On quotient modules – the case of arbitrary multiplicity

Ronald G. Douglas*
Texas A & M University
College Station
Texas 77843
Gadadhar Misra†and Cherian Varughese
Indian Statistical Institute
R. V. College Post
Bangalore 560 059

Abstract: Let $\mathcal{M}$ be a Hilbert module of holomorphic functions defined on a bounded domain $\Omega \subseteq \mathbb{C}^m$. Let $\mathcal{M}_0$ be the submodule of functions vanishing to order k on a hypersurface $\mathcal{Z}$ in Ω. In this paper, we describe the quotient module $\mathcal{M}_{\mathrm{q}}$.

Introduction

If $\mathcal{M}$ is a Hilbert module over a function algebra $\mathcal{A}$ and $\mathcal{M}_0 \subseteq \mathcal{M}$ is a submodule, then determining the quotient module $\mathcal{M}_{\mathrm{q}}$ is an interesting problem, particularly if the function algebra consists of holomorphic functions on a domain Ω in $\mathbb{C}^m$ and $\mathcal{M}$ is a functional Hilbert space. It would be very desirable to describe the quotient module $\mathcal{M}_{\mathrm{q}}$ in terms of the last two terms in the short exact sequence

$$0 \longleftarrow \mathcal{M}_{\mathrm{q}} \longleftarrow \mathcal{M} \xleftarrow{X} \mathcal{M}_0 \longleftarrow 0$$

where X is the inclusion map. For certain modules over the disc algebra, this is related to the model theory of Sz.-Nagy and Foias.

In a previous paper [6], the quotient module was described assuming that the submodule $\mathcal{M}_0$ is the maximal set of functions in $\mathcal{M}$ vanishing on $\mathcal{Z}$, where $\mathcal{Z}$ is an analytic

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submanifold of dimension $m-1$. Two essentially distinct approaches were presented there, the first provides a model for the quotient module as a Bergman kernel Hilbert space while the second uses the notion of tensor product localization introduced in [7] to obtain the hermitian holomorphic line bundle which characterizes the quotient module in $B_1(\Omega)$. Assuming that the module $\mathcal{M}$ is itself defined by a kernel function, the first model mentioned is obtained by describing the kernel function for the quotient module. Although the kernel function for a module is not unique, the geometric methods in [4], [3] allow one to decide when different kernel functions lead to equivalent Hilbert modules. An intrinsic hermitian holomorphic line bundle is the key here since curvature is a complete invariant in the case of line bundles. Localization provides another method to construct this line bundle and hence to obtain the model for the quotient module.

In case $\mathcal{M}_0$ is a submodule determined by the functions in $\mathcal{M}$ which vanish to some higher order on $\mathcal{Z}$, the preceding approach becomes more complicated but this is the subject we consider in this paper. We are able to generalize completely the first part of the results described above by introducing a notion of matrix-valued kernel function which enables us to provide a model for the quotient module. The reason we can't use ordinary kernel functions is that the multiplicity of the zero set shows up in the dimension of the hermitian holomorphic bundle in the complex geometric approach and somehow our model must capture the fact that one is not dealing with a line bundle. While the generalized notion of kernel function accomplishes that, the equivalence problem becomes more complex and not completely resolved. If one considers only the module action of functions on $\mathcal{Z}$, then one could use the $B_k(\mathcal{Z})$ - theory of [4], [3] but the action of functions in the 'normal' variable brings a nilpotent bundle endomorphism into the picture. Our results here are not as definitive and involve approaches using modules corresponding to a resolution of the multiplicity of the zero set analogous to studying the corresponding hermitian holomorphic bundle via a resolution of line bundles. Despite the open questions that remain, our results seem of sufficient interest and the issues raised of such central concern to hermitian algebraic geometry to merit publication.

Our work may have some interesting relationship with earlier work of Martin and Salinas [10]. We have not explored this yet but intend to return to these questions in the near future.

In the following paragraphs, we state the assumptions we make on the Hilbert module $\mathcal{M}$, and the algebra $\mathcal{A}$. The assumptions on the submodule $\mathcal{M}_0$ are stated in the next section after some preliminaries on multiplicity.

We assume that the Hilbert space $\mathcal{M}$ is a functional Hilbert space, that is, it consists of holomorphic functions on a domain (open, connected set) $\Omega \subseteq \mathbb{C}^m$ and that the evaluation functionals on $\mathcal{M}$ are bounded. In addition, we assume that polynomials belong to the Hilbert space $\mathcal{M}$. Consequently, $\mathcal{M}$ admits a reproducing kernel K. We recall that $K : \Omega \times \Omega \to \mathbb{C}$ is holomorphic in the first variable and anti-holomorphic in the second variable. Further, $K(\cdot, w) \in \mathcal{M}$ for each fixed $w \in \Omega$ and $K(z, w) =$

$\overline{K(w,z)}$. Finally, K has the reproducing property

$$\langle h, K(\cdot, w)\rangle = h(w) \text{ for } w \in \Omega,\;\; h \in \mathcal{M}.$$

Since $K(w,w) = \big\langle K(\cdot, w), K(\cdot, w)\big\rangle$, it follows that $K(w,w) \neq 0$ for $w \in \Omega$.

Let $\mathcal{A}(\Omega)$ denote the closure (in the supremum norm on $\bar{\Omega}$, the closure of Ω) of functions holomorphic in a neighborhood of $\bar{\Omega}$. Then $\mathcal{A}(\Omega)$ is a function algebra and consists of continuous functions on $\bar{\Omega}$ which are holomorphic on Ω. In case Ω is polynomially convex, the algebra $\mathcal{A}(\Omega)$ equals the uniform limits of polynomials on $\bar{\Omega}$. A proof of this theorem due to Oka can be found in [9, Proposition 2, p.56]. *We will assume that* Ω *is polynomially convex.*

The module action is assumed to be the natural one, $(f, h) \mapsto f \cdot h$ for $f \in \mathcal{A}(\Omega)$ and $h \in \mathcal{M}$, where $f \cdot h$ denotes pointwise multiplication. The operator associated with this action, that is, $h \mapsto f \cdot h$ for a fixed $f \in \mathcal{A}(\Omega)$ will be denoted by M_f. We assume throughout that the algebra $\mathcal{A}(\Omega)$ acts boundedly on the Hilbert space $\mathcal{M}$. This means that the pointwise product $f \cdot h$ is in $\mathcal{M}$ for each $f \in \mathcal{A}(\Omega)$ and each $h \in \mathcal{M}$. Note that the closed graph theorem ensures the boundedness of the operator $(f, h) \mapsto f \cdot h$ so that $\mathcal{M}$ is a Hilbert module in the sense of [7]. Alternatively, since the polynomials are dense in $\mathcal{A}(\Omega)$, the inequality

$$\|p \cdot h\|_{\mathcal{M}} \leq K \|p\|_\infty \|h\|_{\mathcal{M}}, \;\;\; h \in \mathcal{M} \text{ for all polynomials } p,$$

together with the uniform boundedness principle ensures that $\mathcal{M}$ is a Hilbert module.

It is easy to verify that the adjoint of this action admits $K(\cdot, w)$ as an eigenvector with eigenvalue $\overline{f(w)}$, that is, $M_f^* K(\cdot, w) = \overline{f(w)} K(\cdot, w)$ for $w \in \Omega$.

1 The submodule $\mathcal{M}_0$

Let $\mathcal{Z}$ be an irreducible (hence connected) analytic hypersurface (complex submanifold of dimension $m-1$) in Ω in the sense of [9, Definition 8, p. 17], that is, to every $z^{(0)} \in \mathcal{Z}$, there exists a neighborhood $U \subseteq \Omega$ and a holomorphic map $\varphi : U \to \mathbb{C}$ such that $\frac{\partial \varphi}{\partial z_j}(z^{(0)}) \neq 0$ for some j, $1 \leq j \leq m$ and

$$U \cap \mathcal{Z} = \{z \in U : \varphi(z) = 0\}. \tag{1.1}$$

To a fixed point $z^{(0)} \in \mathcal{Z}$, there corresponds a neighborhood $U \subseteq \Omega$ and local coordinates ([9, Theorem 9, p.17]), $\phi \stackrel{\text{def}}{=} (\phi_1\ (= \varphi), \ldots, \phi_m) : U \subseteq \Omega \to \mathbb{C}^m$ such that $U \cap \mathcal{Z} = \{z \in U : \varphi(z) = 0\}$. We assume that the neighborhood U of $z^{(0)} \in \mathcal{Z}$ has been chosen such that ϕ is biholomorphic on U. Let $V = \phi(U)$.

Lemma 1.1 ([9, p. 33]) *If f is any holomorphic function on U such that $f(z) = 0$ for $z \in U \cap \mathcal{Z}$, then $f(z) = \varphi(z) g(z)$ for some function g holomorphic on U.*

Proof: Fix z in $U \cap \mathcal{Z}$ and consider the power series expansion of $f \circ \phi^{-1}$ at $\phi(z) = 0$ in the local co–ordinates $(\lambda_1\ (= \varphi(z)), \ldots, \lambda_m)$. Since $f \circ \phi^{-1}(0, \lambda_2, \ldots, \lambda_m) \equiv 0$ by hypothesis, it follows that all monomials in the power series expansion must contain λ_1. Hence $f \circ \phi^{-1} = \lambda_1 \tilde{g}$ for some holomorphic function $\tilde{g}$ in a small neighborhood of 0. In other words, in a small neighborhood U_z of z, we have $f = \varphi g$, where $g = \tilde{g} \circ \phi$. For $z' \neq z$ in $U \cap \mathcal{Z}$, we can find an open set $U_{z'}$ such that $f = \varphi g'$. If $U_z \cap U_{z'} \neq \emptyset$, then $\varphi g = f = \varphi g'$ on $U_z \cap U_{z'}$. It follows that $g = g'$ on $U_z \cap U_{z'}$. Hence $f = \varphi g$ for some function g holomorphic on the open set $U_0 = \cup\{U_z : z \in U \cap \mathcal{Z}\}$. This completes the proof since f/φ is holomorphic on $U \backslash (U \cap \mathcal{Z})$ and we have shown that f/φ is holomorphic in the neighborhood U_0 containing $U \cap \mathcal{Z}$. □

In general, a function φ holomorphic on U is a *local defining function* for the submanifold $\mathcal{Z}$ if $\varphi|\,(U \cap \mathcal{Z}) = 0$, and the quotient f/φ is holomorphic in U whenever f is holomorphic in U and $f|\,(U \cap \mathcal{Z}) = 0$. If φ and $\tilde{\varphi}$ are both defining functions, then it follows that both $\varphi/\tilde{\varphi}$ and $\tilde{\varphi}/\varphi$ are holomorphic on U. Hence φ is unique up to multiplication by a nonvanishing holomorphic function on U. For any holomorphic function defined in a neighborhood of z in $\mathcal{Z}$, the order $\mathrm{ord}_{\mathcal{Z},z}(f)$ of the function f at z is defined to be the largest integer p such that $f = \varphi^p g$ for some function g holomorphic in a neighborhood of z. Since $\mathrm{ord}_{\mathcal{Z},z}(f)$ is easily seen to be independent of the point z, we may define the order $\mathrm{ord}_{\mathcal{Z}}(f)$ of the function f to be simply $\mathrm{ord}_{\mathcal{Z},z}(f)$ for some z in $\mathcal{Z}$.

We are now ready to describe the submodule $\mathcal{M}_0$ which will be investigated in this paper. Let

$$\mathcal{M}_0 = \{f \in \mathcal{M} : \mathrm{ord}_{\mathcal{Z}}(f) \geq k\}. \tag{1.2}$$

We give two alternative characterisations of the module $\mathcal{M}_0$. The first of these is a consequence of the following lemma which is proved in the same manner as Lemma 1.1.

Lemma 1.2 ([9, p. 33]) *Let $U \cap \mathcal{Z} = \{z \in U : \varphi(z) = 0\}$ where φ is the defining function of $\mathcal{Z}$. If f is any holomorphic function on U and $\mathrm{ord}_{\mathcal{Z}} f = n$, then $f = \varphi^n g$ for some function g holomorphic on U.*

For $z \in \mathcal{Z}$, we can find a neighborhood U_z and a local defining function φ_z such that $U_z \cap \mathcal{Z} = \{\varphi_z = 0\}$. Thus

$$\mathcal{M}_0 = \{f \in \mathcal{M} : f = \varphi_z^n g, \quad g \text{ holomorphic on } U_z, n \geq k\}. \tag{1.3}$$

For each $z^{(0)}$ in $\mathcal{Z}$, there exists a neighborhood U and local coordinates $(\lambda_1\,(= \varphi), \ldots, \lambda_m)$. Clearly, any function f in $\mathcal{M}_0$ has the factorization $f = \lambda_1^k g$ on V in these co–ordinates. Hence f together with the derivatives $\frac{\partial^\ell f}{\partial \lambda_1^\ell}$ vanishes on $\varphi(\mathcal{Z}) \cap V$ for $1 \leq \ell \leq k-1$. Conversely, we claim that if f together with the derivatives $\frac{\partial^\ell f}{\partial \lambda_1^\ell}$ vanish on $\varphi(\mathcal{Z}) \cap V$ for $1 \leq \ell \leq k-1$, then f is in $\mathcal{M}_0$. To prove this, observe that if f vanishes on $\varphi(\mathcal{Z}) \cap V$, then $f = \lambda_1 g$ by Lemma 1.1 for some holomorphic function g on V. If $\frac{\partial f}{\partial \lambda_1}$ is also zero on $\varphi(\mathcal{Z}) \cap V$, then on the one hand $\frac{\partial f}{\partial \lambda_1} = g + \lambda_1 \frac{\partial g}{\partial \lambda_1}$ and on the other hand $\frac{\partial f}{\partial \lambda_1} = \lambda_1 \tilde{g}$ for some $\tilde{g}$ holomorphic on V. It follows that $g = \lambda_1(\tilde{g} - \frac{\partial g}{\partial \lambda_1})$. Thus $f = \lambda_1^2 g_1$ for some holomorphic function g_1 on V. Proceeding inductively, we assume that if the first ℓ derivatives of f vanishes on $\mathcal{Z}\cap V$ then $f=\lambda_1^{\ell+1}g_\ell$ for some holomorphic function g_ℓ on V. If $\frac{\partial^{\ell+1}f}{\partial\lambda_1^{\ell+1}}$ vanishes on $\varphi(\mathcal{Z})\cap V$ then as before, on the one hand $\frac{\partial^{\ell+1}f}{\partial\lambda_1^{\ell+1}}=g_\ell+\lambda_1 h$ and on the other hand $\frac{\partial^{\ell+1}f}{\partial\lambda_1^{\ell+1}}=\lambda_1\tilde{g}$ for some $\tilde{g},h$ holomorphic on V. This shows that $f=\lambda_1^{\ell+2}g_{\ell+1}$ for some holomorphic function $g_{\ell+1}$ on V. Hence the order of the function f is at least k. Thus we can also describe the module $\mathcal{M}_0$ as

$$\mathcal{M}_0=\{f\in\mathcal{M}:\frac{\partial^\ell f}{\partial\lambda_1^\ell}(\lambda)=0,\ \ \lambda\in V\cap\varphi(\mathcal{Z}),\ \ 0\le\ell\le k-1\}.$$

