CURVATURE INEQUALITIES FOR OPERATORS IN THE COWEN-DOUGLAS CLASS AND LOCALIZATION OF THE WALLACH SET

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ABSTRACT. For any bounded domain Ω in $\mathbb{C}^m$, let $\mathrm{B}_1(\Omega)$ denote the Cowen-Douglas class of commuting m-tuples of bounded linear operators. For an m-tuple $\boldsymbol{T}$ in the Cowen-Douglas class $\mathrm{B}_1(\Omega)$, let $N_{\boldsymbol{T}}(w)$ denote the restriction of $\boldsymbol{T}$ to the subspace $\cap_{i,j=1}^m \ker(T_i - w_iI)(T_j - w_jI)$. This commuting m-tuple $N_{\boldsymbol{T}}(w)$ of $m+1$ dimensional operators induces a homomorphism $\rho_{N_{\boldsymbol{T}}(w)}$ of the polynomial ring $P[z_1, \cdots, z_m]$, namely, $\rho_{N_{\boldsymbol{T}}(w)}(p) = p\big(N_{\boldsymbol{T}}(w)\big)$, $p \in P[z_1, \cdots, z_m]$. We study the contractivity and complete contractivity of the homomorphism $\rho_{N_{\boldsymbol{T}}(w)}$. Starting from the homomorphism $\rho_{N_{\boldsymbol{T}}(w)}$, we construct a natural class of homomorphism $\rho_{N^{(\lambda)}(w)}$, $\lambda > 0$, and relate the properties of $\rho_{N^{(\lambda)}(w)}$ to that of $\rho_{N_{\boldsymbol{T}}(w)}$. Explicit examples arising from the multiplication operators on the Bergman space of Ω are investigated in detail. Finally, it is shown that contractive properties of $\rho_{N_{\boldsymbol{T}}(w)}$ is equivalent to an inequality for the curvature of the Cowen-Douglas bundle $E_{\boldsymbol{T}}$.

1. Introduction

We recall the definition of the well known class of operators $\mathrm{B}_1(\Omega)$ which was introduced in the foundational paper of Cowen and Douglas (cf. [6]). An alternative point of view was discussed in the paper of Curto and Salinas (cf. [8]).

Definition 1.1. The class $\mathrm{B}_n(\Omega)$ consists of m-tuples of commuting bounded operators $\boldsymbol{T} = (T_1, T_2, \ldots, T_m)$ on a Hilbert space $\mathcal{H}$ satisfying the following conditions:

- for $w = (w_1, \ldots, w_m) \in \Omega$, the dimension of the joint kernel $\bigcap_{k=1}^m \ker(T_k - w_kI)$ is n,
- for $w \in \Omega$ and $h \in \mathcal{H}$, the operator $D_{\boldsymbol{T}-wI} : \mathcal{H} \to \mathcal{H} \oplus \ldots \oplus \mathcal{H}$, defined by the rule:

$$D_{\boldsymbol{T}-wI}h = \big((T_1 - w_1I)h, \ldots, (T_m - w_mI)h\big)$$

  has closed range,
- the closed linear span of $\{\bigcap_{k=1}^m \ker(T_k - w_kI) : w \in \Omega\}$ is $\mathcal{H}$.

The commuting m-tuple $N_{\boldsymbol{T}}(w)$ is obtained by restricting T_i to the $(m+1)$ dimensional subspace $\mathcal{N}(w) := \cap_{i,j=1}^m \ker(T_i - w_iI)(T_j - w_jI)$. These commuting m-tuples of finite dimensional operators are included, for instance, in the (generalized) class of examples due to Parrott (cf. [2, 18]). The study of the contractivity and complete contractivity of the homomorphisms induced by these localization operators leads to interesting problems in geometry of finite dimensional Banach spaces (cf. [2, 22]).

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For operators in the the Cowen-Douglas class, there exists a holomorphic choice $\gamma(w)$ of eigenvector at w. The $(1,1)$ - form

$$\sum_{i=1}^{m} \frac{\partial^2}{\partial z_i \bar{\partial} z_j} \log \|\gamma(w)\|^2 dz_i \wedge d\bar{z}_j$$

is then seen to be a complete unitary invariant for these operators. Set

$$\mathcal{K}_{\boldsymbol{T}}(z) := \Big(\!\!\Big(\frac{\partial^2}{\partial z_i \bar{\partial} z_j} \log \|\gamma(w)\|^2 \Big)\!\!\Big).$$

The subspace $\mathcal{N}(w)$ is easily seen to be spanned by the vectors $\{\gamma(w), (\partial_1\gamma)(w), \ldots, (\partial_m\gamma)(w)\}$. The localization $N_{\boldsymbol{T}}(w)$ represented with respect to the orthonormal basis obtained via the Gram-Schmidt process from these vectors, takes the form:

$$\begin{pmatrix} w_1 & \boldsymbol{\alpha}_1(w) \\ 0 & w_1 I_m \end{pmatrix}, \ldots, \begin{pmatrix} w_m & \boldsymbol{\alpha}_m(w) \\ 0 & w_m I_m \end{pmatrix}.$$

Cowen and Douglas, for $m = 2$, had shown that the curvature $\mathcal{K}_{\boldsymbol{T}}(w)$ appears directly in the localization of the operator $\boldsymbol{T}$. Indeed, for all $m \geq 1$, we show that the matrix of inner products $\big(\!\!\big(\langle \boldsymbol{\alpha}_i(w), \boldsymbol{\alpha}_j(w)\rangle\big)\!\!\big)$ equals $\mathcal{K}_{\boldsymbol{T}}(w)^{-1}$.

As is well-known, without loss of generality, a Cowen-Douglas operator $\boldsymbol{T}$ may be thought of as the adjoint of the commuting tuple of multiplication operators by the coordinate functions on a functional Hilbert space. The holomorphic section γ, in this representation, becomes anti-holomorphic, namely $K_w(\cdot)$, where K is the reproducing kernel. One might ask, what properties of the multiplication operators (or the adjoint) are determined by the local operators. Let M be the multiplication (by the coordinate function) operator on a Hilbert space consisting of holomorphic functions on the unit disc $\mathbb{D}$ and possessing a reproducing kernel K. For instance, the contractivity of the homomoprhism, induced by the operator M, of the disc algebra (this is the same as requiring $\|M\| \leq 1$, thanks to the von Neumann inequality) ensures contractivity of the homomorphisms $\rho_{N_{M^*(w)}}$ induced by the local operators $N_{M^*}(w)$, $w \in \mathbb{D}$, which is itself equivalent to an inequality for the curvature $\mathcal{K}_{M^*}$. An example is given in [3] showing that the contractivity of the homomorphisms $\rho_{N_{M^*(w)}}$ need not imply $\|M\| \leq 1$ and that the converse is valid only after imposing some additional conditions. In this paper, for an arbitrary bounded symmetric domain, we construct such examples by exploiting properties typical of the Bergman kernel on these domains.

In another direction, for any positive definite kernel K and a positive real number λ, the function K^λ obtained by polarizing the real analytic function $K(w,w)^\lambda$, defines a Hermitian form, which is not necessarily positive definite. The determination of the Wallach set

$$\mathcal{W}_\Omega := \{\lambda : K^\lambda(z,w) \text{ is positive definite }\}$$

for this kernel is an important problem. The Wallach set was first defined for the Bergman kernel of a bounded symmetric domain. Except in that case (cf. [9]), very little is known about the Wallach set in general. Here, for all $\lambda > 0$, we show that $K^\lambda(z,w)$ is positive definite when restricted to the subspaces $\mathcal{N}^{(\lambda)}(w)$, $w \in \Omega$. This means that for an arbitrary choice of complex numbers $\alpha_0, \ldots, \alpha_m$,

$$\sum_{i,j=0}^{m} \alpha_i \bar{\alpha}_j \big(\partial_i \bar{\partial}_j K^\lambda\big)(w,w) > 0$$

for any $w \in \Omega$ and all $\lambda > 0$. Here ∂_0 is set to be the scalar 1.

For an m-tuple of operator $\boldsymbol{T}$ in $\mathrm{B}_n(\Omega)$, Cowen and Douglas establish the existence of a non-zero holomorphic map $\gamma : \Omega_0 \to \mathcal{H}$ with $\gamma(w)$ in $\bigcap_{k=1}^m \ker(T_k - w_k I)$, w in some open subset Ω_0 of Ω. We fix such an open set and call it Ω. The map γ defines a holomorphic Hermitian vector bundle, say $E_{\boldsymbol{T}}$, on Ω. They show that the equivalence class of the vector bundle $E_{\boldsymbol{T}}$ determines the equivalence class (with respect to unitary equivalence) of the operator $\boldsymbol{T}$ and conversely. The determination of the equivalence class of the operator $\boldsymbol{T}$ in $\mathrm{B}_1(\Omega)$ then is particularly simple since the curvature of the line bundle $E_{\boldsymbol{T}}$

$$-\mathsf{K}(w) = \sum_{i,j=1}^{m} \frac{\partial^2}{\partial w_i \partial \bar{w}_j} \log \|\gamma(w)\|^2 dw_i \wedge d\bar{w}_j$$

is a complete invariant. We reproduce the well-known proof of this fact for the sake of completeness.

Suppose that E is a holomorphic Hermitian line bundle over a bounded domain $\Omega \subseteq \mathbb{C}^m$. Pick a holomorphic frame γ for the line bundle E and let $\Gamma(w) = \langle \gamma_w, \gamma_w \rangle$ be the Hermitian metric. The curvature $(1,1)$ form $\mathsf{K}(w) \equiv 0$ on an open subset $\Omega_0 \subseteq \Omega$ if and only if $\log \Gamma$ is harmonic on Ω_0. Let F be a second line bundle over the same domain Ω with the metric Λ with respect to a holomorphic frame η. Suppose that the two curvatures K_E and K_F are equal on the open subset Ω_0. It then follows that $u = \log(\Gamma/\Lambda)$ is harmonic on this open subset. Thus there exists a harmonic conjugate v of u on Ω_0, which we assume without loss of generality to be simply connected. For $w \in \Omega_0$, define $\tilde{\eta}_w = e^{(u(w)+iv(w))/2}\eta_w$. Then clearly, $\tilde{\eta}_w$ is a new holomorphic frame for F. Consequently, we have the metric $\Lambda(w) = \langle \tilde{\eta}_w, \tilde{\eta}_w \rangle$ for F and we see that

$$\begin{aligned}
\Lambda(w) &= \langle \tilde{\eta}_w, \tilde{\eta}_w \rangle \\
&= \langle e^{(u(w)+iv(w))/2}\eta_w, e^{(u(w)+iv(w))/2}\eta_w \rangle \\
&= e^{u(w)} \langle \eta_w, \eta_w \rangle \\
&= \Gamma(w).
\end{aligned}$$

This calculation shows that the map $U : \eta_w \mapsto \gamma_w$ defines an isometric holomorphic bundle map between E and F. The map, as shown in [7, Theorem 1],

$$U\Big(\sum_{|I|\le n} \alpha_I (\bar{\partial}^I \eta)(w_0)\Big) = \sum_{|I|\le n} \alpha_I (\bar{\partial}^I \gamma)(w_0),\ \alpha_I \in \mathbb{C}, \tag{1.1}$$

where w_0 is a fixed point in Ω and I is a multi-index of length n, is well-defined, extends to a unitary operator on the Hilbert space spanned by the vectors $(\bar{\partial}^I \eta)(w_0)$ and intertwines the two m-tuples of operators in $\mathrm{B}_1(\Omega)$ corresponding to the vector bundles E and F.

