ON THE IRREDUCIBILITY OF A CLASS OF HOMOGENEOUS OPERATORS

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Abstract. In this paper we construct a class of homogeneous Hilbert modules over the disc algebra $\mathcal{A}(\mathbb{D})$ as quotients of certain natural modules over the function algebra $\mathcal{A}(\mathbb{D}^2)$. These quotient modules are described using the jet construction for Hilbert modules. We show that the quotient modules obtained this way, belong to the class $\mathrm{B}_k(\mathbb{D})$ and that they are mutually inequivalent, irreducible and homogeneous.

1. Introduction

Let $\mathcal{M}$ be a Hilbert space. All Hilbert spaces in this paper will be assumed to be complex and separable. Let $\mathcal{A}(\Omega)$ be the natural function algebra consisting of functions holomorphic in a neighborhood of the closure $\bar{\Omega}$ of some open, connected and bounded subset Ω of $\mathbb{C}^m$. The Hilbert space $\mathcal{M}$ is said to be a *Hilbert module* over $\mathcal{A}(\Omega)$ if $\mathcal{M}$ is a module over $\mathcal{A}(\Omega)$ and

$$\|f \cdot h\|_{\mathcal{M}} \leq C \|f\|_{\mathcal{A}(\Omega)} \|h\|_{\mathcal{M}} \text{ for } f \in \mathcal{A}(\Omega) \text{ and } h \in \mathcal{M},$$

for some positive constant C independent of f and h. It is said to be *contractive* if we also have $C \leq 1$.

Fix an inner product on the algebraic tensor product $\mathcal{A}(\Omega) \otimes \mathbb{C}^n$. Let the completion of $\mathcal{A}(\Omega) \otimes \mathbb{C}^n$ with respect to this inner product be the Hilbert space $\mathcal{M}$. Assume that the module action

$$\mathcal{A}(\Omega) \times \mathcal{A}(\Omega) \otimes \mathbb{C}^n \to \mathcal{A}(\Omega) \otimes \mathbb{C}^n$$

extends continuously to $\mathcal{A}(\Omega) \times \mathcal{M} \to \mathcal{M}$. With very little additional assumption on $\mathcal{M}$, we obtain a *quasi-free* Hilbert module (cf. [13]).

The simplest family of modules over $\mathcal{A}(\Omega)$ corresponds to evaluation at a point in the closure of Ω. For $\underline{z}$ in the closure of Ω, we make the one-dimensional Hilbert space $\mathbb{C}$ into the Hilbert module $\mathbb{C}_{\underline{z}}$, by setting $\varphi v = \varphi(z) v$ for $\varphi \in \mathcal{A}(\Omega)$ and $v \in \mathbb{C}$. Classical examples of contractive Hilbert modules are the Hardy and Bergman modules over the algebra $\mathcal{A}(\Omega)$.

Let G be a locally compact second countable group acting transitively on Ω. Let us say that the module $\mathcal{M}$ over the algebra $\mathcal{A}(\Omega)$ is homogeneous if $\varrho(f \circ \varphi)$ is unitarily equivalent to $\varrho(f)$ for all $\varphi \in G$. Here $\varrho : \mathcal{A}(\Omega) \to \mathcal{B}(\mathcal{M})$ is the homomorphism of the algebra $\mathcal{A}(\Omega)$ defined by $\varrho(f)h := f \cdot h$ for $f \in \mathcal{A}(\Omega)$ and $h \in \mathcal{M}$. It was shown in [19] that if the module $\mathcal{M}$ is irreducible and homogeneous then there exists a *projective unitary representation* $U : G \to \mathcal{U}(\mathcal{M})$ such that

$$U_\varphi^* \varrho(f) U_\varphi = \varrho(f \cdot \varphi),\ f \in \mathcal{A}(\Omega),\ \varphi \in G,$$

where $(f \cdot \varphi)(w) = f(\varphi \cdot w)$ for $w \in \Omega$.

A $*$ - homomorphism ϱ of a C^* - algebra $\mathcal{C}$ and a unitary group representation U of G on the Hilbert space $\mathcal{M}$ satisfying the condition as above were first studied by Mackey and were

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called *Systems of Imprimitivity.* Mackey proved the Imprimitivity theorem which sets up a correspondence between induced representations of the group G and the Systems of Imprimitivity. The notion of homogeneity is obtained by compressing the systems of imprimitivities, in the sense of Mackey, to a subspace $\mathcal{N}$ of $\mathcal{M}$ and then restricting to a subalgebra of the C^* - algebra $\mathcal{C}$ (cf.[3]). However, it is not clear if the notion of homogeneity is in some correspondence with holomorphically induced representations, at least when the module $\mathcal{M}$ is assumed to be in $\mathrm{B}_k(\Omega)$.

An alternative description, in the particular case of the disc may be useful. The group of bi-holomorphic automorphisms Möb of the unit disc is $\{\varphi_{\theta,\alpha} : \theta \in [0, 2\pi) \text{ and } \alpha \in \mathbb{D}\}$, where

$$\varphi_{\theta,\alpha}(z) = e^{i\theta}\frac{z-\alpha}{1-\bar{\alpha}z},\ z \in \mathbb{D}. \tag{1.1}$$

As a topological group (with the topology of locally uniform convergence) it is isomorphic to $\mathrm{PSU}(1, 1)$ and to $\mathrm{PSL}(2, \mathbb{R})$.

An operator T from a Hilbert space into itself is said to be *homogeneous* if $\varphi(T)$ is unitarily equivalent to T for all φ in Möb which are analytic on the spectrum of T. The spectrum of a homogeneous operator T is either the unit circle $\mathbb{T}$ or the closed unit disc $\bar{\mathbb{D}}$, so that, actually, $\varphi(T)$ is unitarily equivalent to T for all φ in Möb. We say that a projective unitary representation U of Möb is *associated* with an operator T if

$$\varphi(T) = U_\varphi^* T U_\varphi$$

for all φ in Möb. We have already pointed out that if T is irreducible then it has an associated representation U. It is not hard to see that U is uniquely determined upto unitary equivalence.

Many examples (unitarily inequivalent) of homogeneous operators are known [6]. Since the direct sum (more generally direct integral) of two homogeneous operators is again homogeneous, a natural problem is the classification (up to unitary equivalence) of *atomic homogeneous operators*, that is, those homogeneous operators which can not be written as the direct sum of two homogeneous operators. In this generality, this problem remains unsolved. However, the irreducible homogeneous operators in the Cowen-Douglas class $\mathrm{B}_1(\mathbb{D})$ and $\mathrm{B}_2(\mathbb{D})$ have been classified (cf. [18] and [22]) and all the scalar shifts (not only the irreducible ones) which are homogeneous are known [7, List 4.1, page 312]. Some recent results on classification of homogeneous bundles are in [8] and [15].

Clearly, irreducible homogeneous operators are atomic. Therefore, it is important to understand when a homogeneous operator is irreducible.

There are only two examples of atomic homogeneous operators known which are not irreducible. these are the multiplication operators – by the respective co-ordinate functions – on the Hilbert spaces $L^2(\mathbb{T})$ and $L^2(\mathbb{D})$. Both of these examples happen to be normal operators. We do not know if all atomic homogeneous operators possess an associated projective unitary representation. However, to every homogeneous operator in $\mathrm{B}_k(\mathbb{D})$, there exist an associated representation of the universal covering group of Möb [15, Theorem 4].

It turns out an irreducible homogeneous operator in $\mathrm{B}_2(\mathbb{D})$ is the compression of the tensor product of two homogeneous operators from $\mathrm{B}_1(\mathbb{D})$ (cf. [6]) to a suitable invariant subspace. In the language of Hilbert modules, this is the statement that every homogeneous module in $\mathrm{B}_2(\mathbb{D})$ is obtained as quotient of the tensor product of two homogeneous modules in $\mathrm{B}_1(\mathbb{D})$ by the sub-module of functions vanishing to order 2 on $\triangle \subseteq \mathbb{D}^2$. However, beyond the case of rank 2, the situation is more complicated. The question of classifying homogeneous operators in the class $\mathrm{B}_k(\mathbb{D})$ amounts to classifying holomorphic and Hermitian vector bundles of rank k on the unit disc which are homogeneous. Classification problems such as this one are well known in the representation theory of locally compact second countable groups. However, in that context, there is no Hermitian structure present which makes the classification problem entirely algebraic. A complete classification of homogeneous operators in $\mathrm{B}_k(\mathbb{D})$ may still be possible using techniques from the theory of unitary representations of the Möbius group. Leaving aside, the classification problem of the homogeneous operators in $\mathrm{B}_k(\mathbb{D})$, we show that the "generalized Wilkins examples" (cf. [6]) are irreducible.

If one considers a bounded symmetric domain in $\mathbb{C}^m$, the classification question probably is even more complicated (cf. [4], [1]). Here part of the difficulty lies in the fact that no classification of the irreducible unitary representations of the group $\mathrm{Aut}(\Omega)$, the bi-holomorphic automorphism group of Ω, is known.

In the following section, we discuss reproducing kernels for a functional Hilbert space on a domain $\Omega \subseteq \mathbb{C}^m$ and the m-tuple of multiplication operators $\boldsymbol{M}$ of multiplication by coordinate functions. Although, our applications to the question of irreducibility is only for the multiplication operator $\boldsymbol{M}$ on a functional Hilbert space based on the unit disc $\mathbb{D}$, the more general discussion of this section is not any simpler in the one variable case.

In subsection 2.2, we explain the realization of a m-tuple of operators $\boldsymbol{T}$ in the class $\mathrm{B}_k(\Omega)$ as the adjoint of a m-tuple of multiplication operators $\boldsymbol{M}$ on a Hilbert space of holomorphic functions, on the bounded connected open set $\Omega^* := \{w \in \mathbb{C}^m : \bar{w} \in \Omega\}$, possessing a reproducing kernel K. We point out, as in [11], that the normalized kernel $\tilde{\tilde{K}}$ obtained from the kernel K by requiring that $\tilde{\tilde{K}}(z,0) = 1$, for all $z \in \Omega$, determines the uniatry equivalence class of the m-tuple $\boldsymbol{T}$. We then obtain a criterion for the irreducibility of the m-tuple $\boldsymbol{T}$ in terms of the normalized kernel $\tilde{\tilde{K}}$. Roughly speaking, this says that the m-tuple of operators is irreducible if and only if the coefficients, in the pwer series expansion of $\tilde{\tilde{K}}$, are simultaneously irreducible. Following, [12] and [14], we describe the jet construction for Hilbert modules and discuss some examples.

In section 3, we show that if $\mathcal{H}$ is a Hilbert space of holomorphic functions, on a bounded connected open set Ω and possesses a reproducing kernel K then it admits a natural multiplier representation of the automorphism group of Ω if K is *quasi-invariant*. We show that if K is quasi-inavariant, then the corresponding multiplier representation intertwines $\boldsymbol{M}$ and $\varphi(\boldsymbol{M})$, that is, the m-tuple of multiplication operators $\boldsymbol{M}$ is *homogeneous*.

Our main results on irreducibility of certain class of homogeneous operators is in Section 4. The kernel $B^{(\alpha,\beta)}(z,w) = (1 - z_1\bar{w}_1)^{-\alpha}(1 - z_2\bar{w}_2)^{-\beta}$, $z = (z_1, z_2)$, $w = (w_1, w_2) \in \mathbb{D}^2$, determines a Hilbert module over the function algebra $\mathcal{A}(\mathbb{D}^2)$. We recall the computation of a matrix valued kernel on the unit disc $\mathbb{D}$ using the jet construction for this Hilbert module which consists of holomorphic functions on the unit disc $\mathbb{D}$ taking values in $\mathbb{C}^n$. The multiplication operator on this Hilbert space is then shown to be irreducible by checking that all of the coefficients of the "normalized" matrix valued kernel, obtained from the jet construction, cannot be simultaneously reducible.

In section 5, we show that the kernel obtained from the jet construction is quais-invariant and consequently, the corresponding multiplication operator is homogeneous. This proof involves the verification of a cocycle identity, which in turn, depends on a beutiful identity involving binomial coefficients.

Finally, in section 6, we discuss some examples arising from the jet construction applied to a certain natural family of Hilbert modules over the algebra $\mathcal{A}(\mathbb{D}^3)$. A more systematic study of such examples is to be found in [20].

2. Reproducing Kernels and the Cowen-Douglas class

2.1. Reproducing kernel. Let $\mathcal{L}(\mathbb{F})$ be the Banach space of all linear transformations on a Hilbert space $\mathbb{F}$ of dimension n for some $n \in \mathbb{N}$. Let $\Omega \subset \mathbb{C}^m$ be a bounded open connected set. A function $K : \Omega \times \Omega \to \mathcal{L}(\mathbb{F})$, satisfying

$$\sum_{i,j=1}^{p} \langle K(w^{(i)}, w^{(j)})\zeta_j, \zeta_i\rangle_{\mathbb{F}} \;\geq\; 0,\;\; w^{(1)}, \ldots, w^{(p)} \in \Omega,\;\; \zeta_1, \ldots, \zeta_p \in \mathbb{F},\;\; p > 0 \tag{2.2}$$

is said to be a *non negative definite (nnd) kernel* on Ω. Given such an nnd kernel K on Ω, it is easy to construct a Hilbert space $\mathcal{H}$ of functions on Ω taking values in $\mathbb{F}$ with the property

$$\langle f(w), \zeta\rangle_{\mathbb{F}} = \langle f, K(\cdot, w)\zeta\rangle, \text{ for } w \in \Omega,\; \zeta \in \mathbb{F}, \text{ and } f \in \mathcal{H}. \tag{2.3}$$

The Hilbert space $\mathcal{H}$ is simply the completion of the linear span of all vectors of the form $\mathcal{S} = \{K(\cdot, w)\zeta,\; w \in \Omega,\; \zeta \in \mathbb{F}\}$, where the inner product between two of the vectors from $\mathcal{S}$ is defined by

$$\langle K(\cdot, w)\zeta, K(\cdot, w')\eta\rangle = \langle K(w', w)\zeta, \eta\rangle, \text{ for } \zeta, \eta \in \mathbb{F}, \text{ and } w, w' \in \Omega, \tag{2.4}$$

which is then extended to the linear span $\mathcal{H}^\circ$ of the set $\mathcal{S}$. This ensures the reproducing property (2.3) of K on $\mathcal{H}^\circ$.

Remark 2.1. *We point out that although the kernel K is required to be merely* nnd*, the equation (2.4) defines a positive definite sesqui-linear form. To see this, simply note that $|\langle f(w), \zeta\rangle| = |\langle f, K(\cdot, w)\zeta\rangle|$ which is at most $\|f\|\langle K(w, w)\zeta, \zeta\rangle^{1/2}$ by the Cauchy - Schwarz inequality. It follows that if $\|f\|^2 = 0$ then $f = 0$.*

Conversely, let $\mathcal{H}$ be any Hilbert space of functions on Ω taking values in $\mathbb{F}$. Let $e_w : \mathcal{H} \to \mathbb{F}$ be the evaluation functional defined by $e_w(f) = f(w)$, $w \in \Omega$, $f \in \mathcal{H}$. If e_w is bounded for each $w \in \Omega$ then it admits a bounded adjoint $e_w^* : \mathbb{F} \to \mathcal{H}$ such that $\langle e_w f, \zeta\rangle = \langle f, e_w^*\zeta\rangle$ for all $f \in \mathcal{H}$ and $\zeta \in \mathbb{F}$. A function f in $\mathcal{H}$ is then orthogonal to $e_w^*(\mathcal{H})$ if and only if $f = 0$. Thus $f = \sum_{i=1}^{p} e_{w^{(i)}}^*(\zeta_i)$ with $w^{(1)}, \ldots, w^{(p)} \in \Omega$, $\zeta_1, \ldots, \zeta_p \in \mathbb{F}$, and $p > 0$, form a dense set in $\mathcal{H}$. Therefore we have

$$\|f\|^2 = \sum_{i,j=1}^{p} \langle e_{w^{(i)}} e_{w^{(j)}}^* \zeta_j, \zeta_i\rangle,$$

where $f = \sum_{i=1}^{n} e_{w^{(i)}}^*(\zeta_i)$, $w^{(i)} \in \Omega$, $\zeta_i \in \mathcal{F}$. Since $\|f\|^2 \geq 0$, it follows that the kernel $K(z, w) = e_z e_w^*$ is non-negative definite as in (2.2). It is clear that $K(z, w)\zeta \in \mathcal{H}$ for each $w \in \Omega$ and $\zeta \in \mathbb{F}$, and that it has the reproducing property (2.3).

