On Grothendieck constants

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For any $n \times n$ matrix $A = (a_{ij})$, define

$$\|A\|_{\infty,1} \stackrel{\text{def}}{=} \sup\left\{\sum_{i=1}^{n}\left|\sum_{j=1}^{n} a_{ij}\,\alpha_j\right| : |\alpha_j| \leq 1,\ \alpha_j \text{ scalars.}\right\},$$

and

$$\gamma(A) \stackrel{\text{def}}{=} \sup\left\{\sum_{i=1}^{n}\left\|\sum_{j=1}^{n} a_{i,j}x_j\right\| : \|x_j\| \leq 1,\ x_j \text{ in } \ell^2(n)\right\}.$$

Define the numerical constant

$$K_G(n) \stackrel{\text{def}}{=} \sup\{\gamma(A) : A = A_{n\times n},\ \ \|A\|_{\infty,1} \leq 1\}.$$

The constant $K_G(n)$ depends on the ground field. We shall use $K_G^C(n)$ and $K_G^R(n)$ to distinguish the complex and the real case. The complex constants arise naturally in the study of certain Hilbert modules, see [2].

The fact that $K_G(n)$ remains finite as $n \to \infty$ was established by Grothendieck (cf. [6, Corollary 5.7]). The limit of this sequence is denoted by K_G, and is called the Grothendieck constant. Its exact value is not known.

It should be clear that the value of $K_G(n)$ does not change if in its definition we replace $\ell^2(n)$ by ℓ^2. While it would be quite natural to call this constant the n - dimensional Grothendieck constant, in the literature some other constant $\kappa_G(n)$ has received this name. The definition of $\kappa_G(n)$ is obtained from that of $K_G(n)$ by allowing the size of the matrix A in the supremum to be arbitrary while holding the dimension of the Hilbert space fixed at n. Clearly we have $\kappa_G(n) \geq K_G(n)$, and both approach the Grothendieck constant in the limit.

The constant $\kappa_G(n)$ has been studied in the literature,(see [3] and [4]) though its exact value is not known for any $n > 1$ except for the isolated result $\kappa_G^R(2) = \sqrt{2}$ in [4] . On the other hand $K_G(n)$ has not received any attention so far. In this paper, we intend to show that the constants $K_G(n)$ have their own advantages and surprises. One advantage is that extreme point methods are applicable in their study. The major surprise is that, contrary to what one might expect, the sequence $K_G(n)$ is not strictly increasing with n either in the real or in the complex case. In fact, we show that $K_G^R(2) = K_G^R(3) = \sqrt{2}$ and $K_G^C(1) = K_G^C(2) = 1$. Another surprise is the equality $K_G^R(2) = \kappa_G^R(2)$ as follows from [4].

Remark : Let's call two $n \times n$ matrices A and B equivalent if there are diagonal matrices D_1, D_2 with unimodular diagonal entries, and permutation matrices P_1, P_2 such that $B = D_1 P_1 A P_2 D_2$. Obviously, in this case we have $\|A\|_{\infty,1} = \|B|_{\infty,1}$ and $\gamma(A) = \gamma(B)$. Therefore, in the supremum defining $K_G(n)$, it is enough to run over representatives of equivalence classes rather than over all A.

Again, since the expression within braces in the defintion of $\|A\|_{\infty,1}$, (resp.ly, of $\gamma(A)$) is convex in each α_j (resp.ly in each x_j), we may take this supremum over $|\alpha_j| = 1$ (resp.ly, over $\|x_j\| = 1$) without changing its value.

Theorem 1 $K_G^C(2) = 1$.

The following proof is inspired by a lemma of Arias et al in [1].

Proof : Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be any 2×2 complex matrix. We have to show that $\gamma(A) = \|A\|_{\infty,1}$. By the remark, we may replace A by an equivalent matrix in which any three prescribed entries are non-negative. Thus, we may assume without loss of generality that a, b, c are all non-negative. Let $\bar{\mathbb{D}}$ and $\mathbb{T}$ be the closed unit disc in the complex plane and its boundary, respectively. Let's put $d = e\alpha$, $\alpha \in \mathbb{T}$, $e \geq 0$.

