Contractive and completely contractive modules, matricial tangent vectors and distance decreasing metrics

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Abstract: It is shown that a tangent vector v in $T_\omega\Omega$ determines a finite dimensional Hilbert module over $H^\infty(\Omega)$ and that the module is contractive if and only if $C_{\Omega,\omega}(v)$, the Carathéodory length of v, is less or equal to one. More generally, an element V of $T_\omega\Omega \otimes \mathcal{M}_n$ also determines a finite dimensional Hilbert module over $H^\infty(\Omega)$ and if the norm on $T_\omega\Omega \otimes \mathcal{M}_n$ is understood to be the injective tensor product norm then again the module is contractive if and only if $\|V\|$ is less or equal to one. The question of which contractive modules are completely contractive over $H^\infty(\Omega)$ is discussed in terms of the Carathéodory metric on Ω. The relationship of this question with certian aspects of the theory of tensor products for operator spaces is established.

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1 Introduction

The notion of a Hilbert module over a function algebra was introduced recently by R.G. Douglas (cf. [3]). In this paper we study a class of finite dimensional Hilbert modules over the algebra of bounded analytic functions on a domain $\Omega \subseteq \mathbb{C}^m$. The class of modules we study here have been investigated in a series of papers [1], [7], [8], [9], [10]. Our main objective is to determine when these modules are *contractive*, and among the *contractive* modules, which ones are *completely contractive.* Recent examples of *contractive* modules (over $\mathcal{A}(\mathbb{D}^3)$ due to Parrot (cf. [9]), and over $\mathcal{A}(\mathbb{B}^2)$, [8]) which are not *completely contractive* are in this class.

We show that each tangent vector $v \in T_\omega\Omega$ gives rise to a certain two dimensional module over the algebra $H^\infty(\Omega)$. Further, these modules are *contractive* if and only if they are *completely contractive.* This is an immediate consequence of the distance decreasing property of the Carathéodory metric. Subsequently, we introduce the notion of a *matricial tangent vector,* that is an element of $T_\omega\Omega \otimes \mathcal{M}_n$ and show that each *matricial* tangent vector gives rise to a module of dimension $2n$. However, while the Carathéodory metric on the $T_0(\mathcal{M}_k)_1$ is just the operator norm, the analogue of the Carathéodory metric on the *matricial* tangent space $T_0(\mathcal{M}_k)_1 \otimes \mathcal{M}_n$ is smaller than the usual operator norm (see example 2.1). For this reason, *contractive* modules are not necessarily *completely contractive.*

It turns out that the module determined by a *matricial* tangent vector $V = (V_1, \dots, V_m)$ at $\omega \in \Omega$ is *contractive* if and only if, the induced linear map $\rho : H^\infty(\Omega) \to \mathcal{M}_n$, $\rho(f) = \nabla f(\omega)(V)$ is a contraction. There is a norm on $\mathbb{C}^m$ [10, Proposition 3.1] such that the set $\{\nabla f(\omega) : f \in H^\infty(\Omega), \|f\|_\infty \leq 1\}$ is a unit ball with respect to this norm. Thus, the contractivity of the module is equivalent to the contractivity of $\rho : \mathbb{C}^m \to \mathcal{M}_n$. Similarly, the complete contractivity of such a module is equivalent to contractivity of $\rho^{(k)} : H^\infty(\Omega) \otimes \mathcal{M}_k \to \mathcal{M}_{nk}$, $\rho^{(k)}(F) = DF(\omega)(V)$ for all k. There is a norm on $\mathbb{C}^{mk}$ [10, Proposition 3.1] such that the set $\{DF(\omega) : F \in H^\infty(\Omega) \otimes \mathcal{M}_k, \|F\|_\infty \leq 1\} \subset \mathbb{C}^{mk}$ is a unit ball with respect to this norm. Thus, *complete contractivity* of the module is equivlaent to *contractivity* of $\rho^{(k)} : \mathbb{C}^{mk} \to \mathcal{M}_n$ for all k. An explicit description of these norms was a question raised by Paulsen [loc. cit.]. As a consequence of our duality lemma, this norm is explicitly defined for a domain $\Omega \subset \mathbb{C}^m$. In the particular case when Ω is a product domain $\Omega \times \tilde{\Omega}$, for example, the norm is

$$C^*_{\Omega,\omega} \check{\otimes} \| \ \|_{\mathrm{op}}$$

on $T^*_\omega\Omega \otimes \mathcal{M}_k$.

In section 2, we provide a functorial frame work for dealing with distance decreasing norms, and introduce the notion of a *pullback* and *pushforward* of a given norm with respect to a fixed family of linear maps, which obey a universal norm decreasing property with respect to that family. Indeed, the usual Carathéodory and Kobayashi norms are the classical prototypes of these constructions. These constructions enable us to define norms on *matricial (co)tangent vectors* (see sections 2.1 and 2.3). The pullback and the pushforward norms are dual notions, as we establish in our duality lemma 2.1.

The issue of when *contractive* modules of our class are *completely contractive*, now gets formulated as follows.

For a *matricial* tangent vector $V \in T_\omega\Omega \otimes \mathcal{M}_n$, we define the injective tensor product

$$\check{C}_{\Omega,\omega}(V) = C_{\Omega,\omega}\check{\otimes}\| \ \|_{\rm op}$$

as a pullback norm. The contractivity of the Hilbert module determined by V is equivalent to $\check{C}_{\Omega,\omega}(V) \le 1$ (see Theorem 2.2).

The unit ball $(\mathcal{M}_k)_1$ with respect to operator norm is a homogeneous domain, and has transitive family of bi-holomorphic automorphisms acting on it. Thus, putting the operator norm on the matrix tangent space at the origin uniquely determines a norm on the matrix tangent bundle of $(\mathcal{M}_k)_1$, by requiring these automorphisms to be isometries. Let us call this norm δ. We define the pullback norm $\delta_{\mathcal{L}}$, where $\mathcal{L}$ is the family $\mathcal{D}\Omega^{(k)}(\omega)$ (see 2.10), on the matrix tangent space $T_\omega\Omega \otimes \mathcal{M}_n$. It follows from the Corollary 2.5 that complete contractivity of the module is equivalent to the condition

$$\delta_{\mathcal{L}}(V) \le \check{C}_{\Omega,\omega}(V).$$

In other words, the question of contractivity implying complete contractivity "linearises" for our class of modules, that is, depends only on derivatives of functions in $H^\infty(\Omega)\otimes\mathcal{M}_k$. More precisely, complete contractivity follows if for all $F \in \mathrm{Hol}_\omega(\Omega, (\mathcal{M}_k)_1)$, the map

$$DF(\omega) : (T_\omega\Omega \otimes \mathcal{M}_n, \check{C}_{\Omega,\omega}) \to (T_\omega(\mathcal{M}_k)_1 \otimes \mathcal{M}_n, \delta),$$

(see 2.16), is norm decreasing.

We would like to view this as a generalised Schwarz lemma. A result due to Yau [14] provides a fairly general Schwarz lemma under curvature hypotheses on the domain and target manifolds. In analogy with this result, one should seek geometric conditions on the various different norms on the *matricial* tangent spaces of the domain and target. We note here that as opposed to the situation in [14], our tangent spaces are *matricial* and the norms involved are not necessarily Hermitian, Kahler etc., and so, for example, a suitable notion of curvature would have to be found for such a Schwarz lemma. For instance, the evidence in support of such a lemma is Ando's theorem stating that *contractive* modules over $\mathcal{A}(\mathbb{D}^2)$ are *completely contractive.*

In section 2.6, we introduce the notion of a norm decreasing metric for *matricial* tangent vectors. Let $K_{\Omega,\omega}$ and $C_{\Omega,\omega}$ be the Kobayashi and Carathéodory metric respectively. It is shown that among all such metrics, the injective tensor product norm on

$$((T_\omega\Omega, C_{\Omega,\omega})\check{\otimes}(\mathcal{M}_n, op)),$$

is the smallest, while, the projective tensor product norm on

$$((T_\omega\Omega, K_{\Omega,\omega})\hat{\otimes}(\mathcal{M}_n, op)),$$

is the largest such distance decreasing metric. In a recent paper [1], J. Agler has reproved Lempert's theorem, which states that the Carathéodory and the Kobayashi distances are the same for a convex domain Ω. However, using Parrott's example, it is easy to see that for the tri-disk, the two extremal metrics for *matricial* tangent vectors we have obtained do not agree (see Remark 2.4).

1.1 Tangent vectors

Let Ω be a bounded region in $\mathbb{C}^m$ and $\omega = (\omega_1, \ldots, \omega_m)$ in Ω be an arbitrary but fixed point. In what follows, it will be useful to think of a vector v in $\mathbb{C}^m$ as an element of the tangent space $T_\omega\Omega$. For any pair of complex scalars α, and β, let

$$N(\beta, \alpha) = \begin{bmatrix} \alpha & \beta \\ 0 & \alpha \end{bmatrix} \in \mathcal{M}_2(\mathbb{C}), \tag{1.1}$$

and given a tangent vector $v = (v_1, \ldots, v_m)$ in $T_\omega\Omega$, define the commuting m-tuple

$$N(v, \omega) = (N(v_1, \omega_1), \ldots, N(v_m, \omega_m)). \tag{1.2}$$

Let $\vartheta(\omega)$ be the germs of holomorphic functions at ω. As usual a tangent vector v in $T_\omega\Omega$ acts on any f in $\vartheta(\omega)$ by the rule

$$v(f) = \langle \nabla f(\omega), v \rangle. \tag{1.3}$$

It is not hard to see that the map $\rho_N : \vartheta(\omega) \to \mathcal{M}_2(\mathbb{C})$ defined by

$$\rho_N(f) \stackrel{\mathrm{def}}{=} f(N(v, \omega)) = N(v(f), f(\omega)) \tag{1.4}$$

is a continuous algebra homomorphism coinciding with the evaluation map on $\mathbb{C}[z_1, \ldots, z_m]$ (cf. [7, Proposition 2.2.3]). We can think of $\mathbb{C}^2$ as a module over $\vartheta(\omega)$ via the action $m : \vartheta(\omega) \times \mathbb{C}^2 \to \mathbb{C}^2$

$$m(f, \nu) = \rho_N(f) \cdot \nu, \quad f \in \vartheta(\omega) \text{ and } \nu \in \mathbb{C}^2. \tag{1.5}$$

We will write $\mathbb{C}^2_{N(v,\omega)}$ for this module. Thus, each tangent vector v in $T_\omega\Omega$ determines a module $\mathbb{C}^2_{N(v,\omega)}$ over $\vartheta(\omega)$. In particular, $\mathbb{C}^2_{N(v,\omega)}$ is also a module over $H^\infty(\Omega)$, the algebra of holomorphic functions on Ω. We wish to determine, when the module $\mathbb{C}^2_{N(v,\omega)}$ is *contractive* over $H^\infty(\Omega)$, that is, to determine the set

$$\begin{aligned} \Gamma_{\Omega,\omega} &= \{v \in T_\omega\Omega : \|m(f, \nu)\|_{\ell^2} \leq \|f\|_\infty \|\nu\|_{\ell^2}\} \\ &= \{v \in T_\omega\Omega : \|\rho_N(f)\|_{op} \leq 1,\ f : \Omega \to cl\,\mathbb{D} \text{ holomorphic}\}, \end{aligned} \tag{1.6}$$

where $cl\,\mathbb{D}$ is the closed unit disk in $\mathbb{C}$.

Remark 1.1 Note that, by the maximum modulus principle, $\Gamma_{\Omega,\omega}$ does not change, if we use the open unit disk $\mathbb{D}$ as the target space in 1.6.

We will consider later, more general modules determined by what we call matricial *tangent vectors*.