Remark 2.2. *If we assume that the evaluation functional e_w is surjective then the adjoint e_w^* is injective and it follows that $\langle K(w, w)\zeta, \zeta\rangle > 0$ for all non-zero vectors $\zeta \in \mathbb{F}$.*

There is a useful alternative description of the reproducing kernel K in terms of the orthonormal basis $\{e_k : k \geq 0\}$ of the Hilbert space $\mathcal{H}$. We think of the vector $e_k(w) \in \mathbb{F}$ as a column vector for a fixed $w \in \Omega$ and let $e_k(w)^*$ be the row vector $(\overline{e_k^1(w)}, \ldots, \overline{e_k^n(w)})$. We see that

$$\begin{aligned}
\langle K(z, w)\zeta, \eta\rangle &= \langle K(\cdot, w)\zeta, K(\cdot, z)\eta\rangle \\
&= \sum_{k=0}^{\infty} \langle K(\cdot, w)\zeta, e_k\rangle\langle e_k, K(\cdot, z)\eta\rangle \\
&= \sum_{k=0}^{\infty} \overline{\langle e_k(w), \zeta\rangle}\langle e_k(z), \eta\rangle \\
&= \sum_{k=0}^{\infty} \langle e_k(z)e_k(w)^*\zeta, \eta\rangle,
\end{aligned}$$

for any pair of vectors $\zeta, \eta \in \mathbb{F}$. Therefore, we have the following very useful representation for the reproducing kernel K:

$$K(z,w) = \sum_{k=0}^{\infty} e_k(z)e_k(w)^*, \tag{2.5}$$

where $\{e_k : k \geq 0\}$ is any orthonormal basis in $\mathcal{H}$.

2.2. The Cowen-Douglas class. Let $\boldsymbol{T} = (T_1, \ldots, T_m)$ be a d-tuple of commuting bounded linear operators on a separable complex Hilbert space $\mathcal{H}$. Define the operator $D_{\mathbf{T}} : \mathcal{H} \to \mathcal{H} \oplus \cdots \oplus \mathcal{H}$ by $D_{\mathbf{T}}(x) = (T_1 x, \ldots, T_m x)$, $x \in \mathcal{H}$. Let Ω be a bounded domain in $\mathbb{C}^m$. For $w = (w_1, \ldots, w_m) \in \Omega$, let $\boldsymbol{T} - w$ denote the operator tuple $(T_1 - w_1, \ldots, T_m - w_m)$. Let n be a positive integer. The m-tuple $\boldsymbol{T}$ is said to be in the Cowen-Douglas class $\mathrm{B}_n(\Omega)$ if

(1) ran $D_{\mathbf{T}-w}$ is closed for all $w \in \Omega$
(2) span $\{\ker D_{\mathbf{T}-w} : w \in \Omega\}$ is dense in $\mathcal{H}$
(3) $\dim \ker D_{\mathbf{T}-w} = n$ for all $w \in \Omega$.

This class was introduced in [10]. The case of a single operator was investigated earlier in the paper [9]. In this paper, it is pointed out that an operator T in $\mathrm{B}_1(\Omega)$ is unitarily equivalent to the adjoint of the multiplication operator M on a reproducing kernel Hilbert space, where $(Mf)(z) = zf(z)$. It is not very hard to see that, more generally, a m-tuple $\boldsymbol{T}$ in $\mathrm{B}_n(\Omega)$ is unitarily equivalent to the adjoint of the m-tuple of multiplication operators $\boldsymbol{M} = (M_1, \ldots, M_m)$ on a reproducing kernel Hilbert space [9] and [11, Remark 2.6 a) and b)]. Also, Curto and Salinas [11] show that if certain conditions are imposed on the reproducing kernel then the corresponding adjoint of the m-tuple of multiplication operators belongs to the class $\mathrm{B}_n(\Omega)$.

To a m-tuple $\boldsymbol{T}$ in $\mathrm{B}_n(\Omega)$, on the one hand, one may associate a holomorphic Hermitian vector bundle $E_{\mathbf{T}}$ on Ω (cf. [9]), while on the other hand, one may associate a normalized reproducing kernel K (cf. [11]) on a suitable sub-domain of $\Omega^* = \{w \in \mathbb{C}^m : \bar{w} \in \Omega\}$. It is possible to answer a number of questions regarding the m-tuple of operators $\boldsymbol{T}$ using either the vector bundle or the reproducing kernel. For instance, in the two papers [9] and [10], Cowen and Douglas show that the curvature of the bundle $E_{\mathbf{T}}$ along with a certain number of derivatives forms a complete set of unitary invariants for the operator $\boldsymbol{T}$ while Curto and Salinas [11] establish that the unitary equivalence class of the normalized kernel K is a complete unitary invariant for the corresponding m-tuple of multiplication operators. Also, in [9], it is shown that a single operator in $\mathrm{B}_n(\Omega)$ is reducible if and only if the associated holomorphic Hermitian vector bundle admits an orthogonal direct sum decomposition.

We recall the correspondence between a m-tuple of operators in the class $\mathrm{B}_n(\Omega)$ and the corresponding m-tuple of multiplication operators on a reproducing kernel Hilbert space on Ω.

Let $\boldsymbol{T}$ be a m-tuple of operators in $\mathrm{B}_n(\Omega)$. Pick n linearly independent vectors $\gamma_1(w), \ldots, \gamma_n(w)$ in $\ker D_{\mathbf{T}-w}$, $w \in \Omega$. Define a map $\Gamma : \Omega \to \mathcal{L}(\mathbb{F}, \mathcal{H})$ by $\Gamma(w)\zeta = \sum_{i=0}^{n} \zeta_i \gamma_i(w)$, where $\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{F}$, $\dim \mathbb{F} = n$. It is shown in [9, Proposition 1.11] and [11, Theorem 2.2] that it is possible to choose $\gamma_1(w), \ldots, \gamma_n(w)$, w in some domain $\Omega_0 \subseteq \Omega$, such that Γ is holomorphic on Ω_0. Let $\mathcal{A}(\Omega, \mathbb{F})$ denote the linear space of all $\mathbb{F}$ - valued holomorphic functions on Ω. Define $U_\Gamma : \mathcal{H} \to \mathcal{A}(\Omega_0^*, \mathbb{F})$ by

$$(U_\Gamma x)(w) = \Gamma(w)^* x, \ x \in \mathcal{H}, \ w \in \Omega_0. \tag{2.6}$$

Define a sesqui-linear form on $\mathcal{H}_\Gamma = \operatorname{ran} U_\Gamma$ by $\langle U_\Gamma f, U_\Gamma g \rangle_\Gamma = \langle f, g \rangle$, $f, g \in \mathcal{H}$. The map U_Γ is linear and injective. Hence $\mathcal{H}_\Gamma$ is a Hilbert space of $\mathbb{F}$-valued holomorphic functions on Ω_0^* with inner product $\langle \cdot, \cdot \rangle_\Gamma$ and U_Γ is unitary. Then it is easy to verify the following (cf. [11, Remarks 2.6]).

a) $K(z,w) = \Gamma(\bar{z})^*\Gamma(\bar{w})$, $z, w \in \Omega_0^*$ is the reproducing kernel for the Hilbert space $\mathcal{H}_\Gamma$.
b) $M_i^* U_\Gamma = U_\Gamma T_i$, where $(M_i f)(z) = z_i f(z)$, $z = (z_1, \ldots, z_m) \in \Omega$.

An nnd kernel K for which $K(z, w_0) = I$ for all $z \in \Omega_0^*$ and some $w_0 \in \Omega$ is said to be normalized at w_0.

For $1 \leq i \leq m$, suppose that the operators $M_i : \mathcal{H} \to \mathcal{H}$ are bounded. Then it is easy to verify that for each fixed $w \in \Omega$, and $1 \leq i \leq m$,

$$M_i^* K(\cdot, w)\eta = \bar{w}_i K(\cdot, w)\eta \text{ for } \eta \in \mathbb{F}. \tag{2.7}$$

Differentiating (2.3), we also obtain the following extension of the reproducing property:

$$\langle (\partial_i^j f)(w), \eta \rangle = \langle f, \bar{\partial}_i^j K(\cdot, w)\eta \rangle \text{ for } 1 \leq i \leq m,\ j \geq 0,\ w \in \Omega,\ \eta \in \mathbb{F},\ f \in \mathcal{H}. \tag{2.8}$$

Let $\boldsymbol{M} = (M_1, \ldots, M_m)$ be the commuting m - tuple of multiplication operators and let $\boldsymbol{M}^*$ be the m - tuple $(M_1^*, \ldots, M_m^*)$. It then follows from (2.7) that the eigenspace of the m - tuple $\boldsymbol{M}^*$ at $w \in \Omega^* \subseteq \mathbb{C}^m$ contains the n-dimensional subspace r$an\, K(\cdot, \bar{w}) \subseteq \mathcal{H}$.

One may impose additional conditions on K to ensure that $\boldsymbol{M}$ is in $\mathrm{B}_n(\Omega^*)$. Assume that $K(w,w)$ is invertible for $w \in \Omega$. Fix $w_0 \in \Omega$ and note that $K(z, w_0)$ is invertible for z in some neighborhood $\Omega_0 \subseteq \Omega$ of w_0. Let K_{res} be the restriction of K to $\Omega_0 \times \Omega_0$. Define a kernel function K_0 on Ω_0 by

$$K_0(z,w) = \varphi(z) K(z,w) \varphi(w)^*,\ z, w \in \Omega_0, \tag{2.9}$$

where $\varphi(z) = K_{\mathrm{res}}(w_0, w_0)^{1/2} K_{\mathrm{res}}(z, w_0)^{-1}$. The kernel K_0 is said to be *normalized* at 0 and is characterized by the property $K_0(z, w_0) = I$ for all $z \in \Omega_0$. Let $\boldsymbol{M}_0$ denote the m-tuple of multiplication operators on the Hilbert space $\mathcal{H}$. It is not hard to establish the unitary equivalence of the two m - tuples $\boldsymbol{M}$ and $\boldsymbol{M}_0$ as in (cf. [11, Lemma 3.9 and Remark 3.8]). First, the restriction map $res : f \to f_{res}$, which restricts a function in $\mathcal{H}$ to Ω_0 is a unitary map intertwining the m-tuple $\boldsymbol{M}$ on $\mathcal{H}$ with the m-tuple $\boldsymbol{M}$ on $\mathcal{H}_{\mathrm{res}} = \mathrm{ran}\ res$. The Hilbert space $\mathcal{H}_{\mathrm{res}}$ is a reproducing kernel Hilbert space with reproducing kernel K_{res}. Second, suppose that the m - tuples $\boldsymbol{M}$ defined on two different reproducing kernel Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ are in $\mathrm{B}_n(\Omega)$ and $X : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded operator intertwining these two operator tuples. Then X must map the joint kernel of one tuple in to the other, that is, $XK_1(\cdot, w)\boldsymbol{x} = K_2(\cdot, w)\Phi(w)\boldsymbol{x}$, $\boldsymbol{x} \in \mathbb{C}^n$, for some function $\Phi : \Omega \to \mathbb{C}^{n\times n}$. Assuming that the kernel functions K_1 and K_2 are holomorphic in the first and anti-holomorphic in the second variable, it follows, again as in [11, pp. 472], that Φ is anti-holomorphic. An easy calculation then shows that X^* is the multiplication operator $M_{\bar{\Phi}^{\mathrm{tr}}}$. If the two operator tuples are unitarily equivalent then there exists an unitary operator U intertwining them. Hence U^* must be of the form M_Ψ for some holomorphic function Ψ such that $\overline{\Psi(w)}^{\mathrm{tr}}$ maps the joint kernel of $(\boldsymbol{M} - w)^*$ isometrically onto the joint kernel of $(\boldsymbol{M} - w)^*$ for all $w \in \Omega$. The unitarity of U is equivalent to the relation $K_1(\cdot, w)\boldsymbol{x} = U^* K_2(\cdot, w)\overline{\Psi(w)}^{\mathrm{tr}}\boldsymbol{x}$ for all $w \in \Omega$ and $\boldsymbol{x} \in \mathbb{C}^n$. It then follows that

$$K_1(z,w) = \Psi(z) K_2(z,w) \overline{\Psi(w)}^{\mathrm{tr}}, \tag{2.10}$$

where $\Psi : \Omega_0 \subseteq \Omega \to \mathcal{GL}(\mathbb{F})$ is some holomorphic function. Here, $\mathcal{GL}(\mathbb{F})$ denotes the group of all invertible linear transformations on $\mathbb{F}$.

Conversely, if two kernels are related as above then the corresponding tuples of multiplication operators are unitarily equivalent since

$$M_i^* K(\cdot, w)\zeta = \bar{w}_i K(\cdot, w)\zeta,\ w \in \Omega,\ \zeta \in \mathbb{F},$$

where $(M_i f)(z) = z_i f(z)$, $f \in \mathcal{H}$ for $1 \leq i \leq m$.

Remark 2.3. *We observe that if there is a self adjoint operator X commuting with the m-tuple $\boldsymbol{M}$ on the Hilbert space $\mathcal{H}$ then we must have the relation $\overline{\Phi(z)}^{\mathrm{tr}} K(z,w) = K(z,w)\Phi(w)$ for some anti-holomorphic function $\Phi : \Omega \to \mathbb{C}^{n\times n}$. Hence if the kernel K is normalized then any projection P commuting with the m-tuple $\boldsymbol{M}$ is induced by a constant function Φ such that $\Phi(0)$ is an ordinary projection on $\mathbb{C}^n$.*

In conclusion, what is said above shows that a m-tuple of operators in $\mathrm{B}_n(\Omega^*)$ admits a representation as the adjoint of a m-tuple of multiplication operators on a reproducing kernel Hilbert space of $\mathbb{F}$-valued holomorphic functions on Ω_0, where the reproducing kernel K may be assumed to be normalized. Conversely, the adjoint of the m-tuple of multiplication operators on the reproducing kernel Hilbert space associated with a normalized kernel K on Ω belongs to $\mathrm{B}_n(\Omega^*)$ if certain additional conditions are imposed on K (cf. [11]).

Our interest in the class $\mathrm{B}_n(\Omega)$ lies in the fact that the Cowen-Douglas theorem [9] provides a complete set of unitary invariants for operators which belong to this class. However, these invariants are somewhat intractable. Besides, often it is not easy to verify that a given operator is in the class $\mathrm{B}_n(\Omega)$. Although, we don't use the complete set of invariants that [9] provides, it is useful to ensure that the homogeneous operators that arise from the jet construction are in this class.

2.3. The jet construction. Let $\mathcal{M}$ be a Hilbert module over the algebra $\mathcal{A}(\Omega)$ for Ω a bounded domain in $\mathbb{C}^m$. Let $\mathcal{M}_k$ be the submodule of functions in $\mathcal{M}$ vanishing to order $k > 0$ on some analytic hyper-surface $\mathcal{Z}$ in Ω – the zero set of a holomorphic function φ in $\mathcal{A}(\Omega)$. A function f on Ω is said to vanish to order k on $\mathcal{Z}$ if it can be written $f = \varphi^k g$ for some holomorphic function g. The quotient module $\mathcal{Q} = \mathcal{M} \ominus \mathcal{M}_k$ has been characterized in [12]. This was done by a generalization of the approach in [2] to allow vector-valued kernel Hilbert modules. The basic result in [12] is that $\mathcal{Q}$ can be characterized as such a vector-valued kernel Hilbert space over the algebra $\mathcal{A}(\Omega)|_{\mathcal{Z}}$ of the restriction of functions in $\mathcal{A}(\Omega)$ to $\mathcal{Z}$ and multiplication by φ acts as a nilpotent operator of order k.

For a fixed integer $k > 0$, in this realization, $\mathcal{M}$ consists of $\mathbb{C}^k$-valued holomorphic functions, and there is an $\mathbb{C}^{k\times k}$-valued function $K(z,w)$ on $\Omega \times \Omega$ which is holomorphic in z and anti-holomorphic in w such that

(1) $K(\cdot,w)v$ is in $\mathcal{M}$ for w in Ω and v in $\mathbb{C}^k$;
(2) $\langle f, K(\cdot,w)v\rangle_{\mathcal{M}} = \langle f(w), v\rangle_{\mathbb{C}^k}$ for f in $\mathcal{M}$, w in Ω and v in $\mathbb{C}^k$; and
(3) $\mathcal{A}(\Omega)\mathcal{M} \subset \mathcal{M}$.