Consider the function $\phi : \bar{\mathbb{D}} \to \mathbb{R}_+$, defined by

$$\phi(z) = (a^2 + b^2 + 2ab\mathrm{Re}(z))^{1/2} + (c^2 + e^2 + 2ce\mathrm{Re}(\alpha z))^{1/2}.$$

Note that for $\alpha_1,\ \alpha_2 \in \mathbb{T}$, we have, $|a\alpha_1 + b\alpha_2| + |c\alpha_1 + d\alpha_2| = \phi(z)$ where $z = \overline{\alpha_1}\alpha_2 \in \mathbb{T}$. Also, for unit vectors $x_1,\ x_2 \in \ell^2(2)$, we have $\|ax_1 + bx_2\| + \|cx_1 + dx_2\| = \phi(z)$, with $z = \langle x_2, x_1 \rangle \in \bar{\mathbb{D}}$. Therefore, in view of the remark above, we have $\|A\|_{\infty,1} = \sup\{\phi(z) : z \in \mathbb{T}\}$, and, $\gamma(A) = \sup\{\phi(z) : z \in \bar{\mathbb{D}}\}$. Hence, we have to show that ϕ assumes its maximum on the boundary $\mathbb{T}$. If any of the entries of A is zero, then this is trivial. So, we may assume $a,\ b,\ c,\ e > 0$.

If $\alpha = \pm 1$, ϕ depends on z only through $\mathrm{Re}(z)$, so that it attains its maximum on the boundary. On the other hand, we claim that if $\alpha \neq -1$, then the function ϕ has no local maximum in the interior of $\bar{\mathbb{D}}$. For z in the interior, we can find a positive constant ϵ such that $z + \epsilon\beta$ is in $\mathbb{D}$ for $\beta \in \mathbb{T}$. Given $\alpha \in \mathbb{T}, \alpha \neq -1$, there exists a $\beta \in \mathbb{T}$ such that both $\mathrm{Re}(\beta)$ and $\mathrm{Re}(\alpha\beta)$ are strictly positive. Then, if z is replaced by $z + \epsilon\beta$, both terms in the definition of ϕ increase strictly, so that $\phi(z + \epsilon\beta) > \phi(z)$ and the claim is verified. □

Now we go over to the real case.

Remark : Note that $K_G(n)$ is the maximum of γ on the closed unit ball $\mathbb{B}_n$ of the space $\mathcal{L}(\ell^\infty(n),\ \ell^1(n))$, of operators from $\ell^\infty(n)$ to $\ell^1(n)$, and γ is convex on this ball. Therefore, $K_G(n)$ is the maximum of $\gamma(A)$ as A runs over the set of the extreme points of $\mathbb{B}_n$. Although this is true in the complex case as well, this observation is particularly useful in the real case since here $\mathbb{B}_n$ is a polyhedron with only finitely many extreme points which, at least in principle, can be determined algorithmically for any fixed n. On the other hand, in the complex case we don't know the extreme points of $\mathbb{B}_n$ even for $n = 2$.

Theorem 2 $K_G^R(2) = \sqrt{2}$

Proof: Since $\ell^1(2)$ and $\ell^\infty(2)$ are isometric via the map $U:\ (x_1,x_2)\mapsto(x_1+x_2,x_1-x_2)$ the space $\mathcal{L}(\ell^\infty(2),\ell^1(2))$ is isometric to $\mathcal{L}(\ell^1(2),\ell^\infty(2))$, which in turn is isometric to $\ell^\infty(4)$. Thus, the extreme points in the unit ball $\mathbb{B}_n$ are obtained from the extreme points of the unit ball in $\mathcal{L}(\ell^1(2),\ \ell^\infty(2))$ by applying the isometry $B\mapsto U^{-1}BU^{-1}$. From the $2^4=16$ extreme points of the latter ball one thus obtains as many extreme points of $\mathbb{B}_n$. It turns out that 8 of them are equivalent to $A_1=\begin{pmatrix}1&0\\0&0\end{pmatrix}$ and 8 of them are equivalent to $A_2=\begin{pmatrix}1/2&1/2\\1/2&-1/2\end{pmatrix}$. Thus, $K_G^R(2)$ is the maximum of $\gamma(A_1)$ and $\gamma(A_2)$. A trivial computation gives $\gamma(A_1)=1\ \&\ \gamma(A_2)=\sqrt{2}$. $\Box$

Theorem 3 $K_G^R(3)=\sqrt{2}$.

Proof : 90 extreme points of $\mathbb{B}_3$ are now clearly visible : $3\times3\times2=18$ extreme points equivalent to $B_1=A_1\oplus0$, and $3\times3\times4\times2=72$ extreme points equivalent to $B_2=A_2\oplus0$, where $A_1,\ \ A_2$ are as in the proof of Theorem 2. But in Corollary 6 of [5], Lima proves that $\mathbb{B}_3$ has exactly 90 extreme points. Hence, upto equivalence, $B_1\ \ \&\ \ B_2$ are the only extreme points of $\mathbb{B}_3$. But, clearly, $\gamma(B_i)=\gamma(A_i),\ i=1,2$. $\Box$

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