It is natural to ask what other properties of $\boldsymbol{T}$ are directly reflected in the curvature K. One such property that we explore here is the contractivity and complete contractivity of the homomorphism induced by the m-tuple $\boldsymbol{T}$ via the map $\rho_{\boldsymbol{T}} : f \to f(\boldsymbol{T})$, $f \in \mathcal{O}(\Omega)$, where $\mathcal{O}(\Omega)$ is the set of all holomorphic function in the neighborhood of $\overline{\Omega}$.

It will be useful for us to work with the matrix of the co-efficient of the $(1,1)$ - form defining the curvature K, namely,

$$\mathcal{K}_{\boldsymbol{T}}(w) := -\Big(\Big(\frac{\partial^2}{\partial w_i \partial \bar{w}_j} \log \|\gamma(w)\|^2\Big)\Big)_{i,j=1}^{m}.$$

We recall the curvature inequality from Misra and Sastry cf. [19, Theorem 5.2] and produce a large family of examples to show that the "curvature inequality" does not imply contractivity of the homomorphism $\rho_{\boldsymbol{T}}$.

2. Localization of Cowen-Douglas operators

Fix an operator $\boldsymbol{T}$ in $\mathrm{B}_1(\Omega)$ and let $N_{\boldsymbol{T}}(w)$ be the m-tuple of operators $(N_1(w), \ldots, N_m(w))$, where $N_i(w) = T_i - w_i I|_{\mathcal{N}(w)}$, $i = 1, \ldots, m$. Clearly, $N_i(w)N_j(w) = 0$ for all $1 \le i,j \le m$. Hence the commuting m-tuple $N(w)$ is jointly nilpotent. This m-tuple of nilpotent operators $N_{\boldsymbol{T}}(w)$, has the matrix representation (recall $(T_i - w_i I)\gamma(w) = 0$ and $(T_i - w_i I)(\partial_j \gamma)(w) = \delta_{ij}\gamma(w)$ for $1 \le i,j \le m$) $N_k(w) = \left(\begin{smallmatrix} 0 & e_k \\ 0 & \boldsymbol{0} \end{smallmatrix}\right)$, $k = 1, \ldots, m$. Here $\{e_k\}_{k=1}^m$ is the standard basis of $\mathbb{C}^m$. Representing $N_k(w)$ with respect to an orthonormal basis in $\mathcal{N}(w)$, it is possible to read off the curvature of $\boldsymbol{T}$ at w using the relationship:

$$-\big(\mathcal{K}_{\boldsymbol{T}}(w)^{\mathrm{t}}\big)^{-1} = \big(\!\!\big(\operatorname{tr}\big(N_k(w)\overline{N_j(w)}^{\mathrm{t}}\big)\big)\!\!\big)_{kj=1}^{m} = A(w)^{\mathrm{t}}\overline{A(w)}, \tag{2.1}$$

where the k^{th}-column of $A(w)$ is the vector $\boldsymbol{\alpha}_k$ (depending on w) which appears in the matrix representation of $N_k(w)$ with respect to any choice of an orthonormal basis in $\mathcal{N}(w)$.

This formula is established for a pair of operators in $\mathrm{B}_1(\Omega)$ (cf. [7, Theorem 7]). However, we will verify it for an m-tuple $\boldsymbol{T}$ in $\mathrm{B}_1(\Omega)$ for any $m \geq 1$.

Fix w_0 in Ω. We may assume without loss of generality that $\|\gamma(w_0)\| = 1$. The function $\langle \gamma(w), \gamma(w_0)\rangle$ is invertible in some neighborhood of w_0. Then setting $\hat{\gamma}(w) := \langle \gamma(w), \gamma(w_0)\rangle^{-1}\gamma(w)$, we see that

$$\langle \partial_k \hat{\gamma}(w_0), \gamma(w_0)\rangle = 0,\ k = 1, 2, \ldots, m.$$

Thus $\hat{\gamma}$ is another holomorphic section of E. The norms of the two sections γ and $\hat{\gamma}$ differ by the absolute square of a holomorphic function, that is $\frac{\|\hat{\gamma}(w)\|}{\|\gamma(w)\|} = |\langle \gamma(w), \gamma(w_0)\rangle|$. Hence the curvature is independent of the choice of the holomorphic section. Therefore, without loss of generality, we will prove the claim assuming: for a fixed but arbitrary w_0 in Ω,

(i) $\|\gamma(w_0)\| = 1$,
(ii) $\gamma(w_0)$ is orthogonal to $(\partial_k \gamma)(w_0)$, $k = 1, 2, \ldots, m$.

Let G be the Grammian corresponding to the $m+1$ dimensional space spanned by the vectors

$$\{\gamma(w_0), (\partial_1\gamma)(w_0), \ldots, (\partial_m\gamma)(w_0)\}.$$

This is just the space $\mathcal{N}(w_0)$. Let v, w be any two vectors in $\mathcal{N}(w_0)$. Find $\boldsymbol{c} = (c_0, \ldots, c_m), \boldsymbol{d} = (d_0, \ldots, d_m)$ in $\mathbb{C}^{m+1}$ such that $v = \sum_{i=0}^m c_i \partial_i \gamma(w_0)$ and $w = \sum_{j=0}^m d_j \partial_j \gamma(w_0)$. Here $(\partial_0\gamma)(w_0) = \gamma(w_0)$. We have

$$\begin{aligned}\langle v, w\rangle_G &= \langle \sum_{i=0}^m c_i \partial_i \gamma(w_0), \sum_{j=0}^m d_j \partial_j \gamma(w_0)\rangle \\ &= \langle G^{\mathrm{t}}(w_0)\boldsymbol{c}, \boldsymbol{d}\rangle_{\mathbb{C}^{m+1}} \\ &= \langle (G^{\mathrm{t}})^{\frac{1}{2}}(w_0)\boldsymbol{c}, (G^{\mathrm{t}})^{\frac{1}{2}}(w_0)\boldsymbol{d}\rangle_{\mathbb{C}^{m+1}}.\end{aligned}$$

Let $\{e_i\}_{i=0}^m$ be the standard orthonormal basis for $\mathbb{C}^{m+1}$. Also, let $(G^{\mathrm{t}})^{-\frac{1}{2}}(w_0)e_i := \boldsymbol{\alpha}_i(w_0)$, where $\boldsymbol{\alpha}_i(j)(w_0) = \alpha_{ji}(w_0)$, $i = 0, 1, \ldots, m$. We see that the vectors $\varepsilon_i := \sum_{j=0}^m \alpha_{ji}(\partial_j\gamma)(w_0)$, $i = 0, 1, \ldots, m$, form an orthonormal basis in $\mathcal{N}(w_0)$:

$$\begin{aligned}\langle \varepsilon_i, \varepsilon_l\rangle &= \Big\langle \sum_{j=0}^m \alpha_{ij}\partial_j\gamma(w_0), \sum_{p=0}^m \alpha_{lp}\partial_p\gamma(w_0)\Big\rangle \\ &= \langle (G^{\mathrm{t}})^{-\frac{1}{2}}\boldsymbol{\alpha}_i, (G^{\mathrm{t}})^{-\frac{1}{2}}(w_0)\boldsymbol{\alpha}_l\rangle_G(w_0) \\ &= \delta_{il},\end{aligned}$$

where δ_{il} is the Kornecker delta. Since $N_k\big((\partial_j\gamma)(w_0)\big) = \gamma(w_0)$ for $j = k$ and 0 otherwise, we have $N_k(\varepsilon_i) = \begin{pmatrix} 0 & \boldsymbol{\alpha}_k^{\mathrm{t}} \\ 0 & 0\end{pmatrix}$. Hence

$$\begin{aligned}\operatorname{tr}\big(N_i(w_0)N_j^*(w_0)\big) &= \boldsymbol{\alpha_i}(w_0)^{\mathrm{t}}\overline{\boldsymbol{\alpha}}_j(w_0) \\ &= \big((G^{\mathrm{t}})^{-\frac{1}{2}}(w_0)e_i\big)^{\mathrm{t}}\overline{\big((G^{\mathrm{t}})^{-\frac{1}{2}}(w_0)e_j\big)} \\ &= \langle G^{-\frac{1}{2}}(w_0)e_i, G^{-\frac{1}{2}}e_j(w_0)\rangle = (G^{\mathrm{t}})^{-1}(w_0)_{ij}.\end{aligned}$$

Since the curvature, computed with respect to the holomorphic section γ satisfying the conditions *(i)* and *(ii)*, is of the form

$$\begin{aligned}\mathcal{K}_{\boldsymbol{T}}(w_0)_{ij} &= \frac{\partial^2}{\partial w_i \partial \bar{w}_j} \log \|\gamma(w)\|^2_{|w=w_0} \\ &= \Big(\frac{\|\gamma(w)\|^2\big(\frac{\partial^2 \gamma}{\partial w_i \partial \bar{w}_j}\big)(w) - \big(\frac{\partial \gamma}{\partial w_i}\big)(w)\big(\frac{\partial \gamma}{\partial \bar{w}_j}\big)(w)}{\|\gamma(w)\|^4}\Big)_{|w=w_0} \\ &= \big(\frac{\partial^2 \gamma}{\partial w_i \partial \bar{w}_j}\big)(w_0) = G(w_0)_{ij},\end{aligned}$$

we have verified the claim (2.1).

The following theorem was proved for $m = 2$ in (cf. [7, Theorem 7]). However, for any natural number m, the proof is evident from the preceding discussion.