If we assume that $\mathcal{M}$ is in the class $\mathrm{B}_1(\Omega)$, then it is possible to describe the quotient module via a jet construction along the normal direction to the hypersurface $\mathcal{Z}$. The details are in [12]. In this approach, to every positive definite kernel $K : \Omega\times\Omega \to \mathbb{C}$, we associate a kernel $JK = \left(\!\!\left(\partial_1^i \bar{\partial}_1{}^j K\right)\!\!\right)_{i,j=0}^{k-1}$, where ∂_1 denotes differentiation along the normal direction to $\mathcal{Z}$. Then we may equip

$$J\mathcal{M} = \Big\{\mathbf{f} := \sum_{i=0}^{k-1} \partial_1^i f \otimes \varepsilon_i \in \mathcal{M}\otimes\mathbb{C}^k : f \in \mathcal{M}\Big\},$$

where $\varepsilon_0, \ldots, \varepsilon_{k-1}$ are standard unit vectors in $\mathbb{C}^k$, with a Hilbert space structure via the kernel JK. The module action is defined by $\mathbf{f} \mapsto \mathbb{J}\mathbf{f}$ for $\mathbf{f} \in J\mathcal{M}$, where $\mathbb{J}$ is the array –

$$\mathbb{J} = \begin{pmatrix} 1 & \cdots & \cdots & \cdots & \cdots & 0 \\ \partial_1 & 1 & & & & \vdots \\ \vdots & & \ddots & & & \vdots \\ \vdots & \binom{l}{j}\partial_1^{\ell-j} & & 1 & & \vdots \\ \vdots & & & & \ddots & 0 \\ \partial_1^{k-1} & \cdots & \cdots & \cdots & \cdots & 1 \end{pmatrix}$$

with $0 \le \ell, j \le k-1$. The module $J\mathcal{M}_{|\text{res } \mathcal{Z}}$ which is the restriction of $J\mathcal{M}$ to $\mathcal{Z}$ is then shown to be isomorphic to the quotient module $\mathcal{M} \ominus \mathcal{M}_k$.

We illustrate these results by means of an example. Let $\mathcal{M}^{(\alpha,\beta)}$ be the Hilbert module which corresponds to the reproducing kernel

$$B^{(\alpha,\beta)}(z,w) = \frac{1}{(1-z_1\bar{w}_1)^{\alpha}} \frac{1}{(1-z_2\bar{w}_2)^{\beta}},$$

$(z_1, z_2) \in \mathbb{D}^2$ and $(w_1, w_2) \in \mathbb{D}^2$. Let $\mathcal{M}_2^{(\alpha,\beta)}$ be the subspace of all functions in $\mathcal{M}^{(\alpha,\beta)}$ which vanish to order 2 on the diagonal $\{(z,z) : z \in \mathbb{D}\} \subseteq \mathbb{D} \times \mathbb{D}$. The quotient module $\mathcal{Q} := \mathcal{M}^{(\alpha,\beta)} \ominus \mathcal{M}_2^{(\alpha,\beta)}$ was described in [14] using an orthonormal basis for the quotient module $\mathcal{Q}$. This includes the calculation of the compression of the two operators, $M_1 : f \mapsto z_1 f$ and $M_2 : f \mapsto z_2 f$ for $f \in \mathcal{M}^{(\alpha,\beta)}$, on the quotient module $\mathcal{Q}$ (block weighted shift operators) with respect to this orthonormal basis. These are homogeneous operators in the class $\mathrm{B}_2(\mathbb{D})$ which were first discovered by Wilkins [22].

In [14], an orthonormal basis $\left\{e_p^{(1)}, e_p^{(2)}\right\}_{p=0}^{\infty}$ was constructed in the quotient module $\mathcal{M} \ominus \mathcal{M}_2^{(\alpha,\beta)}$. It was shown that the matrix

$$M_p^{(1)} = \begin{pmatrix} \frac{\binom{-(\alpha+\beta)}{p}^{1/2}}{\binom{-(\alpha+\beta)}{p+1}^{1/2}} & 0 \\ \left(\frac{\beta}{\alpha}\right)^{1/2} \frac{(\alpha+\beta+1)^{1/2}}{\left((\alpha+\beta+p)(\alpha+\beta+p+1)\right)^{1/2}} & \frac{\binom{-(\alpha+\beta+2)}{p-1}^{1/2}}{\binom{-(\alpha+\beta+2)}{p}^{1/2}} \end{pmatrix}$$

represents the operator M_1 which is multiplication by z_1 with respect to the orthonormal basis $\{e_p^{(1)}, e_p^{(2)}\}_{p=0}^{\infty}$. Similarly,

$$M_p^{(2)} = \begin{pmatrix} \frac{\binom{-(\alpha+\beta)}{p}^{1/2}}{\binom{-(\alpha+\beta)}{p+1}^{1/2}} & 0 \\ -\left(\frac{\alpha}{\beta}\right)^{1/2} \frac{(\alpha+\beta+1)^{1/2}}{\left((\alpha+\beta+p)(\alpha+\beta+p+1)\right)^{1/2}} & \frac{\binom{-(\alpha+\beta+2)}{p-1}^{1/2}}{\binom{-(\alpha+\beta+2)}{p}^{1/2}} \end{pmatrix}$$

represents the operator M_2 which is multiplication by z_2 with respect to the orthonormal basis $\{e_p^{(1)}, e_p^{(2)}\}_{p=0}^{\infty}$. Therefore, we see that $Q_1^{(p)} = \frac{1}{2}(M_1^{(p)} - M_2^{(p)})$ is a nilpotent matrix of index 2 while $Q_2^{(p)} = \frac{1}{2}(M_1^{(p)} + M_2^{(p)})$ is a diagonal matrix in case $\beta = \alpha$. These definitions naturally give a pair of operators Q_1 and Q_2 on the quotient module $\mathcal{Q}^{(\alpha,\beta)}$. Let f be a function in the bi-disc algebra $\mathcal{A}(\mathbb{D}^2)$ and

$$f(u_1, u_2) = f_0(u_1) + f_1(u_1)u_2 + f_2(u_1)u_2^2 + \cdots$$

be the Taylor expansion of the function f with respect to the coordinates $u_1 = \frac{z_1+z_2}{2}$ and $u_2 = \frac{z_1-z_2}{2}$. Now the module action for $f \in \mathcal{A}(\mathbb{D}^2)$ in the quotient module $\mathcal{Q}^{(\alpha,\beta)}$ is then given by

$$\begin{aligned} f \cdot h &= f(Q_1, Q_2) \cdot h \\ &= f_0(Q_1) \cdot h + f_1(Q_1) Q_2 \cdot h \\ &\stackrel{\text{def}}{=} \begin{pmatrix} f_0 & 0 \\ f_1 & f_0 \end{pmatrix} \cdot \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}, \end{aligned}$$

where $h = \binom{h_1}{h_2} \in \mathcal{Q}^{(\alpha,\beta)}$ is the unique decomposition obtained from realizing the quotient module as the direct sum $\mathcal{Q}^{(\alpha,\beta)} = \big(\mathcal{M}^{(\alpha,\beta)} \ominus \mathcal{M}_1^{(\alpha,\beta)}\big) \oplus \big(\mathcal{M}_1^{(\alpha,\beta)} \ominus \mathcal{M}_2^{(\alpha,\beta)}\big)$, where $\mathcal{M}_i^{(\alpha,\beta)}$, $i = 1, 2$, are the submodules in $\mathcal{M}^{(\alpha,\beta)}$ consisting of all functions vanishing on $\mathcal{Z}$ to order 1 and 2 respectively.

We now calculate the curvature $\mathcal{K}^{(\alpha,\beta)}$ for the bundle $E^{(\alpha,\beta)}$ corresponding to the metric $B^{(\alpha,\beta)}(\mathbf{u}, \mathbf{u})$, where $\mathbf{u} = (u_1, u_2) \in \mathbb{D}^2$. The curvature $\mathcal{K}^{(\alpha,\beta)}$ is easy to compute:

$$\mathcal{K}^{(\alpha,\beta)}(u_1, u_2) = (1 - |u_1 + u_2|^2)^{-2} \begin{pmatrix} \alpha & \alpha \\ \alpha & \alpha \end{pmatrix} + (1 - |u_1 - u_2|^2)^{-2} \begin{pmatrix} \beta & -\beta \\ -\beta & \beta \end{pmatrix}.$$

The restriction of the curvature to the hyper-surface $\{u_2 = 0\}$ is

$$\mathcal{K}^{(\alpha,\beta)}(u_1, u_2)_{|u_2=0} = (1 - |u_1|^2)^{-2} \begin{pmatrix} \alpha+\beta & \alpha-\beta \\ \alpha-\beta & \alpha+\beta \end{pmatrix},$$

where $u_1 \in \mathbb{D}$. Thus we find that if $\alpha = \beta$, then the curvature is of the form $2\alpha(1 - |u_1|^2)^{-2} I_2$.

We now describe the unitary map which is basic to the construction of the quotient module, namely,

$$h \mapsto \sum_{\ell=0}^{k-1} \partial_1^\ell h \otimes \varepsilon_\ell \Big|_{\text{res } \triangle}$$

for $h \in \mathcal{M}^{(\alpha,\beta)}$. For $k = 2$, it is enough to describe this map just for the orthonormal basis $\{e_p^{(1)}, e_p^{(2)} : p \geq 0\}$. A simple calculation shows that

$$\begin{aligned} e_p^{(1)}(z_1, z_2) &\mapsto \begin{pmatrix} \binom{-(\alpha+\beta)}{p}^{1/2} z_1^p \\ \beta\sqrt{\frac{p}{\alpha+\beta}} \binom{-(\alpha+\beta+1)}{p-1}^{1/2} z_1^{p-1} \end{pmatrix} \\ e_p^{(2)}(z_1, z_2) &\mapsto \begin{pmatrix} 0 \\ \sqrt{\frac{\alpha\beta}{\alpha+\beta}} \binom{-(\alpha+\beta+2)}{p-1}^{1/2} z_1^{p-1} \end{pmatrix}. \end{aligned} \tag{2.11}$$

This allows us to compute the 2×2 matrix-valued kernel function

$$K_{\mathcal{Q}}(\mathbf{z}, \mathbf{w}) = \sum_{p=0}^{\infty} e_p^{(1)}(\mathbf{z}) e_p^{(1)}(\mathbf{w})^* + \sum_{p=0}^{\infty} e_p^{(2)}(\mathbf{z}) e_p^{(2)}(\mathbf{w})^*, \ \mathbf{z}, \mathbf{w} \in \mathbb{D}^2$$

corresponding to the quotient module. Recall that $S(z, w) := (1 - z\bar{w})^{-1}$ is the Szegö kernel for the unit disc $\mathbb{D}$. We set $\mathbb{S}^r(z) := S(z, z)^r = (1 - |z|^2)^{-r}$, $r > 0$. A straight forward computation shows that

$$\begin{aligned} & K_{\mathcal{Q}}(\mathbf{z},\mathbf{z})_{|\text{res }\triangle} \\ = & \begin{pmatrix} \mathbb{S}(z)^{\alpha+\beta} & \beta z\mathbb{S}(z)^{\alpha+\beta+1} \\ \beta\bar{z}\mathbb{S}(z)^{\alpha+\beta+1} & \frac{\beta^2}{\alpha+\beta}\frac{d}{d|z|^2}\big(|z|^2\mathbb{S}(z)^{\alpha+\beta+1}\big)+\frac{\beta\alpha}{\alpha+\beta}\mathbb{S}(z)^{\alpha+\beta+2} \end{pmatrix} \\ = & \big(\!\!\big(\mathbb{S}(z_1)^{\alpha}\partial^i\bar{\partial}^j\mathbb{S}(z_2)^{\beta}{}_{|\text{res }\triangle}\big)\!\!\big)_{i,j=0,1} \\ = & (JK)(\mathbf{z},\mathbf{z})_{|\text{res }\mathcal{Z}},\ \ \mathbf{z}\in\mathbb{D}^2, \end{aligned}$$

where $\triangle=\{(z,z)\in\mathbb{D}^2 : z\in\mathbb{D}\}$. These calculations give an explicit illustration of one of the main theorems on quotient modules from [12, Theorem 3.4].

3. Multiplier representations

Let G be a locally compact second countable (lcsc) topological group acting transitively on the domain $\Omega\subseteq\mathbb{C}^m$. Let $\mathbb{C}^{n\times n}$ denote the set of $n\times n$ matrices over the complex field $\mathbb{C}$. We start with a cocycle J, that is, a holomorphic map $J_g:\Omega\to\mathbb{C}^{n\times n}$ satisfying the cocycle relation

$$J_{gh}(z)=J_h(z)J_g(h\cdot z),\ \text{for all } g,h\in G,\ z\in\Omega, \tag{3.12}$$

Let $\mathrm{Hol}(\Omega,\mathbb{C}^n)$ be the linear space consisting of all holomorphic functions on Ω taking values in $\mathbb{C}^n$. We then obtain a natural (left) action U of the group G on $\mathrm{Hol}(\Omega,\mathbb{C}^n)$:

$$(U_{g^{-1}}f)(z)=J_g(z)f(g\cdot z),\ f\in\mathrm{Hol}(\Omega,\mathbb{C}^n),\ z\in\Omega. \tag{3.13}$$

Let e be the identity element of the group G. Note that the cocycle condition (3.12) implies, among other things, $J_e(z)=J_e(z)^2$ for all $z\in\Omega$.

Let $\mathbb{K}\subseteq G$ be the compact subgroup which is the stabilizer of 0. For h,k in $\mathbb{K}$, we have $J_{kh}(0)=J_h(0)J_k(0)$ so that $k\mapsto J_k(0)^{-1}$ is a representation of $\mathbb{K}$ on $\mathbb{C}^n$.

A positive definite kernel K on Ω defines an inner product on some linear subspace of $\mathrm{Hol}(\Omega,\mathbb{C}^n)$. The completion of this subspace is then a Hilbert space of holomorphic functions on Ω (cf. [2]). The natural action of the group G described above is seen to be unitary for an appropriate choice of such a kernel. Therefore, we first discuss these kernels in some detail.

Let $\mathcal{H}$ be a functional Hilbert space consisting of holomorphic functions on Ω possessing a reproducing kernel K. We will always assume that the m - tuple of multiplication operators $\boldsymbol{M}=(M_1,\ldots,M_m)$ on the Hilbert space $\mathcal{H}$ is bounded. We also define the action of the group G on the space of multiplication operators – $g\cdot M_f=M_{f\circ g}$ for $f\in\mathcal{A}(\Omega)$ and $g\in G$. In particular, we have $g\cdot\boldsymbol{M}=\boldsymbol{M}_g$. We will say that the m-tuple $\boldsymbol{M}$ is *G-homogeneous* if the operator $g\cdot\boldsymbol{M}$ is unitarily equivalent to $\boldsymbol{M}$ for all $g\in G$. $g\mapsto U_{g^{-1}}$ defined in (3.13) leaves $\mathcal{H}$ invariant. The following theorem says that the reproducing kernel of such a Hilbert space must be *quasi invariant* under the G action.

A version of the following Theorem appears in [16] for the unit disc. However, the proof here, which is taken from [16], is for a more general domain Ω in $\mathbb{C}^m$.

Theorem 3.1. *Suppose that $\mathcal{H}$ is a Hilbert space which consists of holomorphic functions on Ω and possesses a reproducing kernel K on which the m - tuple $\boldsymbol{M}$ is irreducible and bounded. Then the following are equivalent.*

1. *The m - tuple $\boldsymbol{M}$ is G-homogeneous.*
2. *The reproducing kernel K of the Hilbert space $\mathcal{H}$ transforms, for some cocycle $J_g:\Omega\to\mathbb{C}^{n\times n}$, according to the rule*

$$K(z,w)=J_g(z)K(g\cdot z,g\cdot w)J_g(w)^*,\ z,w\in\Omega.$$

(3) *The operator* $U_{g^{-1}} : f \mapsto M_{J_g} f \circ g$ *for* $f \in \mathcal{H}$ *is unitary.*

Proof. Assuming that K is quasi-invariant, that is, K satisfies the transformation rule, we see that the linear transformation U defined in (3.13) is unitary. To prove this, note that

$$\begin{aligned}\langle U_{g^{-1}} K(z,w)\boldsymbol{x}, U_{g^{-1}} K(z,w')\boldsymbol{y}\rangle &= \langle J_g(z) K(g\cdot z, w)\boldsymbol{x}, J_g(z) K(g\cdot z, w')\boldsymbol{y}\rangle \\ &= \langle K(z,\tilde{w}) J_g(\tilde{w})^{*-1}\boldsymbol{x}, K(z,\tilde{w}') J_g(\tilde{w}')^{*-1}\boldsymbol{y}\rangle \\ &= \langle K(\tilde{w}',\tilde{w}) J_g(\tilde{w})^{*-1}\boldsymbol{x}, J_g(\tilde{w}')^{*-1}\boldsymbol{y}\rangle \\ &= \langle J_g(\tilde{w}')^{-1} K(\tilde{w}',\tilde{w}) J_g(\tilde{w})^{*-1}\boldsymbol{x}, \boldsymbol{y}\rangle \\ &= \langle K(g\cdot\tilde{w}', g\cdot\tilde{w})\boldsymbol{x}, \boldsymbol{y}\rangle,\end{aligned}$$

where $\tilde{w} = g^{-1}\cdot w$ and $\tilde{w}' = g^{-1}\cdot w'$. Hence

$$\langle K(g\cdot\tilde{w}', g\cdot\tilde{w})\boldsymbol{x}, \boldsymbol{y}\rangle = \langle K(w', w)\boldsymbol{x}, \boldsymbol{y}\rangle.$$

It follows that the map $U_{g^{-1}}$ is isometric. On the other hand, if U of (3.13) is unitary then the reproducing kernel K of the Hilbert space $\mathcal{H}$ satisfies

$$K(z,w) = J_g(z) K(g\cdot z, g\cdot w) J_g(w)^*. \tag{3.14}$$

This follows from the fact that the reproducing kernel has the expansion (2.5) for some orthonormal basis $\{e_\ell : \ell \geq 0\}$ in $\mathcal{H}$. The uniqueness of the reproducing kernel implies that the expansion is independent of the choice of the orthonormal basis. Consequently, we also have $K(z,w) = \sum_{\ell=0}(U_{g^{-1}}e_\ell)(z)(U_{g^{-1}}e_\ell)(w)^*$ which verifies the equation (3.14). Thus we have shown that U is unitary if and only if the reproducing kernel K transforms according to (3.14).