Finally, we may assume that $\frac{\partial\varphi}{\partial z_1}(z)\neq 0$ on the open set U. In this case, $\lambda_1=\varphi$, $\lambda_2=z_2,\dots,\lambda_m=z_m$ is a local coordinate system. However, a simple calculation using the chain rule shows that

$$\begin{pmatrix} 1 & & & & \\ 0 & \frac{\partial\varphi}{\partial z_1}(z) & & & 0 \\ 0 & \frac{\partial^2\varphi}{\partial z_1^2}(z) & (\frac{\partial\varphi}{\partial z_1}(z))^2 & & \\ & & & \ddots & \vdots \\ & \star & & \dots & (\frac{\partial\varphi}{\partial z_1}(z))^{k-1}\end{pmatrix}\begin{pmatrix} f\circ\phi^{-1}(\lambda) \\ \frac{\partial(f\circ\phi^{-1})}{\partial\lambda_1}(\lambda) \\ \frac{\partial^2(f\circ\phi^{-1})}{\partial\lambda_1^2}(\lambda) \\ \vdots \\ \frac{\partial^{k-1}(f\circ\phi^{-1})}{\partial\lambda_1^{k-1}}(\lambda)\end{pmatrix}=\begin{pmatrix} f(z) \\ \frac{\partial f}{\partial z_1}(z) \\ \frac{\partial^2 f}{\partial z_1^2}(z) \\ \vdots \\ \frac{\partial^{k-1}f}{\partial z_1^{k-1}}(z)\end{pmatrix}. \tag{1.4}$$

Thus $\frac{\partial^\ell f}{\partial z_1^\ell}(z)=0$ for $z\in U\cap\mathcal{Z}$, $0\le\ell\le k-1$ if and only if $\frac{\partial^\ell f}{\partial\lambda_1^\ell}(\lambda)=0$ for $\lambda\in V\cap\varphi(\mathcal{Z})$, $0\le\ell\le k-1$ and we obtain the third alternative characterisation of the submodule $\mathcal{M}_0$ simply as

$$\mathcal{M}_0=\{f\in\mathcal{M}:\frac{\partial^\ell f}{\partial z_1^\ell}(z)=0,\ \ z\in U\cap\mathcal{Z},\ \ 0\le\ell\le k-1\}. \tag{1.5}$$

In general, the function ϕ does not define global co–ordinates for Ω. However, if the second Cousin problem is solvable for Ω, then there exists a global defining function (which we will again denote by φ) for the hypersurface $\mathcal{Z}$. This is pointed out in the remark preceding Corollary 3 in [9, p. 34]. In this case, in view of Lemma 1.2, it follows that h belongs to $\mathcal{M}_0$ if and only if it admits a factorization $h=\varphi^n g$ for some holomorphic function g on Ω and $n\ge k$. At this point, we might simply assume that the second Cousin problem is solvable on our domain Ω. However, we show that our module can be localized, that is, it is enough to work with a fixed open set $U\subseteq\Omega$ such that $U\cap\mathcal{Z}=\{z\in U:\varphi(z)=0\}$ for some local defining function φ.

Recall that two Hilbert modules $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ over the algebra $\mathcal{A}(\Omega)$ are said to be equivalent if there is an unitary operator $T:\mathcal{M}\to\widetilde{\mathcal{M}}$ intertwining the two module actions, that is, $f\cdot Th=T(f\cdot h)$ for $f\in\mathcal{A}(\Omega)$ and $h\in\mathcal{M}$. Any operator satisfying the latter condition is said to be a module map.

Let $\mathcal{M}_{\text{res U}}\stackrel{\text{def}}{=}\{f|_U:f\in\mathcal{M}\}$ and $R:\mathcal{M}\to\mathcal{M}_{\text{res U}}$ be the restriction map. If f is in the kernel of the restriction map R, then f must vanish on the open set U implying that f must vanish on all of Ω. Thus the kernel of the restriction map R is trivial. We define an inner product on $\mathcal{M}_{\text{res U}}$ by setting $\langle Rh,Rh\rangle\stackrel{\text{def}}{=}\langle h,h\rangle_{\mathcal{M}}$. This makes R a unitary map. We turn $\mathcal{M}_{\mathrm{res\ U}}$ into a Hilbert module by restricting the original action to the open set U. The fact that $(RM_fR^*)(Rh) = R(f \cdot h) = f \cdot h|_U$ shows that $\mathcal{M}$ is equivalent to the module $\mathcal{M}_{\mathrm{res\ U}}$. Further if $(\mathcal{M}_{\mathrm{res\ U}})_0$ denotes the submodule of functions vanishing on $U \cap \mathcal{Z}$ to at least order k in $\mathcal{M}_{\mathrm{res\ U}}$ and $(\mathcal{M}_{\mathrm{res\ U}})_{\mathrm{q}}$ the corresponding quotient, then $R\mathcal{M}_0 = (\mathcal{M}_{\mathrm{res\ U}})_0$ and $R\mathcal{M}_{\mathrm{q}} = (\mathcal{M}_{\mathrm{res\ U}})_{\mathrm{q}}$. The first of these follows from the characterisation of $\mathcal{M}_0$ we have obtained above and then the second one follows from unitarity of the map R. Hence we may replace, without loss of generality, the module $\mathcal{M}$ by the module $\mathcal{M}_{\mathrm{res\ U}}$. Once we do that, the submodule $(\mathcal{M}_{\mathrm{res\ U}})_0$ may be described as

$$\{h \in \mathcal{M} : h = \varphi^n g,\ g \text{ holomorphic on } U, n \geq k\}.$$

In the following section, we will assume that we have localized our module to a fixed open set U and pretend that $U = \Omega$. In section 3, we describe the quotient module $\mathcal{M}_{\mathrm{q}}$.

2 Reproducing kernels and vector bundles

Let E be a finite dimensional ($\dim E = k$) Hilbert space and $\mathcal{H}$ be a Hilbert space of holomorphic functions from Ω to E. Let $ev_w : \mathcal{H} \to E$ be the evaluation functional defined by $ev_w(f) = f(w)$, for $f \in \mathcal{H}$ and $w \in \Omega$. If ev_w is both bounded and surjective on a Hilbert space $\mathcal{H}$ of holomorphic functions from Ω to E for each $w \in \Omega$, then it is said to be a functional Hilbert space. In this case, $ev_w^* : E \to \mathcal{H}$ is bounded and injective. The function $K : \Omega \times \Omega \to \mathcal{L}(E)$, defined by

$$K(z, w) = ev_z ev_w^*, \quad z, w \in \Omega,$$

is called the reproducing kernel of $\mathcal{H}$. The kernel K has the following reproducing property:

$$\begin{aligned} \left\langle f, K(\cdot, w)\zeta \right\rangle_{\mathcal{H}} &= \left\langle f, ev_w^*(\zeta) \right\rangle_{\mathcal{H}} \\ &= \left\langle ev_w(f), \zeta \right\rangle_E \\ &= \left\langle f(w), \zeta \right\rangle_E. \end{aligned} \tag{2.1}$$

Since $K(\cdot, w)\zeta = (ev_w^*\zeta)(\cdot) \in \mathcal{H}$ for each $w \in \Omega$, it follows that K is holomorphic in the first variable. Also, $K(w, z) = ev_w ev_z^* = (ev_z ev_w^*)^* = K(z, w)^*$. Hence K is anti-holomorphic in the second variable. The reproducing property (2.1) implies that K is uniquely determined.

Clearly, $f \in \mathcal{H}$ is orthogonal to ran ev_w^* if and only if $\left\langle f, ev_w^*\zeta \right\rangle_{\mathcal{H}} = \left\langle f(w), \zeta \right\rangle_E = 0$ for every $\zeta \in E$. Hence $f \perp$ ran ev_w^* for all $w \in \Omega$ if and only if $f = 0$. Hence $\mathcal{H}$ is generated by the subspace $ev_w^*(E)$. Therefore functions f in $\mathcal{H}$ of the form $f = \sum_{j=1}^n ev_{w_j}^*(\zeta_j)$ form a dense linear subspace $\mathcal{H}^0$ of $\mathcal{H}$. For $f \in \mathcal{H}^0$,

$$\|f\|^2 = \Big\langle \sum_{j=1}^n ev_{w_j}^*(\zeta_j), \sum_{j=1}^n ev_{w_j}^*(\zeta_j) \Big\rangle_{\mathcal{H}}$$

$$= \Big\langle \sum_{j,k=1}^{n} ev_{w_k} ev_{w_j}^*(\zeta_j), \zeta_k \Big\rangle_E.$$

Since $\|f\|^2 \geq 0$, it follows that the operator valued kernel $K(z,w) = ev_z ev_w^*$ has the property that

$$\sum_{k,j=1}^{n} \Big\langle K(w_k, w_j)\zeta_j, \zeta_k \Big\rangle_E \;\geq\; 0, \quad w_1, \ldots, w_n \in \Omega, \quad \zeta_1, \ldots, \zeta_n \in E. \tag{2.2}$$

Since ker $ev_w^* = \{0\}$, it follows that

$$\Big\langle K(w,w)\zeta, \zeta \Big\rangle_E = \Big\langle ev_w^*\zeta, ev_w^*\zeta \Big\rangle_{\mathcal{H}} > 0, \quad \text{for all } \zeta \neq 0. \tag{2.3}$$

Any function $K : \Omega \times \Omega \to \mathcal{L}(E)$, holomorphic in the first variable and anti-holomorphic in the second satisfying (2.2) and (2.3) is called a reproducing kernel for $\mathcal{H}$.

The proof of the following theorem is similar to that for a Hilbert space of ordinary scalar valued functions.

THEOREM 2.1 *For any kernel function $K : \Omega \times \Omega \to \mathcal{L}(E)$, it is possible to construct a unique functional Hilbert space $\mathcal{H}$ satisfying*

1. *$\mathcal{H}^0$ is dense in $\mathcal{H}$*
2. *$ev_z : \mathcal{H} \to E$ is bounded for each $z \in \Omega$*
3. *$K(z,w) = ev_z ev_w^*$, $z, w \in \Omega$.*

As a consequence of the uniform boundedness principle, it is easy to see that ev_w is uniformly bounded on $\Omega_0 \subseteq \Omega$ if and only if $\|K(w,w)\|_{E\to E}$ is uniformly bounded on Ω_0. In this case,

$$\begin{aligned} \sup_{w\in\Omega_0} |\Big\langle f(w), \zeta \Big\rangle_E| &= \sup_{w\in\Omega_0} |\Big\langle f, K(\cdot,w)\zeta \Big\rangle_{\mathcal{H}}| \\ &= \|f\| \sup_{w\in\Omega_0} \Big\langle K(\cdot,w)\zeta, K(\cdot,w)\zeta \Big\rangle \;\text{ for } \zeta \in E. \end{aligned}$$

Hence $\sup_{w\in\Omega_0} \|f(w)\|_E \leq \|f\|_{\mathcal{H}} \, \|K(w,w)\|$. Therefore, convergence in $\mathcal{H}$ implies uniform convergence on Ω_0 if $\|K(w,w)\|$ is uniformly bounded. The following lemma is well known (cf. [5]).

(Note that if $f \in \mathcal{H}$ then $f(z)$ is in E, which may be thought of as a linear map $f(z) : \mathbb{C} \to E$, defined by $f(z)(\alpha) = \alpha f(z)$, for $\alpha \in \mathbb{C}$. In the following, $f(z)^*$ merely denotes the adjoint of the linear map $f(z)$.)

LEMMA 2.2 *Let $\mathcal{H}$ be a functional Hilbert space of holomorphic functions taking values in a finite dimensional Hilbert space E and $\{e_n\}_{n=0}^\infty$ be an orthonormal basis for $\mathcal{H}$. The sum $\sum_{n=0}^\infty e_n(z)e_n(w)^*$ converges in $\mathcal{L}(E)$ to $K(z,w)$.*

Of course, the reproducing property of $\sum_{n=0}^\infty e_n(z)e_n(w)^*$ is easy to verify independently. Let $f(z) = \sum_{n=0}^\infty a_n e_n(z)$ be the Fourier series expansion of $f \in \mathcal{H}$. It follows that

$$\begin{aligned}
\Big\langle f(\cdot), \sum_{n=0}^\infty e_n(\cdot)e_n(w)^*\zeta \Big\rangle &= \Big\langle \sum_{n=0}^\infty a_n e_n(\cdot), (\sum_{n=0}^\infty e_n(\cdot)e_n(w)^*)\zeta \Big\rangle \\
&= \sum_{m,n=0}^\infty \overline{(e_m(w)^*\zeta)} \Big\langle a_m e_m(\cdot), e_n(\cdot) \Big\rangle \\
&= \sum_{n=0}^\infty a_m \Big\langle e_m(w), \zeta \Big\rangle \\
&= \Big\langle f(w), \zeta \Big\rangle.
\end{aligned}$$

Since the reproducing kernel K is uniquely determined, it follows that $K(z,w) = \sum_{n=0}^\infty e_n(z)e_n(w)^*$.

Suppose we start with a Hilbert space $\mathcal{H}$ of complex valued holomorphic functions on Ω with a reproducing kernel K. Let ε_ℓ, $\ell = 1, \ldots, k$ be the standard basis vectors for $\mathbb{C}^k$ and ∂_1 denote differentiation with respect to z_1, that is, $\partial_1 = \frac{\partial}{\partial z_1}$. For $h \in \mathcal{H}$, let

$$\boldsymbol{h} = \sum_{\ell=0}^{k-1} \partial_1^\ell h \otimes \varepsilon_{\ell+1}$$

and $J(\mathcal{H}) = \{\boldsymbol{h} : h \in \mathcal{H}\} \subseteq \mathcal{H} \otimes \mathbb{C}^k$. Consider the map $J : \mathcal{H} \to J(\mathcal{H})$ defined by $Jh = \boldsymbol{h}$, for $h \in \mathcal{H}$. Since J is injective, we can define an inner product on $J(\mathcal{H})$

$$\Big\langle J(g), J(h) \Big\rangle_{J(\mathcal{H})} \stackrel{\text{def}}{=} \Big\langle g, h \Big\rangle_{\mathcal{H}}$$

so as to make J unitary. We point out that $J(\mathcal{H}) \neq \mathcal{H} \otimes \mathbb{C}^k$, in general.

Proposition 2.3 *The reproducing kernel $JK : \Omega \times \Omega \to \mathcal{M}_k(\mathbb{C})$ for the Hilbert space $J(\mathcal{H})$ is given by the formula:*

$$(JK)_{\ell,j}(z,w) = (\partial_1^\ell \bar{\partial}_1^j K)(z,w), \quad 0 \le \ell, j \le k-1,$$

where $\bar{\partial}_1 = \frac{\partial}{\partial \bar{w}_1}$ and $\partial_1 = \frac{\partial}{\partial z_1}$ as before.

Proof: Since J is a unitary map, it follows that $\{(Je_n) : n \ge 0\}$ is an orthonormal basis for $J(\mathcal{H})$, where $\{e_n : n \ge 0\}$ is an orthonormal basis for $\mathcal{H}$.