Theorem 2.1. *Two m-tuples of operators $\boldsymbol{T}$ and $\tilde{\boldsymbol{T}}$ in $\mathrm{B}_1(\Omega)$ are unitarily equivalent if and only if $N_k(w)$ and $\tilde{N}_k(w)$ are simultaneously unitarily equivalent for w in some open subset of Ω.*

Proof. Let us fix an arbitrary point w in Ω. In what follows, the dependence on this w is implicit. Suppose that there exists a unitary operator $U : \mathcal{N} \to \widetilde{\mathcal{N}}$ such that $UN_i = \tilde{N}_i U$, $i = 1, \ldots, m$. For $1 \leq i, j \leq m$, we have

$$\begin{aligned}\operatorname{tr}\big(\tilde{N}_i \tilde{N}_j{}^*\big) &= \operatorname{tr}\big(\big(UN_iU^*\big)\big(UN_jU^*\big)^*\big) \\ &= \operatorname{tr}\big(UN_iN_j^*U^*\big) \\ &= \operatorname{tr}\big(N_iN_j^*U^*U\big) \\ &= \operatorname{tr}\big(N_iN_j^*\big).\end{aligned}$$

Thus the curvature of the operators $\boldsymbol{T}$ and $\tilde{\boldsymbol{T}}$ coincide making them unitarily equivalent proving the Theorem in one direction. In the other direction, observe that if the operators $\boldsymbol{T}$ and $\tilde{\boldsymbol{T}}$ are unitarily equivalent then the unitary U given in (1.1) evidently maps $\mathcal{N}$ to $\tilde{\mathcal{N}}$. Thus the restriction of U to the subspace $\mathcal{N}$ intertwines N_k and $\tilde{N}_k$ simultaneously for $k = 1, \cdots, m$. □

As is well-known (cf. [8] and [6]), the m-tuple $\boldsymbol{T}$ in $\mathrm{B}_1(\Omega)$ can be represented as the adjoint of the m-tuple of multiplications M by the co-ordinate functions on a Hilbert space $\mathcal{H}$ of holomorphic functions defined on $\Omega^* = \{\bar{w} \in \mathbb{C}^m : w \in \Omega\}$ possessing a reproducing kernel $K : \Omega^* \times \Omega^* \to \mathbb{C}$ which is holomorphic in the first variable and anti-holomorphic in the second.

In this representation, if we set $\gamma(w) = K(\cdot, \bar{w})$, $w \in \Omega$, then we obtain a natural non-vanishing "holomorphic" map into the Hilbert space $\mathcal{H}$ defined on Ω.

The localization $N_{\boldsymbol{T}}(w)$ obtained from the commuting tuple of operators $\boldsymbol{T}$ defines a a homomorphism $\rho_{N_{\boldsymbol{T}(w)}}$ on the algebra $\mathcal{O}(\Omega)$ of functions, holomorphic in some neighborhood of of the closed set $\bar{\Omega}$, by the rule

$$\rho_{N_{\boldsymbol{T}(w)}}(f) = \begin{pmatrix} f(w) & \nabla f(w) A(w)^{\mathrm{t}} \\ 0 & f(w) I_m \end{pmatrix}, \; f \in \mathcal{O}(\Omega). \tag{2.2}$$

We recall from (cf. [19, Theorem 5.2]) that the contractivity of the homomorphism implies the curvature inequality $\|\big(\mathcal{K}_{\boldsymbol{T}}(w)^{\mathrm{t}}\big)^{-1}\| \leq 1$. Here $\mathcal{K}_{\boldsymbol{T}}(w)$ is thought of as a linear transformation from the normed linear space $(\mathbb{C}^m, C_{\Omega,w})^*$ to the normed linear space $(\mathbb{C}^m, C_{\Omega,w})$, where $C_{\Omega,w}$ is the Carathéodory metric of Ω at w. The operator norm is computed accordingly with respect to these norms.

2.1. **Infinite divisibility.** Let K be a positive definite kernel defined on the domain Ω and let $\lambda > 0$ be arbitrary. Since K^λ is a real analytic function defined on Ω, it admits a power series representation of the form

$$K^\lambda(w,w) = \sum_{I\,J} a_{I\,J}(\lambda)(w-w_0)^I\overline{(w-w_0)}^J$$

in a small neighborhood of a fixed but arbitrary $w_0 \in \Omega$. The polarization $K^\lambda(z,w)$ is the function represented by the power series

$$K^\lambda(z,w) = \sum_{I\,J} a_{I\,J}(\lambda)(z-w_0)^I\overline{(w-w_0)}^J,\ w_0 \in \Omega.$$

It follows that the polarization $K^\lambda(z,w)$ of the function $K(w,w)^\lambda$ defines a Hermitian kernel, that is, $K^\lambda(z,w) = \overline{K(w,z)^\lambda}$. Schur's Lemma (cf. [5]) ensures the positive definiteness of K^λ whenever λ is a natural number. However, it is not necessary that K^λ must be positive definite for all real $\lambda > 0$. Indeed a positive definite kernel K with the property that K^λ is positive definite for all $\lambda > 0$ is called infinitely divisible and plays an important role in studying curvature inequalities (cf. [3, Theorem 3.3]).

Although, K^λ need not be positive definite for all $\lambda > 0$, in general, a related question raised here is relevant to the study of localization of the Cowen-Douglas operators.

Let w_0 in Ω be fixed but arbitrary. Also, fix a $\lambda > 0$. Define the mutual inner product of the vectors

$$\{(\bar{\boldsymbol\partial}^I K^\lambda)(\cdot, w_0) : I = (i_1, \ldots, i_m)\},$$

by the formula

$$\langle (\bar{\boldsymbol\partial}^J K^\lambda)(\cdot, w_0), (\bar{\boldsymbol\partial}^I K^\lambda)(\cdot, w_0)\rangle = \big(\boldsymbol\partial^I \bar{\boldsymbol\partial}^J K^\lambda\big)(w_0, w_0).$$

Now, if K^λ were positive definite, for the λ we have picked, then this formula would extend to an inner product on the linear span of these vectors. The completion of this inner product space is then a Hilbert space, which we denote by $\mathcal{H}^{(\lambda)}$. The reproducing kernel for the Hilbert space $\mathcal{H}^{(\lambda)}$ is easily verified to be the original kernel K^λ. The Hilbert space $\mathcal{H}^{(\lambda)}$ is independent of the choice of w_0.

Now, even if K^λ is not necessarily positive definite, we may ask whether this formula defines an inner product on the $(m+1)$ dimensional space $\mathcal{N}^{(\lambda)}(w)$ spanned by the vectors

$$\{K^\lambda(\cdot, w), (\bar\partial_1 K^\lambda)(\cdot, w), \ldots, (\bar\partial_m K^\lambda)(\cdot, w)\}.$$

An affirmative answer to this question is equivalent to the positive definiteness of the matrix

$$\big(\!\!\big(\, (\partial_i \bar\partial_j K^\lambda)(w,w) \big)\!\!\big)_{i\,j=0}^m.$$

Let $\bar{\boldsymbol\partial}_m^{\rm t} = \big(1, \partial_1, \ldots, \partial_m\big)$ and $\boldsymbol\partial_m$ be its conjugate transpose. Now,

$$\big(\boldsymbol\partial_m \bar{\boldsymbol\partial}_m^{\rm t} K^\lambda\big)(w,w) := \big(\!\!\big(\, (\partial_j \bar\partial_i K^\lambda)(w,w) \big)\!\!\big)_{i\,j=0}^m,\ w \in \Omega \subseteq \mathbb{C}^m.$$

Theorem 2.2. *For a fixed but arbitrary w in Ω, the $(m+1) \times (m+1)$ matrix $\big(\boldsymbol\partial_m \bar{\boldsymbol\partial}_m^{\rm t} K^\lambda\big)(w,w)$ is positive definite.*

Proof. The proof is by induction on m. For $m = 1$ and any positive λ, a direct verification, which follows, shows that

$$\big(\boldsymbol\partial_1 \bar{\boldsymbol\partial}_1^{\rm t} K^\lambda\big)(w,w) := \begin{pmatrix} K^\lambda(w,w) & \partial_1 K^\lambda(w,w) \\ \bar\partial_1 K^\lambda(w,w) & \partial_1\bar\partial_1 K^\lambda(w,w)\end{pmatrix}$$

is positive.

Since $K^\lambda(w,w) > 0$ for any $\lambda > 0$, the verification that $\big(\boldsymbol{\partial}_1\bar{\boldsymbol{\partial}}_1^{\rm t}K^\lambda\big)(w,w)$ is positive definite amounts to showing that $\det\big(\boldsymbol{\partial}_1\bar{\boldsymbol{\partial}}_1^{\rm t}K^\lambda\big)(w,w) > 0$. An easy computation gives

$$\begin{aligned}\det\big(\boldsymbol{\partial}_1\bar{\boldsymbol{\partial}}_1^{\rm t}K^\lambda\big)(w,w) &= \lambda K^{2\lambda-2}(w,w)\big\{K(w,w)(\bar\partial_1\partial_1 K)(w,w) - |\partial_1 K(w,w)|^2\big\}\\ &= \lambda K^{2\lambda}(w,w)\frac{\|K(\cdot,w)\|^2\|(\bar\partial_1 K)(\cdot,w)\|^2 - |\langle K(\cdot,w),(\bar\partial_1 K)(\cdot,w)\,\rangle|^2}{\|K(\cdot,w)\|^4},\end{aligned}$$

which is clearly positive since $K(\cdot,w)$ and $(\bar\partial_1 K)(\cdot,w)$ are linearly independent.