We now show that the m-tuple $\boldsymbol{M}$ is homogeneous if and only if $f \mapsto M_{J_g} f \circ g$ is unitary. The eigenvector at w for $g.\boldsymbol{M}$ is clearly $K(\cdot, g^{-1}\cdot w)$. It is not hard, using the unitary operator U_Γ in (2.6), to see that that $g^{-1}\cdot\boldsymbol{M}$ is unitarily equivalent to $\boldsymbol{M}$ on a Hilbert space $\mathcal{H}_g$ whose reproducing kernel is $K_g(z,w) = K(g\cdot z, g\cdot w)$ and the unitary U_Γ is given by $f \mapsto f\circ g$ for $f \in \mathcal{H}$. However, the homogeneity of the m-tuple $\boldsymbol{M}$ is equivalent to the existence of a unitary operator intertwining the m-tuple of multiplication on the two Hilbert spaces $\mathcal{H}$ and $\mathcal{H}_g$. As we have pointed out in section 2.2, this unitary operator is induced by a multiplication operator M_{J_g}, where J_g is a holomorphic function (depends on g) such that $K_g(z,w) = J_g(z)K(z,w)\overline{J_g(w)}^{\mathrm{tr}}$. The composition of these two unitaries is $f \mapsto M_{J_g} f \circ g$ and is therefore a unitary. □

The discussion below and the Corollary following it is implicit in [16]. Let g_z be an element of G which maps 0 to z, that is $g_z \cdot 0 = z$. We could then try to define possible kernel functions $K : \Omega\times\Omega \to \mathbb{C}^{n\times n}$ satisfying the transformation rule (3.14) via the requirement

$$K(g_z\cdot 0, g_z\cdot 0) = (J_{g_z}(0))^{-1} K(0,0) (J_{g_z}(0)^*)^{-1}, \tag{3.15}$$

choosing any positive operator $K(0,0)$ on $\mathbb{C}^n$ which commutes with $J_k(0)$ for all $k \in \mathbb{K}$. Then the equation (3.15) determines the function K unambiguously as long as $J_k(0)$ is unitary for $k \in \mathbb{K}$. Pick $g \in G$ such that $g\cdot 0 = z$. Then $g = g_z k$ for some $k \in \mathbb{K}$. Hence

$$\begin{aligned}K(g_z k\cdot 0, g_z k\cdot 0) &= (J_{g_z k}(0))^{-1} K(0,0) (J_{g_z k}(0)^*)^{-1} \\ &= \big(J_k(0) J_{g_z}(k\cdot 0)\big)^{-1} K(0,0) \big(J_{g_z}(k\cdot 0)^* J_k(0)^*\big)^{-1} \\ &= (J_{g_z}(0))^{-1}(J_k(0))^{-1} K(0,0) (J_k(0)^*)^{-1}(J_{g_z}(0)^*)^{-1} \\ &= (J_{g_z}(0))^{-1} K(0,0) (J_{g_z}(0)^*)^{-1} \\ &= K(g_z\cdot 0, g_z\cdot 0)\end{aligned}$$

Given the definition (3.15), where the choice of $K(0,0) = A$ involves as many parameters as the number of irreducible representations of the form $k \mapsto J_k(0)^{-1}$ of the compact group $\mathbb{K}$, one can polarize (3.15) to get $K(z,w)$. In this approach, one has to find a way of determining if K is non-negative definite, or for that matter, if $K(\cdot, w)$ is holomorphic on all of Ω for each fixed but arbitrary $w \in \Omega$. However, it is evident from the definition (3.15) that

$$\begin{aligned} K(h\cdot z, h\cdot z) &= J_h(g_z\cdot 0)^{-1} J_{g_z}(0)^{-1} A J_{g_z}(0)^{*-1} (J_h(g_z\cdot 0)^*)^{-1} \\ &= J_h(z)^{-1} K(z,z) J_h(z)^{*-1} \end{aligned}$$

for all $h \in G$. Polarizing this equality, we obtain

$$K(h\cdot z, h\cdot w) = J_h(z)^{-1} K(z,w) J_h(w)^{*-1}$$

which is the identity (3.14). It is also clear that the linear span of the set $\{K(\cdot, w)\zeta : w \in \Omega,\ \zeta \in \mathbb{C}^n\}$ is stable under the action (3.13) of G:

$$g \mapsto J_g(z) K(g\cdot z, w)\zeta = K(z, g^{-1}\cdot w) J_g(g^{-1}w)^{*^{-1}}\zeta,$$

where $J_g(g^{-1}w)^{*-1}\zeta$ is a fixed element of $\mathbb{C}^n$.

Corollary 3.2. *If $J : G \times \Omega \to \mathbb{C}^{n\times n}$ is a cocycle and g_z is an element of G which maps 0 to z then the kernel $K : \Omega \times \Omega \to \mathbb{C}^{n\times n}$ defined by the requirement*

$$K(g_z\cdot 0, g_z\cdot 0) = (J_{g_z}(0))^{-1} K(0,0) (J_{g_z}(0)^*)^{-1}$$

is quasi-invariant, that is, it transforms according to (3.14).

4. Irreducibility

In the section 2.2, we have already pointed out that any Hilbert space $\mathcal{H}$ of scalar valued holomorphic functions on $\Omega \subset \mathbb{C}^m$ with a reproducing kernel K determines a line bundle $\mathcal{E}$ on $\Omega^* := \{\bar{w} : w \in \Omega\}$. The fibre of E at $\bar{w} \in \Omega^*$ is spanned by $K(.,w)$. We can now construct a rank $(n+1)$ vector bundle $J^{(n+1)}\mathcal{E}$ over Ω^*. A holomorphic frame for this bundle is $\{\bar{\partial}_2^l K(.,w) : 0 \le l \le k, w \in \Omega\}$, and as usual, this frame determines a metric for the bundle which we denote by $J^{(n+1)}K$, where

$$J^{(n+1)}K(w,w) = \big(\!\big(\langle \bar{\partial}_2^j K(.,w), \bar{\partial}_2^i K(.,w)\rangle\big)\!\big)_{i,j=0}^n = \big(\!\big(\bar{\partial}_2^j \partial_2^i K(w,w)\big)\!\big)_{i,j=0}^n, w \in \Omega.$$

Recall that the kernel function on $\mathbb{D}^2$, $B^{(\alpha,\beta)} : \mathbb{D}^2 \times \mathbb{D}^2 \longrightarrow \mathbb{C}$ is defined by

$$B^{(\alpha,\beta)}(z,w) = (1 - z_1\bar{w}_1)^{-\alpha}(1 - z_2\bar{w}_2)^{-\beta},$$

for $z = (z_1, z_2) \in \mathbb{D}^2$ and $w = (w_1, w_2) \in \mathbb{D}^2$, $\alpha, \beta > 0$. Take $\Omega = \mathbb{D}^2, K = B^{(\alpha,\beta)}$. Notice that the Hilbert space $\mathcal{M}^{(\alpha,\beta)}$ corresponding to the kernel function $B^{(\alpha,\beta)}$ is the tensor product of the two Hilbert spaces $\mathcal{M}^{(\alpha)}$ and $\mathcal{M}^{(\beta)}$. These are determined by the two kernel functions $B^{(\alpha)}(z,w) = (1 - z\bar{w})^{-\alpha}$ and $B^{(\beta)}(z,w) = (1 - z\bar{w})^{-\beta}$, $z, w \in \mathbb{D}$, respectively.

It follows from [12] that $h_{n+1}(z) = J^{(n+1)}B^{(\alpha,\beta)}(z,z)_{|\mathrm{res}\ \triangle}$ is a metric for the Hermitian anti-holomorphic vector bundle $J^{(n+1)}\mathcal{E}_{|\mathrm{res}\ \triangle}$ over $\triangle = \{(z,z) : z \in \mathbb{D}\} \subseteq \mathbb{D}^2$. However, $J^{(n+1)}\mathcal{E}_{|\mathrm{res}\ \triangle}$ is a Hermitian holomorphic vector bundle over $\triangle^* = \{(\bar{z}, \bar{z}) : z \in \mathbb{D}\}$, that is, $\bar{z}$ is the holomorphic variable in this description. Thus $\partial f = 0$ if and only if f is holomorphic on $\triangle^*$. To restore the usual meaning of ∂ and $\bar{\partial}$, we interchange the roles of z and $\bar{z}$ in the metric which amounts to replacing h_{n+1} by its transpose.

As shown in [12], this Hermitian anti-holomorphic vector bundle $J^{(n+1)}\mathcal{E}_{|\mathrm{res}\ \triangle}$ defined over the diagonal subset $\triangle$ of the bidisc $\mathbb{D}^2$ gives rise to a reproducing kernel Hilbert space $J^{(n+1)}\mathcal{H}$. The reproducing kernel for this Hilbert space is $J^{(n+1)}B^{(\alpha,\beta)}(z,w)$ which is obtained by polarizing $J^{(n+1)}B^{(\alpha,\beta)}(z,z) = h_{n+1}(z)^t$.

Lemma 4.1. *Let α, β be two positive real numbers and $n \geq 1$ be an integer. Let $\mathcal{M}_n$ be the ortho-complement of the subspace of $\mathcal{M}^{(\alpha)} \otimes \mathcal{M}^{(\beta)}$ (viewed as a Hilbert space of analytic functions on the bi-disc $\mathbb{D} \times \mathbb{D}$) consisting of all the functions vanishing to order k on the diagonally embedded unit disc $\triangle := \{(z,z) : z \in \mathbb{D}\}$. The compressions to $\mathcal{M}_n$ of $M^{(\alpha)} \otimes I$ and $I \otimes M^{(\beta)}$ are homogeneous operators with a common associated representation.*

Proof. For each real number $\alpha > 0$, let $\mathcal{M}^{(\alpha)}$ be the Hilbert space completion of the inner product space spanned by $\{f_k : k \in \mathbb{Z}^+\}$ where the f_k's are mutually orthogonal vectors with norms given by

$$\|f_k\|^2 = \frac{\Gamma(1+k)}{\Gamma(\alpha+k)},\ k \in \mathbb{Z}^+.$$

(Upto scaling of the norm, this Hilbert space may be identified, via non-tangential boundary values, with the Hilbert space of analytic functions on $\mathbb{D}$ with reproducing kernel $(z,w) \mapsto (1-z\bar{w})^{-\alpha}$.) The representation D_α^+ lives on $\mathcal{M}^{(\alpha)}$, and is given (at least on the linear span of the f_k's) by the formula

$$D_\alpha^+(\varphi^{-1})f = (\varphi')^{\alpha/2} f \circ \varphi,\ \varphi \in \text{Möb}.$$

Clearly the subspace $\mathcal{M}_n$ is invariant under the Discrete series representation $\pi := D_\alpha^+ \otimes D_\beta^+$ associated with both the operators $M^{(\alpha)} \otimes I$ and $I \otimes M^{(\beta)}$. It is also co-invariant under these two operators. An application of Proposition 2.4 in [5] completes the proof of the lemma. □

The subspace $\mathcal{M}_n$ consists of those functions $f \in \mathcal{M}$ which vanish on $\triangle$ along with their first n derivatives with respect to z_2. As it turns out, the compressions to $\mathcal{M} \ominus \mathcal{M}_n$ of $M^{(\alpha)} \otimes I$ is the multiplication operator on the Hilbert space $J^{(n+1)}\mathcal{H}_{|\text{res}\,\triangle}$ which we denote $M^{(\alpha,\beta)}$. An application of [12, Proposition 3.6] shows that the adjoint M^* of the multiplication operator M is in $\mathrm{B}_{n+1}(\mathbb{D})$.

Theorem 4.2. *The multiplication operator $M := M^{(\alpha,\beta)}$ is irreducible.*

Th proof of this theorem will be facilitated by a series of lemmas which are proved in the sequel. We set, for now, $K(z,w) = J^{(n+1)}B^{(\alpha,\beta)}(z,w)$. Let $\widetilde{K}(z,w) = K(0,0)^{-1/2}K(z,w)K(0,0)^{-1/2}$, so that $\widetilde{K}(0,0) = I$. Also, let $\widetilde{\widetilde{K}}(z,w) = \widetilde{K}(z,0)^{-1}\widetilde{K}(z,w)\widetilde{K}(0,w)^{-1}$. This ensures that $\widetilde{\widetilde{K}}(z,0) = I$ for $z \in \mathbb{D}$, that is, $\widetilde{\widetilde{K}}$ is a normalized kernel. Each of the kernels K, $\widetilde{K}$ and $\widetilde{\widetilde{K}}$ admit a power series expansion, say, $K(z,w) = \sum_{m,p\geq 0} a_{mp}\, z^m \bar{w}^p$, $\widetilde{K}(z,w) = \sum_{m,p\geq 0} \widetilde{a}_{mp}\, z^m \bar{w}^p$, and $\widetilde{\widetilde{K}}(z,w) = \sum_{m,p\geq 0} \widetilde{\widetilde{a}}_{mp}\, z^m \bar{w}^p$ for $z,\, w \in \mathbb{D}$, respectively. Here the coefficients a_{mp} and $\widetilde{a}_{mp}$ and $\widetilde{\widetilde{a}}_{mp}$ are $(n+1)\times(n+1)$ matrices for $m,\, p \geq 0$. In particular, $\widetilde{a}_{mp} = K(0,0)^{-1/2} a_{mp}\, K(0,0)^{-1/2} = a_{00}^{-1/2} a_{mp}\, a_{00}^{-1/2}$ for $m,p \geq 0$. Also, let us write $K(z,w)^{-1} = \sum_{m,p\geq 0} b_{mp}\, z^m \bar{w}^p$ and $\widetilde{K}(z,w)^{-1} = \sum_{m,p\geq 0} \widetilde{b}_{mp}\, z^m \bar{w}^p$, $z,w \in \mathbb{D}$. Again, the coefficients b_{mp} and $\widetilde{b}_{mp}$ are $(n+1)\times(n+1)$ matrices for $m,\, p \geq 0$. However, $\widetilde{\widetilde{a}}_{00} = I$ and $\widetilde{\widetilde{a}}_{m0} = \widetilde{\widetilde{a}}_{0p} = 0$ for $m,\, p \geq 1$.

The following lemma is from [11, Theorem 3.7, Remark 3.8 and Lemma 3.9]. The proof was discussed in section 2.2.

Lemma 4.3. *The multiplication operators on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ with reproducing kernels $K_1(z,w)$ and $K_2(z,w)$ respectively, are unitarily equivalent if and only if $K_2(z,w) = \Psi(z)K_1(z,w)\overline{\Psi(w)}^t$, where Ψ is an invertible matrix-valued holomorphic function.*

The proof of the lemma below appears in [16, Lemma 5.2] and is discussed in section 2.2, see Remark 2.3.

Lemma 4.4. *The multiplication operator M on the Hilbert space $\mathcal{H}$ with reproducing kernel K is irreducible if and only if there is no non-trivial projection P on $\mathbb{C}^{n+1}$ commuting with all the coefficients in the power series expansion of the normalized kernel $\widetilde{\widetilde{K}}(z,w)$.*

We will prove irreducibility of M by showing that only operators on $\mathbb{C}^{n+1}$ which commutes with all the coefficients of $\widetilde{\widetilde{K}}(z,w)$ are scalars. It turns out that the coefficients of $z^k\bar{w}$ for $2 \le k \le n+1$, that is, the coefficients $\widetilde{\widetilde{a}}_{k1}$ for $2 \le k \le n+1$ are sufficient to reach the desired conclusion.