If $\boldsymbol{h}$ is an element of $J(\mathcal{H})$ then it has the expansion $\boldsymbol{h}(z) = \sum_{n=0}^\infty a_n\,(Je_n)(z)$. Also, note that for any $\boldsymbol{x} \in \mathbb{C}^k$, we have

$$\begin{aligned}
(JK)(z,w)\boldsymbol{x} &= \sum_{\ell,j=0}^{k-1} \Big(\sum_{n=0}^\infty (\partial_1^\ell e_n)(z)\, \overline{(\partial_1^j e_n)(w)} \Big) x_j \varepsilon_{\ell+1} \\
&= \sum_{j=0}^{k-1} x_j \Big(\sum_{n=0}^\infty \overline{(\partial_1^j e_n)(w)} (\sum_{\ell=0}^{k-1} (\partial_1^\ell e_n)(z) \varepsilon_{\ell+1}) \Big) \\
&= \sum_{n=0}^\infty \Big\langle \boldsymbol{x}, (Je_n)(w) \Big\rangle_{\mathbb{C}^k} (Je_n)(z)
\end{aligned}$$

Thus

$$\begin{aligned}
\left\langle \boldsymbol{h}, JK(\cdot,w)\boldsymbol{x} \right\rangle_{J(\mathcal{H})} &= \sum_{n=0}^{\infty} a_n \overline{\left\langle \boldsymbol{x}, (Je_n)(w) \right\rangle_{\mathbb{C}^k}} \\
&= \Big\langle \sum_{n=0}^{\infty} a_n (Je_n)(w), \boldsymbol{x} \Big\rangle_{\mathbb{C}^k} \\
&= \left\langle \boldsymbol{h}(w), \boldsymbol{x} \right\rangle_{\mathbb{C}^k}.
\end{aligned}$$

Hence JK has the reproducing property

$$\left\langle \boldsymbol{h}, JK(\cdot,w)\boldsymbol{x} \right\rangle_{J(\mathcal{H})} = \left\langle \boldsymbol{h}(w), \boldsymbol{x} \right\rangle_{\mathbb{C}^k} \text{ for } \boldsymbol{x} \in \mathbb{C}^k \text{ and } \boldsymbol{h} \in J(\mathcal{H}), \tag{2.4}$$

which completes the proof. $\square$

Let $\{\varepsilon_\ell : 1 \leq \ell \leq k\}$ be a set of basis vectors for E. Let $s_\ell(w) = K(\cdot,w)\varepsilon_\ell$. The vectors $s_\ell(w)$ span the range $\mathcal{E}_w$ of $K(\cdot,w) : E \to \mathcal{H}$. Let $\Omega^* = \{w : \bar{w} \in \Omega\}$. The holomorphic frame $w \to \{s_1(\bar{w}), \ldots, s_k(\bar{w})\}$ determines a holomorphically trivial vector bundle $\mathcal{E}$ over Ω^*. The fiber of $\mathcal{E}$ over w is $\mathcal{E}_w = span\{K(\cdot,\bar{w})\varepsilon_\ell : 1 \leq \ell \leq k\}$, $w \in \Omega^*$. An arbitrary section of this bundle is of the form $s = \sum_{\ell=1}^k a_\ell s_\ell$, where $a_\ell, \ell = 1, \ldots, k$, are holomorphic functions on Ω^*. The norm at $w \in \Omega^*$ is determined by

$$\begin{aligned}
\|s(w)\|^2 &= \Big\langle \sum_{\ell=1}^k a_\ell(w) s_\ell(w), \sum_{\ell=1}^k a_\ell(w) s_\ell(w) \Big\rangle_{\mathcal{H}} \\
&= \sum_{\ell,m=1}^k a_\ell(w)\overline{a_m(w)} \left\langle s_\ell(w), s_m(w) \right\rangle_{\mathcal{H}} \\
&= \sum_{\ell,m=1}^k a_\ell(w)\overline{a_m(w)} \left\langle K(\cdot,w)\varepsilon_\ell, K(\cdot,w)\varepsilon_m \right\rangle_{\mathcal{H}} \\
&= \sum_{\ell,m=1}^k a_\ell(w)\overline{a_m(w)} \left\langle K(w,w)\varepsilon_\ell, \varepsilon_m \right\rangle_{E} \\
&= \left\langle K(w,w)^{\mathrm{tr}} a(w), a(w) \right\rangle_E ,
\end{aligned} \tag{2.5}$$

where $a(w) = \sum_{\ell=1}^k a_\ell(w)\epsilon_\ell$ and $K(w,w)^{\mathrm{tr}}$ denotes the transpose of the matrix $K(w,w)$. Since $K(w,w)$ is positive definite and $w \mapsto K(w,w)$ is real analytic, it follows that $K(w,w)$ determines a hermitian metric for the vector bundle $\mathcal{E}$.

Conversely, let $\mathcal{E}$ be a hermitian holomorphic vector bundle with a real analytic metric G on Ω^* and $\{s_1(w), \ldots, s_k(w) : w \in \Omega^*\}$ be a holomorphic frame. Since G is real analytic on Ω^*, we can find a function $\tilde{G} : \Omega^* \times \Omega^* \to \mathbb{C}$ anti-holomorphic in the first variable and analytic in the second such that $\tilde{G}(w,w) = G(w), w \in \Omega^*$. If $\tilde{G}$ is a positive definite kernel on Ω, then it naturally gives rise to a reproducing kernel Hilbert space H, which is spanned by $\{\tilde{G}(\cdot,w)\boldsymbol{x} : w \in \Omega^* \text{ and } \boldsymbol{x} \in \mathbb{C}^k\}$. The inner product on this spanning set is defined by $\left\langle \tilde{G}(\cdot,w)\boldsymbol{x}, \tilde{G}(\cdot,\lambda)\boldsymbol{y} \right\rangle \stackrel{\text{def}}{=} \left\langle \tilde{G}(\lambda,w)\boldsymbol{x}, \boldsymbol{y} \right\rangle$.

The completion with respect to this inner product produces the Hilbert space $\mathcal{H}$ and $\tilde{G}$ is the reproducing kernel for $\mathcal{H}$. Let $s(w) = \sum_{\ell=1}^{k} s_\ell(w)x_\ell(w), w \in \Omega^*$. Then the map $s(w) \mapsto \tilde{G}(\cdot, w)\boldsymbol{x}(w)$ defines a unitary isomorphism between the fiber $\mathcal{E}_w$ and $ran\ \tilde{G}(\cdot, w) \subseteq \mathcal{H}$.

Now consider the operator $M^*_{f,w} : \mathcal{E}_w \to \mathcal{E}_w$ defined by

$$M^*_{f,w}\tilde{G}(\cdot, w)\boldsymbol{x} \stackrel{\text{def}}{=} \overline{f(w)}\tilde{G}(\cdot, w)\boldsymbol{x},$$

where $f \in \mathcal{A}(\Omega)$. This defines an operator $M^*_{f,\underline{w}}$ on the subspace of $\mathcal{H}$ defined by

$$\{\sum_{i=1}^{n} \tilde{G}(\cdot, w_i)\boldsymbol{x}_i : \boldsymbol{x}_i \in \mathbb{C}^k\}, \text{ where } \underline{w} = (w_1, \ldots, w_n), w_i \in \Omega^*.$$

Since $M^*_{f,\underline{w}}$ is an operator on a finite dimensional space, it is bounded and it follows that there exists a positive constant $C_{f,\underline{w}}$ depending on $\{w_1, \ldots, w_n\}$ such that

$$\begin{aligned} \|M^*_{f,\underline{w}}(\sum_{i=1}^{n} \tilde{G}(\cdot, w_i)\boldsymbol{x}_i)\|^2 &= \|\sum_{i=1}^{n} f(w_i)\tilde{G}(\cdot, w_i)\boldsymbol{x}_i\|^2 \\ &= \sum_{i,j=1}^{n} f(w_i)\overline{f(w_j)}\Big\langle \tilde{G}(w_j, w_i)\boldsymbol{x}_i, \boldsymbol{x}_j\Big\rangle \\ &\leq C_{f,\underline{w}} \sum_{i,j=1}^{n} \Big\langle \tilde{G}(w_j, w_i)\boldsymbol{x}_i, \boldsymbol{x}_j\Big\rangle. \end{aligned}$$

We conclude that the above construction defines a bounded map on all of $\mathcal{H}$, if and only if there exists a positive constant C_f independent of $\{w_1, \ldots, w_n\}$ such that the kernel

$$\tilde{G}_f(z, w) \stackrel{\text{def}}{=} (C_f - f(z)\overline{f(w)})\tilde{G}(z, w) \tag{2.6}$$

is non-negative definite. The adjoint of this operator is equal to the multiplication operator M_f on $\mathcal{H}$. Finally, the map $(f, h) \mapsto M_f h$ for $f \in \mathcal{A}(\Omega)$ and $h \in \mathcal{H}$ is uniformly bounded if and only if $C = \sup\{C_f : f \in \mathcal{A}(\Omega), \|f\|_\infty \leq 1\}$ is finite. Thus, $\mathcal{H}$ is a Hilbert module with respect to the natural action of the algebra $\mathcal{A}(\Omega)$ if and only if there exists a positive constant C (independent of f) such that

$$\tilde{G}_f(z, w) = (C - f(z)\overline{f(w)})\tilde{G}(z, w) \tag{2.7}$$

is a non-negative definite kernel for each $f \in \mathcal{A}(\Omega), \|f\|_\infty \leq 1$.

The quotient modules we describe later in this paper turn out to be modules over the algebra $\mathcal{A}(\Omega)$ with respect to a module multiplication quite different from the one described here.

It is possible to associate a jet bundle $J\mathcal{E}$ with a holomorphic hermitian vector bundle $\mathcal{E}$ on Ω. For a holomorphic hermitian bundle $\mathcal{E}$ over a planar domain, the construction of the jet bundle $J\mathcal{E}$ is given in [4]. One may proceed in a similar manner to construct the jet bundle $J\mathcal{E}$ in the multi-variate case. Fortunately, for our application, it is enough to do this for a line bundle $\mathcal{E}$ on Ω along the normal direction to the zero variety $\mathcal{Z} = \{w \in \Omega : w_1 = 0\} \subseteq \Omega$, that is, with respect to the coordinate z_1. In this case, we can adapt the construction given in [4], in a straightforward manner. For the sake of completeness, we give the details of this construction. Let s_0 and s_1 be holomorphic frames for $\mathcal{E}$ on the coordinate patches $\Omega_0 \subseteq \Omega$ and $\Omega_1 \subseteq \Omega$ respectively. That is, s_0(resp. s_1) is a non-vanishing holomorphic section on Ω_0(resp. Ω_1) . Then there is a non-vanishing holomorphic function g on $\Omega_0 \cap \Omega_1$ and $s_0 = gs_1$ there. Let $J(s_\ell) = \sum_{j=0}^{k-1} \frac{\partial_1^j s_\ell}{\partial_1 w^j} \epsilon_{j+1}, \ell = 0, 1$. An easy computation shows that Js_0 and Js_1 transform on $\Omega_0 \cap \Omega_1$ by the rule $J(s_0) = (\mathcal{J}g)J(s_1)$, where $\mathcal{J}$ is the lower triangular operator matrix

$$\mathcal{J} = \begin{pmatrix} 1 & & & & \\ & \ddots & & 0 & \\ \vdots & \binom{\ell}{j}\partial_1^{\ell-j} & 1 & & \\ & & & \ddots & \\ \partial_1^{k-1} & & \dots & & 1 \end{pmatrix}. \tag{2.8}$$

with $0 \le \ell, j \le k-1$.

The components of Js, that is, $s, \partial_1 s, \dots, \partial_1^{k-1} s$, determine a frame for a rank k holomorphic vector bundle $J\mathcal{E}$ on Ω. The transition function with respect to this frame is represented by the matrix $(\mathcal{J}g)^{\mathrm{tr}}$. We will refer to this bundle $J\mathcal{E}$ as the jet bundle associated with $\mathcal{E}$. The hermitian metric $G(w) = \langle s(w), s(w)\rangle_{\mathcal{E}}$ on $\mathcal{E}$ with respect to the frame s on $\mathcal{E}$ induces a hermitian metric JG on $J\mathcal{E}$ such that with respect to the frame Js,

$$(JG)(w) = \begin{pmatrix} G(w) & & \dots & & (\partial^{k-1}G)(w) \\ & \ddots & & & \\ \vdots & & (\partial_1^{\ell}\bar{\partial}_1^{m}G)(w) & & \vdots \\ & & & \ddots & \\ (\bar{\partial}_1^{k-1}G)(w) & & \dots & & (\partial_1^{k-1}\bar{\partial}_1^{k-1}G)(w) \end{pmatrix}. \tag{2.9}$$

We point out that there is no canonical normal to the hypersurface $\mathcal{Z}$. However, the construction of the jet bundle depends on the choice of a normal vector to the hypersurface $\mathcal{Z}$. In the construction outlined above, we have chosen the normal direction to be z_1. Thus, if we take two different normal directions to the zero variety $\mathcal{Z}$, then we can construct two distinct jet bundles. The following proposition explores the relationship between these two jet bundles.

Proposition 2.4 *If φ and $\tilde{\varphi}$ are two different defining functions for $U \cap \mathcal{Z}$ for some open set $U \subseteq \Omega$, then the jet bundles obtained as above are equivalent holomorphic hermitian bundles on U.*

Proof: The fact that both φ and $\tilde{\varphi}$ are defining functions for $\mathcal{Z}$ implies $\frac{\tilde{\varphi}}{\varphi}$ is a non-vanishing holomorphic function on U. Hence if $\frac{\partial \varphi}{\partial z_1} \neq 0$ on U, we can assume (by going to a smaller open set, if necessary) that $\frac{\partial \tilde{\varphi}}{\partial z_1} \neq 0$ on U.

We denote by $L(\varphi)$ the matrix introduced in the equation (1.4). Then the jet bundles constructed using the two defining functions are related by a unitary bundle map represented by the matrix $L(\tilde{\varphi})L(\varphi)^{-1}$. $\Box$

As pointed out above, any Hilbert space $\mathcal{H}$ of holomorphic functions on Ω with a reproducing kernel K determines a line bundle $\mathcal{E}$ on Ω^* whose fiber at $\bar{w} \in \Omega^*$ is spanned by $K(\cdot, w)$. We can now construct a rank k holomorphic vector bundle by the procedure outlined in the previous paragraph. A holomorphic frame for this bundle is $\{K(\cdot, w), \bar{\partial}_1 K(\cdot, w), \ldots, \bar{\partial}_1^{k-1} K(\cdot, w)\}$, and as usual, this frame determines a metric for the bundle by the formula (compare (2.9)):

$$\Big\langle \sum_{j=0}^{k-1} a_j \partial_1^j K(\cdot, w), \sum_{j=0}^{k-1} a_j \partial_1^j K(\cdot, w) \Big\rangle = \sum_{j,\ell=0}^{k-1} a_j \bar{a}_\ell \Big\langle \partial_1^j K(\cdot, w), \partial_1^\ell K(\cdot, w) \Big\rangle.$$

This is the jet bundle $J\mathcal{E}$ associated with $\mathcal{E}$.