Now assume that $\big(\boldsymbol{\partial}_{m-1}\bar{\boldsymbol{\partial}}_{m-1}^{\rm t}K^\lambda\big)(w,w)$ is positive definite. We note that

$$\big(\boldsymbol{\partial}_m\bar{\boldsymbol{\partial}}_m^{\rm t}K^\lambda\big)(w,w) = \begin{pmatrix}\big(\boldsymbol{\partial}_{m-1}\bar{\boldsymbol{\partial}}_{m-1}^{\rm t}K^\lambda\big)(w,w) & \big(\partial_m\bar{\boldsymbol{\partial}}_{m-1}^{\rm t}K^\lambda\big)(w,w)\\ \big(\boldsymbol{\partial}_{m-1}\bar\partial_m K^\lambda\big)(w,w) & (\partial_m\bar\partial_m K^\lambda)(w,w)\end{pmatrix}.$$

Since $\big(\boldsymbol{\partial}_{m-1}\bar{\boldsymbol{\partial}}_{m-1}^{\rm t}K^\lambda\big)(w,w)$ is positive definite by the induction hypothesis and for $\lambda > 0$, we have

$$(\partial_m\bar\partial_m K^\lambda)(w,w) = \lambda K(w,w)^{\lambda-2}\big\{K(w,w)(\partial_m\bar\partial_m K)w,w) + (\lambda-1)|(\bar\partial_m K)(w,w)|^2\big\} > 0,$$

it follows that $\big(\boldsymbol{\partial}_m\bar{\boldsymbol{\partial}}_m^{\rm t}K^\lambda\big)(w,w)$ is positive definite if and only if $\det\big(\big(\boldsymbol{\partial}_m\bar{\boldsymbol{\partial}}_m^{\rm t}K^\lambda\big)(w,w)\big) > 0$ (cf. [3]). To verify this claim, we note

$$\big(\boldsymbol{\partial}_m\bar{\boldsymbol{\partial}}_m^{\rm t}K^\lambda\big)(w,w) = \left(\begin{smallmatrix}K^\lambda(w,w) & B(w,w)\\ B(w,w)^* & D(w,w)\end{smallmatrix}\right),$$

where $D = \big(\!\!\big(\,(\partial_j\bar\partial_i K^\lambda)(w,w)\big)\!\!\big)_{i\,j=1}^m$ and $B = \big(\partial_1 K^\lambda(w,w),\ldots,\partial_m K^\lambda(w,w)\big)$. Recall that (cf. [12])

$$\det\big(\boldsymbol{\partial}_m\bar{\boldsymbol{\partial}}_m^{\rm t}K^\lambda\big)(w,w) = \det\Big(D(w,w) - \frac{B^*(w,w)B(w,w)}{K^\lambda(w,w)}\Big)\det K^\lambda(w,w).$$

Now, following (cf. [3, proposition 2.1(second proof)]), we see that

$$D(w,w) - \frac{B^*(w,w)B(w,w)}{K^\lambda(w,w)} = \lambda K^{2\lambda-2}(w,w)\big(\!\!\big(\,K^2(w,w)(\partial_j\,\bar\partial_i\log K)(w,w)\big)\!\!\big)_{i\,j=1}^m,$$

which was shown to be a Grammian. Thus $D(w,w) - \frac{B^*(w,w)B(w,w)}{K^\lambda(w,w)}$ is a positive definite matrix and hence its determinant is positive. □

The Theorem we have just proved says that if E is a holomorphic Hermitian vector bundle corresponding to a Cowen-Douglas operator, then the first order jet bundle $\mathcal{J}E$ admits a Hermitian structure (cf. [6, Section 4.7]). It also prompts the following definition, which is a localization of the Wallach set to points in Ω.

Definition 2.3. For $\lambda > 0$, and any positive definite kernel K defined on the domain Ω, let $K^\lambda(z,w)$ denote the function obtained by polarizing the real analytic function $K(w,w)^\lambda$. For any two multi indices α and β, let $\alpha \le \beta$ denote the co-lexicographic ordering. Let

$$\mathcal{W}_\Omega(w) := \max\Big\{n\in\mathbb{N}\,\Big|\Big(\!\!\Big(\frac{(\partial_z^\alpha\partial_{\bar w}^\beta K^\lambda)(z,w)}{\alpha!\beta!}\Big)\!\!\Big)_{0\le\alpha,\beta\le\delta},\ |\delta| = n,\text{ is positive definite for all }\lambda\Big\}.$$

Fix a positive definite kernel K on Ω. Following Curto and Salinas [8, Lemma 4.1 and 4.3], for any fixed but arbitrary w in Ω, the Wallach set for K is $[0,\infty)$ if and only if $\mathcal{W}_\Omega(w)$ is not finite. A kernel possessing this property is said to be infinitely divisible. Interestingly enough, the preceding Theorem shows that $\mathcal{W}_\Omega(0) \ge 2$, while the remark, at the end of the paper [3], shows that $\mathcal{W}(0)$ is 3 for the 2×2 matrix unit ball. Since the matrix ball is homogeneous, this number probably doesn't change at other points for the Bergman kernel in that domain. We believe it is important to study the behaviour of $\mathcal{W}_\Omega$ for different positive definite kernels defined on Ω.

3. Bergman Kernel

For any bounded open connected subset Ω of $\mathbb{C}^m$, let $\boldsymbol{B}_\Omega$ denote the Bergman kernel of Ω. This is the reproducing kernel of the Bergman space $\mathbb{A}^2(\Omega)$ consisting of square integrable holomorphic functions on Ω with respect to the volume measure. Consequently, it has a representation of the form

$$\boldsymbol{B}_\Omega(z,w) = \sum_k \varphi_k(z)\overline{\varphi_k(w)}, \tag{3.1}$$

where $\{\varphi_k\}_{k=0}^\infty$ is any orthonormal basis of $\mathbb{A}^2(\Omega)$. This series is uniformly convergent on compact subsets of $\Omega \times \Omega$.

We now exclusively study the case of the Bergman kernel on the unit ball $\mathcal{D}$ (with respect to the usual operator norm) in the linear space of all $r \times s$ matrices $\mathcal{M}_{rs}(\mathbb{C})$. The unit ball $\mathcal{D}$ may be also described as

$$\mathcal{D} = \{Z \in \mathcal{M}_{rs}(\mathbb{C}) : I - ZZ^* \geq 0\}.$$

The Bergman kernel for the domain $\mathcal{D}$ is $\boldsymbol{B}_\mathcal{D}(Z,Z) = \det(I - ZZ^*)^{-p}$, where $p = r + s$. In what follows we give a simple proof of this.

As an immediate consequence of the change of variable formula for integration, we have the transformation rule for the Bergman kernel. We provide the straightforward proof.

Lemma 3.1. *Let Ω and $\tilde{\Omega}$ be two domains in $\mathbb{C}^m$ and $\varphi : \Omega \to \tilde{\Omega}$ be a bi-holomorphic map. Then*

$$\boldsymbol{B}_\Omega(z,w) = J_\mathbb{C}\varphi(z)\overline{J_\mathbb{C}\varphi(w)}\boldsymbol{B}_{\tilde{\Omega}}(\varphi(z),\varphi(w))$$

for all $z, w \in \Omega$, where $J_\mathbb{C}\varphi(w)$ is the determinant of the derivative $D\varphi(w)$.

Proof. Suppose $\{\tilde{\phi}_n\}$ be an orthonormal basis for $\mathbb{A}^2(\tilde{\Omega})$. By change of variable formula, it follows easily that $\phi_n = \{J_\mathbb{C}\varphi(w)\tilde{\phi}_n \circ \varphi\}$, form an orthonormal basis for $\mathbb{A}^2(\Omega)$. Hence,

$$\begin{aligned}\boldsymbol{B}_\Omega(z,w) = \sum_{n=0}^\infty \phi_n(z)\overline{\phi_n(w)} &= \sum_{n=0}^\infty J_\mathbb{C}\varphi(w)(\tilde{\phi}_n \circ \varphi)(z)\overline{J_\mathbb{C}\varphi(w)(\tilde{\phi}_n \circ \varphi)(w)} \\ &= J_\mathbb{C}\varphi(w)\overline{J_\mathbb{C}\varphi(w)}\sum_{n=0}^\infty \tilde{\phi}_n(\varphi(z))\overline{\tilde{\phi}_n(\varphi(w))} \\ &= J_\mathbb{C}\varphi(w)\overline{J_\mathbb{C}\varphi(w)}\boldsymbol{B}_{\tilde{\Omega}}(\varphi(z),\varphi(w))\end{aligned}$$

completing our proof. □

If Ω is a domain in $\mathbb{C}^m$ and the bi-holomorphic automorphism group, $\mathrm{Aut}(\Omega)$ is transitive, then we can determine the Bergman kernel as well as its curvature from its value at 0! A domain with this property is called homogeneous. For instance, the unit ball $\mathcal{D}$ in the linear space of $r \times s$ matrices are homogeneous. If Ω is homogeneous, then for any $w \in \Omega$, there exists an bi-holomorphic automorphism φ_w with the property $\varphi_w(w) = 0$. The following Corollary is an immediate consequence of Lemma 3.1.

Corollary 3.2. *For any homogeneous domain Ω in $\mathbb{C}^m$, we have*

$$\boldsymbol{B}_\Omega(w,w) = J_\mathbb{C}\varphi_w(w)\overline{J_\mathbb{C}\varphi_w(w)}\boldsymbol{B}_\Omega(0,0),\ w \in \Omega.$$

We recall from (cf. [13, Theorem 2]) that for Z, W in the matrix ball $\mathcal{D}$ (of size $r \times s$) and $u \in \mathbb{C}^{r \times s}$, we have

$$D\varphi_W(Z) \cdot u = (I - WW^*)^{\frac{1}{2}}(I - ZW^*)^{-1}u(I - W^*Z)^{-1}(I - W^*W)^{\frac{1}{2}}.$$

In particular, $D\varphi_W(W)\cdot u = (I-WW^*)^{-\frac{1}{2}}u(I-W^*W)^{-\frac{1}{2}}$. Thus $D\varphi_W(W) = (I-WW^*)^{-\frac{1}{2}}\otimes (I-W^*W)^{-\frac{1}{2}}$. We therefore (cf. [11, exercise 8], [10]) have

$$\begin{aligned}\det D\varphi_W(W) &= \big(\det(I-WW^*)^{-\frac{1}{2}}\big)^s\big(\det(I-W^*W)^{-\frac{1}{2}}\big)^r\\ &= \big(\det(I-WW^*)^{-\frac{1}{2}}\big)^{r+s}.\end{aligned}$$

It then follows that

$$J_{\mathbb{C}}\varphi_W(W)\overline{J_{\mathbb{C}}\varphi_W(W)} = \det(I-WW^*)^{-(r+s)},\ W\in\mathcal{D}.$$

With a suitable normalization of the volume measure, we may assume that $\boldsymbol{B}_{\mathcal{D}}(0,0)=1$. With this normalization, we have

$$\boldsymbol{B}_{\mathcal{D}}(W,W) = \det(I-WW^*)^{-(r+s)},\ W\in\mathcal{D}. \tag{3.2}$$

The Bergman kernel $\boldsymbol{B}_\Omega$, where $\Omega = \{(z_1,z_2) : |z_2| \le (1-|z_1|^2)\}\subset\mathbb{C}^2$ is known (cf. [14, Example 6.1.6]):

$$\boldsymbol{B}_\Omega(z,w) = \frac{3(1-z_1\bar w_1)^2+z_2\bar w_2}{\{(1-z_1\bar w_1)^2-z_2\bar w_2\}^3},\ z,w\in\Omega. \tag{3.3}$$

The domain Ω is not homogeneous. However, it is a Reinhadt domain. Consequently, an orthonormal basis consisting of monomials exists in the Bergman space of this domain. We give a very similar example below to show that computing the Bergman kernel in a closed form may not be easy even for very simple Reinhadt domains. We take Ω to be the domain

$$\{(z_1,z_2,z_3) : |z_2|^2\le(1-|z_1|^2)(1-|z_3|^2), 1-|z_3|^2\ge 0\}\subset\mathbb{C}^3.$$

Lemma 3.3. *The Bergman kernel $\boldsymbol{B}_\Omega(z,w)$ for the domain Ω is given by the formula*

$$\sum_{p,m,n=0}^{\infty}\frac{m+1}{4\beta(n+1,m+2)\beta(p+1,m+2)}(z_1\bar w_1)^n(z_2\bar w_2)^m(z_3\bar w_3)^p,$$

where $\beta(m,n)$ is the Beta function.