Lemma 4.5. *The coefficient of $z^k\bar{w}$ is $\widetilde{\widetilde{a}}_{k1} = \sum_{s=1}^{k} \widetilde{b}_{s0}\widetilde{a}_{k-s,1} + \widetilde{a}_{k1}$ for $1 \le k \le n+1$.*

Proof. Let us denote the coefficient of $z^k\bar{w}^l$ in the power series expansion of $\widetilde{\widetilde{K}}(z,w)$ is $\widetilde{\widetilde{a}}_{kl}$ for $k, l \ge 0$. We see that

$$\begin{aligned}\widetilde{\widetilde{a}}_{kl} &= \sum_{s=0}^{k}\sum_{t=0}^{l} \widetilde{b}_{s0}\widetilde{a}_{k-s,l-t}\widetilde{b}_{0t}\\ &= \sum_{s=1}^{k}\sum_{t=1}^{l} \widetilde{a}_{s0}\widetilde{a}_{k-s,l-t}\widetilde{b}_{0t} + \sum_{s=1}^{k} \widetilde{b}_{s0}\widetilde{a}_{k-s,l} + \sum_{t=1}^{l} \widetilde{a}_{k,l-t}\widetilde{b}_{0t} + \widetilde{a}_{kl}\end{aligned}$$

as $\widetilde{a}_{00} = \widetilde{b}_{00} = I$. Also,

$$\begin{aligned}\widetilde{\widetilde{a}}_{k1} &= \sum_{s=1}^{k} \widetilde{b}_{s0}\widetilde{a}_{k-s,0}\widetilde{b}_{01} + \sum_{s=1}^{k} \widetilde{b}_{s0}\widetilde{a}_{k-s,1} + \widetilde{a}_{k0}\widetilde{b}_{01} + \widetilde{a}_{k1}\\ &= \big(\sum_{s=0}^{k} \widetilde{b}_{s0}\widetilde{a}_{k-s,0}\big)\widetilde{b}_{01} + \sum_{s=1}^{k} \widetilde{b}_{s0}\widetilde{a}_{k-s,1} + \widetilde{a}_{k1}\\ &= \sum_{s=1}^{k} \widetilde{b}_{s0}\widetilde{a}_{k-s,1} + \widetilde{a}_{k1}\end{aligned}$$

as $\widetilde{b}_{00} = I$ and coefficient of z^k in $\widetilde{K}(z,w)^{-1}\widetilde{K}(z,w) = \sum_{s=0}^{k} \widetilde{b}_{s0}\widetilde{a}_{k-s,0} = 0$ for $k \ge 1$. ☐

Now we compute some of the coefficients of $K(z,w)$ which are useful in computing $\widetilde{\widetilde{a}}_{k1}$. In what follows, we will compute only the non-zero entries of the matrices involved, that is, *all those entries which are not specified are assumed to be zero.*

Lemma 4.6. $(a_{00})_{kk} = k!(\beta)_k$ *for* $0 \le k \le n$, $(a_{m0})_{r,r+m} = \frac{(m+r)!}{m!}(\beta)_{m+r}$ *and* $(a_{m+1,1})_{r,r+m} = \frac{(m+r)!}{m!}(\beta)_{m+r}\big(\alpha + (1 + \frac{r}{m+1})(\beta + m + r)\big)$ *for* $0 \le r \le n-m, 0 \le m \le n$, *where* $(x)_0 = 1, (x)_d = x(x+1)\dots(x+d-1)$, *for any positive integer d, is the Pochhammer symbol.*

Proof. The coefficient of $z^p\bar{w}^q$ in $J^{(n+1)}B^{(\alpha,\beta)}(z,w)$ is the same as the coefficient of $z^p\bar{z}^q$ in $J^{(n+1)}B^{(\alpha,\beta)}(z,z)$. So, $(a_{00})_{kk}$ = constant term in $\bar{\partial}_2^k\partial_2^k\big(\mathbb{S}(z_1)^\alpha\mathbb{S}(z_2)^\beta\big)|_\triangle$. Now,

$$\begin{aligned}\bar{\partial}_2^k\partial_2^k\big(\mathbb{S}(z_1)^\alpha\mathbb{S}(z_2)^\beta\big)|_\triangle &= \bar{\partial}_2^k\partial_2^k\big(\mathbb{S}(z_1)^\alpha\mathbb{S}(z_2)^\beta\big)|_\triangle\\ &= \mathbb{S}(z_1)^\alpha(\beta)_k\bar{\partial}_2^k\big(\mathbb{S}(z_2)^{\beta+k}\bar{z}_2^k\big)|_\triangle\\ &= \big(\mathbb{S}(z_1)^\alpha(\beta)_k\sum_{l=0}^{k}\binom{k}{l}\bar{\partial}_2^{k-l}(\mathbb{S}(z_2)^{\beta+k})\bar{\partial}_2^l(\bar{z_2}^k)\big)|_\triangle\\ &= \big(\mathbb{S}(z_1)^\alpha(\beta)_k\sum_{l=0}^{k}\binom{k}{l}(\beta+k)_{k-l}\mathbb{S}(z_2)^{\beta+k+(k-l)}{z_2}^{k-l}l!\binom{k}{l}\bar{z_2}^{k-l}\big)|_\triangle,\end{aligned}$$

that is, $\big(a_{00}\big)_{kk} = k!(\beta)_k$ for $0 \le k \le n$.

We see that the coefficient of z^m in $\bar\partial_2^{m+r}\partial_2^r\big(\mathbb{S}(z_1)^\alpha\mathbb{S}(z_2)^\beta\big)|_\triangle$ is $(a_{m0})_{r,r+m}$. Thus

$$\begin{aligned}\bar\partial_2^{m+r}\partial_2^r\big(\mathbb{S}(z_1)^\alpha\mathbb{S}(z_2)^\beta\big)|_\triangle &= \mathbb{S}(z_1)^\alpha(\beta)_r\bar\partial_2^{m+r}\big(\mathbb{S}(z_2)^{\beta+r}\bar z_2^r\big)|_\triangle\\ &= \big(\mathbb{S}(z_1)^\alpha(\beta)_r\sum_{l=0}^{m+r}\tbinom{m+r}{l}\bar\partial_2^{m+r-l}(\mathbb{S}(z_2)^{\beta+r})\bar\partial_2^l(\bar{z_2}^r)\big)|_\triangle\\ &= \big(\mathbb{S}(z_1)^\alpha(\beta)_r\sum_{l=0}^{m+r}\tbinom{m+r}{l}(\beta+r)_{m+r-l}\mathbb{S}(z_2)^{\beta+2r+m-l)}{z_2}^{m+r-l}l!\binom{r}{l}\bar{z_2}^{r-l}\big)|_\triangle.\end{aligned}$$

Therefore, the term containing z^m occurs only when $l=r$ in the sum above, that is, $(a_{m0})_{r,r+m} = (\beta)_r\binom{m+r}{r}(\beta+r)_m r! = \frac{(m+r)!}{m!}(\beta)_{m+r}$, for $0 \le r \le n-m, 0 \le m \le n$.

Coefficient of $z^{m+1}\bar z$ in $\bar\partial_2^{m+r}\partial_2^r\big(\mathbb{S}(z_1)^\alpha\mathbb{S}(z_2)^\beta\big)|_\triangle$ is $(a_{m+1,1})_{r,r+m}$. For any real analytic function f on $\mathbb{D}$, for now, let $\big(f(z,\bar z)\big)_{(p,q)}$ denote the coefficient of $z^p\bar z^q$ in $f(z,\bar z)$. We have

$$\begin{aligned}\big(a_{m+1,1}\big)_{r,r+m} &= \big(\bar\partial_2^{m+r}\partial_2^r\big(\mathbb{S}(z_1)^\alpha\mathbb{S}(z_2)^\beta\big)|_\triangle\big)_{(m+1,1)}\\ &= \Big((\beta)_r\sum_{l=0}^{m+r}\tbinom{m+r}{l}(\beta+r)_{m+r-l}\mathbb{S}(z)^{\alpha+\beta+r+(m+r-l)}z^{m+r-l}l!\binom{r}{l}\bar z^{r-l}\Big)_{(m+1,1)}\end{aligned}$$

The terms containing $z^{m+1}\bar z$ occurs in the sum above, only when $l=r$ and $l=r-1$, that is,

$$\begin{aligned}\big(a_{m+1,1}\big)_{r,r+m} &= \Big((\beta)_r r!\Big(\tbinom{m+r}{r}(\beta+r)_m\mathbb{S}(z)^{\alpha+\beta+m+r}z^m\\ &\qquad +\tbinom{m+r}{r-1}(\beta+r)_{m+1}\mathbb{S}(z)^{\alpha+\beta+m+r+1}z^{m+1}\bar z\Big)\Big)_{(m+1,1)}\\ &= \Big((\beta)_r r!\Big(\frac{(m+r)!}{r!m!}(\beta+r)_m(1+(\alpha+\beta+m+r)|z|^2)z^m\\ &\qquad +\frac{(m+r)!r}{r!(m+1)!}(\beta+r)_{m+1}\mathbb{S}(z)^{\alpha+\beta+m+r+1}z^{m+1}\bar z\Big)\Big)_{(m+1,1)}\\ &= \frac{(m+r)!}{m!}(\beta)_{m+r}\big((\alpha+\beta+m+r)+\frac{r}{m+1}(\beta+m+r)\big)\\ &= \frac{(m+r)!}{m!}(\beta)_{m+r}\big(\alpha+(1+\frac{r}{m+1})(\beta+m+r)\big),\end{aligned}$$

for $0 \le r \le n-m, 0 \le m \le n$, where we have followed the convention: $\binom{p}{q} = 0$ for a negative integer q. ∎

Lemma 4.7. *Let c_{k0} denote $a_{00}^{1/2}\widetilde b_{k0}a_{00}^{1/2}$. For $0 \le r \le n-k$, $0 \le k \le n$, $(c_{k0})_{r,r+k} = \frac{(-1)^k(r+k)!}{k!}(\beta)_{r+k}$.*

Proof. We have $\widetilde K(z,w)^{-1} = a_{00}^{1/2}K(z,w)^{-1}a_{00}^{1/2} = \sum_{mn\ge0}\big(a_{00}^{1/2}\widetilde b_{mn}a_{00}^{1/2}\big)z^m\bar w^n$. Hence $\widetilde b_{mn} = a_{00}^{1/2}b_{mn}a_{00}^{1/2}$ for $m,n \ge 0$. By invertibility of a_{00}, we see that $\widetilde b_{k0}$ and c_{k0} uniquely determine each other for $k \ge 0$. Since $(\widetilde b_{k0})_{k\ge0}$ are uniquely determined as the coefficients of power series expansion of $\widetilde K(z,w)^{-1}$, it is enough to prove that $\sum_{l=0}^m \widetilde a_{m-l,0}\widetilde b_{l0} = 0$ for $1 \le m \le n$. Equivalently, we must show that $\sum_{l=0}^m(a_{00}^{-1/2}a_{m-l,0}a_{00}^{-1/2})(a_{00}^{-1/2}c_{l0}a_{00}^{-1/2}) = 0$ which amounts to showing $a_{00}^{-1/2}\big(\sum_{l=0}^{m} a_{m-l,0}a_{00}^{-1}c_{l0}\big)a_{00}^{-1/2} = 0$ for $1 \leq m \leq n$. It follows from Lemma 4.6 that $(a_{m-l,0})_{r,r+(m-l)} = \frac{(m-l+r)!}{(m-l)!}(\beta)_{m-l+r}$ and $(a_{00})_{rr} = r!(\beta)_r$. Therefore

$$\begin{aligned}(a_{m-l,0}a_{00}^{-1})_{r,r+(m-l)} &= (a_{m-l,0})_{r,r+(m-l)}(a_{00}^{-1})_{r+(m-l),r+(m-l)}\\ &= \frac{(m-l+r)!}{(m-l)!}(\beta)_{m-l+r}\big((m-l+r)!(\beta)_{m-l+r}\big)^{-1}\\ &= \frac{1}{(m-l)!}.\end{aligned}$$

We also have

$$\begin{aligned}(a_{m-l,0}a_{00}^{-1}c_{l0})_{r,r+m} &= (a_{m-l,0}a_{00}^{-1})_{r,r+(m-l)}(c_{l0})_{r+(m-l),r+(m-l)+l}\\ &= \frac{1}{(m-l)!}\frac{(-1)^l(r+m)!}{l!}(\beta)_{r+m}\\ &= \frac{(-1)^l(r+m)!}{(m-l)!l!}(\beta)_{r+m}\end{aligned}$$

for $0 \leq l \leq m, 0 \leq r \leq n-m, 1 \leq m \leq n$. Now observe that

$$\begin{aligned}(\sum_{l=0}^{m} a_{m-l,0}a_{00}^{-1}c_{l0})_{r,r+m} &= (r+m)!(\beta)_{m+r}\sum_{l=0}^{m}\frac{(-1)^l}{(m-l)!l!}\\ &= \frac{(r+m)!}{m!}(\beta)_{m+r}\sum_{l=0}^{m}(-1)^l\binom{m}{l}\\ &= 0,\end{aligned}$$

which completes the proof of this lemma. ∎

Lemma 4.8. *$(\tilde{\tilde{a}}_{k1})_{n-k+1,n}$ is a non-zero real number, for $2 \leq k \leq n+1, n \geq 1$. All other entries of $\tilde{\tilde{a}}_{k1}$ are zero.*

Proof. From Lemma 4.5 and Lemma 4.7, we know that

$$\begin{aligned}\tilde{\tilde{a}}_{k1} &= \sum_{s=1}^{k}\tilde{b}_{s0}\tilde{a}_{k-s,1} + \tilde{a}_{k1}\\ &= \sum_{s=1}^{k}(a_{00}^{-1/2}c_{s0}a_{00}^{-1/2})(a_{00}^{-1/2}a_{k-s,1}a_{00}^{-1/2}) + a_{00}^{-1/2}a_{k1}a_{00}^{-1/2}.\end{aligned}$$

Consequently, $a_{00}^{1/2}\tilde{\tilde{a}}_{k1}a_{00}^{1/2} = \sum_{s=1}^{k} c_{s0}a_{00}^{-1}a_{k-s,1} + a_{k1}$ for $1 \leq k \leq n+1$. By Lemma 4.6 and Lemma 4.7, we have

$$\begin{aligned}(c_{s0}a_{00}^{-1})_{r,r+s} &= (c_{s0})_{r,r+s}(a_{00}^{-1})_{r+s,r+s}\\ &= \frac{(-1)^s(r+s)!}{s!}(\beta)_{r+s}\big((r+s)!(\beta)_{r+s}\big)^{-1}\\ &= \frac{(-1)^s}{s!},\end{aligned}$$

for $0 \leq r \leq n-s, 0 \leq s \leq k, 1 \leq k \leq n+1$.

$$(a_{k-s,1})_{r,r+(k-s-1)} = \frac{(k+r-s-1)!}{(k-s-1)!}(\beta)_{r+k-s-1}\big(\alpha+(1+\frac{r}{k-s})(\beta+r+k-s-1)\big),$$

for $k-s-1 \geq 0, 2 \leq k \leq n+1$. Now,

$$\begin{aligned}
&(c_{s0}a_{00}^{-1}a_{k-s,1})_{r+s,r+s+(k-s-1)} \\
&= (c_{s0}a_{00}^{-1})_{r,r+s}(a_{k-s,1})_{r+s,r+s+(k-s-1)} \\
&= \frac{(-1)^s}{s!}\frac{(r+k-1)!}{(k-s-1)!}(\beta)_{r+k-1}\big(\alpha+(1+\frac{r+s}{k-s})(\beta+r+k-1)\big),
\end{aligned}$$

for $1 \leq s \leq k-1, 0 \leq r \leq n-k+1, 1 \leq k \leq n+1$. Hence

$$(c_{s0}a_{00}^{-1}a_{k-s,1})_{r+s,r+k-1} = \frac{(-1)^s}{s!}\frac{(r+k-1)!}{(k-s-1)!}(\beta)_{r+k-1}\big(\alpha+\frac{k+r}{k-s}(\beta+r+k-1)\big).$$

Since $\overline{K(z,w)}^t = K(w,z)$, it follows that $a_{mn} = \overline{a_{nm}}^t$ for $m,n \geq 0$. Thus, by Lemma 4.6, $(a_{01})_{r+1,r} = (r+1)!(\beta)_{r+1}$, for $0 \leq r \leq n-1$, $(c_{k0}a_{00}^{-1})_{r,r+k} = \frac{(-1)^k}{k!}$, for $0 \leq r \leq n-k, 1 \leq k \leq n+1$ and

$$(c_{k0}a_{00}^{-1}a_{01})_{r,r+k-1} = (c_{k0}a_{00}^{-1})_{r,r+k}(a_{01})_{r+k,r+k-1} = \frac{(-1)^k}{k!}(r+k)!(\beta)_{r+k},$$

$0 \leq r \leq n-k, 1 \leq k \leq n+1$. Now, for $0 \leq r \leq n-k, 2 \leq k \leq n+1$. Since $c_{00} = a_{00}$, we clearly have

$$\begin{aligned}
(a_{00}^{1/2}\tilde{\tilde{a}}_{k1}a_{00}^{1/2})_{r,r+k-1} &= \left(\sum_{s=1}^{k} c_{s0}a_{00}^{-1}a_{k-s,1} + a_{k1}\right)_{r,r+k-1} \\
&= \left(\sum_{s=0}^{k-1} c_{s0}a_{00}^{-1}a_{k-s,1} + c_{k0}a_{00}^{-1}a_{01}\right)_{r,r+k-1} \\
&= \sum_{s=0}^{k-1}\tfrac{(-1)^s(k+r-1)!}{s!(k-s-1)!}(\beta)_{r+k-1}\big(\alpha+\tfrac{k+r}{k-s}(\beta+r+k-1)\big) + \frac{(-1)^k(r+k)!}{k!}(\beta)_{r+k} \\
&= \alpha(\beta)_{r+k-1}\tfrac{(k+r-1)!}{(k-1)!}\sum_{s=0}^{k-1}(-1)^s\tbinom{k-1}{s} + (\beta)_{k+r}\big(\sum_{s=0}^{k-1}\tfrac{(-1)^s(k+r)!}{s!(k-s)!}+\tfrac{(-1)^k(k+r)!}{k!}\big) \\
&= \tfrac{(k+r)!}{k!}(\mathbb{S}(z_2)^{\beta+k+(k-l)}\beta)_{k+r}\sum_{s=0}^{k}(-1)^s\binom{k}{s}.
\end{aligned}$$