The following notation will be very convenient. For fixed ω in Ω and any domain Δ containing 0, we let,

$$\mathrm{Hol}_\omega(\Omega, \Delta) = \{f : \Omega \to \Delta \text{ is holomorphic, and } f(\omega) = 0\}, \tag{1.7}$$

$$\mathrm{Hol}^\omega(\Delta, \Omega) = \{f : \Delta \to \Omega \text{ is holomorphic, and } f(0) = \omega\}, \tag{1.8}$$

and for a normed linear space X, we let $(X)_1$ be the closed unit ball in X. Note that, any holomorphic function $f : \Omega \to \Delta$ induces a linear map $f_* : T_\omega\Omega \to T_{f(\omega)}\Delta$, defined by

$$f_*(v) = (v(f^1), \dots, v(f^m)), \tag{1.9}$$

where $(f^1, \dots, f^m) = f$. In particular, $f_*(v) = v(f)$ is in $T_0\mathbb{D}$ for any f in $\mathrm{Hol}_\omega(\Omega, \mathbb{D})$ and v in $T_\omega\Omega$. It is not hard to see that [7, Lemma 3.2], $\mathbb{C}^2_{N(v,\omega)}$ is *contractive* if and only if

$$\sup\{\|N(f_*(v), f(\omega))\|_{op} : f \in \mathrm{Hol}_\omega(\Omega, \mathbb{D})\} \le 1. \tag{1.10}$$

For any f in $\mathrm{Hol}_\omega(\Omega, \mathbb{D})$,

$$\|N(f_*(v), f(\omega))\|_{op} = \left\| \begin{bmatrix} 0 & f_*(v) \\ 0 & 0 \end{bmatrix} \right\|_{op} = |f_*(v)|. \tag{1.11}$$

The Carathéodory length $C_{\Omega,\,\omega}(v)$ of a tangent vector v in $T_\omega\Omega$ is defind by the formula

$$C_{\Omega,\,\omega}(v) \;=\; \sup\{|f_*(v)| : \; f \in \mathrm{Hol}_\omega(\Omega, \mathbb{D})\}, \tag{1.12}$$

and is a norm for any bounded domain Ω in $\mathbb{C}^m$. It follows that, if $\mathbb{C}^2_{N(v,\omega)}$ is *contractive*, then the Carathéodory norm of the tangent vector v in $T_\omega\Omega$ is at most 1. The indicatrix of Ω at ω is the closed unit ball in $T_\omega\Omega$ with respct to suitable length function on $T_\omega\Omega$. We will write the indicatrix with respect to the Carathéodory norm as $\Gamma(C_{\Omega,\omega})$. Note that,

$$\Gamma_{\Omega,\omega} = \Gamma(C_{\Omega,\omega}). \tag{1.13}$$

For our purposes, it will be necessary to introduce the dual object

$$\mathcal{D}\Omega(\omega) \stackrel{\text{def}}{=} \{\nabla f(\omega) : f \in \mathrm{Hol}_\omega(\Omega, \mathbb{D})\} \tag{1.14}$$

However, the fact that $\mathcal{D}\Omega(\omega) = \Gamma(C^*_{\Omega,\omega})$ will be established in section 2.6.

We refer the reader to the survey article [5] for details on the Carathéodory norm and the indicatrix.

There are some natural modules that can be constructed from $\mathbb{C}^2_{N(v,\omega)}$ as follows. Let $H^\infty(\Omega)\otimes\mathcal{M}_k$ be the algbraic tensor product of $H^\infty(\Omega)$ and the linear space of $k \times k$ matrices, $\mathcal{M}_k$. We think of an element of $H^\infty(\Omega)\otimes\mathcal{M}_k$ as a $\mathcal{M}_k$-valued holomorphic function and declare its norm to be the supremum norm on Ω. Unless, we specify to the contrary, $\mathcal{M}_k$ is to be thought of as a normed linear space with respect to the operator norm. The homomorphism $\rho_N \otimes I_k : H^\infty(\Omega)\otimes\mathcal{M}_k \to \mathcal{M}_{2k}$, which we shall denote by $\rho_N^{(k)}$, makes the k-fold direct sum, $\mathbb{C}^2_{N(v,\omega)} \oplus \cdots \oplus \mathbb{C}^2_{N(v,\omega)}$ a module over $H^\infty(\Omega)\otimes\mathcal{M}_k$ via the action

$$(F, \nu) \to (\rho_N^{(k)})(F)\cdot\nu, \quad \nu \in \mathbb{C}^2_{N(v,\omega)} \oplus \cdots \oplus \mathbb{C}^2_{N(v,\omega)}, \tag{1.15}$$

where the dot on the right indicates usual matrix multipication. If F is in $H^\infty(\Omega)\otimes\mathcal{M}_k$then $F = \left[F^{jl}\right] : \Omega \to \mathcal{M}_k$, and we have

$$\begin{aligned} (\rho_N^{(k)})\left(\left[F^{jl}\right]\right) &= \left[N\left(F^{jl}\right)\right] \\ &= \left[N\left(F^{jl}_*(v), F^{jl}(\omega)\right)\right] \end{aligned} \tag{1.16}$$

We say that the module $\mathbb{C}^2_{N(v,\omega)}$ is *completely contractive* if for each k, the map $\rho_N^{(k)}$ is a contraction. It will be useful to think of $\mathbb{C}^2_{N(v,\omega)} \oplus \cdots \oplus \mathbb{C}^2_{N(v,\omega)}$ as module with respect to a different but equivalent action. For any pair of linear transformations T and A in $\mathcal{M}_k$, let

$$N(T, A) = \begin{bmatrix} A & T \\ 0 & A \end{bmatrix} \tag{1.17}$$

By applying suitable row and column operations, we can write

$$\begin{aligned} (\rho_N^{(k)})(F) &= \left[N\left(F_*^{jl}(v), F^{jl}(\omega)\right)\right] \\ &\simeq \begin{bmatrix} F(\omega) & \left(F_*^{jl}(v)\right) \\ 0 & F(\omega) \end{bmatrix} \\ &= N(F_*(v), F(\omega)), \end{aligned} \tag{1.18}$$

where $F_* : T_\omega\Omega \to T_{F(\omega)}\mathcal{M}_k$ is the induced map.

Note that if we write F in $\mathrm{Hol}_\omega(\Omega, (\mathcal{M}_k)_1)$ as

$$F(z) = (z - \omega_1)F_1 + \cdots + (z - \omega_m)F_m + \cdots, F_\ell \in \mathcal{M}_k \tag{1.19}$$

then

$$F_*(v) = v_1F_1 + \cdots + v_mF_m. \tag{1.20}$$

The module structure on $\mathbb{C}^2_{N(v,\omega)} \oplus \cdots \oplus \mathbb{C}^2_{N(v,\omega)}$ over $H^\infty(\Omega)\otimes\mathcal{M}_k$ determined by the action

$$m^{(k)} : (F, \nu) \to N(F_*(v), F(\omega)) \cdot \nu \tag{1.21}$$

is isomorphic to the original one, via a unitary module map. Thus, the contractivity $\rho_N^{(k)}$ is equivalent to that of $m^{(k)}$. Once again, it can be shown [8, Lemma 1.6] that $\rho_N^{(k)}$ (or, for that matter $m^{(k)}$) is a contraction if and only if

$$\sup\left\{\|N(F_*(v), F(\omega))\|_{op} : F \in \mathrm{Hol}_\omega(\Omega, (\mathcal{M}_k)_1)\right\} \le 1. \tag{1.22}$$

However, for F in $\mathrm{Hol}_\omega(\Omega, (\mathcal{M}_k)_1)$

$$\|N(F_*(v), F(\omega))\|_{op} = \left\|\begin{bmatrix} 0 & F_*(v) \\ 0 & 0 \end{bmatrix}\right\|_{op} = \|F_*(v)\|_{op}. \tag{1.23}$$

The following is a version of Schwarz Lemma.

LEMMA 1.1 *If Ω in $\mathbb{C}^m$ is the unit ball with respect to some norm $\|\cdot\|_\Omega$ on $\mathbb{C}^m$, then the indicatrix $\Gamma(C_{\Omega,0})$ is Ω.*

Proof: Recall that, the indicatrix $\Gamma(C_{\Omega,0})$ is

$$\begin{aligned} \Gamma(C_{\Omega,0}) &= \{v \in T_0\Omega : C_{\Omega,0}(v) \le 1\} \\ &= \{v \in T_0\Omega : |\langle\nabla f(0), v\rangle| \le 1,\ f \in \mathrm{Hol}_0(\Omega, \mathbb{D})\}. \end{aligned}$$

Here, we are thinking of $\nabla f(0)$ as a co-tangent vector in $T_0^*\Omega$. One form of Schwarz lemma [12, p.161], implies that $\nabla f(0)$ is a linear function in $\mathrm{Hol}_0(\Omega, \mathbb{D})$, that is, $\nabla f(0)$ is a linear functional of norm at most 1, on $(\mathbb{C}^m, \|\cdot\|_\Omega)$. On the other hand, any linear functional of norm at most 1, on $(\mathbb{C}^m, \|\cdot\|_\Omega)$, is in $\mathrm{Hol}_0(\Omega, \mathbb{D})$. Thus,

$$\mathcal{D}\Omega(0) = \{\nabla f(0) : f \in \mathrm{Hol}_0(\Omega, \mathbb{D})\} = (\mathbb{C}^m, \|\cdot\|_\Omega^*)_1 \,.$$

It now follows that $C_{\Omega,0}(V) \leq 1$ if and only if v is in $(\mathbb{C}^m, \|\cdot\|_\Omega^{**})_1$. Since, $(\mathbb{C}^m, \|\cdot\|_\Omega)^{**}$ and $(\mathbb{C}^m, \|\cdot\|_\Omega)$ are isometrically isomorphic, the proof is complete.

While the proof of the following theorem is not difficult, we wish to emphasize that the statement of the theorem is equivalent to the distance decreasing property of the Carathéodory metric.

Theorem 1.1 *Every contractive module $\mathbb{C}^2_{N(v,\omega)}$ is completely contractive.*

Proof: We have to show that for any v in $\Gamma_{\Omega,\omega} = \Gamma(C_{\Omega,\omega})$

$$\|F_*(v)\| \leq 1, \; F \in \mathrm{Hol}_\omega(\Omega, (\mathcal{M}_k)_1).$$

The Carathéodory metric is distance decreasing, that is, for any holomorphic map $f : \Omega \to \widetilde{\Omega}$,

$$C_{\widetilde{\Omega}, f(\omega)}(f_*(v)) \leq C_{\Omega,\,\omega}(v).$$

In particular, we have

$$C_{(\mathcal{M}_k)_1,0}(F_*(v)) \leq C_{\Omega,\,\omega}(v).$$

A trivial consequence of the previous lemma is that, $C_{\Omega,0}(v) = \|v\|_\Omega$ for any ball $((\mathbb{C}^m, \|\cdot\|_\Omega))_1$. In particular, $C_{(\mathcal{M}_k)_1,0}(V) = \|V\|_{op}$, and for v in $\Gamma(C_{\Omega,\omega})$, we have

$$\|F_*(v)\|_{op} = C_{(\mathcal{M}_k)_1,0}(F_*(v)) \leq C_{\Omega,\,\omega}(v) \leq 1.$$

The proof is now complete.

Let K be a compact subset of $\mathbb{C}^m$ and $\mathcal{A}(\mathrm{int}\ K)$, be the algebra of holomorphic functions on int K, which extend continuously to K. Contractive modules over $\mathcal{A}(\mathrm{int}\ K)$ correspond in a one to one manner to m-tuples of Hilbert space operators, which admit K as a spectral set (cf. [3]). Recently, J. Agler has introduced the notion of a spectral domain for an m-tuple of operaors. Contractive Hilbert modules over $H^\infty(\Omega)$ correspond in a one to one manner to m-tuples, which admit Ω as a spectral domain. Thus Theorem 1.2 is the limiting case of Theorem 1.9 of [1]. While, Agler suggests that a proof can be obtained by limiting arguments, we have included a direct proof of Theorem 1.1, both to introduce some basic techniques and to emphasize the complex geometric language.