On the other hand, the Hilbert space $J\mathcal{H}$ together with its kernel function JK defined in Proposition 2.3 defines a rank k hermitian holomorphic bundle on Ω^* (see discussion preceding equation (2.5)). That these two constructions yield equivalent hermitian holomorphic bundles is a consequence of the fact that J is a unitary map from $\mathcal{H}$ onto $J\mathcal{H}$.

Our interest in describing this connection between a functional Hilbert space and the associated bundle lies in a theorem due to Cowen and Douglas [4] which states that local equivalence of these associated bundles determines the unitary equivalence class of the multiplication tuple. Since the curvature determines the equivalence class of a line bundle, this theorem becomes particularly useful in that case.

In this paper, we start with a Hilbert space $\mathcal{M}$ consisting of holomorphic functions on $\Omega \subseteq \mathbb{C}^m$. We assume that $\mathcal{M}$ admits a reproducing kernel K satisfying the positive definiteness condition in (2.7). Then the multiplication operators on this Hilbert space induce a map $\mathcal{A}(\Omega) \times \mathcal{M} \to \mathcal{M}$ given by $(f, h) \mapsto M_f h$, $f \in \mathcal{A}(\Omega), h \in \mathcal{M}$ which is bounded. Consequently, we have an action of $\mathcal{A}(\Omega)$ on the Hilbert space $\mathcal{M}$, which makes it a module over $\mathcal{A}(\Omega)$. Let $\mathcal{M}_0$ be the submodule of all functions in $\mathcal{M}$ vanishing to order k on some hypersurface $\mathcal{Z}$ and let $\mathcal{M}_\mathrm{q}$ be the quotient module. *The main goal of this paper is to understand the quotient module* $\mathcal{M}_\mathrm{q}$. This means, we wish to describe the quotient module in some canonical manner and possibly find unitary invariants. While we succeed in our first objective, we have not been able to make much headway in the second one.

The reason for describing the construction of a jet bundle lies in the fact that the quotient module gives rise to a rank k bundle over $\mathcal{Z}$ which is the jet bundle $J\mathcal{E}$ associated with $\mathcal{E}$ restricted to $\mathcal{Z}$ together with a bundle map J_f, for every $f \in \mathcal{A}(\Omega)$. The bundle maps J_f, in case $k > 1$, are not necessarily trivial. (We say that a bundle map is trivial if it is multiplication by a scalar, when restricted to any fiber of the bundle.) The complex geometric approach developed in [4], is applicable to Hilbert modules which necessarily give rise to holomorphic bundles together with bundle maps J_f which are trivial. If two such bundles $\mathcal{E}$ and $\tilde{\mathcal{E}}$ are equivalent via the bundle map $\Theta : \mathcal{E} \to \tilde{\mathcal{E}}$, then it is shown that there exists an unitary $U_\Theta : \mathcal{M} \to \widetilde{\mathcal{M}}$. Since the action of $\mathcal{A}(\Omega)$ in that treatment is scalar on each fiber of the respective bundles and Θ is a bundle map, an unitary module map is obtained. Although the quotient module in our case gives rise to a rank k bundle over $\mathcal{Z}$, the action of the algebra $\mathcal{A}(\Omega)$ is no longer scalar on the fiber. Hence, even if we obtain an unitary map $U : \mathcal{M}_{\rm q} \to \widetilde{\mathcal{M}}_{\rm q}$ using techniques from [4], we have to ensure further that this is a module map. We have not been able to find necessary and sufficient conditions for this. (In a previous paper [6], we assumed that $\mathcal{M}_0$ is the submodule consisting of all functions vanishing on a hypersurface $\mathcal{Z}$. In that case, the quotient module gives rise to a line bundle on $\mathcal{Z}$ and the module action is scalar on each fiber. Hence the complex geometric approach of [4] applies.)

Now, we give a construction which may be thought of as associating a k th order jet $J\mathcal{M}$ to a Hilbert module $\mathcal{M}$ of holomorphic functions on a bounded domain $\Omega \subseteq \mathbb{C}^m$ with a reproducing kernel K. In the preceding paragraphs, we have not only constructed the Hilbert space $J\mathcal{M}$ but also described the kernel function JK. To complete this construction, we only need the module action on $J\mathcal{M}$.

Define the action of the algebra $\mathcal{A}(\Omega)$ on $J\mathcal{M}$ by

$$f \cdot \boldsymbol{h} = \sum_{\ell=0}^{k-1} \Big(\sum_{j=0}^{\ell} \binom{\ell}{j} \partial_1^{\ell-j} f \partial_1^j h \Big) \otimes \varepsilon_{\ell+1},\ h \in \mathcal{M} \text{ and } f \in \mathcal{A}(\Omega).$$

This action is best described in terms of the matrix $\mathcal{J}$ defined in equation (2.8), where $\mathcal{J}_{\ell,j} = \binom{\ell}{j}\partial_1^{\ell-j}$ and $(\mathcal{J}f)_{\ell,j} = \binom{\ell}{j}\partial_1^{\ell-j}(f)$, $0 \leq \ell \leq j \leq k-1$. If J_f denotes the module action $(f, \boldsymbol{h}) \mapsto f \cdot \boldsymbol{h}$, then we find that $J_f(\boldsymbol{h}) = (\mathcal{J}f)(\boldsymbol{h})$. Using the Leibnitz formula, we obtain

$$\begin{aligned} J(f \cdot h) &= \sum_{\ell=0}^{k-1} \partial_1^{\ell}(f \cdot h) \otimes \varepsilon_{\ell+1} \\ &= \sum_{\ell,j=0}^{k-1} \binom{\ell}{j} \partial_1^{\ell-j} f \partial_1^j h \otimes \varepsilon_{\ell+1} \\ &= (\mathcal{J}f)(\boldsymbol{h}). \end{aligned}$$

It follows that $\mathcal{M}$ and $J\mathcal{M}$ are equivalent modules via the module map J.

The elements of $J\mathcal{M}$ vanishing on $\mathcal{Z}$ form a submodule of $J\mathcal{M}$. Let

$$(J\mathcal{M})_0 = \{\boldsymbol{h} \in J\mathcal{M} : \boldsymbol{h}(z) = 0 \text{ for } z \in \mathcal{Z}\},$$

and let $(J\mathcal{M})_{\rm q}$ denote the quotient module of $J\mathcal{M}$ by the sub-module $(J\mathcal{M})_0$. Let $X : \mathcal{M}_0 \to \mathcal{M}$ and $X_0 : (J\mathcal{M})_0 \to J\mathcal{M}$ be the inclusion maps.

The following proposition shows not only that the two modules $\mathcal{M}$ and $J\mathcal{M}$ are equivalent but also that $\mathcal{M}_0$ and $\mathcal{M}_{\rm q}$ are equivalent to $(J\mathcal{M})_0$ and $(J\mathcal{M})_{\rm q}$ respectively.

Proposition 2.5 *The following diagram of two short exact sequences is commutative.*

$$\begin{array}{ccccccccc} 0 & \longleftarrow & \mathcal{M}_{\rm q} & \longleftarrow & \mathcal{M} & \xleftarrow{X} & \mathcal{M}_0 & \longleftarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longleftarrow & (J\mathcal{M})_{\rm q} & \longleftarrow & J\mathcal{M} & \xleftarrow{X_0} & (J\mathcal{M})_0 & \longleftarrow & 0 \end{array}$$

Proof: It is clear that J maps $\mathcal{M}_0$ onto $(J\mathcal{M})_0$ and hence it follows that they are equivalent. The fact that J is unitary and onto implies that it maps the orthogonal compliment $\mathcal{M}_{\rm q}$ onto $(J\mathcal{M})_{\rm q}$. Hence the quotient modules $\mathcal{M}_{\rm q}$, $(J\mathcal{M})_{\rm q}$ are equivalent. □

The proof of the following lemma is similar to the first part of the proof of Theorem 1.4 in [6].

Lemma 2.6 *The reproducing kernel $K_0(z,w)$ for the submodule $\mathcal{M}_0$ is of the form $\varphi^k(z)\chi(z,w)\overline{\varphi^k(w)}$, where $\chi(w,w)\neq 0$, $w\in\Omega$.*

Proof: Recall that $\mathcal{M}_0$ is the space of all functions which admit φ^k as a factor. Let $\{e_n^{(0)} : n\geq 0\}$ be an orthonormal basis for $\mathcal{M}_0$. The reproducing kernel has the expansion $K_0(z,w)=\sum_{n=0}^\infty e_n^{(0)}(z)\overline{e_n^{(0)}(w)}$. Since $e_n^{(0)}(z)=\varphi^k(z)g_n(z)$ for each n, it follows that $K_0(z,w)=\varphi^k(z)\chi(z,w)\overline{\varphi^k(w)}$, where $\chi(z,w)=\sum_{n=0}^\infty g_n(z)\overline{g_n(w)}$. The reproducing property of $K_0(\cdot,w)$ implies that $K_0(w,w)$ does not vanish on $\Omega\backslash\mathcal{Z}$. It follows that $\chi(w,w)\neq 0$ off the set $\mathcal{Z}\cap\Omega$. We point out, in fact, that $\chi(w,w)$ is never zero on Ω. If $\chi(w,w)=0$ for some $w\in\mathcal{Z}$, then $\sum_{n=0}^\infty |g_n(w)|^2=0$. It follows that $g_n(w)=0$ for each n. This in turn would mean the order of the zero at w for each $f\in\mathcal{M}_0$ is strictly greater than k. This contradiction proves our assertion. □

Calculations similar to the ones leading up to equation (2.4) show that

$$\left(J_0K\right)_{i,j}(z,w)=(\partial_1^i\bar{\partial}_1^jK_0)(z,w),\quad 0\leq i,j\leq k-1, \tag{2.10}$$

is the reproducing kernel for $(J\mathcal{M})_0$. Let $(J\chi)_{i,j}(z,w)=(\partial_1^i\bar{\partial}_1^j\chi)(z,w)$, $0\leq i,j\leq k-1$. Then the preceding lemma yields the factorization

$$J_0K(z,w)=(\mathcal{J}\varphi^k)(z)J\chi(z,w)(\mathcal{J}\varphi^k)(w)^*, \tag{2.11}$$

where $\mathcal{J}$ is the operator matrix defined in (2.8).

3 The quotient module $\mathcal{M}_{\rm q}$

The fact that $\mathcal{M}_{\rm q}$ is equivalent to $(J\mathcal{M})_{\rm q}$ was pointed out in the previous section. We record this as a separate proposition along with a computational proof. These computations will be useful later.

Proposition 3.1 *The quotient modules $\mathcal{M}_{\rm q}$ and $(J\mathcal{M})_{\rm q}$ are equivalent.*

Proof: We begin by pointing out that $\bar{\partial}_1^\ell K(\cdot,w)$ is in the Hilbert space $\mathcal{M}$ for $0\leq\ell\leq k-1$. Hence if $h\in\mathcal{M}$ has the expansion $\sum_{n=0}^\infty a_ne_n(\cdot)$ in terms of an orthonormal basis $\{e_n : n\geq 0\}$ and $K(\cdot,w)=\sum_{n=0}^\infty e_n(\cdot)\overline{e_n(w)}$, then we have

$$\begin{aligned}\Big\langle\sum_{n=0}^\infty a_ne_n(\cdot),\bar{\partial}_1^\ell K(\cdot,w)\Big\rangle &= \Big\langle\sum_{n=0}^\infty a_ne_n(\cdot),\sum_{n=0}^\infty e_n(\cdot)\overline{\partial_1^\ell e_n(w)}\Big\rangle\\ &= \sum_{n=0}^\infty a_n\partial_1^\ell e_n(w)\\ &= (\partial_1^\ell h)(w)\quad . \end{aligned}\tag{3.1}$$

If h in $\mathcal{M}$ is orthogonal to all the vectors in the set

$$D = \{\bar{\partial}_1^j K(\cdot, w) : 0 \le j \le k-1,\ w \in \mathcal{Z}\}\ ,$$

then $(\partial_1^j h)(w)$ must vanish for $0 \le j \le k-1$ and $w \in \mathcal{Z}$. Also, the reproducing property implies that $\{\bar{\partial}_1^j K(\cdot, w) : 0 \le j \le k-1\}$ are orthogonal to $\mathcal{M}_0$. It follows that D is a spanning set for $\mathcal{M}_{\mathrm{q}}$. Notice that for $0 \le j \le k-1$, we have

$$J(\bar{\partial}_1^j K(\cdot, w)) = \sum_{\ell=0}^{k-1} \partial_1^\ell \bar{\partial}_1^j K(\cdot, w) \otimes \varepsilon_{\ell+1}.$$

Recalling the fact that $(JK)_{\ell,j} = (\partial_1^\ell \bar{\partial}_1^j K)$, we find that $(J\mathcal{M})_{\mathrm{q}}$ is spanned by the set of vectors

$$JD = \{JK(\cdot, w)\varepsilon_{\ell+1} : 0 \le \ell \le k-1,\ \ w \in \mathcal{Z}\}.$$

It is clear from equation (2.4) that

$$\Big\langle \boldsymbol{h}, JK(\cdot, w)\boldsymbol{x} \Big\rangle = \bar{x}_1 h(w) + \cdots + \bar{x}_k \partial_1^{k-1} h(w)$$

vanishes for all $\boldsymbol{x} \in \mathbb{C}^k$ and $w \in \mathcal{Z}$ if and only if $\boldsymbol{h}$ is in $(J\mathcal{M})_0$. Consequently, the set of vectors JD spans $(J\mathcal{M})_{\mathrm{q}}$. Hence $J(\mathcal{M}_{\mathrm{q}}) = (J\mathcal{M})_{\mathrm{q}}$. We have $(M_f^* - \overline{f(w)})K(\cdot, w) = 0$, for $f \in \mathcal{A}(\Omega)$. Differentiating this equation by $\overline{\partial_1}$, we see that $M_f^* \bar{\partial}_1 K(\cdot, w) = \overline{f(w)}\bar{\partial}_1 K(\cdot, w) + \overline{\partial_1 f(w)} K(\cdot, w)$. By induction, we find that

$$M_f^* \bar{\partial}_1^\ell K(\cdot, w) = \sum_{j=0}^{\ell} \binom{\ell}{j} \bar{\partial}_1^{\ell-j} K(\cdot, w) \overline{\partial_1^j f(w)}, \quad 0 \le \ell \le k-1. \tag{3.2}$$

If we can verify the equation $JM_f^* = J_f^* J$ on the set D, then the proof will be complete.

We note that $J\bar{\partial}_1^j K(\cdot, w) = JK(\cdot, w)\varepsilon_{j+1}$, $0 \le j \le k-1$. Hence using equation (3.2), we find that

$$\begin{aligned} J(M_f^* \bar{\partial}_1^\ell K(\cdot, w)) &= \sum_{j=0}^{\ell} \binom{\ell}{j} J(\bar{\partial}_1^{\ell-j} K(\cdot, w)) \overline{\partial_1^j f(w)} \\ &= \sum_{j=0}^{\ell} \binom{\ell}{j} JK(\cdot, w)\varepsilon_{j+1}, \overline{\partial_1^j f(w)}, \quad 0 \le \ell \le k-1. \end{aligned} \tag{3.3}$$

The fact that

$$J_f^* JK(\cdot, w) \cdot \boldsymbol{x} = JK(\cdot, w)(\mathcal{J}f)(w)^* \cdot \boldsymbol{x}, \quad \boldsymbol{x} \in \mathbb{C}^k, w \in \Omega \tag{3.4}$$

is established in the next lemma.