Therefore $(a_{00}^{1/2}\widetilde{\widetilde{a}}_{k1}a_{00}^{1/2})_{r,r+k-1}=0$. Now, $c_{00}=a_{00}$ and $(c_{k0}a_{00}^{-1}a_{01})_{n-k+1,n}=0$ for $2\leq k\leq n+1$. Hence

$$
\begin{aligned}
(a_{00}^{1/2}\widetilde{\widetilde{a}}_{k1}a_{00}^{1/2})_{n-k+1,n} &= \Big(\sum_{s=1}^{k} c_{s0}a_{00}^{-1}a_{k-s,1}+a_{k1}\Big)_{n-k+1,n}\\
&= \Big(\sum_{s=0}^{k-1} c_{s0}a_{00}^{-1}a_{k-s,1}\Big)_{n-k+1,n}\\
&= \sum_{s=0}^{k-1}\tfrac{(-1)^s(k+(n-k+1)-1)!}{s!(k-s-1)!}(\beta)_n\Big(\alpha+\tfrac{k+(n-k+1)}{k-s}(\beta+n)\Big)\\
&= n!(\beta)_n\Big(\alpha\sum_{s=0}^{k-1}\tfrac{(-1)^s}{s!(k-1-s)!}+(n+1)(\beta+n)\sum_{s=0}^{k-1}\tfrac{(-1)^s}{s!(k-s)!}\Big)\\
&= n!(\beta)_n\Big(\tfrac{\alpha}{(k-1)!}\sum_{s=0}^{k-1}(-1)^s\tbinom{k-1}{s}+\tfrac{(n+1)(\beta+n)}{k!}\sum_{s=0}^{k}(-1)^s\tbinom{k}{s}-\tfrac{(-1)^k(n+1)(\beta+n)}{k!}\Big)\\
&= 0+0-n!(\beta)_n\tfrac{(-1)^k(n+1)(\beta+n)}{k!}\\
&= \tfrac{(-1)^{k+1}(n+1)!(\beta)_{n+1}}{k!},\ \text{ for } 2\leq k\leq n+1.
\end{aligned}
$$

Since a_{00} is a diagonal matrix with positive diagonal entries, $\widetilde{\widetilde{a}}_{k1}$ has the form as stated in the lemma, for $2\leq k\leq n+1, n\geq 1$. □

Here is a simple lemma from matrix theory which will be useful for us in the sequel.

Lemma 4.9. *Let $\{A_k\}_{k=0}^{n-1}$ are $(n+1)\times(n+1)$ matrices such that $(A_k)_{kn}=\lambda_k\neq 0$ for $0\leq k\leq n-1$, $n\geq 1$. If $AA_k=A_kA$ for some $(n+1)\times(n+1)$ matrix $A=(\!(a_{ij})\!)_{i,j=0}^{n}$ for $0\leq k\leq n-1$, then A is upper triangular with equal diagonal entries.*

Proof. $(AA_k)_{in}=a_{ik}(A_k)_{kn}=a_{ik}\lambda_k$ and $(A_kA)_{kj}=(A_k)_{kn}a_{nj}=\lambda_ka_{nj}$ for $0\leq i,j\leq n, 0\leq k\leq n-1$. Putting $i=k$ and $j=n$, we have $(AA_k)_{kn}=a_{kk}\lambda_k$ and $(A_kA)_{kn}=\lambda_ka_{nn}$. By hypothesis we have $a_{kk}\lambda_k=\lambda_ka_{nn}$. As $\lambda_k\neq 0$, this implies that $a_{kk}=a_{nn}$ for $0\leq k\leq n-1$, which is same as saying that A has equal diagonal entries. Now observe that $(A_kA)_{ij}=0$ if $i\neq k$ for $0\leq j\leq n$, which implies that $(A_kA)_{in}=0$ if $i\neq k$. By hypothesis this is same as $(AA_k)_{in}=a_{ik}\lambda_k=0$ if $i\neq k$. This implies $a_{ik}=0$ if $i\neq k, 0\leq i\leq n, 0\leq k\leq n-1$, which is a stronger statement than saying A is upper triangular. □

Lemma 4.10. *If an $(n+1)\times(n+1)$ matrix A commutes with $\widetilde{\widetilde{a}}_{k1}$ and $\widetilde{\widetilde{a}}_{1k}$ for $2\leq k\leq n+1, n\geq 1$, then A is a scalar.*

Proof. It follows from Lemma 4.8 and Lemma 4.9 that if A commutes with $\widetilde{\widetilde{a}}_{k1}$ for $2\leq k\leq n+1$, then A is upper triangular with equal diagonal entries. As the entries of $\widetilde{\widetilde{a}}_{k1}$ are real, $(\widetilde{\widetilde{a}}_{1k})=(\widetilde{\widetilde{a}}_{k1})^t$. If A commutes with $\widetilde{\widetilde{a}}_{1k}$ for $2\leq k\leq n+1$, then by a similar proof as in Lemma 4.9, it follows that A is lower triangular with equal diagonal entries. So A is both upper triangular and lower triangular with equal diagonal entries, hence A is a scalar. □

This sequence of Lemmas put together constitutes a proof of Theorem 1.

For homogeneous operators in the class $\mathrm{B}_1(\mathbb{D})$, we have a proof of reducibility that avoids the normalization of the kernel. This proof makes use of the fact that if such an operator is reducible then each of the direct summands must belong to the class $\mathrm{B}_1(\mathbb{D})$. We give a precise formulation of this phenomenon along with a proof below. Let K be a positive definite kernel on $\mathbb{D}^2$ and $\mathcal{H}$ be the corresponding Hilbert space. Assume that the pair (M_1, M_2) on $\mathcal{H}$ is in $\mathrm{B}_1(\mathbb{D}^2)$. The operator M^* is the adjoint of the multiplication operator on Hilbert space $J^{(2)}\mathcal{H}_{|\mathrm{res}\ \triangle}$ which consists of $\mathbb{C}^2$ - valued holomorphic function on $\mathbb{D}$ and possesses the reproducing kernel $J^{(2)}K(z,w)$. The operator M^* is in $\mathrm{B}_2(\mathbb{D})$ (cf. [12, Proposition 3.6]).

Proposition 4.11. *The operator M^* on Hilbert space $J^{(2)}\mathcal{H}_{|\mathrm{res}\ \triangle}$ is irreducible.*

Proof. If possible, let M^* be reducible, that is, $M^* = T_1 \oplus T_2$ for some $T_1, T_2 \in \mathrm{B}_1(\mathbb{D})$ by [9, Proposition 1.18], which is same as saying by [9, Proposition 1.18] that the associated bundle E_{M^*} is reducible. A metric on the associated bundle E_{M^*} is given by $h(z) = J^{(2)}K(z,z)^t$. So, there exists a holomorphic change of frame $\psi : \mathbb{D} \longrightarrow GL(2,\mathbb{C})$ such that $\overline{\psi(z)}^t h(z)\psi(z) = \begin{pmatrix} h_1(z) & 0 \\ 0 & h_2(z) \end{pmatrix}$ for $z \in \mathbb{D}$, where h_1 and h_2 are metrics on the associated line bundles E_{T_1} and E_{T_2} respectively. So $\psi(z)^{-1}\mathcal{K}_h(z)\psi(z) = \begin{pmatrix} \mathcal{K}_{h_1}(z) & 0 \\ 0 & \mathcal{K}_{h_2}(z) \end{pmatrix}$, where $\mathcal{K}_h(z) = \bar\partial(h^{-1}\partial h)(z)$ is the curvature of the bundle E_{M^*} with respect to the metric h and $\mathcal{K}_{h_i}(z)$ are the curvatures of the bundles E_{T_i} for $i = 1, 2$ as in [9, pp. 211]. A direct computation shows that $\mathcal{K}_h(z) = \begin{pmatrix} \alpha & -2\beta(\beta+1)(1-|z|^2)^{-1}\bar z \\ 0 & \alpha + 2\beta + 2 \end{pmatrix}(1-|z|^2)^{-2}$. Thus the matrix $\psi(z)$ diagonalizes $\mathcal{K}_h(z)$ for $z \in \mathbb{D}$. It follows that $\psi(z)$ is determined, that is, the columns of $\psi(z)$ are eigenvectors of $\mathcal{K}_h(z)$ for $z \in \mathbb{D}$. These are uniquely determined upto multiplication by non-vanishing scalar valued functions f_1 and f_2 on $\mathbb{D}$. Now one set of eigenvectors of $\mathcal{K}_h(z)$ is given by $\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\beta\bar z \\ 1-|z|^2 \end{pmatrix}\}$ and it is clear that there does not exist any non-vanishing scalar valued function f_2 on $\mathbb{D}$ such that $f_2(z)\begin{pmatrix} -\beta\bar z \\ 1-|z|^2 \end{pmatrix}$ is an eigenvector for $\mathcal{K}_h(z)$ whose entries are holomorphic functions on $\mathbb{D}$. Hence there does not exist any holomorphic change of frame $\psi : \mathbb{D} \longrightarrow GL(2,\mathbb{C})$ such that $\overline{g}^t hg = \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}$ on $\mathbb{D}$. Hence M^* is irreducible. □

Theorem 4.12. *The operators $T = M^{(\alpha,\beta)}$ and $\widetilde{T} := M^{(\widetilde{\alpha},\widetilde{\beta})}$ are unitarily equivalent if and only if $\alpha = \widetilde{\alpha}$ and $\beta = \widetilde{\beta}$.*

One of the implications is trivial. To prove the other implication, recall that [12, Proposition 3.6] $T, \widetilde{T} \in \mathrm{B}_{n+1}(\mathbb{D})$. It follows from [9] that if $T, \widetilde{T} \in \mathrm{B}_{n+1}(\mathbb{D})$ are unitarily equivalent then the curvatures $\mathcal{K}_T, \mathcal{K}_{\widetilde{T}}$ of the associated bundles E_T and $E_{\widetilde{T}}$ respectively, are unitarily equivalent as matrix-valued real-analytic functions on $\mathbb{D}$. In particular, this implies that $\mathcal{K}_T(0)$ and $\mathcal{K}_{\widetilde{T}}(0)$ are unitarily equivalent. Therefore, we compute $\mathcal{K}_T(0)$ and $\mathcal{K}_{\widetilde{T}}(0)$. Let $\mathcal{K}_T(h)$ denote the curvature of the bundle E_T with respect to the metric $h(z) := \widetilde{\widetilde{K}}(z,z)^t$.

Lemma 4.13. *The curvature $\mathcal{K}_T(h)(0)$ at 0 of the bundle E_T equals the coefficient of $z\bar z$ in the normalized kernel $\widetilde{\widetilde{K}}$, that is, $\mathcal{K}_T(h)(0) = \widetilde{\widetilde{a}}_{11}^t$.*

Proof. The curvature of the bundle E_T with respect to the metric $h(z) := \widetilde{\widetilde{K}}(z,z)^t$ is $\mathcal{K}_T(h) = \bar\partial(h^{-1}\partial h)$. If $h(z) = \sum_{m,n\ge 0} h_{mn} z^m \bar z^n$, then $h_{mn} = \widetilde{\widetilde{a}}_{mn}^t$ for $m, n \ge 0$. So, $h_{00} = I$ and $h_{m0} = h_{0n} = 0$ for $m, n \ge 1$. Hence $\mathcal{K}_T(h)(0) = \bar\partial h^{-1}(0)\partial h(0) + h^{-1}(0)\bar\partial\partial h(0) = (\bar\partial h^{-1}(0))h_{10} + h_{00}^{-1}h_{11} = h_{11} = \widetilde{\widetilde{a}}_{11}^t$. □

Lemma 4.14. $\big(\mathcal{K}_T(0)\big)_{ii} = \alpha$, *for $i = 0, \ldots, n-1$ and* $\big(\mathcal{K}_T(0)\big)_{nn} = \alpha + (n+1)(\beta + n)$ *for* $n \ge 1$.

Proof. From Lemma 4.13 and Lemma 4.5 we know that $\mathcal{K}_T(0) = \widetilde{\widetilde{a}}_{11}^{\,t} = \big(\widetilde{a}_{11} + \widetilde{b}_{10}\widetilde{a}_{01}\big)^t$. Thus $\mathcal{K}_T(0)$ is the transpose of $a_{00}^{-1/2}(a_{11} + c_{10}a_{00}^{-1}a_{01})a_{00}^{-1/2}$ by Lemma 4.7. Now, by Lemma 4.6 and Lemma 4.7, $\big(c_{10}\big)_{r,r+1} = -(r+1)!(\beta)_{r+1}$ for $0 \le r \le n-1$, $\big(a_{00}\big)_{rr} = r!(\beta)_r$, $\big(a_{11}\big)_{rr} = r!(\beta)_r\big(\alpha + (r+1)(\beta + r)\big)$ for $0 \le r \le n$ and $\big(a_{01}\big)_{r+1,r} = (r+1)!(\beta)_{r+1}$ for $0 \le r \le n-1$. Therefore, $\big(c_{10}a_{00}^{-1}a_{01}\big)_{rr} = -(r+1)!(\beta)_{r+1}$ for $0 \le r \le n-1$. Also, $\big(a_{11} + c_{10}a_{00}^{-1}a_{01}\big)_{rr} = \alpha r!(\beta)_{r+1}$ for $0 \le r \le n-1$, and $\big(a_{11} + c_{10}a_{00}^{-1}a_{01}\big)_{nn} = n!(\beta)_n(\alpha + (n+1)(\beta + n))$. Finally, $\mathcal{K}_T(h)(0) = \widetilde{\widetilde{a}}_{11}^{\,t} = \widetilde{\widetilde{a}}_{11}$, as $\widetilde{\widetilde{a}}_{11}$ is a diagonal matrix with real entries. In fact, $\big(\mathcal{K}_T(0)\big)_{ii} = \alpha$, for $i = 0, \ldots, n-1$ and $\big(\mathcal{K}_T(0)\big)_{nn} = \alpha + (n+1)(\beta + n)$. ∎

We now see that M and $\widetilde{M}$ are unitarily equivalent implies that $\alpha = \widetilde{\alpha}$ and $\alpha + (n+1)(\beta + n) = \widetilde{\alpha} + (n+1)(\widetilde{\beta} + n)$, that is, $\alpha = \widetilde{\alpha}$ and $\beta = \widetilde{\beta}$.

5. Homogeneity of the operator $M^{(\alpha,\beta)}$

Theorem 5.1. *The multiplication operator $M := M^{(\alpha,\beta)}$ on $J^{(n+1)}\mathcal{H}$ is homogeneous.*

This theorem is a particular case of the Lemma 4.1. A proof first appeared in [6, Theorem 5.2.]. We give an alternative proof of this Theorem by showing that that the kernel is quasi-invariant, that is,

$$K(z,w) = J_{\varphi^{-1}}(z)K\big(\varphi^{-1}(z), \varphi^{-1}(w)\big)\overline{J_{\varphi^{-1}}(w)}^t$$

for some cocycle

$$J : \text{Möb} \times \mathbb{D} \longrightarrow \mathbb{C}^{(n+1)\times(n+1)},\ \varphi \in \text{Möb},\ z,\ w \in \mathbb{D}.$$

First we prove that $K(z,z) = J_{\varphi^{-1}}(z)K\big(\varphi^{-1}(z), \varphi^{-1}(z)\big)\overline{J_{\varphi^{-1}}(z)}^t$ and then polarize to obtain the final result. We begin with a series of lemmas.