2 Matricial Tangent Vectors

In this section, we consider modules determined by *matricial tangent vectors*, that is, an element V of $T_\omega\Omega \otimes \mathcal{M}_n$. Note that, we may write $V \in T_\omega\Omega \otimes \mathcal{M}_n$ either as

$$V = \sum_{i,j=1}^{n} v^{ij} \otimes E_{ij}, \quad v^{ij} \in T_\omega\Omega, \tag{2.1}$$

where, E_{ij} is the usual matrix unit, or by setting $V^l = \left[v_l^{ij}\right]$, we have

$$V = \sum_{l=1}^{m} e_l \otimes V^l, \quad V^l \in \mathcal{M}_n, \tag{2.2}$$

where, e_l is the standard basis vector in $\mathbb{C}^m$. If $f : \Omega \to \tilde{\Omega}$ is holomorphic, then the induced map $f_* \otimes I_n : T_\omega\Omega \otimes \mathcal{M}_n \to T_\omega\tilde{\Omega} \otimes \mathcal{M}_n$, is defined by

$$\begin{aligned}(f_* \otimes I_n)(V) &= (f_* \otimes I_n)(\sum e_l \otimes V^l) \\ &= \sum f_*(e_l) \otimes V^l \\ &= \sum((\langle \nabla f^j(\omega), e_l\rangle)_{j=1}^{m}) \otimes V^l. \end{aligned}\tag{2.3}$$

If $f_l = (\frac{\partial}{\partial z_l} f)(\omega)$, then

$$(f_* \otimes I_n)(V) = \sum f_l \otimes V^l. \tag{2.4}$$

For V in $T_\omega\Omega \otimes \mathcal{M}_n$, we will write $f_*(V)$ instead of $(f_* \otimes I_n)(V)$. The map $\rho_N : \vartheta(\omega) \to \mathcal{M}_{2n}(\mathbb{C})$, defined by (see, 1.2 and 1.17)

$$\begin{aligned}\rho_N(f) &\stackrel{\text{def}}{=} f(N(V, \omega_1 I_n), \ldots, N(V, \omega_m I_n)) \\ &= N(f_*(V), f(\omega)I_n), \qquad f \in \vartheta(\omega), \end{aligned}\tag{2.5}$$

is a continuous algebra homomorphism [9, Proposition 2.3], coinciding with the evaluation map on polynomials. Let $\mathbb{C}^{2n}_{N(V,\omega)}$ be the module over $H^\infty(\Omega)$ determined by this action. As before the k-fold directsum, $\mathbb{C}^{2n}_{N(V,\omega)} \oplus \cdots \oplus \mathbb{C}^{2n}_{N(V,\omega)}$ is a module over $H^\infty(\Omega) \otimes \mathcal{M}_k$, via the action determined by

$$\begin{aligned}(\rho_N^{(k)})\left(\left[F^{jl}\right]\right) &= \left[F^{jl}(N(V, \omega_1 I_n)), \ldots, N(V, \omega_m I_n)\right] \\ &= \left[N\left(F_*^{jl}(V), F^{jl}(\omega)I_n\right)\right], \end{aligned}\tag{2.6}$$

where, $F = \left[F^{jl}\right]$ is in $H^\infty(\Omega) \otimes \mathcal{M}_k$. However, after suitable row and column operations, we find that

$$\begin{aligned}(\rho_N^{(k)})\left(\left[F^{jl}\right]\right) &= \left[N\left(F_*^{jl}(V), F^{jl}(\omega)I_n\right)\right] \\ &\simeq \begin{bmatrix} I_n \otimes F(\omega) & F_*^{jl}(V) \\ 0 & I_n \otimes F(\omega) \end{bmatrix} \\ &= N(F_*(V), I_n \otimes F(\omega)); \end{aligned}\tag{2.7}$$

$$F_*(V) = F_1 \otimes V_1 + \cdots F_m \otimes V_m, \quad \text{where} \quad F_i = (\frac{\partial}{\partial z_i} F^{jl})(\omega) \ (\text{ see } 2.4).$$

Thus, for F in $H^\infty(\Omega) \otimes \mathcal{M}_k$ and ν in $\mathbb{C}^{2n}_{N(V,\omega)} \oplus \cdots \oplus \mathbb{C}^{2n}_{N(V,\omega)}$, the module structure on the k-fold directsum of $\mathbb{C}^{2n}_{N(V,\omega)}$ determined by either of the actions 2.6, or 2.8, are isomorphic. We will without loss of generality, consider the k-fold direct sum of $\mathbb{C}^{2n}_{N(V,\omega)}$ as a module over $H^\infty(\Omega) \otimes \mathcal{M}_k$ via the action induced by $N(F_*(V), I_n \otimes F(\omega))$, and set

$$(\rho_N^{(k)})(F) = N(F_*(V), I_n \otimes F(\omega)). \tag{2.8}$$

Even in this generality, it can be shown that [9, Lemma 3.3] , $\rho_N^{(k)}$ is *contractive* for any $k \geq 1$, if and only if

$$\begin{aligned} & \sup \{ \|N(F_*(V),0)\|_{op} : F \in \mathrm{Hol}_\omega(\Omega, (\mathcal{M}_k)_1) \} \\ = \;& \sup\{\|F_*(V)\|_{op} : F \in \mathrm{Hol}_\omega(\Omega, (\mathcal{M}_k)_1) \} \leq 1. \end{aligned} \tag{2.9}$$

QUESTION 2.1 *When is a contractive module $\mathbb{C}^{2n}_{N(V,\omega)}$ over $H^\infty(\Omega)$, completely contractive?*

REMARK 2.1 To answer this question it would be helpful to define

$$\mathcal{D}\Omega^{(k)}(\omega) \stackrel{\text{def}}{=} \{DF(\omega) : F \in \mathrm{Hol}_\omega(\Omega, (\mathcal{M}_k)_1)\}. \tag{2.10}$$

It turns out that $\mathcal{D}\Omega^{(k)}(\omega)$ is a unit ball with respect to some norm (cf. [10, Proposition 3.1]). We will later determine this norm explicitly (see Corollary 2.1 and 2.2). For a fixed but arbitrary *matricial* tangent vector $V \in T_\omega\Omega \otimes \mathcal{M}_n$, set

$$\rho(f) \stackrel{\text{def}}{=} f_*(V), \text{ and } \rho^{(k)}(F) \stackrel{\text{def}}{=} F_*(V),$$

for $f \in H^\infty(\Omega)$, and $F \in H^\infty(\Omega) \otimes \mathcal{M}_k$, respectively. Question 2.1, is answered by determining the norm

$$\|\rho^{(k)}\| = \sup\{\|\rho^{(k)}(F)\|_{op} : F \in \mathrm{Hol}_\omega(\Omega, (\mathcal{M}_k)_1)\}\|. \tag{2.11}$$

First, we consider the case $k = 1$, and try to describe the set (Recall, 1.6)

$$\begin{aligned} \check{\Gamma}_{\Omega,\omega} &= \{V \in T_\omega\Omega \otimes \mathcal{M}_n : \|\rho_N(f)\|_{op} \leq 1, f : \Omega \to \mathbb{D} \text{ is holomorphic}\} \\ &= \{V \in T_\omega\Omega \otimes \mathcal{M}_n : \|f_*(V)\|_{op} \leq 1, f \in \mathrm{Hol}_\omega(\Omega, \mathbb{D})\}. \end{aligned} \tag{2.12}$$

To imitate the proof of theorem 1.1, it would then seem natural to define

$$\check{C}_{\Omega,\omega}(V) = \sup\{\|f_*(V)\|_{op} : f \in \mathrm{Hol}_\omega(\Omega, \mathbb{D})\}. \tag{2.13}$$

It turns out (see Proposition 2.2), $\check{C}_{\Omega,\omega}(V)$ is distance decreasing. However, the following simple example, shows that $\check{C}_{(\mathcal{M}_k)_1,0}(V)$, is not always equal to $\|V\|_{op}$.

EXAMPLE 2.1 Let

$$V = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Note that, $\|V\|_{op} = \sqrt{2}$, while

$$\check{C}_{\Omega,\omega}(V) = \sup\left\{\left|\sum_{i,j=1,2} a_{ij}V_{ij}\right| : |tr\,[a_{ij}]^*| \leq 1\right\}.$$

If x and y are unit vectors in $\mathbb{C}^2$, then

$$\left\langle\left(\sum_{i,j=1,2} a_{i,j}V_{i,j}\right)x, y\right\rangle = tr\left([a_{ij}]\,[\langle V_{ij}x, y\rangle]^*\right).$$

Since $\|[\langle V_{ij}x, y\rangle]^*\|_{op} \leq 1$, it follows that $\check{C}_{\Omega,\omega}(V) \leq 1$.

2.1 Pullbacks

Let $\mathcal{V}$ be a finite dimensional vector space, and $\mathcal{W}$ be a finite dimensional normed linear space. Let $\mathcal{L} \subset \mathrm{Hom}(\mathcal{V}, \mathcal{W})$ be some family of linear maps. Define a function $\| \;\|_{\mathcal{L}} : \mathcal{V} \to \mathbb{R}_+$ by

$$\|v\|_{\mathcal{L}} = \sup\{\|Lv\| : L \in \mathcal{L}\} \tag{2.14}$$

It is clear that

$$\|\alpha v\|_{\mathcal{L}} = |\alpha| \|v\|_{\mathcal{L}},\ \alpha \in \mathbb{C},$$

and

$$\begin{aligned} \|v_1 + v_2\|_{\mathcal{L}} &= \sup\{\|Lv_1 + Lv_2\| : L \in \mathcal{L}\} \\ &\leq \sup\{\|Lv_1\| + \|Lv_2\| : L \in \mathcal{L}\} \\ &\leq \|v_1\|_{\mathcal{L}} + \|v_2\|_{\mathcal{L}}, \end{aligned}$$

since

$$\|Lv_k\| \leq \|v_k\|_{\mathcal{L}} \text{ for } k = 1, 2 \text{ and } L \in \mathcal{L}.$$

It is easy to see that $\cap\{\ker L : L \in \mathcal{L}\} = \{0\}$ if and only if $\| \;\|_{\mathcal{L}}$ is a norm.

Proposition 2.1 *Let us assume that that $\| \;\|_{\mathcal{L}}$ is a norm. Then*

$$\{v : \|v\|_{\mathcal{L}} \leq 1\} = \cap_{L \in \mathcal{L}} \{L^{-1}(w : \|w\| \leq 1)\},$$

where $\{w : \|w\| \leq 1\}$ is the unit ball in $\mathcal{W}$.

Proof:
$$\begin{aligned} \|v\|_{\mathcal{L}} \leq 1 &\Leftrightarrow \|Lv\| \leq 1 \text{ for all } L \in \mathcal{L} \\ &\Leftrightarrow Lv \in \{w : \|w\| \leq 1\} \text{ for all } L \in \mathcal{L} \\ &\Leftrightarrow v \in L^{-1}\{w : \|w\| \leq 1\} \text{ for all } L \in \mathcal{L} \\ &\Leftrightarrow v \in \cap_{L \in \mathcal{L}} L^{-1}\{w : \|w\| \leq |\}. \end{aligned}$$

2.2 Examples of $\mathcal{L}$, where $\| \;\|_{\mathcal{L}}$ is a Norm

Example 2.2 Let

$$\begin{aligned} \mathcal{W} &= T_0(\mathcal{M}_k)_1 \cong (\mathcal{M}_k, op), \\ \mathcal{V} &= T_\omega(\Omega),\ \text{and} \\ \mathcal{L} &= \{DF(\omega) : F \in \ \mathrm{Hol}\,_\omega(\Omega, (\mathcal{M}_k)_1)\} \subset\ \mathrm{Hom}\ (\mathcal{V}, \mathcal{W}). \end{aligned}$$

Then $\| \;\|_{\mathcal{L}}$ is a *norm.*

It is enough to show that for some $v \in T_\omega\Omega, v \neq 0$, there exists F in $\mathrm{Hol}_\omega(\Omega, (\mathcal{M}_k)_1)$ such that $DF(\omega)(v) = F_*(v) \neq 0$.