The equations (3.3) and (3.4) together imply that $JM_f^* = J_f^* J$ on the set D. This completes the proof. □

LEMMA 3.2 *Let $\mathcal{M}$ be a Hilbert module of holomorphic functions on Ω over the algebra $\mathcal{A}(\Omega)$ with reproducing kernel K. Let $J\mathcal{M}$ be the associated module of jets with reproducing kernel JK. The adjoint of the module action J_f on $JK(\cdot,w)\boldsymbol{x}$, $\boldsymbol{x}\in\mathbb{C}^k$ is given by*

$$J_f^* JK(\cdot,w)\cdot\boldsymbol{x} = JK(\cdot,w)(\mathcal{J}f)(w)^*\cdot\boldsymbol{x}.$$

Proof: We find that for $\boldsymbol{h}=\sum_{j=0}^{k-1}\partial_1^j h\otimes\varepsilon_{j+1}\in J(\mathcal{M})$ and $\boldsymbol{x}=(x_1,\ldots,x_k)\in\mathbb{C}^k$,

$$\begin{aligned}\Big\langle J_f\boldsymbol{h}, JK(\cdot,w)\cdot\boldsymbol{x}\Big\rangle &= \Big\langle J_f(\boldsymbol{h})(w),\boldsymbol{x}\Big\rangle\\ &= \Big\langle (\mathcal{J}f)(w)\cdot(\boldsymbol{h})(w),\boldsymbol{x}\Big\rangle\\ &= \Big\langle (\boldsymbol{h})(w),(\mathcal{J}f)(w)^*\boldsymbol{x}\Big\rangle_{\mathbb{C}^k}\\ &= \Big\langle \boldsymbol{h}, JK(\cdot,w)(\mathcal{J}f)(w)^*\cdot\boldsymbol{x}\Big\rangle.\end{aligned}$$

This calculation completes the proof. □

We consider the Hilbert space $(J\mathcal{M})_{\rm res}$ obtained by restricting the functions in $J\mathcal{M}$ to the set $\mathcal{Z}$, that is,

$$(J\mathcal{M})_{\rm res}=\{\boldsymbol{h}_0 \text{ holomorphic on } \mathcal{Z}:\boldsymbol{h}_0=\boldsymbol{h}|_{\mathcal{Z}} \text{ for some } \boldsymbol{h}\in J\mathcal{M}\}.$$

The norm of $\boldsymbol{h}_0\in(J\mathcal{M})_{\rm res}$ is

$$\|\boldsymbol{h}_0\|=\inf\{\|\boldsymbol{h}\|:\boldsymbol{h}|_{\mathcal{Z}}=\boldsymbol{h}_0 \text{ for } \boldsymbol{h}\in J\mathcal{M}\},$$

and the module action is obtained by restricting the map $(f,\boldsymbol{h}|_{\mathcal{Z}})\to J_f\boldsymbol{h}$ in both the arguments to $\mathcal{Z}$, that is,

$$\begin{aligned}(f,\boldsymbol{h}|_{\mathcal{Z}}) &\to (J_f\cdot\boldsymbol{h})|_{\mathcal{Z}}\\ &\qquad = \mathcal{J}f|_{\mathcal{Z}}\cdot\boldsymbol{h}|_{\mathcal{Z}}.\end{aligned}$$

Aronszajn [1, p. 351] shows that the restriction map R is unitary, on a functional Hilbert space consisting of scalar valued holomorphic functions. This proof was reproduced in [6]. However, his proof goes through for the vector valued case as well. The restriction map can be used to show that the reproducing kernel $JK(\cdot,w)_{\rm res}$ for $(J\mathcal{M})_{\rm res}$ is $JK_{\rm res}(\cdot,w)=K(\cdot,w)|_{\mathcal{Z}}$, $w\in\mathcal{Z}$.

THEOREM 3.3 *Let $\mathcal{M}$ be a Hilbert module over the algebra $\mathcal{A}(\Omega)$ and $\mathcal{M}_0$ be the submodule of functions h such that $\partial_1^j h$ vanish on $\mathcal{Z}$ for $0\le j\le k-1$. The module $(J\mathcal{M})_{\rm res}$ is equivalent to the quotient module $(J\mathcal{M})_{\rm q}$.*

Proof: Since the reproducing kernel $K_0(\cdot,w)$ admits the factorization (2.6), it follows that $J_0K(z,w)$ vanishes for all $w\in\mathcal{Z}$. However, $JK(\cdot,w)=J_{\rm q}K(\cdot,w)+J_0K(\cdot,w)$, where $J_{\rm q}K(\cdot,w)$ denotes the reproducing kernel for the quotient module $(J\mathcal{M})_{\rm q}$. Hence it follows that $J_{\mathrm{q}}K(\cdot, w) = JK(\cdot, w)$ for $w \in \mathcal{Z}$. Note that for $\sum_{j=0}^{k-1} \partial_1^j h \otimes \varepsilon_{j+1} \in (J\mathcal{M})_{\mathrm{q}}$ and $w \in \mathcal{Z}$, we have

$$\begin{aligned} \sum_{j=0}^{k-1} \partial_1^j h \otimes \varepsilon_{j+1} &\xrightarrow{R} \sum_{j=0}^{k-1} \Big\langle \partial_1^j h \otimes \varepsilon_{j+1}, JK_{\mathrm{q}}(\cdot, w)\varepsilon_{j+1} \Big\rangle \varepsilon_{j+1} \\ &= \sum_{j=0}^{k-1} \partial_1^j h(w)\varepsilon_{j+1}. \end{aligned}$$

Hence R is the restriction map on $(J\mathcal{M})_{\mathrm{q}}$. Since $J_{\mathrm{q}}K(\cdot, w) = JK(\cdot, w)$ for $w \in \mathcal{Z}$, it follows that $R(J_{\mathrm{q}}K(\cdot, w)) = JK|_{\mathrm{res}}(\cdot, w)$. Besides, R is injective on $(J\mathcal{M})_{\mathrm{q}}$. We may therefore define an inner product on $R\{(J\mathcal{M})_{\mathrm{q}}\}$ so as to make R an isometry. For $\boldsymbol{h} \in (J\mathcal{M})_{\mathrm{q}}$ and $w \in \mathcal{Z}$, then it follows that

$$\Big\langle R\boldsymbol{h}, JK|_{\mathrm{res}}(\cdot, w)\boldsymbol{x} \Big\rangle = \Big\langle \boldsymbol{h}, J_{\mathrm{q}}K(\cdot, w)\boldsymbol{x} \Big\rangle = \Big\langle \boldsymbol{h}(w), \boldsymbol{x} \Big\rangle.$$

Thus the reproducing kernel for the space $R\{(J\mathcal{M})_{\mathrm{q}}\}$ is $JK|_{\mathrm{res}}(\cdot, w),\ w \in \mathcal{Z}$. By our construction R is an isometry from $R\{(J\mathcal{M})_{\mathrm{q}}\}$ onto the Hilbert space $(J\mathcal{M})_{\mathrm{res}}$.

We point out that (see (3.4))

$$J_{f\circ i}^* JK_{\mathrm{res}}(\cdot, w)\boldsymbol{x} = JK_{\mathrm{res}}(\cdot, w)(\mathcal{J}f)(w)^*\boldsymbol{x}, \tag{3.5}$$

where $i : \mathcal{Z} \to \Omega$ is the inclusion map and $w \in \mathcal{Z}$.

We only need to verify that $R : (J\mathcal{M})_{\mathrm{q}} \to (J\mathcal{M})_{\mathrm{res}}$ is a module map, that is, $J_{f\circ i} \cdot R(\boldsymbol{h}) = RP(J_f \cdot \boldsymbol{h})$ for all $\boldsymbol{h} \in (J\mathcal{M})_{\mathrm{q}}$, where P denotes the orthogonal projection onto the space $(J\mathcal{M})_{\mathrm{q}}$. Note that for $w \in \mathcal{Z}$, we have

$$\begin{aligned} \Big\langle \boldsymbol{h}, J_f^* P J_{\mathrm{q}}K(\cdot, w)\boldsymbol{x} \Big\rangle &= \Big\langle \boldsymbol{h}, J_f^* JK(\cdot, w)\boldsymbol{x} \Big\rangle \\ &= \Big\langle \boldsymbol{h}, JK(\cdot, w)(\mathcal{J}f)(w)^*\boldsymbol{x} \Big\rangle \\ &= \Big\langle (\boldsymbol{h})(w), (\mathcal{J}f)(w)^*\boldsymbol{x} \Big\rangle. \end{aligned}$$

¿From this calculation, it follows that

$$\Big\langle PJ_f \cdot \boldsymbol{h}, J_{\mathrm{q}}K(\cdot, w)\boldsymbol{x} \Big\rangle = \Big\langle (\mathcal{J}f)(w)\, Jh(w), \boldsymbol{x} \Big\rangle. \tag{3.6}$$

Further, for $\boldsymbol{h} \in (J\mathcal{M})_{\mathrm{q}}$, we have

$$\begin{aligned} \Big\langle J_{f\circ i} \cdot R\boldsymbol{h}, JK_{\mathrm{res}}(\cdot, w)\boldsymbol{x} \Big\rangle &= \Big\langle R\boldsymbol{h}, J_{f\circ i}^* JK_{\mathrm{res}}(\cdot, w)\boldsymbol{x} \Big\rangle \\ &= \Big\langle R\boldsymbol{h}, JK_{\mathrm{res}}(\cdot, w)(\mathcal{J}f)(w)^*\boldsymbol{x} \Big\rangle \\ &= \Big\langle \boldsymbol{h}, JK_{\mathrm{q}}(\cdot, w)(\mathcal{J}f)(w)^*\boldsymbol{x} \Big\rangle \\ &= \Big\langle PJ_f\boldsymbol{h}, J_{\mathrm{q}}K(\cdot, w)\boldsymbol{x} \Big\rangle \\ &= \Big\langle PJ_f\boldsymbol{h}, R^* JK_{\mathrm{res}}(\cdot, w)\boldsymbol{x} \Big\rangle \\ &= \Big\langle RPJ_f\boldsymbol{h}, JK_{\mathrm{res}}(\cdot, w)\boldsymbol{x} \Big\rangle. \end{aligned}$$

This calculation verifies that R is a module map and the proof is complete. $\Box$

A special case of the following theorem was worked out by B. Bagchi and the second author.

THEOREM 3.4 *The quotient module $\mathcal{M}_{\rm q}$ is equivalent to the module $(J\mathcal{M})_{\rm res}$.*

Proof: We have already shown that $\mathcal{M}_{\rm q}$ and $(J\mathcal{M})_{\rm q}$ are equivalent modules. Now that we have proved $(J\mathcal{M})_{\rm q}$ and $(J\mathcal{M})_{\rm res}$ are also equivalent, it follows that $\mathcal{M}_{\rm q}$ and $(J\mathcal{M})_{\rm res}$ are equivalent. $\Box$

At this point, it may seem a little unnatural to consider $(J\mathcal{M})_{\rm res}$ as a module over the algebra $\mathcal{A}(\Omega)$. We describe an alternative point of view.

Let

$$J\mathcal{A}(\Omega) = \{\mathcal{J}f : f \in \mathcal{A}(\Omega)\} \subseteq \mathcal{A}(\Omega) \otimes \mathcal{M}_k(\mathbb{C}). \tag{3.7}$$

A multiplication on $J\mathcal{A}(\Omega)$ is obtained by defining the product $(\mathcal{J}f \cdot \mathcal{J}g)(z) \stackrel{\rm def}{=} (\mathcal{J}f)(z) \cdot (\mathcal{J}g)(z)$, where $\cdot$ is the usual matrix product. The algebra $J\mathcal{A}(\Omega)$ acts naturally on $J\mathcal{M}$ via the map $(Jf, \boldsymbol{h}) \mapsto (\mathcal{J}f)\boldsymbol{h}$ (see equation (2.8)). The restriction of the algebra $J\mathcal{A}(\Omega)$ to the hypersurface $\mathcal{Z}$ will be denoted by $J\mathcal{A}(\Omega)_{\rm res}$. Indeed, restricting the map $(Jf, \boldsymbol{h}) \mapsto (\mathcal{J}f)\boldsymbol{h}$ in both arguments to the hypersurface $\mathcal{Z}$, it is easy to see that $(J\mathcal{M})_{\rm res}$ is a module over the restriction algebra $J\mathcal{A}(\Omega)_{\rm res}$. The inclusion map $i : \mathcal{Z} \to \Omega$ induces a map $i^* : J\mathcal{A}(\Omega) \to J\mathcal{A}(\Omega)_{\rm res}$ defined by $i^*(Jf) = (Jf) \circ i$. Finally, if we think of $(J\mathcal{M})_{\rm res}$ as a module over the algebra $J\mathcal{A}(\Omega)_{\rm res}$, then we may push it forward to a module $i_*(J\mathcal{M})_{\rm res}$ over the algebra $J\mathcal{A}(\Omega)$ via the map

$$(Jf, \boldsymbol{h}|_{\mathcal{Z}}) \to i^*(Jf) \cdot \boldsymbol{h}|_{\mathcal{Z}}, \quad \boldsymbol{h} \in J\mathcal{M},\ Jf \in J\mathcal{A}(\Omega).$$

Thus, we may push forward the module $(J\mathcal{M})_{\rm res}$, thought of as a module over the algebra $J\mathcal{A}(\Omega)_{\rm res}$, to a module over the algebra $J\mathcal{A}(\Omega)$, which can then be thought of as a module over $\mathcal{A}(\Omega)$.

Now, consider the module of holomorphic functions on $\mathcal{Z}$ taking values in $\mathbb{C}^k$ over the algebra $\mathcal{A}_k(\mathcal{Z})$, where

$$\mathcal{A}_k(\mathcal{Z}) = \{\mathcal{J}\boldsymbol{f} = \begin{pmatrix} f_0 & & & & \\ & \ddots & & 0 & \\ \vdots & \binom{\ell}{j} f_{\ell-j} & f_0 & & \\ & & & \ddots & \\ f_{k-1} & & \cdots & & f_0 \end{pmatrix} : \boldsymbol{f} = (f_0, \ldots, f_{k-1}) \in \mathcal{A}(\mathcal{Z}) \otimes \mathbb{C}^k\}.$$

If $\boldsymbol{h}$ is an arbitrary element of the module of holomorphic functions on $\mathcal{Z}$ taking values in $\mathbb{C}^k$, then the module action is given by the usual matrix multiplication $\mathcal{J}\boldsymbol{f} \cdot \boldsymbol{h}$. It is clear that $J\mathcal{A}(\Omega)_{\rm res} = \mathcal{A}_k(\mathcal{Z})$.