Proof. Let $\{(z_1)^n(z_2)^m(z_3)^p\}_{n,m,p=1}^\infty$ be the orthonormal basis for the Bergman space $\mathbb{A}^2(\Omega)$. Now,

$$\begin{aligned}\|(z_1)^n(z_2)^m(z_3)^p\|^2 &= \int_0^{2\pi}d\theta_1 d\theta_2 d\theta_3\int_0^1 r_1^{(2n+1)}dr_1\int_0^1 r_3^{(2p+1)}dr_3\int_0^{\sqrt{(1-r_1^2)(1-r_2^2)}} r_2^{(2m+1)}dr_2\\ &= 8\pi^3\int_0^1 r_1^{(2n+1)}dr_1\int_0^1 r_3^{(2p+1)}dr_3\frac{(1-r_1^2)^{(m+1)}(1-r_2^2)^{(m+1)}}{2m+2}\\ &= \frac{\pi^3}{m+1}\int_0^1 s_1^n(1-s_1)^{(m+1)}ds_1\int_0^1 s_2^p(1-s_2)^{(m+1)}ds_2 \qquad (3.4)\end{aligned}$$

where $r_1^2=s_1$ and $r_2^2=s_2$. Since $\beta(n,m)=\int_0^1 r^{(n-1)}(1-r)^{(m-1)}dr$, therefore equation (3.4)is equal to

$$\|(z_1)^n(z_2)^m(z_3)^p\|^2 = \frac{\pi^3}{m+1}\beta(n+1,m+2)\beta(p+1,m+2).$$

From equation (3.4), it follows that $\|1\|^2\pi^3\beta(1,2)\beta(1,2)=\frac{\pi^3}{4}$. We normalize the volume measure in an appropriate manner to ensure

$$\|(z_1)^n(z_2)^m(z_3)^p\|^2 = \frac{4}{m+1}\beta(n+1,m+2)\beta(p+1,m+2).$$

Having computed an orthonormal basis for the Bergman space, we can complete the the computation of the Bergman kernel using the infinite expansion (3.1). ∎

The Proposition following the Lemma (a change of variable formula from (cf. [23, The chain rule 1.3.3]) given below provides the transformation rule for the Bergman metric (cf. [15, proposition 1.4.12]).

Lemma 3.4. *Suppose Ω is in $\mathbb{C}^m, F = (f_1, \ldots, f_n)$ maps Ω into $\mathbb{C}^n, g$ maps the range of F into $\mathbb{C}$, and $f_1, \ldots, f_n, g$ are of class $\mathcal{C}^2$. If*

$$h = g \circ F = g(f_1, \ldots, f_n)$$

then, for $1 \le i, j \le m$ and $z \in \Omega$,

$$\big(\overline{D}_j D_i h\big)(z) = \sum_{k=1}^{n} \sum_{l=1}^{n} \big(\overline{D}_l D_k h\big)(w) \overline{D_j f_l}(z) D_i f_k(z),$$

where $\overline{D}_j \bar{f}_l = \overline{D_j f_l}(z)$.

Proposition 3.5. *Let Ω and $\tilde{\Omega}$ be two domain in $\mathbb{C}^m$ and $\varphi : \Omega \to \tilde{\Omega}$ is bi-holomorphic map. Then*

$$\mathcal{K}_{\boldsymbol{B}_\Omega}(w) = \big(D\varphi\big)(w)^{\mathrm{t}} \mathcal{K}_{\boldsymbol{B}_{\widetilde{\Omega}}}(\varphi(w)) \overline{\big(D\varphi\big)(w)}, w \in \Omega,$$

where $\mathcal{K}_{\boldsymbol{B}_\Omega}(w) := \frac{\partial^2}{\partial w_i \partial \bar{w}_j} \log \boldsymbol{B}_\Omega(w, w)$.

Proof. For any holomorphic function φ defined on Ω, we have $\frac{\partial}{\partial w_i \partial \bar{w}_j} \log |J_{\mathbb{C}}\varphi(w)|^2 = 0$. Combining this with Lemma 3.1, we get

$$\begin{aligned} \frac{\partial^2}{\partial w_i \partial \bar{w}_j} \log \boldsymbol{B}_{\widetilde{\Omega}}(\varphi(w), \varphi(w)) &= \frac{\partial^2}{\partial w_i \partial \bar{w}_j} \log |J_{\mathbb{C}}\varphi(w)|^{-2} \boldsymbol{B}_\Omega(w, w) \\ &= -\frac{\partial^2}{\partial w_i \partial \bar{w}_j} \log |J_{\mathbb{C}}\varphi(w)|^2 + \frac{\partial^2}{\partial w_i \partial \bar{w}_j} \boldsymbol{B}_\Omega(w, w) \\ &= \frac{\partial^2}{\partial w_i \partial \bar{w}_j} \boldsymbol{B}_\Omega(w, w). \end{aligned}$$

Also by Lemma 3.4 with $g(z) = \log \boldsymbol{B}_{\widetilde{\Omega}}(z, z)$ and $F = f$ we have,

$$\frac{\partial^2}{\partial w_i \partial \bar{w}_j} \log \boldsymbol{B}_{\widetilde{\Omega}}(\varphi(w), \varphi(w)) = \sum_{k,l=1}^{n} \frac{\partial \varphi_k}{\partial w_i}(w) \frac{\partial^2}{\partial w_k \partial \bar{z}_l} \log \boldsymbol{B}_{\widetilde{\Omega}}(z, z)(\varphi(w), \varphi(w)) \frac{\partial \varphi_l}{\partial w_j}(w).$$

Hence

$$\begin{aligned} &\left(\!\!\left(\frac{\partial^2}{\partial w_i \partial \bar{w}_j} \log \boldsymbol{B}_{\widetilde{\Omega}}(\varphi(w), \varphi(w))\right)\!\!\right)_{ij} \\ = \;& \left(\!\!\left(\frac{\partial \varphi_k}{\partial w_i}(w)\right)\!\!\right)_{ik} \left(\!\!\left(\frac{\partial^2}{\partial z_k \partial \bar{z}_l} \log \boldsymbol{B}_{\widetilde{\Omega}}(z, z)(\varphi(w), \varphi(w))\right)\!\!\right)_{kl} \left(\!\!\left(\frac{\partial \varphi_l}{\partial w_j}(w)\right)\!\!\right)_{lj} \\ = \;& \big(D\varphi\big)(w)^{\mathrm{t}} \mathcal{K}_{\boldsymbol{B}_{\widetilde{\Omega}}}(\varphi(w)) \overline{\big(D\varphi\big)(w)}. \end{aligned}$$

Therefore we have the desired transformation rule for the Bergman metric, namely,

$$\mathcal{K}_{\boldsymbol{B}_\Omega}(w) = \big(D\varphi\big)(w)^{\mathrm{t}} \mathcal{K}_{\boldsymbol{B}_{\widetilde{\Omega}}}(\varphi(w)) \overline{\big(D\varphi\big)(w)}, \; w \in \Omega.$$

$\square$

As a consequence of this transformation rule, a formula for the Bergman metric at an arbitrary w in Ω is obtained from its value at 0. The proof follows from the transitivity of the automorphism group.

Corollary 3.6. *For a homogeneous domain Ω, pick a a bi-holomorphic automorphism φ_w of Ω with $\varphi_w(w) = 0$, $w \in \Omega$, we have*

$$\mathcal{K}_{\boldsymbol{B}_\Omega}(w) = \big(D\varphi_w(w)\big)^{\mathrm{t}}\mathcal{K}_{\boldsymbol{B}_\Omega}(0)\overline{D\varphi_w(w)}$$

for all $w \in \Omega$.

For the matrix ball $\mathcal{D}$, as is well-known (cf. [9]), $\boldsymbol{B}_{\mathcal{D}}^\lambda$ is not necessarily positive definite for all $\lambda > 0$. However, as we have pointed out before, the space $\mathcal{N}^{(\lambda)}(w)$ has a natural inner product induced by $\boldsymbol{B}_{\mathcal{D}}^\lambda$. Thus we explore properties of $\boldsymbol{B}_{\mathcal{D}}^\lambda$ for all $\lambda > 0$. In what follows, we will repeatedly use the transformation rule for $\boldsymbol{B}_\Omega^\lambda$ which is an immediate consequence of the transformation rule for $\boldsymbol{B}_\Omega$, namely,

$$\mathcal{K}_{\boldsymbol{B}_\Omega^\lambda}(w) = \lambda\mathcal{K}_{\boldsymbol{B}_\Omega}(w) = \lambda D\varphi_w(w)^{\mathrm{t}}\mathcal{K}_{\boldsymbol{B}_\Omega}(0)\overline{D\varphi_w(w)} \tag{3.5}$$

for $w \in \Omega$ and $\lambda > 0$.

To compute the Bergman metric, we begin with a Lemma on the Taylor expansion of the determinant. To facilitate its proof, for Z in $\mathcal{M}_{rs}(\mathbb{C})$, we write $Z = \begin{pmatrix} Z_1 \\ \vdots \\ Z_r \end{pmatrix}$, with $Z_i = (z_{i1}, \ldots, z_{is})$, $i = 1, \ldots, r$. In this notation,

$$I - ZZ^* = \begin{pmatrix} 1-\|Z_1\|^2 & -\langle Z_1,Z_2\rangle & \cdots & -\langle Z_1,Z_r\rangle \\ \vdots & \vdots & \vdots & \vdots \\ -\langle Z_r,Z_1\rangle & -\langle Z_r,Z_2\rangle & \cdots & 1-\|Z_r\|^2 \end{pmatrix},$$

where $\|Z_i\|^2 = \sum_{j=1}^s |z_{ij}|^2$, $\langle Z_i, Z_j\rangle = \sum_{k=1}^s z_{ik}\bar{z}_{jk}$. Set $X_{ij} = \langle Z_i, Z_j\rangle$, $1 \le i, j \le r$.