So take a linear functional $\lambda : T_\omega(\Omega) \to \mathbb{C}$ such that $\lambda(v) \neq 0$. Say $\lambda = (\lambda_1 \ldots \lambda_m)$. Then the linear map

$$\tilde{\lambda} : T_\omega(\Omega) \to \mathcal{M}_k = T_0((\mathcal{M}_k)_1),\ \tilde{\lambda}(v) = \lambda(v)\ \mathrm{Id}.$$

satisfies $\tilde{\lambda}(v) \neq 0$. If we define $\tilde{F}$ by

$$\tilde{F}(z) = \tilde{\lambda}[(z-\omega)] = (z_1 - \omega_1)\lambda_1 \text{ Id} + \cdots + (z_m - \omega_m)\lambda_m \text{ Id},$$

then $\tilde{F} \in \text{ Hol }_\omega(\mathbb{C}^m, \mathcal{M}_k)$, and clearly $D\tilde{F}(\omega) = \tilde{\lambda}$. However, $\tilde{F}$ does not necessarily map Ω into $(\mathcal{M}_k)_1$. Since Ω is bounded, we can take $C = \max_{z \in \Omega} |\tilde{F}(z)| < \infty$, and $F = \frac{1}{c}\tilde{F}$ satisfies $DF(\omega)v = \frac{1}{c}D\tilde{F}(\omega)v = \frac{1}{c}\tilde{\lambda}(v) \neq 0$. So we are done.

Let X and Y be finite dimensional normed vector spaces. It is possible to construct various norms on the algebraic tensor product $X \otimes Y$ using the norms on X and Y. One way is to introduce a norm, which is independant of the representation of the equivalence class is to assign to $\sum_{i=1}^n x_i \otimes y_i$ the norm it recieves when regarded as an operator from X^* to Y, that is

$$\| \sum_{i=1}^n x_i \otimes y_i \check{\|} = \sup \left\{ \left\| \sum_{i=1}^n \varphi(x_i) y_i \right\| : \varphi \in X^*, \|\varphi\| = 1 \right\}. \tag{2.15}$$

The norm $\| \ \check{\|}$ is called the *injective tensor product norm.*

Example 2.3 Let

$$\begin{aligned} \mathcal{W} &= T_0(\mathcal{M}_k)_1 \otimes (\mathcal{M}_n) \cong (\mathcal{M}_{kn}, op), \\ \mathcal{V} &= T_\omega(\Omega) \otimes (\mathcal{M}_n, op), \text{ and} \\ \mathcal{L} &= \{DF(\omega) \otimes Id : F \in Hol_\omega(\Omega, (\mathcal{M}_k)_1)\} \end{aligned}$$

For any bounded domain Ω, $\| \ \|_{\mathcal{L}}$ is a *norm* as well.

The following is true for vector-spaces of finite dimension. Suppose $L : \mathcal{V} \to \mathcal{W}$ then for a fixed vector space X, consider

$$(L \otimes Id) : \mathcal{V} \otimes X \to \mathcal{W} \otimes X$$

and note that

$$0 \to \ker L \to \mathcal{V} \xrightarrow{L} \mathcal{W}$$

is exact and $\otimes X$ is an exact functor

$$0 \to (\ker L) \otimes X \to \mathcal{V} \otimes X \xrightarrow{L \otimes Id_X} \mathcal{W} \otimes X$$

is also exact. So $\ker(L \otimes Id_X) = (\ker L) \otimes X$. Thus,

$$\begin{aligned} &\cap \ \ker\{DF \otimes Id_X : F \in Hol_\omega(\Omega, (\mathcal{M}_k)_1)\} \\ &= \ \cap\{\ker DF : F \in Hol_\omega(\Omega, (\mathcal{M}_k)_1)\} \otimes X \\ &= \ 0. \end{aligned}$$

by the preceeding example.

Remark 2.2 This example suggests the following definition. For $V \in \mathcal{V}$, let

$$\check{C}^k_{\Omega,\omega}(V) \stackrel{\text{def}}{=} \|V\|_{\mathcal{L}}, \tag{2.16}$$

where $\mathcal{L}$ is the same set of linear maps as in the preceding example. Note that, for $m \geq 2$, we can not apply the maximum principle to conclude that the norm $\check{C}^k_{\Omega,\omega}$, does not change if we use only those holomorphic function on Ω, which take their values in the open unit ball of $k \times k$ matrices (Recall 1.1). However, Given any $r > 1$, we observe that,

$$\mathrm{Hol}_\omega(\Omega, (\mathcal{M}_k)_1^\circ) \subset \mathrm{Hol}_\omega(\Omega, (\mathcal{M}_k)_1) \subset \mathrm{Hol}_\omega(\Omega, r(\mathcal{M}_k)_1^\circ).$$

If we call these families $\mathcal{L}^\circ$, $\mathcal{L}$ and $r\mathcal{L}^\circ$, then the corresponding norms satisfy

$$\|\ \|_{\mathcal{L}^\circ} \leq \|\ \|_{\mathcal{L}} \leq \|\ \|_{r\mathcal{L}^\circ} = r\|\ \|_{\mathcal{L}^\circ},$$

for all $r > 1$, and we have equality everywhere, by letting $r \to 1$. Thus, $\check{C}^k_{\Omega,\omega}$ is the same whether we use an open or closed matrix ball as the target. The above family $\mathcal{L}$ gives a sort of Carathéodory norm with respect to $\mathrm{Hol}_\omega(\Omega, (\mathcal{M}_k)_1)$ for *matricial* tangent vectors as $\|\ \|_{\mathcal{L}}$ (compare 2.13). We obtain the usual Carathédory metric by taking n=k=1.

The next proposition about $\|\ \|_{\mathcal{L}}$ is a universal functorial property that characterises $\|\ \|_{\mathcal{L}}$.

Definition 2.1 We say some arbitrary norm $\|\ \|_a$ on $\mathcal{V}$ is $\mathcal{L}$ - *distance decreasing* if $\mathcal{L} \subset (\mathcal{L}(\mathcal{V}, \mathcal{W}))_1$,that is

$$\|Lv\| \leq \|v\|_a \text{ for all } L \in \mathcal{L}.$$

Proposition 2.2 *$\|\ \|_{\mathcal{L}}$ is the smallest $\mathcal{L}$ - distance decreasing norm on $\mathcal{V}$.*

Proof: It is trivial to check that $\|\ \|_{\mathcal{L}}$ is $\mathcal{L}$ - distance decreasing by definition.
The norm $\|\ \|_a$ is $\mathcal{L}$ - distance decreasing if and only if

$$\|Lv\| \leq \|v\|_a \text{ for all } L \in \mathcal{L}\ v \in \mathcal{V}.$$

Equivalently, $\sup\{\|Lv\| : L \in \mathcal{L}\} \leq \|v\|_a$, that is, $\|v\|_{\mathcal{L}} \leq \|v\|_a$.

The least distance decreasing property of various Caratheodory norms (Examples 2.2 and 2.3) follows for holomorphic maps (see 2.22 and 2.24).

2.3 Pushforwards

There is a dual notion to $\|\ \|_{\mathcal{L}}$, namely $\|\ \|^{\mathcal{L}}$, which is defined as follows.

As before, let $\mathcal{V}$ and $\mathcal{W}$ be finite dimensional vector spaces with a norm on $\mathcal{W}$ with the only difference that $\mathcal{L} \subset \mathrm{Hom}(\mathcal{W}, \mathcal{V})$. Define a function $\|\ \|^{\mathcal{L}} : \mathcal{V} \to \mathbb{R}_+$ by

$$\begin{aligned} \|\lambda\|^{\mathcal{L}} &= \inf\{\|\mu\| : L\mu = \lambda \text{ for some } L \in \mathcal{L}, \mu \in \mathcal{W}\} \\ &= \inf\{\|\mu\| : \mu \in \mathcal{L}^{-1}(\lambda)\},\ \text{where } \mathcal{L}^{-1}(\lambda) \stackrel{def}{=} \cup\{L^{-1}(\lambda) : L \in \mathcal{L}\}, \end{aligned} \tag{2.17}$$

for $\lambda \in \mathcal{V}$. If no L with $L\mu = \lambda$, exists, define $\|\lambda\|^{\mathcal{L}} = 0$ note $\|0\|^{\mathcal{L}} = 0$, and also for $c \neq 0$,

$$\begin{aligned}
\|c\lambda\|^{\mathcal{L}} &= \inf\{\|\mu\| : L\mu = c\lambda \text{ for some } \mu \in \mathcal{W} \text{ and } L \in \mathcal{L}\} \\
&= \inf\{|c|\|\frac{1}{c}\mu\| : L(\frac{1}{c}\mu) = \lambda \text{ for some } L \in \mathcal{L},\ \frac{1}{c}\mu \in \mathcal{W}\} \\
&= |c| \inf\{\|\frac{1}{c}\mu\| : L(\frac{1}{c}\mu) = \lambda \text{ for some } L \in \mathcal{L},\ \frac{1}{c}\mu \in \mathcal{W}\} \\
&= |c| \inf\{\|\mu'\| : L(\mu') = \lambda,\ \mu' \in \mathcal{W}, L \in \mathcal{L}\} \\
&= |c|\|\lambda\|^{\mathcal{L}}
\end{aligned}$$

EXAMPLE 2.4 Let $\mathcal{W} = T_0(\mathbb{D}), \mathcal{V} = T_\omega(\Omega)$, and

$$\mathcal{L} = \{Df(0) : f \in \mathrm{Hol}^\omega(\mathbb{D}, \Omega)\}.$$

Note that the norm $\| \|^{\mathcal{L}}$ is the Kobayashi norm $K_{\Omega,\omega}$ on $T_\omega(\Omega)$ (cf. [5]).

We now find sufficient conditions on $\mathcal{L}$ so that $\| \|^{\mathcal{L}}$ is a norm.

HYPOTHESIS 2.1 We list, for $\mathcal{L} \subset \mathrm{Hom}(\mathcal{W}, \mathcal{V})$ the following conditions

(i) there exists a distinguished vector $J \in \mathcal{W}$ with $\|J\| = 1$ such that for each $\mu \in \mathcal{W}$ with $\|\mu\| = 1$, there exists a linear endomorphism (of $\mathcal{W}$) R_μ which is of operator norm 1 and $R_\mu(J) = \mu$.

(ii) If $L \in \mathcal{L}$ and R_μ as in (i) then $L \circ R_\mu \in \mathcal{L}$.

(iii) For any $\lambda \in \mathcal{V}$, there exists $L \in \mathcal{L}, \mu \in \mathcal{W}$ such that

$$L\mu = \lambda \text{ and } \|\lambda\|^{\mathcal{L}} = \|\mu\|$$

(iv) $\mathcal{L}$ is *convex*, that is, for $c_1, c_2 \in \mathbb{R}_+, L_1, L_2 \in \mathcal{L}$ we have

$$\frac{c_1L_1 + c_2L_2}{c_1 + c_2} \in \mathcal{L}.$$

PROPOSITION 2.3 *If $\mathcal{L}$ satisfies Hypothesis 2.1 and $J \in \mathcal{W}$ be the distinguished unit vector guaranteed by (i), thereof, then*

(a) for any $\lambda \in \mathcal{V}$, there exists an $L \in \mathcal{L}$ with $L(cJ) = \lambda$, where $c = \|\lambda\|^{\mathcal{L}}$.

(b) $\| \|^{\mathcal{L}}$ is a norm.

Proof of (a): By (iii), there exists μ such that $\|\mu\| = c = \|\lambda\|^{\mathcal{L}}$ and $L\mu = \lambda$. But $\frac{1}{c}\mu$ is a unit vector, so $\frac{1}{c}\mu = R_\mu(J)$ by (i) . So $\mu = cR_\mu J = R_\mu(cJ)$. But then $L\mu = (L \circ R_\mu)(cJ)$ and by (ii) $L \circ R_\mu \in \mathcal{L}$. So we are done. By definition $c = \|\lambda\|^{\mathcal{L}}$.

Proof of (b): By part (a), there exists $L \in \mathcal{L}$ such that

$$L(cJ) = \lambda \text{ and } |c| = \|cJ\| = \|\lambda\|^{\mathcal{L}}.$$

So $\|\lambda\|^{\mathcal{L}} = 0$, which implies $c = 0$, and in turn $\lambda = 0$. *Thus* $\| \|^{\mathcal{L}}$ is *positive definite.*

To prove *triangle-inequality*, again by Proposition 2.3 for $\lambda_1, \lambda_2 \in \mathcal{V}$, there exists $c_1, c_2 \in \mathbb{R}_+$ with

$$L_i(c_i J) = \lambda_i \text{ and } c_i = \|\lambda_i\|^{\mathcal{L}} > 0$$

But then

$$c_i L_i(J) = \lambda_i \Rightarrow \frac{(c_1 L_1 + c_2 L_2)(J)}{(c_1 + c_2)} = \frac{\lambda_1 + \lambda_2}{(c_1 + c_2)}$$

But $L = \frac{c_1 L_1 + c_2 L_2}{c_1 + c_2} \in \mathcal{L}$ by (iv) of achieving property. This means

$$\frac{\|\lambda_1 + \lambda_2\|^{\mathcal{L}}}{c_1 + c_2} \leq \|J\| = 1 \Rightarrow \|\lambda_1 + \lambda_2\|^{\mathcal{L}} \leq c_1 + c_2 = \|\lambda_1\|^{\mathcal{L}} + \|\lambda_2\|^{\mathcal{L}}.$$

This completes the proof.