The preceding discussion together with Theorem 3.4 implies that the quotient modules that arise in our context (for fixed Ω, $\mathcal{Z}$ and k) are modules of holomorphic functions on $\mathcal{Z}$ taking values in $\mathbb{C}^k$ over the algebra $\mathcal{A}_k(\mathcal{Z})$. However, the algebra $\mathcal{A}(\mathcal{Z})$ sits inside $\mathcal{A}_k(\mathcal{Z})$ as diagonal elements. Hence any module over the algebra $\mathcal{A}_k(\mathcal{Z})$ is also a module over $\mathcal{A}(\mathcal{Z})$. We can therefore ask, if we restrict the action to this smaller algebra, then whether the module lies in the class $B_k(\mathcal{Z})$ (cf. [7]). In particular, we ask if the quotient module equipped with the action of the smaller algebra $\mathcal{A}(\mathcal{Z})$ lies in the class $B_k(\mathcal{Z})$.

The discussions so far have led us to consider the following classes of modules:
(i) $Mod(\Omega)$: These are Hilbert spaces of holomorphic functions on $\Omega \subseteq \mathbb{C}^m$ (with bounded evaluation functionals) which are modules over the algebra $\mathcal{A}(\Omega)$.
(ii) $Mod_0^{(k)}(\Omega, \mathcal{Z})$: Submodules of modules in $Mod(\Omega)$ consisting of functions which vanish to order k on a hypersurface $\mathcal{Z} \subseteq \Omega$.
(iii) $Quot_k(\Omega, \mathcal{Z})$: Quotients of modules in $Mod(\Omega)$ by submodules of the type described in (ii). Of course, these are modules over $\mathcal{A}(\Omega)$.
(iv) $Mod_k(\mathcal{Z})$: These modules are reproducing kernel Hilbert spaces of $\mathbb{C}^k$ valued holomorphic functions on $\mathcal{Z}$. They are modules over the algebra $\mathcal{A}_k(\mathcal{Z})$ which is a homomorphic image of $\mathcal{A}(\Omega)$ as indicated earlier.

A significant part of the foregoing discussion has been devoted to showing that given a module in $Quot_k(\Omega, \mathcal{Z})$, there corresponds a unitarily equivalent module in $Mod_k(\mathcal{Z})$.

The converse question may be termed a 'dilation question' since, in the language of reproducing kernels, it corresponds to the following:

Assume we are given a module in $Mod_k(\mathcal{Z})$ with (matrix valued) reproducing kernel $\mathbb{K}$. Then, does there exist a module in $Mod(\Omega)$ (as in (i) above) with (scalar) reproducing kernel K such that $\mathbb{K} = JK_{\text{res } \mathcal{Z}}$?

Let $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ be two Hilbert modules over the algebra $\mathcal{A}(\Omega)$ with reproducing kernels K and $\tilde{K}$ respectively. Assume further that both $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ are in $B_1(\Omega)$ (cf. [7]).

As pointed out earlier, these modules give rise to trivial holomorphic hermitian bundles $\mathcal{E}$ and $\tilde{\mathcal{E}}$ on Ω^*. The assumption that $\mathcal{M}$ is in the class $B_1(\Omega)$ implies, in particular,

1. $\mathcal{E}_w = \cap\{\ker(M_\ell^* - \bar{w}_\ell), 1 \leq \ell \leq k\}$, where $\mathcal{E}_w$ is the fiber of the holomorphic bundle $\mathcal{E}$ at $w \in \Omega^*$ and M_ℓ is the operator of multiplication by w_ℓ
2. $\dim \mathcal{E}_w = 1$,
3. $\vee\{\mathcal{E}_w : w \in \Omega^*\} = \mathcal{M}$.

The holomorphic frame for the bundle $\mathcal{E}$ (respectively, $\tilde{\mathcal{E}}$) is $s(w) = K(\cdot, \bar{w})$ (respectively, $\tilde{s}(w) = \tilde{K}(\cdot, \bar{w})$) for $w \in \Omega^*$. Similarly, the hermitian metric for the bundle $\mathcal{E}$ (respectively, $\tilde{\mathcal{E}}$) is $K(\bar{w}, \bar{w})$ (respectively, $\tilde{K}(\bar{w}, \bar{w})$) for $w \in \Omega^*$.

If $T : \widetilde{\mathcal{M}} \to \mathcal{M}$ is a module map (T is a bounded operator intertwining the two module actions), then $T^* \ker(M_f^* - w) \subseteq \ker(\tilde{M}_f^* - w)$. Hence $T^* K(\cdot, w) = \psi(w)\tilde{K}(\cdot, w)$, for some function $\psi : \Omega \to \mathbb{C}$. If this operator is to be bounded then

$$\Big(\Big(\overline{\psi(w_i)}\psi(w_j)\tilde{K}(w_j, w_i)\Big)\Big) \leq C\Big(\Big(K(w_j, w_i)\Big)\Big),$$

for every finite set $\{w_1, \ldots, w_n\} \subseteq \Omega$ and some positive constant C. Moreover,

$$
\begin{aligned}
(Tf)(w) = \big\langle Tf, K(\cdot, w)\big\rangle &= \big\langle f, T^* K(\cdot, w)\big\rangle \\
&= \big\langle f, \psi(w)\tilde{K}(\cdot, w)\big\rangle = \overline{\psi(w)} f(w)
\end{aligned}
$$

In particular, $\bar{\psi}$ must be also holomorphic since both $T\tilde{h} = \bar{\psi}\tilde{h}$ and $\tilde{h}$ are holomorphic.

We obtain a bundle map $\Psi : \tilde{\mathcal{E}} \to \mathcal{E}$ which is merely multiplication by $\overline{\psi(w)}$ on the fiber $\tilde{\mathcal{E}}_w$, $w \in \Omega^*$.

If T is also unitary, that is, $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ are equivalent modules, then $e_n = T\tilde{e}_n$ is an orthonormal basis for $\mathcal{M}$ whenever $\tilde{e}_n$ is an orthonormal basis for $\widetilde{\mathcal{M}}$. Therefore the reproducing kernel K is of the form

$$
\begin{aligned}
K(z, w) &= \sum_{n=0}^{\infty} (T\,\tilde{e}_n)(z)\overline{(T\,\tilde{e}_n)(w)} \\
&= \sum_{n=0}^{\infty} \overline{\psi(z)}\tilde{e}_n(z)\overline{\tilde{e}_n(w)}\psi(w) \\
&= \overline{\psi(z)}\tilde{K}(z, w)\psi(w).
\end{aligned}
$$

Finally, note that if T is unitary then $\psi(w) \neq 0$ for $w \in \Omega$.

This implies that the bundle map Ψ is isometric, that is, $K(z, w) = \overline{\psi(z)}\tilde{K}(w, w)\psi(w)$, $z, w \in \Omega$.

THEOREM 3.5 *If $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ are equivalent modules in $B_1(\Omega)$ and $\mathcal{M}_0$, $\widetilde{\mathcal{M}}_0$ are the submodules of functions vanishing to order k on $\mathcal{Z}$, then the quotient modules $\mathcal{M}_{\mathrm{q}}$ and $\widetilde{\mathcal{M}}_{\mathrm{q}}$ are also equivalent.*

Proof: It follows from the preceding discussion that there exists a holomorphic function $\eta : \Omega \to \mathbb{C}$ ($\eta = \bar{\psi}$) such that $M_\eta : \widetilde{\mathcal{M}} \to \mathcal{M}$ is a unitary module map. We also have the unitary module maps $J : \mathcal{M} \to J\mathcal{M}$ and $\tilde{J} : \widetilde{\mathcal{M}} \to J\widetilde{\mathcal{M}}$. The composite map $JM_\eta\tilde{J}^* : J\widetilde{\mathcal{M}} \to J\mathcal{M}$ is $\tilde{\boldsymbol{h}} \mapsto J(\eta\tilde{\boldsymbol{h}})$. Or, in other words, $JM_\eta J^* = \mathcal{J}\eta$.

It is, of course, as easy to verify directly that this map is a unitary module map. Since η does not vanish on Ω, the submodule $(J\widetilde{\mathcal{M}})_0$ is mapped onto $(J\mathcal{M})_0$. Hence the quotient modules $(J\widetilde{\mathcal{M}})_{\mathrm{q}}$ and $(J\mathcal{M})_{\mathrm{q}}$ must also be equivalent. But these latter modules are equivalent to $\widetilde{\mathcal{M}}_{\mathrm{q}}$ and $\mathcal{M}_{\mathrm{q}}$ respectively by Lemma 3.1. This completes the proof. □

A simple example to illustrate the ideas above is the following:

Let $\mathcal{M}$ be the Hardy space $H^2(\mathbb{D}^2)$. If $\mathbb{D}^2$ is parameterized by coordinates $z = (z_1, z_2) \in \mathbb{C}^2$, we choose $\mathcal{Z}$ to be the hypersurface defined by $z_1 = 0$, that is, $\mathcal{Z}$ is a disc parameterized by the co-ordinate z_2. $\mathcal{M}_0$ is the set of functions in $H^2(\mathbb{D}^2)$ which vanish up to order k on $\mathcal{Z}$.

The quotient module can then be identified with $H^2(\mathbb{D}) \otimes \mathbb{C}^k$ ($\mathbb{D}$ here is parameterized by z_2). If we think of an element of $H^2(\mathbb{D}) \otimes \mathbb{C}^k$ as a vector of functions from

$H^2(\mathbb{D})$, the module actions of $z_1, z_2 \in \mathcal{A}(\mathbb{D}^2)$ are defined by the following matrices:

$$z_1 \mapsto \begin{pmatrix} 0 & & & & & \\ 1 & 0 & & & & 0 \\ & 1 & 0 & & & \\ & & & \ddots & & \\ & 0 & & & 0 & \\ & & & & 1 & 0 \end{pmatrix}$$

$$z_2 \mapsto \begin{pmatrix} z_2 & & & & \\ & z_2 & & & 0 \\ & & z_2 & & \\ & 0 & & \ddots & \\ & & & & z_2 \end{pmatrix}$$

Notice that z_2 (and, in fact, any function of z_2) acts by a scalar. This, as mentioned in the introduction, is a general feature, that is, the action of $\mathcal{A}(\mathcal{Z}) \subseteq \mathcal{A}_k(\mathcal{Z})$ defined above on $(J\mathcal{M})_{\mathrm{q}}$ is a scalar action. The quotient module is naturally a module over the algebra $\mathcal{A}_k(\mathcal{Z})$. As pointed out earlier, we can look at the restricted action of the algebra $\mathcal{A}(\mathcal{Z}) \subseteq \mathcal{A}_k(\mathcal{Z})$ on the quotient module and ask whether it lies in the class $B_k(\mathcal{Z})$ as a module over this smaller algebra. That this is true for the Hardy space example discussed above is a special case of a more general theorem.

Proposition 3.6 *Assume that the reproducing kernel K has a diagonal power series expansion, that is, for $z, w \in \Omega$*

$$K(z,w) = \sum_{\alpha \geq 0} A_\alpha (z - Z)^\alpha \overline{(w - W)}^\alpha \quad ,$$

for some $Z, W \in \Omega$ with $Z_1 = 0 = W_1$, where $\alpha = (\alpha_1, \cdots, \alpha_m) \in \mathbb{Z}^m$. Then the quotient module (restricted to a module over $\mathcal{A}(\mathcal{Z})$) lies in $B_k(\mathcal{Z})$ if $\mathcal{M} \in B_1(\Omega)$ as we have assumed.

Proof: It is easy to see, in this case, that JK_{res} also has a diagonal Taylor expansion. Further, the Taylor coefficients (which are now $k \times k$ matrices) are themselves diagonal.

Explicitly, if $\tilde{z} = (z_2, \cdots, z_m)$, $\tilde{w} = (w_2, \cdots, w_m)$, $\tilde{Z} = (Z_2, \cdots, Z_m)$, $\tilde{W} = (W_2, \cdots, W_m)$, $\mu = (\mu_2, \cdots, \mu_m) \in \mathbb{Z}^{m-1}$, then JK_{res} has the Taylor expansion

$$JK_{\mathrm{res}}(\tilde{z}, \tilde{w}) = \sum_{\mu \geq 0} D_\mu (\tilde{z} - \tilde{Z})^\mu \overline{(\tilde{w} - \tilde{W})}^\mu,$$

where D_μ is the (diagonal) matrix given by

$$(D_\mu)_{ij} = A_{i,\mu_2,\cdots,\mu_m} \delta_{ij}.$$

It follows, therefore (see [5]), that the co-ordinate functions of $\mathcal{Z}$ act on $(J\mathcal{M})_q$ by weighted shift operators with weights determined by the D_μ's.

We then apply Theorem 5.4 in [5] in two stages. Since $\mathcal{M} \in B_1(\Omega)$ and K has a diagonal Taylor expansion, the Taylor coefficients satisfy the inequality in part (b) of that theorem. Consequently, the weights referred to above satisfy a corresponding (operator) inequality. Another application of the same theorem (using the latter inequality) ensures that $(J\mathcal{M})_q$ (as a module over $\mathcal{A}(\mathcal{Z})$) lies in $B_k(\mathcal{Z})$. □

If, in fact, the quotient modules (as modules over $\mathcal{A}(\mathcal{Z})$) lie in $B_k(\mathcal{Z})$, we have the following possible approach to the equivalence question:

If two quotient modules are equivalent, they must be equivalent as modules over the subalgebra $\mathcal{A}(\mathcal{Z})$. The latter then becomes a question of equivalence in $B_k(\mathcal{Z})$. This question has been studied in [4].

For a complete answer to the equivalence question, we need to determine when there is, among all the unitaries that implement the equivalence in $B_k(\mathcal{Z})$, one that intertwines the (nilpotent) action of functions depending only on the 'normal' co-ordinate. This question can be studied in a series of steps as follows:

Notice, firstly, that the action of z_1^p is given by a $(k-p)$-step nilpotent operator. The requirement that the unitary which describes the $B_k(\mathcal{Z})$ equivalence must intertwine these powers of z_1 translates into a sequence of conditions on the unitary. (For instance, in the case $k = 2$, where only the first power of z_1 is relevant, this requires that the unitary is upper triangular with equal entries on the diagonal.)

We are thus led to the following vector bundle picture. If the quotient module lies in $B_k(\mathcal{Z})$, there is naturally associated a (rank k) bundle on $\mathcal{Z}$. However, this bundle now comes equipped with a collection of subbundles which together determine a flag on each fiber. The full equivalence of the quotient modules is then characterised in terms of the equivalence of these 'flag bundles'.

Equivalence of flag bundles, at least formally like these, is considered and characterized by Martin and Salinas [10, Theorem 4.5]. We hope to explore possible implications of their work for ours at a later time.

4 Module tensor products

The module action on $J\mathcal{M}$ defined by $(f, \boldsymbol{h}) \mapsto (\mathcal{J}f)\boldsymbol{h}$, $f \in \mathcal{A}(\Omega), \boldsymbol{h} \in J\mathcal{M}$, naturally induces a module action on $\mathbb{C}^k$ which is merely given by the map $(f, \boldsymbol{x}) \mapsto (\mathcal{J}f)(w)\boldsymbol{x}$, $\boldsymbol{x} \in \mathbb{C}^k$, $w \in \Omega$. The k - dimensional vector space $\mathbb{C}^k$ equipped with this module action will be denoted by $\mathbb{C}^k_w$. We point out that this action is somewhat different from the one introduced in [3].

