariant by the action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$.

Let ω denote the Kähler form on $\mathbb{H}$ given by the Poincaré metric. So, we have

$$\mathcal{K}(\nabla) \,=\, K_E\otimes\omega\,, \tag{7.1}$$

where K_E is a $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-invariant smooth section of $\mathrm{End}(E)$. For any point $z\,\in\,\mathbb{H}$, consider the eigenvalues (along with multiplicities) of $K_E(z)\,\in\,\mathrm{End}(E_z)$. Since K_E is $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-invariant, these eigenvalues are independent of the choice of the point z. Therefore, they can be considered as numerical invariants for $(E\,,h)$.

We will describe $\mathcal{K}(\nabla)$ in terms of the data for $(E\,,h)$ given by Theorem 5.1.

Let $\nabla_{\mathcal{E}}$ denote the Chern connection of the holomorphic Hermitian vector bundle $(\mathcal{E}\,,\mathcal{H})$ constructed in Eqn. (4.6) from $(E\,,h)$. Therefore, $\nabla^{0,1}_{\mathcal{E}}$ is the Dolbeault operator $\overline{\partial}_{\mathcal{E}}$ on $\mathcal{E}$ (see Eqn. (4.9)).

Let

$$\widetilde{\nabla}^{1,0} \,:=\, (\gamma_E^{-1}\otimes \mathrm{Id}_{\Omega^{1,0}_{\mathbb{H}}})\circ\nabla^{1,0}\circ\gamma_E\,:\,\mathcal{E}\,\longrightarrow\,\mathcal{E}\otimes\Omega^{1,0}_{\mathbb{H}} \tag{7.2}$$

be the differential operator, where $\nabla^{1,0}$ is the $(1\,,0)$-component of the Chern connection ∇ on $(E\,,h)$, and γ_E is the smooth isomorphism constructed in Eqn. (4.7). We know that

$$\overline{\partial}^{\phi}_E \,:=\, (\gamma_E^{-1}\otimes \mathrm{Id}_{\Omega^{0,1}_{\mathbb{H}}})\circ\overline{\partial}_E\circ\gamma_E \,=\, \overline{\partial}_{\mathcal{E}}+\Theta \tag{7.3}$$

(see Eqn. (4.8) and Eqn. (4.9)). Since the homomorphism γ_E is an isometry, the connection $\gamma^*E_{\nabla}\,=\,\widetilde{\nabla}^{1,0}+\overline{\partial}^{\phi}_E$ on $\mathcal{E}$ is preserves its Hermitian structure $\mathcal{H}$. Now from Eqn. (7.3), and the fact the connection $\nabla_{\mathcal{E}}$ is Hermitian, it follows that

$$\widetilde{\nabla}^{1,0} \,=\, \nabla^{1,0}_{\mathcal{E}}-\Theta^*\,,$$

where $\nabla^{1,0}_{\mathcal{E}}$ is the $(1\,,0)$-component of the Chern connection $\nabla_{\mathcal{E}}$ on $\mathcal{E}$.

In other words, we have

$$\gamma_E^*\nabla \,=\, \nabla_{\mathcal{E}}-\Theta^*+\Theta\,.$$

Therefore, the curvature $\mathcal{K}(\gamma_E^*\nabla)$ of $\gamma_E^*\nabla$ has the following expression:

$$\mathcal{K}(\gamma_E^*\nabla) \,=\, \gamma_E^*\mathcal{K}(\nabla) \,=\, \mathcal{K}(\nabla_{\mathcal{E}})-\Theta^*\wedge\Theta-\Theta\wedge\Theta^*\,, \tag{7.4}$$

where $\mathcal{K}(\nabla_{\mathcal{E}})$ is the curvature of the connection $\nabla_{\mathcal{E}}$.

The curvature $\mathcal{K}(\nabla_{\mathcal{E}})$ has the following expression:

$$\mathcal{K}(\nabla_{\mathcal{E}}) \,=\, -\bigoplus_{\lambda\in\mathbb{R}}\lambda\cdot\mathrm{Id}_{\mathcal{E}_\lambda}\cdot\omega\,,$$

where ω is the Kähler form on $\mathbb{H}$ for the Poincaré metric (here the decomposition of $\mathcal{E}$ given in Eqn. (4.6) is used). Indeed, this follows from the fact that the decomposition in Eqn. (4.6) is an orthogonal decomposition into holomorphic subbundles.

Let $(H = \bigoplus_{\lambda\in\mathbb{R}} H_\lambda \,, \rho)$ be the data for $(E\,,h)$ given by Theorem 5.1. Using Eqn. (7.4) and the above expression for $\mathcal{K}(\nabla_{\mathcal{E}})$ it follows that the $\gamma_E^*\mathcal{K}(\nabla)$ corresponds to the endomorphism

$$T \,=\, -\bigoplus_{\lambda\in\mathbb{R}}((\rho_\lambda)^*\rho_\lambda - \rho_{\lambda+2}(\rho_{\lambda+2})^* + \lambda\cdot \mathrm{Id}_{H_\lambda})$$

of H, where ρ_λ is the restriction of ρ to H_λ.

It may be noted that the eigenvalues of the operator

$$\widetilde{\rho} := \bigoplus_{\lambda\in\mathbb{R}}(\rho_\lambda)^*\rho_\lambda$$

on H are also numerical invariants of $(E\,,\rho)$.

We will now give an example to show that these numerical invariants and the numerical invariants given by the eigenvalues of K_E (constructed in Eqn. (7.1)) do not determine $(E\,,h)$. Let a, b and x be any three real numbers. Consider

$$\rho = \left(\begin{array}{cc|cc|cc} 0 & 0 & a & 0 & 0 & 0\\ 0 & 0 & 0 & b & 0 & 0\\ \hline 0 & 0 & 0 & 0 & x & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ \hline 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \text{ and } \rho' = \left(\begin{array}{cc|cc|cc} 0 & 0 & a & 0 & 0 & 0\\ 0 & 0 & 0 & b & 0 & 0\\ \hline 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & x & 0\\ \hline 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Here $H = H_0\bigoplus H_2\bigoplus H_4$ with $H_0 = H_2 = H_4 = \mathbb{C}^2$ equipped with the standard metric. Since $\rho^3 = 0$ and ${\rho'}^3 \neq 0$, it follows that ρ cannot be unitarily equivalent to ρ'. Furthermore, $\rho\rho^*$ as well as $\rho^*\rho$ are diagonal with eigenvalues $a^2, b^2, x^2, 0$. Similarly, $\rho'{\rho'}^*$ and ${\rho'}^*\rho'$ are diagonal with the same set of eigenvalues $a^2, b^2, x^2, 0$. The eigenvalues of $\rho\rho^* - \rho^*\rho$ are $a^2, b^2, x^2 - a^2, -b^2, -x^2, 0$ while those of $\rho'{\rho'}^* - {\rho'}^*\rho'$ are $a^2, b^2, -a^2, x^2 - b^2, -x^2, 0$. These two sets of eigenvalues are equal for $a = b$.

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