PROPOSITION 2.4 *Let $\mathcal{L}$ satisfy Hypothesis 2.1. Then the unit ball of $\| \|^{\mathcal{L}}$ is described by*

$$\begin{aligned}\{\lambda \in \mathcal{V} : \|\lambda\|^{\mathcal{L}} \leq 1\} &= \mathcal{L}\{\mu \in \mathcal{W} : \|\mu\| \leq 1\} \\ &\stackrel{\text{def}}{=} \cup_{L \in \mathcal{L}} L((\mathcal{W})_1)\end{aligned}$$

Proof: If $\|\lambda\|^{\mathcal{L}} \leq 1$, then

$$\lambda = L\mu \text{ for } \mu \in \mathcal{W} \text{ and } \|\mu\| = \|\lambda\|^{\mathcal{L}} \leq 1$$

by *(iii)* of *Hypothesis* 2.1. Thus, $\lambda \in \cup_{L \in \mathcal{L}} L((\mathcal{W})_1)$.

Conversely, if $\lambda \in \cup_{L \in \mathcal{L}} L((\mathcal{W})_1)$, then

$$\lambda = L\mu \text{ for some } \mu \in \mathcal{W},\ \|\mu\| \leq 1 \text{ and } L \in \mathcal{L},$$

which implies

$$\inf\{\|\mu\| : L\mu = \lambda, \mu \in \mathcal{W}, L \in \mathcal{L}\} \leq \|\mu\| \leq 1$$

Thus, $\|\lambda\|^{\mathcal{L}} \leq 1$.

2.4 EXAMPLES OF $\mathcal{L}$ SATISFYING *Hypothesis* 2.1

EXAMPLE 2.5 Let $\Omega \subset \mathbb{C}^m$ be a bounded domain, $\omega \in \Omega$. Let

$$\begin{aligned}\mathcal{W} &= T_0^*(\mathbb{D}) \cong \mathbb{C} \text{ (usual cotangent spaces)}, \\ \mathcal{V} &= T_\omega^*(\Omega), \text{ and} \\ \mathcal{L} &= \{(Df(\omega))^* : \text{ where } f \in \text{ Hol}_{\,\omega}(\Omega, \mathbb{D})\}.\end{aligned}$$

On $\mathcal{W}$ put the obvious norm $| \, |$. The family $\mathcal{L}$ satisfies *Hypothesis* 2.1

(i) of *Hypothesis* 2.1 is satisfied (for the distinguished vector $1 \in \mathbb{C}$), since the unitary group of S^1 acts transitively on $S^1 = \{\alpha : |\alpha| = 1\}$.

Note $(Df(\omega))^*(\mu) = \lambda$ implies

$$\begin{aligned}\lambda(v) &= \mu((Df(\omega))_* v)\\ &= \mu(\sum_i \frac{\partial f}{\partial z_i}(\omega) v_i)\\ &= \sum_i \mu \frac{\partial f}{\partial z_i}(\omega) v_i\end{aligned}$$

So that $(Df(\omega))^*$ is the linear functional

$$\mu \mapsto (\mu \frac{\partial f}{\partial z_1}, \dots, \mu \frac{\partial f}{\partial z_m}) \text{ on } T^*_\omega(\Omega) = \mathcal{V}$$

by definition .

(ii) is clearly satisfied, for if $\alpha \in$ Iso $(\mathcal{W}), \alpha \in \mathbb{C},$ and $|\alpha| = 1$, then

$$(Df(\omega))^* \circ \alpha = D(\alpha f(\omega))^* \text{ for } f \in \text{ Hol }_\omega(\Omega, \mathbb{D}),$$

and clearly, $\alpha f \in$ Hol $_\omega(\Omega, \mathbb{D})$.

(iv) is satisfied since

$$\frac{c_1(Df_1(\omega))^* + c_2(Df_2(\omega))^*}{c_1 + c_2} = (Df(\omega))^*,$$

where $f = \frac{c_1 f_1 + c_2 f_2}{c_1 + c_2}, c_i \geq 0$. But

$$|f| \leq \frac{(c_1 + c_2) \max_{i=1,2} |f_i|}{c_1 + c_2} = 1.$$

So $f \in$ Hol $_\omega(\Omega, \mathbb{D})$.

It only remains to prove (iii). We have to use a normal family argument, which applies to any bounded region $\Omega \subset \mathbb{C}^m$.

Let $r = \|\lambda\|^{\mathcal{L}}$. Then it is not hard to show that there exists a *linear map* $\ell : \mathbb{C}^m \to \mathbb{C}, \ell(\omega) = 0$ and $z \to \sum(z_k - \omega_k)\ell_k$ such that $\lambda = \ell^*(1)$. By choosing α satisfying $\alpha\ell : \Omega \to \mathbb{D}$, we get $\lambda = (\alpha\ell)^*(\frac{1}{\alpha})$ so that $\lambda = (D(\alpha\ell))^*(\frac{1}{\alpha})$ This shows that the set $\mathcal{L}^{-1}(\lambda)$ is non-empty. So suppose given $f_k \in$ Hol $_\omega(\Omega, \mathbb{D})$ with $(Df_k(\omega))^*(\mu_k) = \lambda$ and $\|\mu_k\| = r + \epsilon_k$, $\epsilon_k \to 0$. By extracting subsequence, since Hol $_\omega(\Omega, \mathbb{D})$ is a normal family, $f_k \to f$ uniformly on compact sets for some $f \in$ Hol $_\omega(\Omega, \mathbb{D})$, and so $Df_k \to Df$ by Montel's Theorem. Similarly, μ_k is a bounded sequence, so $\mu_k \to \mu$ by taking subsequence. Now,

$$\lambda = (Df_k(\omega))^*(\mu_k) = (Df)^*(\mu)$$

and $\|\mu\| = \lim \|\mu_k\| = r + \lim \epsilon_k = r$. This proves (iii), and we are done.

Corollary 2.1 $\{\Lambda \in T^*_\omega(\Omega) : \|\Lambda\|^{\mathcal{L}} \leq 1\} = \{\nabla f(\omega) : f \in \text{Hol }_\omega(\Omega, \mathbb{D})\}$.

Proof: For any $f \in \mathrm{Hol}_\omega(\Omega, \mathbb{D})$, using Proposition 2.4, we have

$$\begin{aligned}
\{\Lambda \in T^*_\omega(\Omega) : \|\Lambda\|^{\mathcal{L}} \le 1\} &= \mathcal{L}(w : |w| \le 1) \\
&= \{(Df(\omega)^*)(\mu) : |\mu| \le 1\} \\
&= \{(\mu \frac{\partial f}{\partial z_1}(\omega), \ldots, \mu \frac{\partial f}{\partial z_m}(\omega)) : |\mu| \le 1\} \\
&= \{(\frac{\partial f}{\partial z_1}(\mu f)(\omega), \ldots, \frac{\partial}{\partial z_m}(\mu f)(\omega)) : |\mu| \le 1\} \\
&= \{\nabla(\mu f)(\omega) : |\mu| \le 1\},
\end{aligned}$$

which is clearly equal to $\{\nabla f(\omega) : f \in \mathrm{Hol}\,_\omega(\Omega, \mathbb{D})\}$.

We note that for $A = \sum_{i,j} E_{ij} \otimes A_{ij} \in \mathcal{M}_k{}^* \otimes \mathcal{M}_n \equiv \mathrm{Hom}\ (\mathcal{M}_k, \mathcal{M}_n)$, we have

$$\begin{aligned}
(DF(\omega)^* \otimes \mathrm{Id})(A) &= \left(\sum_{i,j} \frac{\partial F_{ij}}{\partial z_1}(\omega)\, A_{ij}, \ldots, \sum_{i,j} \frac{\partial F_{ij}}{\partial z_m}(\omega)\, A_{ij} \right) \\
&= A \circ DF(\omega) \otimes \mathrm{Id}.
\end{aligned}$$

We will in analogy with 2.8, write $(DF(\omega)^* \otimes \mathrm{Id})$ as F^*.

EXAMPLE 2.6 Let

$$\begin{aligned}
\mathcal{W} &= \mathrm{Hom}\ ((\mathcal{M}_k, op), (\mathcal{M}_k, op)) \cong (\mathcal{M}_k, tr) \check{\otimes} (\mathcal{M}_k, op), \\
\mathcal{V} &= T^*_\omega(\Omega) \otimes (\mathcal{M}_k, op) \cong \mathrm{Hom}(T_\omega(\Omega), (\mathcal{M}_k, op)), \text{ and} \\
\mathcal{L} &= \{(DF(\omega))^* : F \in \mathrm{Hol}\,_\omega(\Omega, (M_k)_1)\}.
\end{aligned}$$

Then $\mathcal{L}$ satisfies *Hypothesis* 2.1.

Since $DF(\omega)$ is in $\mathrm{Hom}(T_\omega(\Omega), \mathcal{M}_k)$, and A is in $\mathrm{Hom}(\mathcal{M}_k, \mathcal{M}_n)$; we see that, $A \circ DF(\omega)$ belongs to $\mathrm{Hom}(T_\omega\Omega, \mathcal{M}_k) = \mathcal{V}$. Further more, if Ω is a bounded domain, and Δ is a bounded set in $\mathbb{C}^n$, then $\mathrm{Hol}_\omega(\Omega, \Delta)$ is an equi-continuous family of maps. Furthermore, if every bounded subset of Δ is relatively compact, then $\mathrm{Hol}_\omega(\Omega, \Delta)$ is a normal family (cf. [13, Lemma 1.1(iii) and Lemma 1.4]).

Proof of (i): If $\mu \in \mathrm{Hom}\ (\mathcal{M}_k, \mathcal{M}_k) = \mathcal{W}$ is an operator of norm 1, $R_\mu(Id) = \mu$, where $R_\mu : \mathrm{Hom}\ (\mathcal{M}_k, \mathcal{M}_k) \to \mathrm{Hom}\ (\mathcal{M}_k, \mathcal{M}_k)$ is right multiplication by μ, so the distinguished element J can be taken as $Id \in \mathrm{Hom}(\mathcal{M}_k, \mathcal{M}_k)$.

Proof of (ii): Suppose $R_\mu \in \mathrm{Hom}\ (\mathcal{M}_k, \mathcal{M}_k)$ with $\|R_\mu\|_{op,op} \le \|\mu\|_{op} \le 1$ then

$$\begin{aligned}
(F^* \circ R_\mu)(A) &= (R_\mu A) \circ DF(\omega) = A \circ \mu \circ DF(\omega) \\
&= A \circ D(\mu \circ F)(\omega) \\
&= A \circ DF'(\omega) = (F')^*(A),
\end{aligned}$$

where $F' = \mu \circ F$. But $F : \Omega \to (\mathcal{M}_k)_1$ and $\|\mu \circ F\|_{op} \le \|\mu\|_{op}$, so $\|F'\| \le \|F\|$, and $F' : \Omega \to (\mathcal{M}_k)_1$. This implies (ii).

Proof of (*iii*): Normal family argument, we won't repeat the proof.