The curvature $\mathcal{K}_{\boldsymbol{B}_\mathcal{D}}(0)$ of the Bergman kernel, which is often called the Bergman metric, is easily seen to be p times the $rs \times rs$ identity as a consequence of the following Lemma. The value of the curvature $\mathcal{K}_{\boldsymbol{B}_\mathcal{D}}(W)$ at an arbitrary point W is then easy to write down using the homogeneity of the unit ball $\mathcal{D}$.

Lemma 3.7. *The determinant* $\det(I - ZZ^*) = 1 - \sum_{i=1}^r \|Z_i\|^2 + P(X)$, *where* $P(X) = \sum_{|\ell|\ge 2} p_\ell X^\ell$ *with*

$$X^\ell := X_{11}^{\ell_{11}} \ldots X_{1r}^{\ell_{1r}} \ldots X_{r1}^{\ell_{r1}} \ldots X_{rr}^{\ell_{rr}}.$$

Proof. The proof is by induction on r. For $r = 1$ we have $\det(I - ZZ^*) = 1 - \|Z\|^2$. Therefore in this case , $P = 0$ and we are done. For $r = 2$, we have

$$\det(I - ZZ^*) = \det\begin{pmatrix} 1-\|Z_1\|^2 & -\langle Z_1,Z_2\rangle \\ -\langle Z_2,Z_1\rangle & 1-\|Z_2\|^2 \end{pmatrix}.$$

For $r = 2$, a direct verification shows that the $\det(I - ZZ^*)$ is equal to $1 - \sum_{i=1}^2 \|Z_i\|^2 + P(X)$, where $P(X) = X_{11}X_{22} - |X_{12}|^2$. The decomposition

$$I - ZZ^* = \left(\begin{array}{cccc|c} 1-\|Z_1\|^2 & -\langle Z_1, Z_2\rangle & \cdots & -\langle Z_1, Z_{r-1}\rangle & -\langle Z_1, Z_r\rangle \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -\langle Z_{r-1}, Z_1\rangle & -\langle Z_{r-1}, Z_2\rangle & \cdots & 1-\|Z_{r-1}\|^2 & -\langle Z_{r-1}, Z_r\rangle \\ \hline -\langle Z_r, Z_1\rangle & -\langle Z_r, Z_2\rangle & \cdots & -\langle Z_r, Z_{r-1}\rangle & 1-\|Z_r\|^2 \end{array}\right)$$

is crucial to our induction argument. Let A_{ij}, $i, j = 1, 2$, denote the blocks in this decomposition. By induction hypothesis, we have

$$\det A_{11} = 1 - \sum_{i=2}^r \|Z_i\|^2 + Q(X),$$

where $Q(X) = \sum_{|\ell|\geq 2} q_\ell X^\ell$. Since $\det(A_{22} - A_{21}A_{11}^{-1}A_{12})$ is a scalar, it follows that

$$\begin{aligned}\det(I - ZZ^*) &= (A_{22} - A_{21}A_{11}^{-1}A_{12}) \det A_{11}\\ &= A_{22}\det A_{11} - A_{21}\big(\det A_{11}\big)A_{11}^{-1}A_{12}\\ &= A_{22}\det A_{11} - A_{21}\big(\operatorname{Adj}(A_{11})\big)A_{12},\end{aligned}$$

where, as usual, $\operatorname{Adj}(A_{11})$ denotes the transpose of the matrix of co-factors of A_{11}. Clearly, $A_{21}\big(\operatorname{Adj}(A_{11})\big)A_{12}$ is a sum of $(r-1)^2$ terms. Each of these is of the form $X_{k1}a_{jk}X_{1j}$, where a_{jk} denotes the (j,k) entry of $\operatorname{Adj}(A_{11})$. It follows that any one term in the sum $A_{21}\big(\operatorname{Adj}(A_{11})\big)A_{12}$ is some constant multiple of X^ℓ with $|\ell| \geq 2$. Furthermore,

$$A_{22}\det A_{11} = 1 - \sum_{i=1}^{r} \|Z_i\|^2 + \|Z_r\|^2 \sum_{i=1}^{r-1} \|Z_i\|^2 + Q(X)(1 - \|Z_r\|^2).$$

Putting these together, we see that

$$\det(I - ZZ^*) = 1 - \sum_{i=1}^{r} \|Z_i\|^2 + P(X),$$

where $P(X) = X_{rr}\sum_{i=1}^{r-1} X_{ii} + Q(X)(1 - X_{rr}) - A_{21}\big(\operatorname{Adj}(A_{11})A_{12}$ completing the proof. □

Let $\mathcal{K}_{\boldsymbol{B}_\mathcal{D}}(Z)$ be the curvature (some times also called the Bergman metric) of the Bergman kernel $\boldsymbol{B}_\mathcal{D}(Z,Z)$. Set $w_1 = z_{11}, \ldots, w_s = z_{1s}, \ldots, w_{rs-s+1} = z_{r1}, \ldots, w_{rs} = z_{rs}$. The formula for the Bergman metric given below is due to Koranyi (cf. [16]).

Theorem 3.8. *$\mathcal{K}_{\boldsymbol{B}_\mathcal{D}}(0) = pI$, where I is the $rs \times rs$ identity matrix.*

Proof. Lemma 3.7 says that

$$\log \boldsymbol{B}_\mathcal{D}(Z) = -p\log\big(1 - \sum_{i=1}^{r} \|Z_i\|^2 + P(X)\big).$$

It now follows that $\big(\frac{\partial^2}{\partial w_i \partial \bar{w}_j}\log \boldsymbol{B}_\mathcal{D}\big)(0) = 0$, $i \neq j$. On the other hand, $\big(\frac{\partial^2}{\partial w_i \partial \bar{w}_i}\log \boldsymbol{B}_\mathcal{D}\big)(0) = p$, $i = 1, \ldots, rs$. □

In consequence, for the matrix ball $\mathcal{D}$, which is a homogeneous domain in $\mathbb{C}^{r\times s}$, we record separately, the transformation rule:

$$\begin{aligned}\big(\mathcal{K}_{\boldsymbol{B}_\mathcal{D}}(W)^{\mathrm{t}}\big)^{-1} &= \big(D\varphi_W(W)\big)^{-1}\big(\mathcal{K}_{\boldsymbol{B}_\mathcal{D}}(0)^{\mathrm{t}}\big)^{-1}\big(\overline{D\varphi_W(W)}^{\,\mathrm{t}}\big)^{-1}\\ &= \frac{1}{p}\big(\overline{D\varphi_W(W)}^{\,\mathrm{t}} D\varphi_W(W)\big)^{-1},\ W \in \mathcal{D},\end{aligned} \tag{3.6}$$

where $p = r + s$.

4. Curvature inequalities

4.1. **The Euclidean Ball.** Let Ω be a homogeneous domain and $\theta_w : \Omega \to \Omega$ be a bi-holomorphic automorphism of Ω with $\theta_w(w) = 0$. The linear map $D\theta_w(w) : (\mathbb{C}^m, C_{\Omega,w}) \to (\mathbb{C}^m, C_{\Omega,0})$ is a contraction by definition. Since θ_w is invertible, $D\theta_w^{-1}(0) : (\mathbb{C}^m, C_{\Omega,0}) \to (\mathbb{C}^m, C_{\Omega,w})$ is also a contraction. However, since $D\theta_w^{-1}(0) = D\theta_w(w)^{-1}$, it follows that $D\theta_w(w)$ must be an isometry. We paraphrase the Theorem from (cf. [19, Theorem 5.2]) slightly.

Lemma 4.1. *If Ω is a homogeneous domain and θ_w is a bi-holomorphic automorphism with $\theta_w(w) = 0$, then $\|A(w)^{\mathrm{t}}\|_{\ell^2 \to C_{\Omega,w}} \leq 1$ if and only if $\|A(0)^{\mathrm{t}}\|_{\ell^2 \to C_{\Omega,0}} \leq 1$.*

Proof. As before, let $\boldsymbol{D}_w\Omega := \{Df(w) : f \in \mathrm{Hol}_w(\Omega, \mathbb{D})\}$. The map $\varphi \mapsto \varphi \circ \theta_w(w)$ is injective from $\mathrm{Hol}_0(\Omega, \mathbb{D})$ onto $\mathrm{Hol}_w(\Omega, \mathbb{D})$. Therefore,

$$\begin{aligned}\boldsymbol{D}_w\Omega &= \{D(f \circ \theta_w)(w) : f \in \mathrm{Hol}_0(\Omega, \mathbb{D})\}\\ &= \{Df(0)D\theta_w(w) : f \in \mathrm{Hol}_0(\Omega, \mathbb{D})\}\\ &= \{u \cdot D\theta_w(w) : u \in \boldsymbol{D}_0\Omega\}\end{aligned}$$

This is another way of saying that $D\theta_w(w)$ is an isometry.

$$\begin{aligned}\sup_{v \in \boldsymbol{D}_w\Omega} \|A(w)^{\mathrm{t}} v\| &= \sup_{u \in \boldsymbol{D}_0\Omega} \|A(w)^{\mathrm{t}} D\theta_w(w) u\|\\ &= \sup_{u \in \boldsymbol{D}_0\Omega} \|A(0)^{\mathrm{t}} u\|,\end{aligned}$$

where we have set $A(0)^{\mathrm{t}} := A(w)^{\mathrm{t}} D\theta_w(w)$. Thus we have shown

$$\begin{aligned}\{A(w)^{\mathrm{t}} : \|A(w)^{\mathrm{t}}\|_{\ell^2 \to C_{\Omega,w}} \leq 1\} &= \{A(0)^{\mathrm{t}} D\theta_w(w)^{-1} : \|A(0)^{\mathrm{t}}\|_{\ell^2 \to C_{\Omega,w}}\}\\ &= \{A(0)^{\mathrm{t}} D\theta_w^{-1}(0) : \|A(0)^{\mathrm{t}}\|_{\ell^2 \to C_{\Omega,w}}\}.\end{aligned}$$