The module tensor product $J\mathcal{M} \otimes_{\mathcal{A}(\Omega)} \mathbb{C}^k_w$ is the orthogonal complement of the following subspace $\mathcal{N}$ in the Hilbert space $J\mathcal{M} \otimes \mathbb{C}^k_w$. In fact, the subspace $\mathcal{N}$ is left invariant by both $J_f \otimes I$ and $I \otimes (\mathcal{J}f)(w)$. The module action is obtained by compressing either $J_f \otimes I$ or $I \otimes (\mathcal{J}f)(w)$ to the orthocomplement of the subspace

$$\begin{aligned}\mathcal{N} = \operatorname{span}\Big\{ \sum_{\ell=1}^{k} (J_f \cdot \boldsymbol{h}_\ell \otimes \varepsilon_\ell - \boldsymbol{h}_\ell \otimes (\mathcal{J}f)(w) \cdot \varepsilon_\ell)&: \\ \boldsymbol{h}_\ell \in J\mathcal{M},\ \varepsilon_\ell \in \mathbb{C}^k \text{ are standard basis vectors and } & f \in \mathcal{A}(\Omega), 1 \le \ell \le k \Big\}.\end{aligned}$$

The compression of $I \otimes (\mathcal{J}f)(w)$ and $J_f \otimes I$ to $\mathcal{N}^\perp$ are equal. Let $J_f \otimes_{\mathcal{A}(\Omega)} I$ denote this compression. The subspace $\mathcal{N}^\perp \subseteq J\mathcal{M} \otimes \mathbb{C}^k_w$, together with the action defined by the operator $J_f \otimes_{\mathcal{A}(\Omega)} I$, is a module over $\mathcal{A}(\Omega)$, which is the localization $J\mathcal{M} \otimes_{\mathcal{A}(\Omega)} \mathbb{C}^k_w$ of the module $\mathcal{M}$. The adjoint operators $J_f^* \otimes I$ and $I \otimes (\mathcal{J}f)(w)^*$ leave the subspace $\mathcal{N}^\perp$ invariant and both of them equal $(J_f \otimes_{\mathcal{A}(\Omega)} I)^*$. We will calculate $I \otimes (\mathcal{J}f)(w)^*$ on $J\mathcal{M} \otimes_{\mathcal{A}(\Omega)} \mathbb{C}^k_w$, which will then be related to the earlier work in [4].

Lemma 4.1 *The module tensor product $J\mathcal{M} \otimes_{\mathcal{A}(\Omega)} \mathbb{C}^k_w$ is spanned by the vectors e_p in $J\mathcal{M} \otimes \mathbb{C}^k_w$, where*

$$e_p(w) = \sum_{\ell=1}^{p} a_{p,\ell} JK(\cdot, w)\varepsilon_{p-\ell+1} \otimes \varepsilon_\ell,\ a_{p,\ell} = \frac{(p-1)!}{(p-\ell)!(\ell-1)!},\ 1 \le p \le k.$$

The module action $J_f^* \otimes I : J\mathcal{M} \otimes_{\mathcal{A}(\Omega)} \mathbb{C}^k_w \to J\mathcal{M} \otimes_{\mathcal{A}(\Omega)} \mathbb{C}^k_w$ is given by $(J_f^* \otimes I)(e_p) = \sum_{\ell=1}^p \Big\langle (\mathcal{J}f)(w)^*\varepsilon_p, \varepsilon_\ell \Big\rangle e_\ell$.

Proof: During the course of this proof, $w \in \Omega$ is fixed and we write e_p for $e_p(w)$. By assuming that constants are in $\mathcal{M}$, we have ensured that the rank of the module $\mathcal{M}$ is 1. Then Lemma 5.11 from [7] implies that the dimension of $J\mathcal{M} \otimes_{\mathcal{A}(\Omega)} \mathbb{C}^k_w$ is at most k. Therefore, if $e_p \perp \mathcal{N}$ for $1 \le p \le k$ then it will follow that $\{e_p : 1 \le p \le k\}$ span $J\mathcal{M} \otimes_{\mathcal{A}(\Omega)} \mathbb{C}^k_w$ (since the e_p's are linearly independent). We find that

$$\begin{aligned}
&\Big\langle \sum_{\ell=1}^{k} J_f \cdot \boldsymbol{h}_\ell \otimes \varepsilon_\ell - \boldsymbol{h}_\ell \otimes (\mathcal{J}f)(w)\cdot \varepsilon_\ell, e_p \Big\rangle \\
&= \sum_{j=1,\ell=1}^{p,k} a_{p,j} \Big\{ \Big\langle \boldsymbol{h}_\ell, J_f^* JK(\cdot,w)\varepsilon_{p-j+1} \Big\rangle \Big\langle \varepsilon_\ell, \varepsilon_j \Big\rangle - \Big\langle \boldsymbol{h}_\ell, JK(\cdot,w)\varepsilon_{p-j+1} \Big\rangle \Big\langle \varepsilon_\ell, (\mathcal{J}f)(w)^*\varepsilon_j \Big\rangle \Big\} \\
&= \sum_{\ell=1}^{p} a_{p,\ell} \Big\langle \boldsymbol{h}_\ell, JK(\cdot,w)(\mathcal{J}f)(w)^*\varepsilon_{p-\ell+1} \Big\rangle - \sum_{j=1,\ell=1}^{p,k} a_{p,j} \Big\langle \boldsymbol{h}_\ell(w), \varepsilon_{p-j+1} \Big\rangle \Big\langle (\mathcal{J}f)(w)\varepsilon_\ell, \varepsilon_j \Big\rangle \\
&= \sum_{\ell=1}^{p} a_{p,\ell} \Big\langle \boldsymbol{h}_\ell(w), (\mathcal{J}f)(w)^*\varepsilon_{p-\ell+1} \Big\rangle - \sum_{j=1,\ell=1}^{p,k} a_{p,j} \Big\langle \boldsymbol{h}_\ell(w), \varepsilon_{p-j+1} \Big\rangle \Big\langle (\mathcal{J}f)(w)\varepsilon_\ell, \varepsilon_j \Big\rangle \\
&= \sum_{\ell=1}^{p} \Big\{ \sum_{j=1}^{p} \Big\langle \boldsymbol{h}_\ell(w), \varepsilon_j \Big\rangle \Big(a_{p,\ell} \Big\langle (\mathcal{J}f)(w)\varepsilon_j, \varepsilon_{p-\ell+1} \Big\rangle - a_{p,p-j+1} \Big\langle (\mathcal{J}f)(w)\varepsilon_\ell, \varepsilon_{p-j+1} \Big\rangle \Big) \Big\}.
\end{aligned}$$

It is easy to verify that $a_{p,p-j+1}\Big\langle (\mathcal{J}f)(w)\varepsilon_\ell, \varepsilon_{p-j+1} \Big\rangle = a_{p,\ell}\Big\langle (\mathcal{J}f)(w)\varepsilon_j, \varepsilon_{p-\ell+1} \Big\rangle$. Hence it follows that $e_p \perp \mathcal{N}$, $1 \le p \le k$.

Now we calculate $I \otimes (\mathcal{J}f)(w)^*$ which is equal to $J_f^* \otimes I$ on $\mathcal{N}^\perp$.

$$\begin{aligned}
(I \otimes (\mathcal{J}f)(w)^*)(e_p) &= \sum_{\ell=1}^{p} a_{p,\ell} JK(\cdot,w)\varepsilon_{p-\ell+1} \otimes (\mathcal{J}f)(w)^*\varepsilon_\ell \\
&= \sum_{\ell,j=1}^{p,\ell} a_{p,\ell} \Big\langle (\mathcal{J}f)(w)^*\varepsilon_\ell, \varepsilon_j \Big\rangle JK(\cdot,w)\varepsilon_{p-\ell+1} \otimes \varepsilon_j
\end{aligned}$$

$$= \sum_{j=0}^{p-1}\sum_{\ell=1}^{p-j} a_{p,\ell+j}\Big\langle(\mathcal{J}f)(w)^*\varepsilon_{\ell+j},\varepsilon_\ell\Big\rangle JK(\cdot,w)\varepsilon_{p-j-\ell+1}\otimes\varepsilon_\ell.$$

We verify that

$$\begin{aligned} a_{p,\ell+j}\Big\langle(\mathcal{J}f)(w)^*\varepsilon_{\ell+j},\varepsilon_\ell\Big\rangle &= \frac{(p-1)!}{(p-\ell-j)!(\ell+j-1)!}\frac{(\ell+j-1)!}{j!(\ell-1)!}(\partial^j f)(w) \\ &= a_{p-j,\ell}\frac{(p-1)!}{(p-j-1)!}\frac{(\partial^j f)(w)}{j!}. \end{aligned}$$

Consequently,

$$\begin{aligned} (I\otimes(\mathcal{J}f)(w)^*)(e_p) &= \sum_{j=0}^{p-1}\frac{(\partial^j f)(w)}{j!}\frac{(p-1)!}{(p-j-1)!}e_{p-j} \\ &= \sum_{\ell=1}^{p}\Big\langle(\mathcal{J}f)(w)^*\varepsilon_p,\varepsilon_\ell\Big\rangle e_\ell. \end{aligned}$$

This completes the proof. □

We emphasize that $I\otimes\mathcal{J}f(w)^*$ equals $(J_f\otimes_{\mathcal{A}(\Omega)}I)^*$. Recall that the vectors $s_p = JK(\cdot,w)\varepsilon_p$ span the fiber $J\mathcal{E}_w$ at $w\in\Omega^*$ of the bundle $J\mathcal{E}$ associated with the module $J\mathcal{M}$. The module action on this fiber was shown to be

$$\begin{aligned} J_f^* s_p &= JK(\cdot,w)(\mathcal{J}f)(w)^*\varepsilon_p \\ &= \sum_{\ell=1}^{p}\Big\langle(\mathcal{J}f)(w)^*\varepsilon_p,\varepsilon_\ell\Big\rangle s_\ell. \end{aligned} \tag{4.1}$$

The localizations $J\mathcal{M}\otimes_{\mathcal{A}(\Omega)}\mathbb{C}_w^k$ also give rise to a hermitian holomorphic bundle $J_{\rm loc}\mathcal{E}$ over Ω^* via the holomorphic frame $\{e_p(w) : w\in\Omega^*,\ 1\le p\le k\}$. The preceding lemma says that there is a natural (adjoint) action of the algebra $\mathcal{A}(\Omega)$ on each fiber $\vee\{e_p(w) : w\in\Omega^*,\ 1\le p\le k\}$, namely, $e_p(w)\to\sum_{\ell=1}^p\Big\langle(\mathcal{J}f)(w)^*\varepsilon_p,\varepsilon_\ell\Big\rangle e_\ell(w)$. However, the natural metric $\Big(\!\Big(\big\langle e_p,e_q\big\rangle\Big)\!\Big)_{p,q=1}^k$ on $J_{\rm loc}\mathcal{E}$ is not the same as that of $J\mathcal{E}$ - although they are related. To unravel this relationship, we set

$$\Big(\!\Big(J_{\rm loc}K(z,w)\Big)\!\Big)_{p,q} = \Big\langle e_q(z),e_p(w)\Big\rangle,\ 1\le p,q\le k,$$

and observe that

$$\begin{aligned} \Big\langle e_q(w),e_p(z)\Big\rangle &= \sum_{\ell=1}^{p}\sum_{m=1}^{q} a_{p,\ell}a_{q,m}\Big\langle JK(\cdot,w)\varepsilon_{q-\ell+1},JK(\cdot,z)\varepsilon_{p-m+1}\Big\rangle\delta_{\ell m} \\ &= \sum_{\ell=1}^{k} a_{p,\ell}a_{q,\ell}\Big(\!\Big(J_{k-\ell+1}K(z,w)\Big)\!\Big)_{p-\ell+1,q-\ell+1} \\ &= \sum_{\ell=1}^{k} a_{m+\ell-1,\ell}a_{n+\ell-1,\ell}\Big(\!\Big(J_{k-\ell+1}K(z,w)\Big)\!\Big)_{m,n},\ 1\le n,m\le k-l+1, \end{aligned}$$

where

$$\Big(\Big(J_{k-\ell+1}K(z,w)\Big)\Big)_{p-\ell+1,q-\ell+1} = \begin{cases} \Big(\Big(JK(z,w)\Big)\Big)_{p-\ell+1,q-\ell+1} & \text{if } p-\ell+1 \geq 1 \\ & \text{and } q-\ell+1 \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

It follows that

$$\begin{aligned} JK_{\mathrm{loc}}(z,w) &\stackrel{\mathrm{def}}{=} \Big(\Big(\Big\langle e_p(w), e_q(z)\Big\rangle\Big)\Big) \\ &= \sum_{\ell=1}^{k} D(\ell) J_{k-\ell+1}K(z,w)D(\ell), \end{aligned} \tag{4.2}$$

where $D(\ell)$ is a diagonal matrix with diagonal entries $D(\ell)_{m,m} = a_{m+\ell-1,\ell}$.

Now consider the map $J_{k-\ell+1} : \mathcal{M} \to \mathcal{M} \otimes \mathbb{C}^k$ given by $h \to \sum_{p=0}^{k-\ell} a_{p+\ell-1,\ell}\partial_1^p h \otimes \varepsilon_{p+\ell}$. Since $J_{k-\ell+1}$ is injective, we may choose an inner product on $J_{k-\ell+1}\mathcal{M}$ which makes $J_{k-\ell+1}$ an unitary map. As before (see Proposition 2.3), it is easy to see that the reproducing kernel for this Hilbert space is $D(\ell)J_{k-\ell+1}K(z,w)D(\ell)$. It follows that $D(\ell)J_{k-\ell+1}K(z,w)D(\ell)$ is positive definite. Consequently, $J_{\mathrm{loc}}K(w,w) = \sum_{\ell=1}^k D(\ell)J_{k-\ell+1}K(z,w)D(\ell)$ is positive definite. Hence there is a functional Hilbert space $J_{\mathrm{loc}}\mathcal{M}$ consisting of holomorphic functions on Ω associated with the vector bundle $J_{\mathrm{loc}}\mathcal{E}$ with reproducing kernel $J_{\mathrm{loc}}K$ as described in Section 2.

Since $J_{k-\ell+1}\mathcal{M} \cap J_{k-m+1}\mathcal{M} = \{0\}$, $1 \leq \ell \neq m \leq k$, we can form the internal direct sum $\oplus_{\ell=1}^k J_{k-\ell+1}\mathcal{M}$.