Proof of (*iv*): Since $\frac{c_1F_1^*+c_2F_2^*}{c_1+c_2} = \frac{(c_1F_1+c_2F_2)^*}{c_1+c_2}$, same proof as previous example shows that

$$F' = \frac{c_1F_1 + c_2F_2}{c_1 + c_2} \in \text{ Hol }_\omega(\Omega, (\mathcal{M}_k)_1).$$

COROLLARY 2.2 *Let $\mathcal{W}, \mathcal{V}$, and $\mathcal{L}$ be as in Example 2.6. Then*

$$\{\Lambda : \|\Lambda\|^{\mathcal{L}} \leq 1\} = \{DF(\omega) : F \in Hol_\omega(\Omega, (\mathcal{M}_k)_1)\}.$$

Proof: Exactly as in Corollary 2.1, with the identity map in $\text{Hom}(\mathcal{M}_k, \mathcal{M}_k)$ replacing 1 as the distinguished element J. Note that

$$F^*(Id) = (\frac{\partial F}{\partial z_1}, \ldots, \frac{\partial F}{\partial z_m}).$$

DEFINITION 2.2 An arbitrary norm $\| \|^a$ on $\mathcal{V}$ is called $\mathcal{L}$ - distance decreasing if for all $L \in \mathcal{L}$ we have

$$\|L\mu\|^a \leq \|\mu\| \text{ for all } \mu \in \mathcal{W}.$$

PROPOSITION 2.5 *The norm $\| \|^{\mathcal{L}}$ is the* largest *$\mathcal{L}$ -distance decreasing norm on $\mathcal{V}$.*

(This Proposition is dual to Proposition 2.2)

Proof: Again, $\| \|^{\mathcal{L}}$ is $\mathcal{L}$ - distance decreasing by definition. Suppose $\lambda \in \mathcal{W}$ and $L\mu = \lambda$ for $L \in \mathcal{L}$ then $\|\lambda\|^a = \|L\mu\|^a \leq \|\mu\|$. So $\|\lambda\|^a \leq \inf\{\|\mu\| : L\mu = \lambda \text{ for some } L \in \mathcal{L}\}$, and $\|\lambda\|^a \leq \|\lambda\|^{\mathcal{L}}$.

It is easy to check that the largest distance decreasing property of the Kobayashi norm (see 2.22) follows from the above.

For any $z \in X \otimes Y$ define the *projective tensor product norm* as

$$\|z\hat{\|} = \inf\left\{\sum_{i=1}^{n} \|x_i\|\,\|y_i\| : x_i \in X, y_i \in Y, z = \sum_{i=1}^{n} x_i \otimes y_i\right\} \tag{2.18}$$

The next example is that of a family resulting from tensoring a family satisfying *Hypothesis* 2.1 with the Id operator.

EXAMPLE 2.7 Let

$$\begin{array}{rcl} \mathcal{W} & = & T_0(\mathbb{D})\hat{\otimes}(\mathcal{M}_n, op) \cong (\mathcal{M}_n, op) \\ \mathcal{V} & = & T_\omega(\Omega) \otimes (\mathcal{M}_n) \\ \mathcal{L} & = & \{Df(0) \otimes Id : f \in \text{Hol}^\omega(\mathbb{D}, \Omega)\} \end{array}$$

and assume that the family $\mathcal{G} = \{Df(0) : f \in \text{Hol}^\omega(\mathbb{D}, \Omega)\}$ satisfies *Hypothesis* 2.1 (see Example 2.4), and that $\| \|^{\mathcal{L}}$ is a *pseudonorm.* Then $\| \|^{\mathcal{L}}$ is a norm and equal to the *projective* tensor product norm $\hat{K}_{\Omega,\omega} \overset{\text{def}}{=} K_{\Omega,\omega}\hat{\otimes}\| \|_{op}$ on the matrix tangent space $\mathcal{V}$. (Note that $K_{\Omega,\omega}$ is defined in Example 2.4)

First let $v \otimes X$ be an indecomposable tensor in $\mathcal{V}$. Since $\mathcal{G}$ is assumed to be achieving, we have a scalar $\mu \in T_0(\mathbb{D}) = \mathbb{C}$ such that $K_{\Omega,\omega}(v) =| \mu |$ and $Df(0)(\mu) = v$. Thus $(Df(0) \otimes Id)(\mu X) = v \otimes X$, and since

$$\|\mu X\|_{op} =| \mu | \|X\|_{op} = K_{\Omega,\omega}(v)\|X\|_{op}$$

we have that

$$\|v \otimes X\|^{\mathcal{L}} \leq K_{\Omega,\omega}(v)\|X\|_{op}$$

which implies that if $Y \in \mathcal{V}$ with $Y = \sum_i v_i \otimes X_i$ we have, because $\| \|^{\mathcal{L}}$ is a pseudonorm, that

$$\|Y\|^{\mathcal{L}} \leq \sum_i K_{\Omega,\omega}(v_i)\|X_i\|_{op}$$

so that

$$\|Y\|^{\mathcal{L}} \leq \hat{K}_{\Omega,\omega}(Y)$$

because of the definition of projective tensor norm. On the other hand, it is trivially checked that the norm $\hat{K}_{\Omega,\omega}$ is $\mathcal{L}$ - distance decreasing, so that by Proposition 2.5 above, we have this norm majorised by $\| \|^{\mathcal{L}}$. The latter is thus a norm.

2.5 Duality Principle

Let $\mathcal{W}$ be a normed linear space, and $\mathcal{W}^*$ be the dual linear space with the dual norm. Let

$$\mathcal{L} \subset \mathrm{Hom}(\mathcal{V}, \mathcal{W}), \qquad \text{so that} \qquad \mathcal{L}^* \subset \mathrm{Hom}(\mathcal{W}^*, \mathcal{V}^*), \tag{2.19}$$

where $\mathcal{L}^* = ($ dual of $\mathcal{L})$, $\mathcal{V}^* =$ dual of $\mathcal{V}$.

Theorem 2.1 (Basic duality principle) *Assume that $\| \|_{\mathcal{L}}$ and $\| \|^{\mathcal{L}^*}$ are norms. Then* $(\| \|_{\mathcal{L}})^* = \| \|^{\mathcal{L}^*}$.

Proof: First we claim $(\| \|_{\mathcal{L}})^* \leq \| \|^{\mathcal{L}^*}$.

We recall Proposition 2.5 which says $\| \|^{\mathcal{L}^*}$ is the *largest* $\mathcal{L}^*$ - distance decreasing norm on $\mathcal{V}^*$. So it is just enough to show that $(\| \|_{\mathcal{L}})^*$ is $\mathcal{L}^*$ - distance decreasing.

Let $L^* \in \mathcal{L}^*$(so $L \in \mathcal{L}$). Then for $\mu \in \mathcal{W}^*$

$$\begin{aligned}(\|L^*\mu\|_{\mathcal{L}})^* &\stackrel{\text{def}}{=} \sup_{\|v\|_{\mathcal{L}} \leq 1} \frac{|\langle L^*\mu, v\rangle|}{\|v\|_{\mathcal{L}}} \\ &= \sup_{\|v\|_{\mathcal{L}} \leq 1} \frac{|\langle \mu, Lv\rangle|}{\|v\|_{\mathcal{L}}}\end{aligned}$$

but $\| \|_{\mathcal{L}}$ is $\mathcal{L}$ - distance decreasing, so $\|v\|_{\mathcal{L}} \geq \|Lv\|$. So

$$(\|L^*\mu\|_{\mathcal{L}})^* \leq \sup_{\|v\|_{\mathcal{L}} \leq 1} \frac{|\langle \mu, Lv\rangle|}{\|Lv\|}.$$

Now, if $\|v\|_{\mathcal{L}} \leq 1, \|Lv\| \leq 1$, then $\{Lv : \|v\|_{\mathcal{L}} \leq 1\} \subset \{w \in \mathcal{W} : \|w\| \leq 1\}$. So that,

$$(\|L^*\mu\|_{\mathcal{L}})^* \leq \sup_{\|w\| \leq 1} \frac{|\langle \mu, w\rangle|}{\|w\|} \stackrel{\text{def}}{=} \|\mu\| \text{ (as an element of } \mathcal{W}^*).$$

This proves what we wanted.

Next, we *claim*

$$\|\ \|_{\mathcal{L}} \leq (\|\ \|^{\mathcal{L}^*})^*.$$

(This would show $\|\ \|_{\mathcal{L}}^* \geq \|\ \|^{\mathcal{L}^*}$, the other inequality we need). We will show that $(\|\ \|_{\mathcal{L}}^*)^*$ is $\mathcal{L}$-distance decreasing, and since $\|\ \|_{\mathcal{L}}$ is least $\mathcal{L}$ - distance decreasing by Proposition 2.2, we would be done.

Use proposition 2.4, to observe that $\{\|\lambda\|^{\mathcal{L}^*} \leq 1\} = \mathcal{L}^*\{\mu : \|\mu\| \leq 1\}$

$$\begin{aligned}(\|v\|^{\mathcal{L}^*})^* &= \sup_{\|\lambda\|^{\mathcal{L}^*}\leq 1} \frac{|\langle \lambda, v\rangle|}{\|\lambda\|^{\mathcal{L}^*}} \\ &= \sup_{\|\mu\|\leq 1,\, L^*\in\mathcal{L}^*} \frac{|\langle L^*\mu, v\rangle|}{\|L^*\mu\|^{\mathcal{L}^*}} \\ &\geq \sup_{\|\mu\|\leq 1,\, L\in\mathcal{L}} \frac{|\langle \mu, Lv\rangle|}{\|\mu\|}, (\text{ because } \|L^*\mu\|^{\mathcal{L}^*} \leq \|\mu\|) \\ &= \|Lv\|^{**} = \|Lv\| \text{ for all } L \in \mathcal{L}\end{aligned}$$

Thus, $\|Lv\| \leq \|v\|^{\mathcal{L}^{**}}$ for all $L \in \mathcal{L}$.

This shows $\|\ \|^{\mathcal{L}^{**}}$ is $\mathcal{L}$-distance decreasing, and the proof is complete.

Remark 2.3 This theorem generalises to any arbitrary family $\mathcal{L}$. In this situation $\|\|_{\mathcal{L}}$ and $\|\ \|^{\mathcal{L}}$ are quasi and pseudo norms respectively. This theorem therefore contains Proposition 6 of [6] as a special case.

2.6 Distance decreasing metrics

Following Royden [11, p.397], a hyperbolic infinitesimal metric of order n is an asignment of a norm δ_Ω on the *matricial* tangent space $T_\Omega \otimes \mathcal{M}_n$, $\Omega \subseteq \mathbb{C}^m$ such that

$$\delta_D(\omega, V) = (1 - \|\omega\|^2)^{-1}\|V\|_{op} \tag{2.20}$$

and for any holomorphic function $f : \Omega \to \tilde{\Omega}$,

$$\delta_{\tilde{\Omega}}(f(\omega), f_*(V)) \leq \delta_\Omega(\omega, V). \tag{2.21}$$

Note that if $\mathcal{L}^{(k)} = \{DF(\omega) \otimes Id : F \in \mathrm{Hol}_\omega(\Omega, (\mathcal{M}_k)_1)\}$ and δ is hyperbolic infinitesimal metric then δ is $\mathcal{L}^{(1)}$ - distance decreasing, that is, for all $f \in \mathrm{Hol}_\omega(\Omega, \mathbb{D})$ we have $\|f_*(V)\|_{op} \leq \delta(V)$. Similarly, if $\mathcal{L}_{(k)} = \{DF(\omega) \otimes Id : F \in \mathrm{Hol}^\omega((\mathcal{M}_k)_1, \Omega)\}$, then δ is $\mathcal{L}_{(1)}$ - distance decreasing. Thus, in view of Propositions 2.5 and 2.2, we have the inequalities

$$\check{C}_{\Omega,\omega}(V) \leq \delta(V) \leq \hat{K}_{\Omega,\omega}(V). \tag{2.22}$$

Further, if we define as in example 2.3 (compare 2.16)

$$\hat{K}^k_{\Omega,\omega}(V) \stackrel{\text{def}}{=} \|V\|^{\mathcal{L}_{(k)}}, \tag{2.23}$$

where

$$\begin{aligned}
\mathcal{W} &= (T_0((\mathcal{M}_k)_1)\hat{\otimes}(\mathcal{M}_n, op)) \\
\mathcal{V} &= T_\omega(\Omega) \otimes (\mathcal{M}_n) \\
\mathcal{L}_{(k)} &= \{DF(0) \otimes Id : F \in \mathrm{Hol}^\omega((\mathcal{M}_k)_1, \Omega)\}
\end{aligned}$$

then we have the following inequalities at the level of *matricial* tangent vectors

$$\check{C}_{\Omega,\omega} \leq \cdots \leq \check{C}^j_{\Omega,\omega} \leq \cdots \leq \hat{K}^k_{\Omega,\omega} \leq \cdots \leq \hat{K}_{\Omega,\omega}. \tag{2.24}$$

Note that the maps $\mathbb{D} \underset{\pi}{\overset{i}{\rightleftarrows}} (\mathcal{M}_k)_1$, defined by

$$z \xrightarrow{i} \text{ diag } (z, \ldots, z) \text{ and } (z_{j\ell}) \xrightarrow{\pi} z_{11}$$

satisfy $\pi \circ i = id$. For $f \in \mathrm{Hol}_\omega(\Omega, \mathbb{D})$, define $F = i \circ f \in \mathrm{Hol}_\omega(\Omega, (\mathcal{M}_k)_1)$. Since $\pi \circ F = f$, the map $f \to F$ is injective. Similarly, for $g \in \mathrm{Hol}^\omega(\mathbb{D}, \Omega)$, define $G = g \circ \pi \in \mathrm{Hol}^\omega((\mathcal{M}_k)_1, \Omega)$. Since $G \circ i = g$, the map $g \to G$ is injective. This proves the first and the last inequality in 2.24.