The proof is now complete since $D\theta_w(w)$ is an isometry. □

We note that if $\|A(w)^{\mathrm{t}}\|_{\ell^2 \to C_{\Omega,w}} \leq 1$, then

$$\begin{aligned}\|\big(\mathcal{K}_{\boldsymbol{T}}(w)^{\mathrm{t}}\big)^{-1}\|_{C^*_{\Omega,w} \to C_{\Omega,w}} &= \|A(w)^{\mathrm{t}} \overline{A(w)}\|_{C^*_{\Omega,w} \to C_{\Omega,w}}\\ &\leq \|A(w)^{\mathrm{t}}\|_{\ell^2 \to C_{\Omega,w}} \|\overline{A(w)}\|_{C^*_{\Omega,w} \to \ell^2}\\ &= \|A(w)^{\mathrm{t}}\|^2_{\ell^2 \to C_{\Omega,w}} \leq 1, \qquad (4.1)\end{aligned}$$

which is the curvature inequality of (cf. [19, Theorem 5.2]). For a homogeneous domain Ω, using the transformation rules in Corollary 3.6 and the equation (3.6), for the curvature $\mathcal{K}$ of the Bergman kernel $\boldsymbol{B}_\Omega$, we have

$$\begin{aligned}\|\big(\mathcal{K}_{\boldsymbol{T}}(w)^{\mathrm{t}}\big)^{-1}\|_{C^*_{\Omega,w} \to C_{\Omega,w}} &= \Big\|\big(D\theta_w(w)^{\mathrm{t}} \mathcal{K}(0) \overline{D\theta_w(w)}\big)^{\mathrm{t}-1}\Big\|_{C^*_{\Omega,w} \to C_{\Omega,w}}\\ &= \Big\|D\theta_w(w)^{-1} \big(\mathcal{K}(0)^{\mathrm{t}}\big)^{-1} \overline{D\theta_w(w)^{-1}}^{\mathrm{t}}\Big\|_{C^*_{\Omega,w} \to C_{\Omega,w}}\\ &= \Big\|D\theta_w(w)^{-1} A(0)^{\mathrm{t}} \overline{A(0) D\theta_w(w)^{-1}}^{\mathrm{t}}\Big\|_{C^*_{\Omega,w} \to C_{\Omega,w}}\\ &\leq \Big\|D\theta_w(w)^{-1} A(0)^{\mathrm{t}}\Big\|^2_{\ell^2 \to C_{\Omega,w}} = \Big\|A(0)^{\mathrm{t}}\Big\|^2_{\ell^2 \to C_{\Omega,0}} \qquad (4.2)\end{aligned}$$

since $D\theta_w(w)^{-1}$ is an isometry. For the Euclidean ball $\mathbb{B} := \mathbb{B}^n$, the inequality for the curvature is more explicit. In the following, we set $\mathfrak{B}(w,w) := \big(\boldsymbol{B}_{\mathbb{B}}(w,w)\big)^{-\frac{1}{n+1}}$. Thus polarizing $\mathfrak{B}$, we have $\mathfrak{B}(z,w) = \big(1 - \langle z, w\rangle\big)^{-1}$, $z, w \in \mathbb{B}$. The inequality appearing below (cf. [19]) is a point-wise inequality with respect to the usual ordering of Hermitian matrices.

Theorem 4.2. *Let θ_w is a bi-holomorphic automorphism of $\mathbb{B}$ such that $\theta_w(w) = 0$. If $\rho_{\boldsymbol{T}}$ is contractive homomorphism of $\mathcal{O}(\mathbb{B})$ induced by the localization $N_{\mathbf{T}}(w)$, $\boldsymbol{T} \in \mathrm{B}_1(\mathbb{B})$, then*

$$\mathcal{K}_{\boldsymbol{T}}(w) \leq -\overline{D\theta_w(w)}^{t} D\theta_w(w) = \mathcal{K}_{\mathfrak{B}}(w),\ w \in \mathbb{B}$$

Proof. The equation (4.1) combined with the equality $\mathcal{C}_{\mathbb{B},0} = \|\cdot\|_{\ell^2}$ and the contractivity of $\rho_{\boldsymbol{T}}$ implies that $\|D\theta_w(w)A(w)^t\|_{\ell_2\to\ell_2} \leq 1$. Hence

$$\begin{aligned} I - D\theta_w(w)A(w)^t\overline{A(w)}\,\overline{D\theta_w(w)}^{\,\mathrm{t}} \geq 0 \quad &\Leftrightarrow \quad (D\theta_w(w))^{-1}\big(\overline{D\theta_w(w)}^{\,\mathrm{t}}\big)^{-1} - A(w)^t\overline{A(w)} \geq 0\\ &\Leftrightarrow \quad A(w)^t\overline{A(w)} \leq (D\theta_w(w))^{-1}\big(\overline{D\theta_w(w)}^{\,\mathrm{t}}\big)^{-1}\\ &\Leftrightarrow \quad \big(-\mathcal{K}_{\boldsymbol{T}}(w)^{\mathrm{t}}\big)^{-1} \leq \big(\overline{D\varphi_w(w)}^{\,\mathrm{t}}D\varphi_w(w)\big)^{-1}. \end{aligned}$$

Since $-\big(\mathcal{K}_{\boldsymbol{T}}(w)^{\mathrm{t}}\big)^{-1}$ and $\big(\overline{D\theta_w(w)}^{\mathrm{t}}D\theta_w(w)\big)^{-1}$ are positive definite matrices, it follows (cf. [4]) that $\mathcal{K}_{\boldsymbol{T}}(w) \leq -\overline{D\theta_w(w)}^{\mathrm{t}}D\theta_w(w) = \mathcal{K}_{\mathfrak{B}}(w)$. □

This inequality generalizes the curvature inequality obtained in (cf. [17, Corollary 1.2′]) for the unit disc. However, assuming that $\mathcal{K}_{\mathfrak{B}^{-1}K}(w)$ is a non-negative kernel defined on the ball $\mathbb{B}$ implies $(\mathfrak{B}(w))^{-1}K(w)$ is a non-negative kernel on $\mathbb{B}$ (cf. [3, Theorem 4.1]), indeed, it must be infinitely divisible. This stronger assumption on the curvature amounts to the factorization of the kernel $K(z,w) = \mathfrak{B}(z,w)\tilde{K}(z,w)$ for some positive definite kernel $\tilde{K}$ on the ball $\mathbb{B}$ with the property: $\big(\mathfrak{B}(z,w)\tilde{K}(z,w)\big)^{\lambda}$ is non-negative definite for all $\lambda > 0$.

For $\lambda > 0$, the polarization of the function $\boldsymbol{B}(w,w)^{\lambda}$ defines a positive definite kernel $\boldsymbol{B}^{\lambda}(z,w)$ on the ball $\mathbb{B}$ (cf. [1, Proposition 5.5]). We note that $\mathcal{K}_{\boldsymbol{B}^{\lambda}}(w) \leq \mathcal{K}_{\mathfrak{B}}(w)$ if and only if $\mathcal{K}_{\boldsymbol{B}^{\lambda}}(0) \leq \mathcal{K}_{\mathfrak{B}}(0) = -I$. Since $\mathcal{K}_{\boldsymbol{B}^{\lambda}}(0) = -\lambda(n+1)I$, it follows that $\mathcal{K}_{\boldsymbol{B}^{\lambda}}(w) \leq \mathcal{K}_{\mathfrak{B}}(w)$ if and only if $\lambda \geq \frac{1}{n+1}$. Thus whenever $\lambda \geq \frac{1}{n+1}$, we have the point-wise curvature inequality for $\boldsymbol{B}^{\lambda}(w,w)$. However, since the operator of multiplication by the co-ordinate functions on the Hilbert space corresponding to the kernel $\boldsymbol{B}^{\lambda}(w,w)$, is not even a contraction for $\frac{1}{n+1} \leq \lambda < \frac{n}{n+1}$, the induced homomorphism can't be contractive. We therefore conclude that the curvature inequality does not imply the contractivity of ρ whenever $n > 1$. For $n = 1$, an example illustrating this (for the unit disc) was given in (cf. [3, page 2]). Thus the contractivity of the homomorphism induced by the commuting tuple of the local operators $N_{\mathbf{T}}(w)$, for $\boldsymbol{T} \in \mathrm{B}_1(\mathbb{B})$, does not imply the contractivity of the homomorphism induced by the commuting tuple of operators $\boldsymbol{T}$.

4.2. **The matrix ball.** Recall that $\mathcal{N}^{(\lambda)}(w)$ is the $m+1$ dimensional space spanned by the vectors $\boldsymbol{B}^{\lambda}_{\mathcal{D}}(\cdot,w), \bar{\partial}_1\boldsymbol{B}^{\lambda}_{\mathcal{D}}(\cdot,w)\ldots,\bar{\partial}_m\boldsymbol{B}^{\lambda}_{\mathcal{D}}(\cdot,w)$. On this space, there exists a canonical m-tuple of jointly commuting nilpotent operators, namely,

$$N_i(w)\big(\bar{\partial}_j\boldsymbol{B}^{\lambda}_{\mathcal{D}}(\cdot,w)\big) = \begin{cases}\boldsymbol{B}^{\lambda}_{\mathcal{D}}(\cdot,w) & \text{if } i=j\\ 0 & \text{otherwise}\end{cases}.$$

We recall that the positive function $\boldsymbol{B}^{\lambda}_{\mathcal{D}}, \lambda > 0$, defines an inner product on the finite dimensional space $\mathcal{N}^{(\lambda)}(w)$ for all $\lambda > 0$ irrespective of whether $\boldsymbol{B}^{\lambda}_{\mathcal{D}}$ is positive definite on the matrix ball $\mathcal{D}$ or not. Let $N^{(\lambda)}(w)$ denote the commuting m-tuple of matrices $(N_1(w)+w_1I,\ldots,N_m(w)+w_mI)$ represented with respect to some orthonormal basis in $\mathcal{N}^{(\lambda)}(w)$. If $\boldsymbol{B}^{\lambda}_{\mathcal{D}}$ happens to be positive definite for some $\lambda > 0$ (this is the case if λ is a natural number), then $N^{(\lambda)}$ is nothing but the restriction of the adjoint of the multiplication operators induced by the coordinate functions to the subspace $\mathcal{N}^{(\lambda)}(w)$ in the Hilbert space determined by the positive definite kernel $\boldsymbol{B}^{\lambda}_{\mathcal{D}}$. In this section, we exclusively study the contractivity of the homomorphism $\rho_{_{N^{(\lambda)}(w)}}$ induced by the commuting m-tuples $N^{(\lambda)}(w)$.