PROPOSITION 4.2 *The reproducing kernel for the Hilbert space $\oplus_{\ell=1}^k J_{k-\ell+1}\mathcal{M}$ is*

$$\sum_{\ell=1}^{k} D(\ell)J_{k-\ell+1}K(z,w)D(\ell)$$

and

$$J_{\mathrm{loc}}\mathcal{M} = \oplus_{\ell=1}^k J_{k-\ell+1}\mathcal{M}.$$

Proof: Let $\{e_{\ell,j} : j \in \mathbb{N}\}$ be an orthonormal basis for the Hilbert space $J_{k-\ell+1}\mathcal{M}$. It follows that $\{e_{\ell,j} : j \in \mathbb{N}, 1 \leq \ell \leq k\}$ is an orthonormal basis for the Hilbert space $\oplus_{\ell=1}^k J_{k-\ell+1}\mathcal{M}$. Therefore (see Lemma 2.2), the reproducing kernel for this Hilbert space is

$$\sum_{\ell=1}^{k}\sum_{j=0}^{\infty} e_{\ell,j}(z)e_{\ell,j}(w)^* = \sum_{\ell=1}^{k} J_{k-\ell+1}K(z,w).$$

Since the reproducing kernel uniquely determines a Hilbert space, and both $J_{\mathrm{loc}}\mathcal{M}$ and $\oplus_{\ell=1}^k J_{k-\ell+1}\mathcal{M}$ have the same reproducing kernel, it follows that $J_{\mathrm{loc}}\mathcal{M} = \oplus_{\ell=1}^k J_{k-\ell+1}\mathcal{M}$. □

Recall that $(J_f^* \otimes I)(e_p) = \sum_{\ell=1}^p \Big\langle (\mathcal{J}f)(w)^*\varepsilon_p, \varepsilon_\ell\Big\rangle e_\ell$. As discussed in Section 2, the span of $\{e_p(w) : w \in \Omega^*\}$ and the range of $J_{\mathrm{loc}}K(\cdot, w)$ are isomorphic via the map

$\sum_{p=1}^{k} e_p(w)x_p \to K(\cdot, w)\boldsymbol{x}$. Thus the algebra $\mathcal{A}(\Omega)$ acts on the linear span of the vectors $\{J_{\rm loc}K(\cdot, w)\varepsilon_\ell : 1 \leq \ell \leq k\}$, via $(\mathcal{J}f)(w)^*$. The Hilbert space $J_{\rm loc}\mathcal{M}$ is the closed linear span of ran$K(\cdot, w)$. Thus we find that $\mathcal{A}(\Omega)$ acts on a dense linear subspace of $J_{\rm loc}\mathcal{M}$. Since we are assuming that $\mathcal{M}$ is a Hilbert module over $\mathcal{A}(\Omega)$, it follows that M_f defines a bounded module map via $(f, h) \to M_f h$. However, the fact that $J_{k-\ell+1}$ is unitary implies $J_{k-\ell+1}M_f J_{k-\ell+1}^*$ defines a bounded module map on $J_{k-\ell+1}\mathcal{M}$, for $1 \leq \ell \leq k$. Consequently, the sum $\sum_{\ell=1}^{k} J_{k-\ell+1}M_f J_{k-\ell+1}^*$ defines a bounded module map on $J_{\rm loc}\mathcal{M}$. We observe that the action of $\mathcal{A}(\Omega)$ obtained this way is identical (on a dense subspace) with the one we have obtained via the localization. Thus, equipped with this module action, $J_{\rm loc}\mathcal{M}$ becomes a bounded module over the algebra $\mathcal{A}(\Omega)$.

We find that in the orthogonal decomposition of $J_{\rm loc}\mathcal{M}$, described in Proposition 4.2, the first piece, namely, $J_k\mathcal{M}$ equals $J\mathcal{M}$, and that the first term in the sum (4.2) is JK. Besides, the module action described in the previous paragraph leaves the subspace $J_k\mathcal{M}$ invariant. Finally, the action of $\mathcal{A}(\Omega)$ restricted to the sub-module $J_k\mathcal{M}$ is the same as J_f. This discussion proves the following theorem.

THEOREM 4.3 *The modules $P(J_{\rm loc}\mathcal{M})$ and $J\mathcal{M}$ are isomorphic, where P is the orthogonal projection of $J_{\rm loc}\mathcal{M}$ on $J_k\mathcal{M}$. The quotient module $\mathcal{M}_{\rm q}$ is isomorphic to the module $RP(J_{\rm loc}\mathcal{M})$, where R denotes restriction to the hypersurface $\mathcal{Z}$.*

In the case $k = 1$, which was discussed in [6], it was possible to identify the quotient module without the auxiliary construction involving the jets. Indeed, the quotient module was obtained from the localization simply by restricting. However, we have seen above that if $k > 1$, then to construct the quotient module, we cannot simply restrict the bundle obtained from the localization. Let $Gr(J_{\rm loc}\mathcal{M}, k)$ denote the Grassmanian manifold of rank k, the set of all k - dimensional subspaces of $J_{\rm loc}\mathcal{M}$. As pointed out in [4, section 2], the bundle $J_{\rm loc}\mathcal{E}$ is the pull-back of the tautological bundle $S(J_{\rm loc}\mathcal{M}, k)$ on the Grassmanian $Gr(J_{\rm loc}\mathcal{M}, k)$ via the map $t : \Omega^* \to G(J_{\rm loc}\mathcal{M}, k)$, $t(w) = J_{\rm loc}K(\cdot, w)\boldsymbol{x}$, $\boldsymbol{x} \in \mathbb{C}^k$. Clearly, the projection operator $P : J_{\rm loc}\mathcal{M} \to J_k\mathcal{M}$ induces a map, which we denote again by P, from $Gr(J_{\rm loc}\mathcal{M}, k)$ to $Gr(J_k\mathcal{M}, k)$. The pull-back of the tautological bundle $S(J_k\mathcal{M}, k)$ on the Grassmanian $Gr(J_k\mathcal{M}, k)$ under the map $P \circ t : \Omega^* \to Gr(J_k\mathcal{M}, k)$ will be denoted by $PJ_{\rm loc}\mathcal{E}$. Similarly, we obtain the bundles $J_{\rm loc,0}\mathcal{E}$ and $J_{\rm loc,q}\mathcal{E}$ from the localization of the modules $(J\mathcal{M})_0$ and $(J\mathcal{M})_{\rm q}$ respectively. As shown in section 2, there exists a holomorphic hermitian bundle $J\mathcal{E}$ associated with the module $J\mathcal{M}$. What we have established above is the fact that the bundle $J\mathcal{E}$ is identical with the bundle $P(J_{\rm loc}\mathcal{E})$. The fact that $\mathcal{M}_{\rm q}$ is isomorphic to $(J\mathcal{M})_{\rm res}$ shows that the bundle associated with the quotient module $\mathcal{M}_{\rm q}$ is the restriction of $PJ_{\rm loc}\mathcal{E}$ to the zero variety $\mathcal{Z}$.

Again, in the case $k = 1$, it was shown in [6], that invariants for the quotient module may be defined via the map

$$X(w) \stackrel{\rm def}{=} X \otimes_{\mathcal{A}(\Omega)} I : \mathcal{M}_0 \otimes_{\mathcal{A}(\Omega)} \mathbb{C}_w \to \mathcal{M} \otimes_{\mathcal{A}(\Omega)} \mathbb{C}_w,$$

where $X : \mathcal{M}_0 \to \mathcal{M}$ is the inclusion. One of the key results in that paper was that if $\mathcal{K}(K)$ and $\mathcal{K}(K_0)$ represent the curvatures determined by the metrics K and K_0

respectively, then the alternating sum

$$\sum_{i,j=1}^{m} \tfrac{\partial^2}{\partial w_i \partial \bar{w}_j}(X(w)^* X(w)) dw_i \wedge d\bar{w}_j - \mathcal{K}(K_0) + \mathcal{K}(K)$$

represents the fundamental class $[\mathcal{Z}]$. Even in the case $k > 1$, if we localize using the one dimensional module $\mathbb{C}_w$, then it is easy to see, using the factorization (2.6), that the alternating sum defined above represents $k[\mathcal{Z}]$.

It was hoped that we will be able to find an analogue of the alternating sum discussed above corresponding to each of the localizations using higher dimensional modules $\mathbb{C}_w^j$, $1 \leq j \leq k$. Some results of Donaldson (cf. [11], page 24, eqn.(1.9.1)) helped in the identification of such an alternating sum. We suspect that it is enough to consider the case $j = k$. This remark is justified, to some extent, by the theorem at the end of this paper.

For any module map $X : (J\mathcal{M})_0 \to J\mathcal{M}$, define

$$\tilde{X}(w) \stackrel{\text{def}}{=} X \otimes_{\mathcal{A}(\Omega)} I : (J\mathcal{M})_0 \otimes_{\mathcal{A}(\Omega)} \mathbb{C}_w^k \to J\mathcal{M} \otimes_{\mathcal{A}(\Omega)} \mathbb{C}_w^k,$$

where $\tilde{X}(w)$ is obtained by first restricting the map $X \otimes I$ to $(J\mathcal{M})_0 \otimes_{\mathcal{A}(\Omega)} \mathbb{C}_w^k$ and then compressing to $J\mathcal{M} \otimes_{\mathcal{A}(\Omega)} \mathbb{C}_w^k$. However, we can restrict $\tilde{X}(w)$ further to $P((J\mathcal{M})_0 \otimes_{\mathcal{A}(\Omega)} \mathbb{C}_w^k)$ and then compress to $P(J\mathcal{M} \otimes_{\mathcal{A}(\Omega)} \mathbb{C}_w^k)$. Let $X(w) = P\tilde{X}(w)R$, where R denotes the restriction. Assume that X is the inclusion map. Then $X(w)^*$ is represented by the identity matrix with respect to the basis $\{J_0 K(\cdot, w)\varepsilon_i : 1 \leq i \leq k\}$ in $(J\mathcal{M})_0 \otimes_{\mathcal{A}(\Omega)} \mathbb{C}_w^k$ and the corresponding basis $\{JK(\cdot, w)\varepsilon_i : 1 \leq i \leq k\}$ in $J\mathcal{M} \otimes_{\mathcal{A}(\Omega)} \mathbb{C}_w^k$. Let V and W be finite dimensional vector spaces with inner products P and Q respectively, that is, $\big\langle u, v \big\rangle_V = \bar{v}^{\text{tr}} P u$ and $\big\langle w, z \big\rangle_W = \bar{z}^{\text{tr}} Q w$. If $T : (V, P) \to (W, Q)$ is a linear map with matrix representation τ then $T^* = P^{-1} \bar{\tau}^{\text{tr}} Q$. From this remark, it follows that the $X(w)$ is represented by the matrix $JK(w,w)^{-1} J_0 K(w,w)$. Therefore, $X(w)^* X(w) = JK(w,w)^{-1} J_0 K(w,w)$.

Let $\mathcal{E}$ be a holomorphic bundle with a hermitian metric H. Let $\theta(H)$ denote the unique connection compatible with the metric H and let ∂_H denote the covariant differentiation with respect to the metric connection $\theta(H)$. In matrix notation, $\partial_H(X) = \partial X + X\theta(H) - \theta(H)X$. Let H_0 be another metric on $\mathcal{E}$ and set $h = H^{-1} H_0$. We note that

$$\begin{aligned}
\theta(H_0) - \theta(H) &= H_0^{-1} \partial H_0 - H^{-1} \partial H \\
&= h^{-1} H^{-1} \partial (Hh) - H^{-1} \partial H \\
&= h^{-1} H^{-1} \Big((\partial H) h + H \partial h \Big) - H^{-1} \partial H \\
&= h^{-1} H^{-1} \partial H h + h^{-1} \partial h - H^{-1} \partial H \\
&= h^{-1} \Big\{ H^{-1} \partial H h + \partial h - h H^{-1} \partial H \Big\} \\
&= h^{-1} \partial_H h.
\end{aligned}$$

It follows that

$$-\bar{\partial}\Big(h^{-1} \partial_H h\Big) = \mathcal{K}(H_0) - \mathcal{K}(H),$$

where $\mathcal{K}(H) = -\partial(H^{-1}\bar{\partial}H)$ (resp. $\mathcal{K}(H_0)$) denotes the curvature of the bundle $\mathcal{E}$ with respect to the metric H (resp. H_0).

THEOREM 4.4 *The alternating sum*

$$-\bar{\partial}\Big((X^*X)^{-1}\partial_{JK}(X^*X)\Big) + \mathcal{K}(JK) - \mathcal{K}(J_0K)$$

is zero when evaluated on any open set $U \subseteq \Omega$ which does not intersect $\mathcal{Z}$.

Proof: Recall that the adjoint of the map $X \otimes_{\mathcal{A}(\Omega)} I : (J\mathcal{M})_0 \otimes_{\mathcal{A}(\Omega)} \mathbb{C}_w^k \to J\mathcal{M} \otimes_{\mathcal{A}(\Omega)} \mathbb{C}_w^k$ identifies (as holomorphic bundles) the two bundles $P(J_{\rm loc}\mathcal{E}) = J\mathcal{E}$ and $P(J_{\rm loc}\mathcal{E})_0 = (J\mathcal{E})_0$ obtained from localizing $(J\mathcal{M})_0$. Let $\mathbb{E}$ denote either of these bundles. Since the identification via $X^* \otimes_{\mathcal{A}(\Omega)} I$ is not isometric, it is clear that $\mathbb{E}$ has two natural metrics on it. One of these is the metric JK and the other is J_0K. The proof is completed by setting $H_0 = J_0K$ and $H = JK$. In this case, $h(w) = X(w)^*X(w)$, where, as before, $X(w) = P(X \otimes_{\mathcal{A}(\Omega)} I)R$. □

Unfortunately, we have not been able to evaluate the alternating sum in the theorem above, as a $(1,1)$ form with distributional coefficients on all of Ω. Nevertheless, we are able to evaluate a certain alternating sum obtained naturally by considering the determinant bundles. Recall that to any rank k bundle $\mathcal{E}$, we may associate a line bundle $\det\mathcal{E}$, called the determinant bundle. If $g_{\rm UV}$ denotes the transition functions for $\mathcal{E}$, then the transition functions $\det g_{\rm UV}$ determine $\det\mathcal{E}$. If G denotes the metric on $\mathcal{E}$, then the metric on the determinant bundle $\det\mathcal{E}$ is $\det G$.

Consider the bundle $\det\mathbb{E}$ with the two metrics $\det JK$ and $\det J_0K$. Clearly, the metric $\det J_0K$ vanishes on $\mathcal{Z}$. However, the curvature of $\det\mathbb{E}$ with respect to $\det J_0K$ can be calculated on any open subset of Ω which does not intersect $\mathcal{Z}$. Since the coefficient of the curvature form is a real analytic function on Ω, it is enough to calculate it on any open set. The factorization (2.6) implies that the curvature $\mathcal{K}(\det J_0K) = \sum_{i,j=1}^m \frac{\partial^2}{\partial w_i \partial \bar{w}_j} \log\det(\mathcal{J}\chi) dw_i \wedge d\bar{w}_j$. Also, $\det(X^*X) = (\det JK)^{-1}|\varphi|^{2k}\det(\mathcal{J}\chi)$. It follows that the alternating sum in the following theorem is nothing but k times the current

$$\sum_{i,j=1}^m \frac{\partial^2}{\partial w_i \partial \bar{w}_j} \log|\varphi|^2 dw_i \wedge d\bar{w}_j.$$

Since the Poincare-Lelong formula [8, p. 388] asserts that the current displayed above represents the fundamental class $[\mathcal{Z}]$ of the hypersurface $\mathcal{Z}$, the proof of the theorem below is complete.

THEOREM 4.5 *The alternating sum*

$$-\bar{\partial}\partial\log(\det X^*X) + \mathcal{K}(\det JK) - \mathcal{K}(\det J_0K)$$

represents $k[\mathcal{Z}]$.

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