To prove the middle inequality, we show that $\check{C}^j_{\Omega,\omega}$ is $\mathcal{L}_{(k)}$ - distance decreasing for all k. Recall that, in this case $\mathcal{W} = T_0(\mathcal{M}_k)_1\hat{\otimes}\mathcal{M}_n$ and $\mathcal{V} = T_\omega\Omega \otimes \mathcal{M}_n$. Let
$L \in \mathcal{L}_{(k)} = \{DF \otimes Id : F \in \mathrm{Hol}^\omega((\mathcal{M}_k)_1, \Omega)\}$. Then for $W \in \mathcal{W}$ and $V = L(W) = g_*W, g \in \mathrm{Hol}^\omega((\mathcal{M}_k)_1, \Omega)$, we have

$$\begin{aligned}
\|L(W)\|_{\mathcal{L}^{(j)}} &= \check{C}^j_{\Omega,\omega}(V) \\
&= \sup\{\|F_*(V)\|_{op} : F \in \mathrm{Hol}(\Omega, (\mathcal{M}_j)_1)\} \\
&= \sup\{\|F_*g_*(W)\|_{op} : F \in \mathrm{Hol}(\Omega, (\mathcal{M}_j)_1)\} \\
&\leq \sup\{\|(Fh)_*(W)\|_{op} : F \in \mathrm{Hol}(\Omega, (\mathcal{M}_j)_1), h \in \mathrm{Hol}^\omega((\mathcal{M}_k)_1, \Omega)\} \\
&\leq \sup\{\|G_*W\|_{op} : G \in \mathrm{Hol}_0((\mathcal{M}_k)_1, (\mathcal{M}_j)_1)\} \\
&\leq \sup\{\|G_*W\hat{\|} : G \in \mathrm{Hol}_0((\mathcal{M}_k)_1, (\mathcal{M}_j)_1)\}.
\end{aligned}$$

Recall that (see 2.8), $G_*W = (DG(0) \otimes I)(W)$. The Schwarz lemma applied to the two unit balls $(\mathcal{M}_k)_1$ and $(\mathcal{M}_j)_1$ says that $DG(0)$ is a contraction. If $W = \sum_{i=1}^n A_i \otimes B_i$ is any representation of the *matricial* tangent vector W, then we have

$$\begin{aligned}
\|G_*(W)\hat{\|} &\leq \sum \|DG(0)A_i\|_{op}\|B_i\|_{op} \\
&\leq \sum \|A_i\|_{op}\|B_i\|_{op}
\end{aligned}$$

Thus, $\sup\{\|G_*W\hat{\|} : G \in \mathrm{Hol}_0((\mathcal{M}_k)_1, (\mathcal{M}_j)_1)\} \leq \|W\hat{\|}$. This completes the proof of the inequalities 2.24.

We now go back to our basic question (Question 2.1).

Theorem 2.2 $\check{\Gamma}_{\Omega,\omega} \stackrel{\text{def}}{=} \{V \in T_\omega\Omega \otimes \mathcal{M}_n : \check{C}_{\Omega,\omega}(V) \leq 1\} = \Gamma(\check{C}_{\Omega,\omega})$

Proof: Note that, Corollary 2.1 identifies the set $\mathcal{D}\Omega(\omega)$ with the unit ball with respect to the norm $\|\ \|^{\mathcal{L}}$, where $\mathcal{L} = \mathcal{D}\Omega(\omega)$. Next, Theorem 2.1 identifies this ball as the unit ball in the co-tangent space $T_\omega^*\Omega$ with respect to the dual of the Carthédory norm. This completes the proof

COROLLARY 2.3 *If* $\mathbb{C}^{2n}_{N(V,\omega)}$ *is contractive, then*

$$\sup\{\check{C}_{(\mathcal{M}_k)_1,0}(F_*(V)) : F \in Hol_\omega(\Omega, (\mathcal{M}_k)_1)\} \leq 1,$$

for $k = 1, 2, \ldots$.

The proof of this corollary is the same as that of Theorem 1.1, once we note that $\check{C}_{\Omega,\omega}$ is a $\mathcal{L}^{(1)}$ - distance decreasing metric. In fact, as we have seen, it is the least such metric.

Note that, $\check{C}_{\Omega,\omega}(V)$ is the injective tensor product norm of V as an element of $(T_\omega\Omega, C_{\Omega,\omega}) \otimes (\mathcal{M}_n, op)$. Thus, $V \in \check{\Gamma}_{\Omega,\omega}$ if and only if $V : \Gamma C^*_{\Omega,\omega} \to (\mathcal{M}_k)_1$.

COROLLARY 2.4 $\mathbb{C}^{2n}_{N(V,\omega)}$ *is contractive over* $H^\infty(\Omega)$ *if and only if*

$$V \in ((\mathcal{M}_n, op)\check{\otimes}(T_\omega\Omega, C_{\Omega,\omega})_1)\,.$$

The proof follows directly from theorem 2.2.

COROLLARY 2.5 *Every contractive module* $\mathbb{C}^{2n}_{N(V,\omega)}$ *over* $H^\infty(\Omega)$ *is completely contractive if and only if*

$$F_* : \Gamma(\check{C}_{\Omega,\omega}) \to ((T_0\mathcal{M}_k \otimes \mathcal{M}_n), op))_1 \cong (\mathcal{M}_{kn}, op)_1,$$

for all k*, and* F *in* $Hol_\omega(\Omega, (\mathcal{M}_k)_1)$*. Or, equivalently, if and only if for all*

$$\check{C}^1_{\Omega,\omega} = \check{C}^2_{\Omega,\omega} = \cdots = \check{C}^k_{\Omega,\omega} = \cdots,$$

$k = 1, 2, \ldots$.

This corollary is merely the statement that *contractive* modules are *completely contractive* if and only $\rho^{(k)}$ is a contraction (in the sense of 2.11) for $k = 1$ implies it remains a contraction for all $k > 1$.

COROLLARY 2.6 $\mathbb{C}^{2n}_{N(V,\omega)}$ *is contractive over* $H^\infty(\Omega)$ *if and only if* $\mathbb{C}^{2n}_{N(V,0)}$ *is contractive over* $H^\infty(\Gamma(C_{\Omega,\omega}))$.

REMARK 2.4 Note that if $\check{C}_{\Omega,\omega} = \hat{K}_{\Omega,\omega}$, then every *contractive* module $\mathbb{C}^2_{N(v,\omega)}$ is *completely contractive* over $H^\infty(\Omega)$. However, there are examples due to Parrott (cf. [9], in fact $\check{C}_{\Omega,\omega} \neq \check{C}^2_{\Omega,\omega}$) of *contractive* modules of this type over the tri-disk, which are not *completely contractive*. This shows that the two extremal metrics in 2.24 can not be equal.

3 Operator spaces

In this section, we relate our discussion on *contractive*, and completely contractive modules over $H^\infty(\Omega)$ to that of the theory of abstract *Operator Spaces.*

Let X be a vector space over $\mathbb{C}$, and let $\mathcal{M}_n(X)$ be the vector space of $n \times n$ matrices with entries from X. The vector space X is *matrix normed* if each $(\mathcal{M}_n(X), \|\cdot\|_n)$ is a normed linear space such that

1. For every B in $\mathcal{M}_n(X)$, 0 in $\mathcal{M}_m(X)$, $\|B \oplus 0\|_{n+m} = \|B\|_n$,
2. For B in $\mathcal{M}_n(X)$, A, C in $\mathcal{M}_n$, $\|ABC\| \leq \|A\|\,\|B\|_n\|C\|$.

Among the matricially normed spaces, there are some matrix norm structures, which are ℓ^∞-matricially normed. Such spaces are called operator spaces (cf. [10]).

Definition 3.1 If X and Y are matrix normed spaces and $\varphi : X \to Y$ is linear, then we define, $\varphi^{(k)} : \mathcal{M}_n(X) \to \mathcal{M}_n(Y)$ via $\varphi^{(k)}((x_{ij})) = (\varphi(x_{ij}))$. We say that, $\|\varphi\|_{cb} = \sup_k \|\varphi^{(k)}\|$ is the cb-norm of φ. The map φ is said to be *completely contractive*, if $\varphi^{(k)}$ is a contraction for each k.

There are many natural ways in which we may matricially norm a vector space X (cf. [2]). However, we single out one particular matrix norm structure on a vector space X. Let $(X, \|\cdot\|)$, be any vector space, declare the norm $\|\cdot\|_n$ by identifying $\mathcal{M}_n(X)$ with the injective tensor product $(X, \|\cdot\|)\check{\otimes}(\mathcal{M}_n, op)$.

Let $X = (T^*_\omega\Omega, C^*_{\Omega,\omega})$ and $\mathcal{D}\Omega^{(k)}(\omega) \subset \mathcal{M}_k(X)$ be the unit ball with respect to the norm $\|\ \|^{\mathcal{L}}$, where $\mathcal{L} = \mathrm{Hol}_\omega(\Omega, (\mathcal{M}_k)_1)$. This gives X an operator space structure [10, Proposition 3.2]. It is shown in [10] that if $\Omega = \mathbb{B}$ is the open unit ball with respect to some norm in $\mathbb{C}^m$, then

$$\mathcal{D}\mathbb{B}^{(k)}(0) = (T^*_{\mathbb{B},0}\check{\otimes}(\mathcal{M}_k, op))_1.$$

If $\mathbb{B}$ is homogeneous, then this result remains valid for an arbitrary point $\omega \in \mathbb{B}$. We have a similar result for product domains.

Theorem 3.1 *For any two domains $\Omega_1, \Omega_2 \subset \mathbb{C}$, let $\Omega = \Omega_1 \times \Omega_2$, be the product domain, and $\omega = (\omega_1, \omega_2) \in \Omega$. Then*

$$\mathcal{D}\Omega^{(k)}(\omega) = (T^*_{\Omega,\omega}\check{\otimes}(\mathcal{M}_k, op))_1.$$

Proof: For $j = 1, 2$, let F_{Ω_j,ω_j} be the Ahlfors functions (cf. [4, Theorem 1.6]) for the domains Ω_j at the points $\omega_j \in \Omega_j$. The indicatrices $\Gamma_{\Omega_j,\omega_j}$ are disks of radius $r_j = DF_{\Omega_j,\omega_j}(\omega_j)$. Let $G_j = r_j^{-1}F_{\Omega_j,\omega_j}$. Then $G_j \to \Gamma_{\Omega_j,\omega_j}$ and the derivative $DG_j(\omega_j) = 1$. It is of course, easy to see that

$$\mathcal{D}\Omega^{(k)}(\omega) \subset (T^*_{\Omega_1\times\Omega_2,(\omega_1,\omega_2)}\check{\otimes}(\mathcal{M}_k, op))_1.$$

To verify the opposite inclusion, let $\Lambda \in (T^*_{\Omega_1\times\Omega_2,(\omega_1,\omega_2)}\check{\otimes}(\mathcal{M}_k, op))_1$ be arbitrary. Since

$$C_{\Omega_1\times\Omega_2,(\omega_1,\omega_2)} = \max\{C_{\Omega_1,\omega_1}, C_{\Omega_2,\omega_2}\},$$

the indicatrix

$$\Gamma_{\Omega_1\times\Omega_2,(\omega_1,\omega_2)} = \Gamma_{\Omega_1,\omega_1} \times \Gamma_{\Omega_2,\omega_2},$$

and it follows that

$$G = (G_1, G_2) : \Omega_1 \times \Omega_2 \to \Gamma_{\Omega_1\times\Omega_2,(\omega_1,\omega_2)}.$$

Note that,

$$\Lambda \circ G : \Omega_1 \times \Omega_2 \to (\mathcal{M}_k)_1,\ (\Lambda \circ G)(\omega_1, \omega_2) = 0.$$

The derivative $(DG)(\omega_1, \omega_2) = Id$, and therefore

$$D(\Lambda \circ G)(\omega_1, \omega_2) = \Lambda.$$

Thus,

$$\Lambda \in \mathcal{D}\Omega^{(k)}(\omega)$$

This completes the proof.