Lemma 5.2. *Suppose that $J : \text{Möb} \times \mathbb{D} \longrightarrow \mathbb{C}^{(n+1)\times(n+1)}$ is a cocycle. Then the following are equivalent*

(1) $K(z,z) = J_{\varphi^{-1}}(z)K\big(\varphi^{-1}(z), \varphi^{-1}(z)\big)\overline{J_{\varphi^{-1}}(z)}^t$ *for all $\varphi \in$ Möb and $z \in \mathbb{D}$;*
(2) $K(0,0) = J_{\varphi^{-1}}(0)K\big(\varphi^{-1}(0), \varphi^{-1}(0)\big)\overline{J_{\varphi^{-1}}(0)}^t$ *for all $\varphi \in$ Möb.*

Proof. One of the implications is trivial. To prove the other implication, note that

$$\begin{aligned} J_{\varphi_1^{-1}}(0)K\big(\varphi_1^{-1}(0), \varphi_1^{-1}(0)\big)\overline{J_{\varphi_1^{-1}}(0)}^t &= K(0,0) \\ &= J_{\varphi_2^{-1}}(0)K\big(\varphi_2^{-1}(0), \varphi_2^{-1}(0)\big)\overline{J_{\varphi_2^{-1}}(0)}^t \end{aligned}$$

for any $\varphi_1, \varphi_2 \in$Möb and $z \in \mathbb{D}$. Now pick $\psi \in$Möb such that $\psi^{-1}(0) = z$ and taking $\varphi_1 = \psi, \varphi_2 = \psi\varphi$ in the previous identity we see that

$$\begin{aligned} &J_{\psi^{-1}}(0)K\big(\psi^{-1}(0), \psi^{-1}(0)\big)\overline{J_{\psi^{-1}}(0)}^t \\ &= J_{\varphi^{-1}\psi^{-1}}(0)K\big(\varphi^{-1}\psi^{-1}(0), \varphi^{-1}\psi^{-1}(0)\big)\overline{J_{\varphi^{-1}\psi^{-1}}(0)}^t \\ &= J_{\psi^{-1}}(0)J_{\varphi^{-1}}(\psi^{-1}(0))K\big(\varphi^{-1}(z), \varphi^{-1}(z)\big)\overline{J_{\varphi^{-1}}(\psi^{-1}(0))}^t\overline{J_{\psi^{-1}}(0)}^t \end{aligned}$$

for $\varphi \in \text{Möb}, z \in \mathbb{D}$. Since $J_{\psi^{-1}}(0)$ is invertible, it follows from the equality of first and third quantities that

$$K\big(\psi^{-1}(0), \psi^{-1}(0)\big) = J_{\varphi^{-1}}(\psi^{-1}(0))K\big(\varphi^{-1}\psi^{-1}(0), \varphi^{-1}\psi^{-1}(0)\big)\overline{J_{\varphi^{-1}}(\psi^{-1}(0))}^t.$$

This is the same as $K(z,z) = J_{\varphi^{-1}}(z)K\big(\varphi^{-1}(z), \varphi^{-1}(z)\big)\overline{J_{\varphi^{-1}}(z)}^t$ by the choice of ψ. The proof of this lemma is therefore complete. ∎

Let $\mathcal{J}_{\varphi^{-1}}(z) = (J_{\varphi^{-1}}(z)^t)^{-1}, \varphi \in$Möb, $z \in \mathbb{D}$, where X^t denotes the transpose of the matrix X. Clearly, $(J_{\varphi^{-1}}(z)^t)^{-1}$ satisfies the cocycle property if and only if $\mathcal{J}_{\varphi^{-1}}(z)$ does and they uniquely determine each other. It is easy to see that the condition

$$K(0,0) = J_{\varphi^{-1}}(0)K\big(\varphi^{-1}(0), \varphi^{-1}(0)\big)\overline{J_{\varphi^{-1}}(0)}^t$$

is equivalent to

$$h\big(\varphi^{-1}(0)\big) = \overline{\mathcal{J}_{\varphi^{-1}}(0)}^t h(0)\mathcal{J}_{\varphi^{-1}}(0), \tag{5.16}$$

where $h(z)$ is the transpose of $K(z,z)$ as before. It will be useful to define the two functions

(i) $c : \text{Möb}\times\mathbb{D} \longrightarrow \mathbb{C}$, $c(\varphi^{-1}, z) = (\varphi^{-1})'(z)$;
(ii) $p : \text{Möb}\times\mathbb{D} \longrightarrow \mathbb{C}$, $p(\varphi^{-1}, z) = \frac{\overline{ta}}{1+\overline{ta}z}$

for $\varphi_{t,a} \in$ Möb, $t \in \mathbb{T}$, $a \in \mathbb{D}$. We point out that the function c is the well-known cocycle for the group Möb.

Lemma 5.3. *With notation as above, we have*

(a) $\varphi_{t,a}^{-1} = \varphi_{\bar{t},-ta}$
(b) $\varphi_{s,b}\varphi_{t,a} = \varphi_{\frac{s(t+\bar{a}b)}{1+ta\bar{b}}, \frac{a+\bar{t}b}{1+ta\bar{b}}}$
(c) $c(\varphi^{-1}, \psi^{-1}(z))c(\psi^{-1}(z)) = c(\varphi^{-1}\psi^{-1}, z)$ *for* $\varphi, \psi \in$Möb, $z \in \mathbb{D}$
(d) $p(\varphi^{-1}, \psi^{-1}(z))c(\psi^{-1}, z) + p(\psi^{-1}, z) = p(\varphi^{-1}\psi^{-1}, z)$ for $\varphi, \psi \in$Möb, $z \in \mathbb{D}$.

Proof. The proof of (a) is a mere verification. We note that

$$\varphi_{s,b}(\varphi_{t,a}(z)) = s\frac{t\frac{z-a}{1-\bar{a}z} - b}{1 - \bar{b}t\frac{z-a}{1-\bar{a}z}} = s\frac{tz - ta - b + \bar{a}bz}{1 - \bar{a}z - t\bar{b}z + ta\bar{b}} = \frac{s(t+\bar{a}b)}{1+ta\bar{b}}\frac{z - \frac{ta+b}{t+\bar{a}b}}{1 - \frac{\bar{a}+t\bar{b}}{1+ta\bar{b}}z},$$

which is (b). The chain rule gives (c). To prove (d), we first note that for $\varphi = \varphi_{t,a}$ and $\psi = \varphi_{s,b}$, if $\psi^{-1}\varphi^{-1} = \varphi_{t',a'}$ for some $(t', a') \in \mathbb{T}\times\mathbb{D}$ then

$$\overline{t'a'} = \frac{\bar{s}(\bar{t} + a\bar{b})}{1+\overline{ta}b}\frac{\bar{a}+t\bar{b}}{1+ta\bar{b}} = \frac{\bar{s}(\bar{b}+\overline{ta})}{1+\overline{ta}b}.$$

It is now easy to verify that

$$\begin{aligned} p(\varphi^{-1}, \psi^{-1}(z))c(\psi^{-1}, z) + p(\psi^{-1}, z) &= \frac{\overline{ta}}{1+\overline{ta}\psi_{s,b}^{-1}(z)}\frac{\bar{s}(1-|b|^2)}{(1+\overline{sb}z)^2} + \frac{\overline{sb}}{1+\overline{sb}z} \\ &= \frac{\bar{s}(\bar{b}+\overline{ta})}{1+\overline{ta}b + \bar{s}(\bar{b}+\overline{ta})z} \\ &= \frac{\bar{s}\frac{\bar{b}+\overline{ta}}{1+\overline{ta}b}}{1+\frac{\bar{s}(\bar{b}+\overline{ta})}{1+\overline{ta}b}z} \\ &= p(\varphi^{-1}\psi^{-1}, z). \end{aligned}$$

□

Let

$$\big(\mathcal{J}_{\varphi^{-1}}(z)\big)_{ij} = c(\varphi^{-1}, z)^{-\frac{\alpha+\beta}{2}-n}\frac{(\beta)_j}{(\beta)_i}\binom{j}{i}c(\varphi^{-1}, z)^{n-j}p(\varphi^{-1}, z)^{j-i} \tag{5.17}$$

for $0 \le i \le j \le n$.

Lemma 5.4. $J_{\varphi^{-1}}(z)$ *defines a cocycle for the group Möb.*

Proof. To say that $J_{\varphi^{-1}}(z)$ satisfies the cocycle property is the same as saying $\mathcal{J}_{\varphi^{-1}}(z)$ satisfies the cocycle property, which is what we will verify. Thus we want to show that $\big(\mathcal{J}_{\psi^{-1}}(z)\mathcal{J}_{\varphi^{-1}}(\psi^{-1}(z))\big)_{ij} = \big(\mathcal{J}_{\varphi^{-1}\psi^{-1}}(z)\big)_{ij}$ for $0 \leq i,j \leq n$. We note that $\mathcal{J}_{\varphi^{-1}}(z)$ is upper triangular, as the product of two upper triangular matrices is again upper triangular, it suffices to prove this equality for $0 \leq i \leq j \leq n$. Clearly, we have

$$
\begin{aligned}
&\big(\mathcal{J}_{\psi^{-1}}(z)\mathcal{J}_{\varphi^{-1}}(\psi^{-1}(z))\big)_{ij} = \sum_{k=i}^{j}\big(\mathcal{J}_{\psi^{-1}}(z)\big)_{ik}\big(\mathcal{J}_{\varphi^{-1}}(\psi^{-1}(z))\big)_{kj}\\
&= c(\psi^{-1},z)^{-\frac{\alpha+\beta}{2}-n}c(\varphi^{-1},\psi^{-1}(z))^{-\frac{\alpha+\beta}{2}-n}\sum_{k=i}^{j}\Big(\frac{(\beta)_k}{(\beta)_i}\binom{k}{i}c(\psi^{-1},z)^{n-k}\\
&p(\psi^{-1},z)^{k-i}\frac{(\beta)_j}{(\beta)_k}\binom{j}{k}c(\varphi^{-1},\psi^{-1}(z))^{n-j}p(\varphi^{-1},\psi^{-1}(z))^{j-k}\Big)\\
&= c(\varphi^{-1}\psi^{-1},z)^{-\frac{\alpha+\beta}{2}-n}\frac{(\beta)_j}{(\beta)_i}c(\psi^{-1},z)^{n-j}c(\varphi^{-1},\psi^{-1}(z))^{n-j}\\
&\qquad\sum_{k=i}^{j}\frac{j!}{i!(k-i)!(j-k)!}c(\psi^{-1},z)^{j-k}p(\varphi^{-1},\psi^{-1}(z))^{j-k}p(\psi^{-1},z)^{k-i}\\
&= c(\varphi^{-1}\psi^{-1},z)^{-\frac{\alpha+\beta}{2}-n}\frac{(\beta)_j}{(\beta)_i}\binom{j}{i}c(\varphi^{-1}\psi^{-1},z)^{n-j}\\
&\qquad\sum_{k=i}^{j}\binom{j-i}{k-i}c(\psi^{-1},z)^{j-k}p(\varphi^{-1},\psi^{-1}(z))^{j-k}p(\psi^{-1},z)^{k-i}\\
&= c(\varphi^{-1}\psi^{-1},z)^{-\frac{\alpha+\beta}{2}-n}\frac{(\beta)_j}{(\beta)_i}\binom{j}{i}c(\varphi^{-1}\psi^{-1},z)^{n-j}\\
&\qquad\sum_{k=0}^{j-i}\binom{j-i}{k}c(\psi^{-1},z)^{(j-i)-k}p(\varphi^{-1},\psi^{-1}(z))^{(j-i)-k}p(\psi^{-1},z)^{k}\\
&= c(\varphi^{-1}\psi^{-1},z)^{-\frac{\alpha+\beta}{2}-n}\frac{(\beta)_j}{(\beta)_i}\binom{j}{i}c(\varphi^{-1}\psi^{-1},z)^{n-j}\\
&\qquad\qquad\Big(c(\psi^{-1},z)p(\varphi^{-1},\psi^{-1}(z))+p(\psi^{-1},z)\Big)^{j-i}\\
&= c(\varphi^{-1}\psi^{-1},z)^{-\frac{\alpha+\beta}{2}-n}\frac{(\beta)_j}{(\beta)_i}\binom{j}{i}c(\varphi^{-1}\psi^{-1},z)^{n-j}p(\varphi^{-1}\psi^{-1},z)^{j-i}\\
&= \big(\mathcal{J}_{\varphi^{-1}\psi^{-1}}(z)\big)_{ij},
\end{aligned}
$$

for $0 \leq i \leq j \leq n$. The penultimate equality follows from Lemma 5.3. □

We need the following beautiful identity to prove (5.16). We provide two proofs, the first one is due to C. Verughese and the second is due to B. Bagchi.

Lemma 5.5. *For nonnegative integers $j \geq i$ and $0 \leq k \leq i$, we have*

$$\sum_{l=0}^{i-k}(-1)^l(l+k)!\binom{i}{l+k}\binom{j}{l+k}\binom{l+k}{l}(a+j)_{i-l-k} = k!\binom{i}{k}\binom{j}{k}(a+k)_{i-k},$$

for all $a \in \mathbb{C}$.

Proof. Here is the first proof due to C. Verughese: For any integer $i \geq 1$ and $a \in \mathbb{C} \setminus \mathbb{Z}$, we have

$$\begin{aligned}
&\sum_{l=0}^{i-k}(-1)^l (l+k)! \binom{i}{l+k}\binom{j}{l+k}\binom{l+k}{l}(a+j)_{i-l-k} \\
&= \frac{i!j!}{k!\Gamma(a+j)}\sum_{l=0}^{i-k}\frac{(-1)^l}{l!(i-k-l)!}\frac{\Gamma(a+j+i-l-k)}{\Gamma(j-l-k+1)} \\
&= \frac{i!j!}{k!(i-k)!\Gamma(a+j)\Gamma(1-a-i)}\sum_{l=0}^{i-k}(-1)^l \tbinom{i-k}{l} B(a+j+i-k-l, 1-a-i) \\
&= \frac{i!j!}{k!(i-k)!\Gamma(a+j)\Gamma(1-a-i)}\sum_{l=0}^{i-k}(-1)^l \tbinom{i-k}{l}\int_0^1 t^{a+j+i-k-l-1}(1-t)^{-a-i}dt \\
&= \frac{i!j!}{k!(i-k)!\Gamma(a+j)\Gamma(1-a-i)}\int_0^1 \sum_{l=0}^{i-k}(-1)^l \tbinom{i-k}{l} t^{a+j+i-k-l-1}(1-t)^{-a-i}dt \\
&= \frac{i!j!}{k!(i-k)!\Gamma(a+j)\Gamma(1-a-i)}\int_0^1 (1-t)^{-a-i}t^{a+j-1}\Big(\sum_{l=0}^{i-k}(-1)^l \tbinom{i-k}{l} t^{i-k-l}\Big)dt \\
&= \frac{i!j!}{k!(i-k)!\Gamma(a+j)\Gamma(1-a-i)}\int_0^1 (1-t)^{-a-i}t^{a+j-1}(t-1)^{i-k}dt \\
&= \frac{(-1)^{i-k}i!j!}{k!(i-k)!\Gamma(a+j)\Gamma(1-a-i)}\int_0^1 (1-t)^{-a-i}t^{a+j-1}(1-t)^{i-k}dt \\
&= \frac{(-1)^{i-k}i!j!}{k!(i-k)!\Gamma(a+j)\Gamma(1-a-i)}B(a+j, 1-a-k) \\
&= \frac{(-1)^{i-k}i!j!}{k!(i-k)!\Gamma(a+j)\Gamma(1-a-i)}\frac{\Gamma(a+j)\Gamma(1-a-k)}{\Gamma(1+j-k)} \\
&= \frac{(-1)^{i-k}i!j!}{k!(i-k)!\Gamma(1-a-i)}\frac{\Gamma(1-a-k)}{(j-k)!} \\
&= k!\binom{i}{k}\binom{j}{k}\frac{\Gamma(1-a)}{(-1)^i\Gamma(1-a-i)}\frac{(-1)^k\Gamma(1-a-k)}{\Gamma(1-a)} \\
&= k!\binom{i}{k}\binom{j}{k}\frac{\Gamma(1-a)}{\cos i\pi\Gamma(1-a-i)}\frac{\cos k\pi\Gamma(1-a-k)}{\Gamma(1-a)} \\
&= k!\binom{i}{k}\binom{j}{k}\frac{\Gamma(1-a)\Gamma(a+i)}{\cos i\pi\Gamma(1-a-i)\Gamma(a+i)}\frac{\cos k\pi\Gamma(1-a-k)\Gamma(a+k)}{\Gamma(a+k)\Gamma(1-a)} \\
&= k!\binom{i}{k}\binom{j}{k}\frac{\Gamma(1-a)\Gamma(a+i)\sin(a+i)\pi}{\pi\cos(i\pi)}\frac{\pi\cos k\pi}{\sin(a+k)\pi\Gamma(a+k)\Gamma(1-a)} \\
&= k!\binom{i}{k}\binom{j}{k}\frac{\Gamma(a+i)}{\Gamma(a+k)}.
\end{aligned}$$

Since we have an equality involving a polynomial of degree $i-k$ for all a in $\mathbb{C} \setminus \mathbb{Z}$, it follows that the equality holds for all $a \in \mathbb{C}$.