4 Appendix on The quotient norm for matrix tangent vectors

On Ω, we clearly have the holomorphic vector bundles

$$\begin{aligned} \mathcal{L}\left((\mathcal{M}_n, tr), T\Omega\right) &\cong (\mathcal{M}_n, tr)^* \otimes T\Omega, \text{ and} \\ \mathcal{L}\left(T\Omega, (\mathcal{M}_n, op)\right) &\cong T^*\Omega \otimes (\mathcal{M}_n, op), \end{aligned} \tag{4.1}$$

where the tensor products and linear maps are fibrewise.

Definition 4.1 We define,

$$\begin{aligned} \mathcal{E}_{\omega,n} &= \mathcal{L}\left((\mathcal{M}_n, tr), (T_\omega\Omega, C_{\Omega,\omega})\right) \cong (T_\omega\Omega, C_{\Omega,\omega}) \check{\otimes} (\mathcal{M}_n, op), \text{ and} \\ \mathcal{E}_{\Omega,n} &= \amalg\left\{(\omega, V) : \omega \in \Omega, V \in (\mathcal{E}_{\omega,n})_1\right\} \end{aligned}$$

Similarly, there is a dual definition,

$$\begin{aligned} \mathcal{E}^*_{\omega,n} &= \mathcal{L}\left((T_\omega\Omega, C_{\Omega,\omega}), (\mathcal{M}_n, op)\right) \cong \left(T^*_\omega\Omega, C^*_{\Omega,\omega}\right) \check{\otimes} (\mathcal{M}_n, op), \text{ and} \\ \mathcal{E}^*_{\Omega,n} &= \amalg\left\{(\omega, \Lambda) : \omega \in \Omega, \Lambda \in \left(\mathcal{E}^*_{\omega,n}\right)_1\right\} \end{aligned}$$

If $f : \Omega \to \tilde{\Omega}$, $f(\omega) = 0$, and f is holomorphic, then

$$f_* : \Gamma(C_{\Omega,\omega}) \to \Gamma(C_{\tilde{\Omega},f(\omega)}). \tag{4.2}$$

and by 4.2, we obtain the map, $f_* : \mathcal{E}_{\omega,n} \to \mathcal{E}_{f(\omega),n}$ defined by

$$f_*(V) = \nabla f(\omega) \circ V. \tag{4.3}$$

Similarly, there is a pullback, $f^* : \mathcal{E}^*_{f(\omega),n} \to \mathcal{E}^*_{\omega,n}$ defined by

$$f^*(V) = \Lambda \circ \nabla f(\omega). \tag{4.4}$$

These maps induce functorially maps $\tilde{f} : \mathcal{E}_{\Omega,n} \to \mathcal{E}_{\tilde{\Omega},n}$ and $f_\# : \mathcal{E}^*_{\tilde{\Omega},n} \to \mathcal{E}^*_{\Omega,n}$. Where, the case $k = 1$ is the standard push foroward of tangent vectors or pullback of co-tangent vectors. The corresponding bundle diagrams are

$$\begin{array}{ccc} \tilde{f} : \mathcal{E}_{\Omega,n} & \to & \mathcal{E}_{\tilde{\Omega},n} \\ \pi \downarrow & & \downarrow \\ f : \Omega & \to & \tilde{\Omega} \end{array} \quad \text{and} \quad \begin{array}{ccc} f_\# : \mathcal{E}^*_{\Omega,n} & \leftarrow & \mathcal{E}^*_{\tilde{\Omega},n} \\ \pi \downarrow & & \downarrow \\ f : \Omega & \to & \tilde{\Omega} \end{array} \tag{4.5}$$

For notational convenience, we think of a point in $\mathcal{E}_{\Omega,n}$ as a pair (z, Z). We obtain the map $\tilde{f} : \mathcal{E}_{\Omega,n} \to \mathcal{E}_{\tilde{\Omega},n}$ by setting

$$\tilde{f}(z, Z) = (f(z), f_*(z) \circ Z). \tag{4.6}$$

We have called an element of $\mathcal{E}_{\omega,n}$, a *matricial tangent vector at* ω. There is a way to view these as *ordinary tangent vectors* to $\mathcal{E}_{\Omega,n}$. Indeed, $\Omega \hookrightarrow \mathcal{E}_{\Omega,n}$ and $\Omega \hookrightarrow \mathcal{E}^*_{\Omega,n}$ as the zero section. Let j be these inclusions.

PROPOSITION 4.1 *There is a split exact sequence,*

$$0 \to T\Omega \xrightarrow{j_*} T\mathcal{E}_{\Omega,n}|_\Omega \to T\Omega \otimes (\mathcal{M}_n, tr)^* \to 0,$$

which identifies the matricial *tangent bundle as the normal bundle to* $j(\Omega)$ *in* $\mathcal{E}_{\Omega,n}$.

Proof: If U is a co-ordinate chart around ω, we have

$$\begin{aligned} T_\omega(\mathcal{E}_{\Omega,n}) = T_\omega(\pi^{-1}(U)) &\cong T_\omega(U \times \mathrm{Hom}(\mathcal{M}_k, T_\omega(\Omega)) \\ &\cong T_\omega(U) \oplus \mathrm{Hom}(\mathcal{M}_k, T_\omega(\Omega)) \\ &\cong T_\omega(\Omega) \oplus \mathrm{Hom}(\mathcal{M}_k, T_\omega(\Omega)) \end{aligned}$$

This completes the proof.

The map, $\tilde{f}_* : \Gamma(C_{\mathcal{E}_{\Omega,n},(z,Z)}) \to \Gamma(C_{\mathcal{E}_{\tilde{\Omega},n},\tilde{f}(z,Z)})$ is obtained by

$$\tilde{f}_*(v, V) = (f_*(v), f_*(z) \circ V). \tag{4.7}$$

Let $((z_1, \ldots, z_m), (Z^1_{ij}, \ldots, Z^m_{ij}))$, be a cordinate system in $\mathcal{E}_{\Omega,n}$, and $f : \mathcal{E}_{\Omega,n} \to \mathbb{D}$ be holomorphic, with $f(\omega, 0) = 0$. Write,

$$f(z, Z) = \sum_{l=1}^{m} (z_l - \omega_l) f^l + \sum_{i,j=1}^{n} \sum_{l=1}^{m} Z^l_{ij} f^{ij}_l \tag{4.8}$$

Identifying, $(\mathcal{E}_{\omega,n})_1$ as the fibre at ω of $\mathcal{E}_{\Omega,n}$ we see that $C_{\mathcal{E}_{\Omega,n},(\omega,0)}(0, V) \le C_{(\mathcal{E}_{\omega,n})_1,(\omega,0)}(0, V)$, and using the distance decreasing property, 4.2 of the Carathéodory metric on $\mathcal{E}_{\Omega,n}$, we obtain

$$\begin{aligned} |f_*(0, V)| &\le C_{\mathcal{E}_{\Omega,n},(\omega,0)}(0, V) \\ &\le C_{(\mathcal{E}_{\omega,n})_1,0}(V) = \check{C}_{\Omega,\omega,}(V). \end{aligned} \tag{4.9}$$

Let us put $F_l = \left[F_l^{ij}\right], l = 1, \ldots, m$. Thus,

$$|tr \sum F_l \cdot V^l| = |f_*(0, V)| \leq \check{C}_{\Omega,\omega}(V). \tag{4.10}$$

In particular, if $V^l = v_l X$, $l = 1, \ldots, m$, $(v_1, \ldots, v_m) \in \Gamma(C_{\Omega,\omega})$ and $X \in (\mathcal{M}_n)_1$, then $\check{C}_{\Omega,\omega}(V) \leq 1$. Observe that,

$$|tr(v_1 F_1 + \cdots + v_m F_m) \cdot X| \leq 1, \text{ for } X \in (\mathcal{M}_n)_1, \tag{4.11}$$

that is,

$$\|(v_1 F_1 + \cdots + v_m F_m)\|_{op} \leq \|(v_1 F_1 + \cdots + v_m F_m)\|_{tr} \leq 1. \tag{4.12}$$

Definition 4.2 Let $q : T\mathcal{E}_{\Omega,n} \to T\Omega \otimes \mathcal{M}_n$, be the quotient map, that is, $q(v, V) = V$. Define, the quotient norm of V as

$$\|V\|_q = \inf \{C_{\mathcal{E}_{\Omega,n},(\omega,0)}(v, V) : q(v, V) = V\}.$$

Theorem 4.1 *The quotient norm and the injective tensor norm for a* matricial *tangent vector are the same.*

Proof: Let (ω, W) be a fixed but arbitrary point in $\mathcal{E}_{\Omega,n}$, and f be in $\mathrm{Hol}_\omega(\Omega, \mathbb{D})$. We obtain, a map (as in 4.7), $\mathcal{E}_{\Omega,n} \xrightarrow{\tilde{f}} \mathcal{E}_{D,n} \xrightarrow{\tilde{\pi}} (\mathcal{M}_n)_1$,

$$\tilde{\pi} \circ \tilde{f}(z, Z) = \tilde{\pi}(f(z), f_*(z) \circ Z) = f_*(z) \circ Z. \tag{4.13}$$

For (v, V) in $T_{(\omega,0)}\mathcal{E}_{\Omega,n}$, we have

$$(\tilde{\pi} \circ \tilde{f})_*(v, V) = f_*(\omega)(V) \tag{4.14}$$

$$\begin{aligned}
\check{C}_{\Omega,\omega}(V) &= \sup \{\|f_* V\|_{op} : f \in \mathrm{Hol}_\omega(\Omega, \mathbb{D})\} \\
&= \sup \{\|(\tilde{\pi} \circ \tilde{f})_*(v, V)\|_{op} : f \in \mathrm{Hol}_\omega(\Omega, \mathbb{D})\} \\
&\leq \sup \{\|F_*(v, V)\|_{op} : F \in \mathrm{Hol}_\omega(\mathcal{E}_{\Omega,n}, (\mathcal{M}_n)_1\} \\
&= \sup \{\|F_*(v, V)\|_{op} : F \in \mathrm{Hol}_\omega(\mathcal{E}_{\Omega,n}, D\}, \\
&\leq \inf_v C_{\mathcal{E}_{\Omega,n},(\omega,0)}(v, V) = \|V\|_q.
\end{aligned}$$

The last inequality is valid, since every map $f : \mathcal{E}_{\Omega,n} \to \mathbb{D}$ descends to a map $F = (F_1, \ldots, F_m) : \Gamma(C_{\Omega,\omega}) \to (\mathcal{M}_n)_1$ in view of 4.12. On the other hand,

$$\begin{aligned}
\|V\|_q = \inf C_{\mathcal{E}_{\Omega,n},(\omega,0)}(v, V) &\leq C_{\mathcal{E}_{\Omega,n},(\omega,0)}(0, V) \\
&= \sup \{\|f_*(0, V)\| : f \in \mathrm{Hol}_\omega(\mathcal{E}_{\Omega,n}, \mathbb{D})\}, \\
&\leq \check{C}_{\Omega,\omega}(V) \text{ (recall 4.9).}
\end{aligned}$$

This completes the proof.

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