Here is another proof due to B. Bagchi: Since $\binom{-x}{n} = \frac{-x(-x-1)\cdots(-x-n+1)}{n!} = (-1)^n\binom{x+n-1}{n}$ and $(x)_n = x(x+1)\cdots(x+n-1) = n!\binom{x+n-1}{n}$, it follows that

$$\begin{aligned}
&\sum_{l=0}^{i-k}(-1)^l(l+k)!\binom{i}{l+k}\binom{j}{l+k}\binom{l+k}{l}(a+j)_{i-l-k}\\
&= \frac{i!j!}{k!}\sum_{l=0}^{i-k}\frac{(-1)^l}{l!(i-k-l)!(j-k-l)!}(i-k-l)!\binom{a+j+i-k-l-1}{i-k-l}\\
&= \frac{i!j!}{k!(j-k)!}\sum_{l=0}^{i-k}\frac{(-1)^l(j-k)!}{l!(j-k-l)!}(-1)^{i-k-l}\binom{-a-j}{i-k-l}\\
&= i!\binom{j}{k}(-1)^{i-k}\sum_{l=0}^{i-k}\binom{j-k}{l}\binom{-a-j}{i-k-l}\\
&= i!\binom{j}{k}(-1)^{i-k}\binom{-a-k}{i-k}\\
&= i!\binom{j}{k}(-1)^{i-k}(-1)^{i-k}\binom{a+i-1}{i-k}\\
&= k!\binom{i}{k}\binom{j}{k}(a+k)_{i-k},
\end{aligned}$$

where the equality after the last summation symbol follows from Vandermonde's identity. □

Lemma 5.6. *For $\varphi \in$Möb and $\mathcal{J}_{\varphi^{-1}}(z)$ as in (5.17),*

$$h\big(\varphi^{-1}(0)\big) = \overline{\mathcal{J}_{\varphi^{-1}}(0)}^t h(0)\mathcal{J}_{\varphi^{-1}}(0).$$

Proof. It is enough to show that

$$h\big(\varphi^{-1}(0)\big)_{ij} = \big(\overline{\mathcal{J}_{\varphi^{-1}}(0)}^t h(0)\mathcal{J}_{\varphi^{-1}}(0)\big)_{ij}, \text{ for } 0 \le i \le j \le n.$$

Let $\varphi = \varphi_{t,z}$, $t \in \mathbb{T}$, and $z \in \mathbb{D}$. Since $\big(h\big(\varphi^{-1}(0)\big)\big)_{ij} = \big(h(z)\big)_{ij}$, it follows that

$$\begin{aligned}
\big(h\big(\varphi^{-1}(0)\big)\big)_{ij} &= \bar\partial_2^i\partial_2^j\big(\mathbb{S}(z_1)^\alpha\mathbb{S}(z_2)^\beta\big)|_\triangle\\
&= (\beta)_j\mathbb{S}(z_1)^\alpha\bar\partial_2^i\big(\mathbb{S}(z_2)^{\beta+j}\bar z_2^j\big)|_\triangle\\
&= (\beta)_j\mathbb{S}(z_1)^\alpha\sum_{r=0}^{i}\binom{i}{r}\bar\partial_2^{(i-r)}\big(\mathbb{S}(z_2)^{\beta+j}\big)\bar\partial_2^r(\bar z_2^j)|_\triangle\\
&= (\beta)_j\mathbb{S}(z_1)^\alpha\sum_{r=0}^{i}\binom{i}{r}(\beta+j)_{i-r}\mathbb{S}(z_2)^{\beta+j+(i-r)}z_2^{i-r}r!\binom{j}{r}\bar z_2^{j-r}|_\triangle\\
&= (\beta)_j\mathbb{S}(z)^{\alpha+\beta+i+j}\,\bar z^{j-i}\sum_{r=0}^{i}r!\binom{i}{r}\binom{j}{r}(\beta+j)_{i-r}\mathbb{S}(z)^{-r}|z|^{2(i-r)},
\end{aligned}$$

for $i \leq j$. Clearly, $\big(\mathcal{J}_{\varphi^{-1}}(0)\big)_{ij} = c(\varphi^{-1},0)^{-\frac{\alpha+\beta}{2}-n}\frac{(\beta)_j}{(\beta)_i}\binom{j}{i}c(\varphi^{-1},0)^{n-j}p(\varphi^{-1},0)^{j-i}$ and $h(0)_{ii} = i!(\beta)_i$, $0 \leq i \leq j \leq n$. We have

$$\begin{aligned}
\big(\overline{\mathcal{J}_{\varphi^{-1}}(0)}^t h(0)\mathcal{J}_{\varphi^{-1}}(0)\big)_{ij} &= \sum_{k=0}^{j}\big(\overline{\mathcal{J}_{\varphi^{-1}}(0)}^t h(0)\big)_{ik}\big(\mathcal{J}_{\varphi^{-1}}(0)\big)_{kj} \\
&= \sum_{k=0}^{i}\sum_{k=0}^{j}\big(\overline{\mathcal{J}_{\varphi^{-1}}(0)}^t\big)_{ik}\big(h(0)\big)_{kk}\big(\mathcal{J}_{\varphi^{-1}}(0)\big)_{kj} \\
&= \sum_{k=0}^{\min(i,j)}\big(\overline{\mathcal{J}_{\varphi^{-1}}(0)}^t\big)_{ik}\big(h(0)\big)_{kk}\big(\mathcal{J}_{\varphi^{-1}}(0)\big)_{kj}.
\end{aligned}$$

Now, for $0 \leq i \leq j \leq n$,

$$\begin{aligned}
\sum_{k=0}^{\min(i,j)}\big(\overline{\mathcal{J}_{\varphi^{-1}}(0)}^t\big)_{ik}\big(h(0)\big)_{kk}\big(\mathcal{J}_{\varphi^{-1}}(0)\big)_{kj} &= |c(\varphi^{-1},0)|^{-\alpha-\beta-2n} \\
&\quad \sum_{k=0}^{i}\Big(\frac{(\beta)_i}{(\beta)_k}\binom{i}{k}\overline{c(\varphi^{-1},0)}^{n-i}\overline{p(\varphi^{-1},0)}^{i-k}k!(\beta)_k\frac{(\beta)_j}{(\beta)_k} \\
&\qquad \binom{j}{k}c(\varphi^{-1},0)^{n-j}p(\varphi^{-1},0)^{j-k}\Big) \\
&= \mathbb{S}(z)^{\alpha+\beta+2n}\sum_{k=0}^{i}\frac{k!(\beta)_i(\beta)_j}{(\beta)_k}\binom{i}{k}\binom{j}{k} \\
&\qquad \big(t\mathbb{S}(z)\big)^{-n+i}(tz)^{i-k}\big(\bar{t}\mathbb{S}(z)\big)^{-n+j}(\overline{tz})^{j-k} \\
&= (\beta)_j\mathbb{S}(z)^{\alpha+\beta+i+j}\sum_{k=0}^{i}k!\binom{i}{k}\binom{j}{k}\frac{(\beta)_i}{(\beta)_k}z^{i-k}\bar{z}^{j-k} \\
&= (\beta)_j\mathbb{S}(z)^{\alpha+\beta+i+j}\bar{z}^{j-i}\sum_{k=0}^{i}k!\binom{i}{k}\binom{j}{k}\frac{(\beta)_i}{(\beta)_k}|z|^{2(i-k)}.
\end{aligned}$$

Clearly, to prove the desired equality we have to show that

$$\sum_{r=0}^{i}r!\binom{i}{r}\binom{j}{r}(\beta+j)_{i-r}\mathbb{S}(z)^{-r}|z|^{2(i-r)} = \sum_{k=0}^{i}k!\binom{i}{k}\binom{j}{k}\frac{(\beta)_i}{(\beta)_k}|z|^{2(i-k)} \tag{5.18}$$

for $0 \leq i \leq j \leq n$. But

$$\begin{aligned}
&\sum_{r=0}^{i}r!\binom{i}{r}\binom{j}{r}(\beta+j)_{i-r}(1-|z|^2)^r|z|^{2(i-r)} \\
&\quad = \sum_{r=o}^{i}r!\binom{i}{r}\binom{j}{r}(\beta+j)_{i-r}\sum_{l=0}^{r}(-1)^l\binom{r}{l}|z|^{2l}|z|^{2(i-r)} \\
&\quad = \sum_{l=0}^{i}\sum_{r=l}^{i}(-1)^l r!\binom{i}{r}\binom{j}{r}\binom{r}{l}(\beta+j)_{i-r}|z|^{2(i-(r-l))} \\
&\quad = \sum_{l=0}^{i}\sum_{r=0}^{i-l}(-1)^l(r+l)!\binom{i}{r+l}\binom{j}{r+l}\binom{r+l}{l}(\beta+j)_{i-r-l}|z|^{2(i-r)}.
\end{aligned}$$

For $0 \leq k \leq i-l$, the coefficient of $|z|^{2(i-k)}$ in the left hand side of (5.18) is

$$\sum_{l=0}^{i}(-1)^l(k+l)!\binom{i}{k+l}\binom{j}{k+l}\binom{k+l}{l}(\beta+j)_{i-k-l},$$

which is the same as

$$\sum_{l=0}^{i-k}(-1)^l(k+l)!\binom{i}{k+l}\binom{j}{k+l}\binom{k+l}{l}(\beta+j)_{i-k-l},$$

for $0 \leq l \leq i-k \leq i$. So, to complete the proof we have to show that

$$\sum_{l=0}^{i-k}(-1)^l(k+l)!\binom{i}{k+l}\binom{j}{k+l}\binom{k+l}{l}(\beta+j)_{i-k-l} = k!\binom{i}{k}\binom{j}{k}\frac{(\beta)_i}{(\beta)_k},$$

for $0 \leq k \leq i, i \leq j$. But this follows from Lemma 5.5. $\square$

6. The case of the tri-disc $\mathbb{D}^3$

We discuss the jet construction for $\mathbb{D}^3$. Let $K : \mathbb{D}^3 \times \mathbb{D}^3 \longrightarrow \mathbb{C}$ be a reproducing kernel. Following the jet construction of [12], we define

$$J^{(1,1)}K(z,w) = \begin{pmatrix} K(z,w) & \partial_2 K(z,w) & \partial_3 K(z,w) \\ \bar\partial_2 K(z,w) & \partial_2\bar\partial_2 K(z,w) & \bar\partial_2\partial_3 K(z,w) \\ \bar\partial_3 K(z,w) & \partial_2\bar\partial_3 K(z,w) & \bar\partial_3\partial_3 K(z,w) \end{pmatrix},\ z,\, w \in \mathbb{D}^3.$$

As before, to retain the usual meaning of ∂ and $\bar\partial$ we replace $J^{(1,1)}K(z,w)$ by its transpose. For simplicity of notation, we let $G(z,w) := J^{(1,1)}K(z,w)^t$. In this notation, choosing the kernel function K on $\mathbb{D}^3$ to be

$$K(z,w) = (1-z_1\bar w_1)^{-\alpha}(1-z_2\bar w_2)^{-\beta}(1-z_3\bar w_3)^{-\gamma},$$

we have

$$G(z,w) = \begin{pmatrix} (1-z\bar w)^2 & \beta z(1-z\bar w) & \gamma z(1-z\bar w) \\ \beta\bar w(1-z\bar w) & \beta(1+\beta z\bar w) & \beta\gamma z\bar w \\ \gamma\bar w(1-z\bar w) & \beta\gamma z\bar w & \gamma(1+\gamma z\bar w) \end{pmatrix}(1-z\bar w)^{-\alpha-\beta-\gamma-2},$$

for $z, w \in \mathbb{D},\ \alpha,\beta,\gamma > 0$.

Theorem 6.1. *The adjoint of the multiplication operator M^* on the Hilbert space of $\mathbb{C}^3$ valued holomorphic functions on $\mathbb{D}^3$ with reproducing kernel G is in* $\mathrm{B}_3(\mathbb{D})$*. It is homogeneous and reducible. Moreover, M^* is unitarily equivalent to $M_1^* \oplus M_2^*$ for a pair of irreducible homogeneous operators M_1^* and M_2^* from* $\mathrm{B}_1(\mathbb{D})$*.*

Proof. Although homogeneity of M^* follows from [6, Theorem 5.2.], we give an independent proof using the ideas we have developed in this note. Let

$$\tilde{\tilde{G}}(z,w) = G(0,0)^{1/2}G(z,0)^{-1}G(z,w)G(0,w)^{-1}G(0,0)^{1/2}.$$

Evidently, $\tilde{\tilde{G}}(z,0) = I$, that is, $\tilde{\tilde{G}}$ is a normalized kernel. The form of $\tilde{\tilde{G}}(z,w)$ for $z, w \in \mathbb{D}$ is $(1-z\bar w)^{-\alpha-\beta-\gamma-2}$ times

$$\begin{pmatrix} \begin{matrix}(1-z\bar w)^2-(\beta+\gamma)(1-z\bar w)z\bar w \\ +(\beta+\gamma)(1+\beta+\gamma)z^2\bar w^2\end{matrix} & -\sqrt{\beta}(1+\beta+\gamma)z^2\bar w & -\sqrt{\gamma}(1+\beta+\gamma)z^2\bar w \\ -\sqrt{\beta}(1+\beta+\gamma)z\bar w^2 & 1+\beta z\bar w & \sqrt{\beta\gamma}z\bar w \\ -\sqrt{\gamma}(1+\beta+\gamma)z\bar w^2 & \sqrt{\beta\gamma}z\bar w & 1+\gamma z\bar w \end{pmatrix}.$$

Let $U = \frac{1}{\sqrt{\beta+\gamma}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{\beta} & \sqrt{\gamma} \\ 0 & -\sqrt{\gamma} & \sqrt{\beta} \end{pmatrix}$ which is unitary on $\mathbb{C}^3$. By a direct computation, we see that the equivalent normalized kernel $U\tilde{\tilde{G}}(z,w)\overline{U}^t$ is equal to the direct sum $G_1(z,w) \oplus G_2(z,w)$, where $G_2(z,w) = (1-z\bar{w})^{-\alpha-\beta-\gamma-2}$ and

$$G_1(z,w) = \begin{pmatrix} \begin{smallmatrix}(1-z\bar{w})^2-(\beta+\gamma)(1-z\bar{w})z\bar{w} \\ +(\beta+\gamma)(1+\beta+\gamma)z^2\bar{w}^2\end{smallmatrix} & -\sqrt{\beta+\gamma}(1+\beta+\gamma)z^2\bar{w} \\ -\sqrt{\beta+\gamma}(1+\beta+\gamma)z\bar{w}^2 & 1+(\beta+\gamma)z\bar{w} \end{pmatrix}(1-z\bar{w})^{-\alpha-\beta-\gamma-2}.$$

It follows that M^* is unitarily equivalent to a reducible operator by an application of Lemma 4.3, that is, M^* is reducible. If we replace β by $\beta+\gamma$ in Theorem 4.2 take $n = 1$, then

$$K(z,w) = \begin{pmatrix} (1-z\bar{w})^2 & (\beta+\gamma)z(1-z\bar{w}) \\ (\beta+\gamma)\bar{w}(1-z\bar{w}) & (\beta+\gamma)(1+(\beta+\gamma)z\bar{w}) \end{pmatrix}(1-z\bar{w})^{-\alpha-\beta-\gamma-2},$$

for $z, w \in \mathbb{D}$. We observe that

$$G_1(z,w) = K(0,0)^{1/2}K(z,0)^{-1}K(z,w)K(0,w)^{-1}K(0,0)^{1/2}$$

and $G_1(z,0) = I$, as is to be expected. The multiplication operator corresponding to G_1, which we denote by M_1, is unitarily equivalent to $M^{(\alpha,\beta+\gamma)}$ by Lemma 4.3. Hence it is in $\mathrm{B}_2(\mathbb{D})$ by [12, Proposition 3.6]. Since both homogeneity and irreducibility are invariant under unitary equivalence it follows, by an easy application of Lemma 4.3, Theorem 4.2 and Theorem 5.1 that M_1^* is a irreducible homogeneous operator in $\mathrm{B}_2(\mathbb{D})$. Irreducibility of M_1^* also follows from Proposition 4.11. Let M_2 be the multiplication operator on the Hilbert space of scalar valued holomorphic functions with reproducing kernel G_2. Again, M_2^* is in $\mathrm{B}_1(\mathbb{D})$. The operator M_2 is irreducible by [9, corollary 1.19]. Homogeneity of M_2^* was first established in [17], see also [22]. An alternative proof is obtained when we observe that $\Gamma : \text{Möb} \times \mathbb{D} \longrightarrow \mathbb{C}$, where $\Gamma_{\varphi^{-1}}(z) = \big((\varphi^{-1})'(z)\big)^{\frac{\alpha+\beta+\gamma}{2}+1}$ is a cocycle such that $G_2(z,w) = \Gamma_{\varphi^{-1}}(z)G_2\big(\varphi^{-1}(z),\varphi^{-1}(w)\big)\overline{\Gamma_{\varphi^{-1}}(z)}$ for $z, w \in \mathbb{D}, \varphi \in$Möb. Now we conclude that M^* is homogeneous as it is unitarily equivalent to the direct sum of two homogeneous operators. Also M^* is in $\mathrm{B}_3(\mathbb{D})$ being the direct sum of two operators from the Cowen-Douglas class. □

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