We set $\mathcal{K}^{(\lambda)}(w) := \mathcal{K}_{\boldsymbol{B}^{\lambda}_{\mathcal{D}}}(w)$, $w \in \mathcal{D}$. If the homomorphism $\rho_{_{N^{(\lambda)}(w)}}$ is contractive for some $\lambda > 0$, then for this λ, we have:$\|\big(\mathcal{K}^{(\lambda)^{\mathrm{t}}}\big)^{-1}(0)\| \leq 1$. Like the Euclidian Ball, we study several implications of the curvature inequality in this case, as well.

Theorem 4.3. *For $\lambda > 0$, we have $\|\big(\mathcal{K}^{(\lambda)^{\mathrm{t}}}\big)^{-1}(0)\|_{C^*_{\mathcal{D},0}\to C_{\mathcal{D},0}} = \frac{1}{\lambda p}$, $p = r+s$.*

Proof. We have shown that $\big(\mathcal{K}^{\mathrm{t}}\big)^{-1}(0) = \frac{1}{p} I_{rs}$. Since $C_{\mathcal{D},0}$ is the operator norm on $(\mathcal{M})_{rs}$ and consequently $C^*_{\mathcal{D},0}$ is the trace norm, it follows that $\|I_{rs}\|_{C^*_{\mathcal{D},0}\to C_{\mathcal{D},0}} \leq 1$. This completes the proof. □

The following Theorem provides a necessary condition for the contractivity of the homomorphism induced by the commuting tuple of the local operators $N^{(\lambda)}(w)$.

Theorem 4.4. *If the homomorphism $\rho_{_{N^{(\lambda)}(w)}}$ is contractive, then $\nu \geq 1$, where $\nu = \lambda p$.*

Proof. The matrix unit ball $\mathcal{D}$ is homogenous. Let $\theta_w(w)$ be the bi-holomorphic automorphism of $\mathcal{D}$ with $\theta_w(w) = 0$. We have seen that $A(w)^{\mathrm{t}} = A(0)^{\mathrm{t}} D\theta_w^{-1}(0)$. Since $D\theta_w^{-1}(0)$ is an isometry, therefore the contractivity of $\rho_{_{N^{(\lambda)}(0)}}$ implies that contractivity of $\rho_{_{N^{(\lambda)}(w)}}$, $w \in \Omega$, see Lemma 4.1. The contractivity of $\rho_{_{N^{(\lambda)}(w)}}$ is equivalent to $\|A(0)^{\mathrm{t}}\|_{\ell^2\to C_{\mathcal{D},0}} \leq 1$. Therefore the contractivity of $\rho_{_{N^{(\lambda)}(w)}}$, for some $w \in \mathcal{D}$, implies $\|\big(\mathcal{K}^{(\lambda)^{\mathrm{t}}}\big)^{-\frac{1}{2}}(0)\|_{C^*_{\mathcal{D},0}\to C_{\mathcal{D},0}} \leq 1$. Theorem 4.3 shows that $\nu \geq 1$. □

If $\lambda > 0$ is picked such that $\boldsymbol{B}^{\lambda}_{\mathcal{D}}$ is positive definite, then Arazy and Zhang (cf. [1, Proposition 5.5]) prove that the homomorphism induced by the commuting tuple of multiplication operators on the twisted Bergman space $\mathbb{A}^{(\lambda)}(\mathcal{D})$ is bounded (k-spectral) if and only if $\nu \geq s$.

It follows that if $1 \leq \nu < s$, then the homomorphism induced by the commuting tuple of multiplication operators is not contractive on twisted Bergman space $\mathbb{A}^{(\lambda)}(\mathcal{D})$. While the homomorphism $\rho_{_{N^{(\lambda)}(w)}}$, $w \in \Omega$, is contractive on the finite dimensional Hilbert space $\mathcal{N}^{(\lambda)}(w)$. This is equivalent to the curvature inequality for $\nu \geq 1$. However, for $1 \leq \nu < s$, the rs-tuple of multiplication operators on twisted Bergman space $\mathbb{A}^{(\lambda)}(\mathcal{D})$ is not contractive. This shows that the curvature inequality is not sufficient for contractivity of the homomorphism induced by the commuting tuple of multiplication operators on the twisted Bergman spaces $\mathbb{A}^{(\lambda)}(\mathcal{D})$, when $1 \leq \nu < s$ and $n > 1$.

We have seen that any commuting tuple of operators $\boldsymbol{T}$ in $\mathrm{B}_1(\mathcal{D})$ induces a homomorphism $\rho_{_{N^{(\lambda)}(w)}} : \mathcal{O}(\mathcal{D}) \to \mathcal{L}(\mathbb{C}^{rs+1})$, $\lambda > 0$, as in the first paragraph of this subsection. Indeed, what we have said applies equally well to a generalized Bergman kernel, in the language of Curto and Salinas or to a commuting tuple of operators in the Cowen-Douglas class. We note that $\rho_{_{N^{(\lambda)}(w)}} \otimes I_{rs} : \mathcal{O}(\mathcal{D}) \otimes \mathcal{M}_{rs} \to \mathcal{L}(\mathcal{N}(w)) \otimes \mathcal{M}_{rs}$ is given by the formula

$$(\rho_{_{N^{(\lambda)}(w)}} \otimes I_{rs})(P) := \begin{pmatrix} P(w) \otimes I_{rs} & DP(w)\cdot N(w) \\ 0 & P(w)\otimes I_{rs} \end{pmatrix},$$

where

$$DP(w)\cdot N(w) = \partial_1 P(w) \otimes N_1(w) + \ldots + \partial_d P(w) \otimes N_{rs}(w). \tag{4.3}$$

The contractivity of $\rho_{_{N^{(\lambda)}(w)}} \otimes I_{rs}$, as shown in (cf. [20, Theorem 1.7] and [22, Theorm 4.2]), is equivalent to the contractivity of the operator

$$\|\partial_1 P(w) \otimes N_1(w) + \ldots + \partial_d P(w) \otimes N_{rs}(w)\|_{\mathrm{op}} \leq 1.$$

Let $P_{\mathbf{A}}$ be the matrix valued polynomial in rs variables:

$$P_{\mathbf{A}}(z) = \sum_{i=1}^{r}\sum_{j=1}^{s} z_{ij} E_{ij},$$

where E_{ij} be the $r \times s$ matrices whose (i,j) entries are 1 and other entries are 0. Let $V = \begin{pmatrix} V_1 \\ \vdots \\ V_{rs} \end{pmatrix}$ be the $rs \times rs$ matrix, where

$$V_1 = (v_{11}, 0, \ldots, 0_{sr}), \ldots, V_{sr} = (0, \ldots, 0, \ldots, v_{sr}).$$

We compute the norm of $(\rho_{N^{(\lambda)}(w)} \otimes I_{rs})(P_{\mathbf{A}})$.

Theorem 4.5. *For $\rho_{N^{(\lambda)}(w)} \otimes I_{rs}$ as above, we have*

$$\|(\rho_{N^{(\lambda)}(w)} \otimes I_{rs})(P_{\mathbf{A}})\|^2 = \max\{\sum_{i=1}^{s} |v_{1i}|^2, \ldots, \sum_{i=1}^{s} |v_{ri}|^2\}.$$

Proof. We have

$$\begin{aligned} \|\big(\rho_{N^{(\lambda)}(w)} \otimes I_{rs}\big)(P_{\mathbf{A}})\|^2 &= \|V_1 \otimes E_{11} + \ldots + V_s \otimes E_{1s} + V_{s+1} \otimes E_{21} + \ldots + V_{rs} \otimes E_{rs}\|^2 \\ &= \left\| \begin{pmatrix} V_1 & \ldots & V_s \\ \vdots & \vdots & \vdots \\ V_{rs-s+1} & \ldots & V_{rs} \end{pmatrix} \right\|^2 = \left\| \begin{pmatrix} W_1 \\ \vdots \\ W_r \end{pmatrix} \right\|^2, \end{aligned}$$

where $W_i = \big(V_{is-s+1}, \ldots, V_{is}\big)$. It is easy to see that $W_i W_j^* = 0$ for $i \neq j$. Furthermore, $W_i W_i^* = \sum_{j=1}^{s} |v_{ij}|^2$. Hence we have

$$\|(\rho_{N^{(\lambda)}(w)} \otimes I_{rs})(P_{\mathbf{A}})\|^2 = \max\{\sum_{i=1}^{s} |v_{1i}|^2, \ldots, \sum_{i=1}^{s} |v_{ri}|^2\}$$

completing the proof of the theorem. □

Even for the small class of homomorphisms $\rho_{N^{(\lambda)}(w)}$ discussed here, finding the cb norm of $\rho_{N^{(\lambda)}(w)}$ is not easy. However, we determine when $\|(\rho_{N^{(\lambda)}(w)} \otimes I_{rs})(P_{\mathbf{A}})\|^2 \leq 1$. This gives a necessary condition for the complete contractivity of $\rho_{N^{(\lambda)}(w)}$.

Theorem 4.6. *If $\|(\rho_{N^{(\lambda)}(w)} \otimes I_{rs})(P_{\mathbf{A}})\|^2 \leq 1$, then $\nu \geq s$.*

Proof. By Theorem 4.5 we have

$$\|(\rho_{N^{(\lambda)}(w)} \otimes I_{rs})(P_{\mathbf{A}})\|^2 = \max\{\sum_{i=1}^{s} |v_{1i}|^2, \ldots, \sum_{i=1}^{s} |v_{ri}|^2\}.$$

Since $|v_{ij}|^2 = \frac{1}{\nu}, 1 \leq i \leq r, 1 \leq j \leq s$, it is immediate that $\|(\rho_{N^{(\lambda)}(w)}(w) \otimes I_{rs})(P_{\mathbf{A}})\|^2 \leq 1$ implies $\nu \geq s$ completing the proof of the theorem. □

As a consequence, it follows that if $1 \leq \nu < s$, then the homomorphism induced by the commuting tuple of the local operators $N^{(\lambda)}(w)$ is not completely contractive.

4.3. **More examples.** We have discussed the Bergman kernel $\boldsymbol{B}_\Omega(w,w)$ for the domain $\Omega = \{(z_1, z_2) : |z_2| \leq (1 - |z_1|^2)\} \subset \mathbb{C}^2$. The curvature $\mathcal{K}_{\boldsymbol{B}_\Omega}(w) = \sum_{i,j=1}^{2} T_{ij}(w) dw_i \wedge d\bar{w}_j$ of the Bergman kernel $\boldsymbol{B}_\Omega(w,w)$ is (cf.[14, Example 6.2.1]):

$$T_{11}(w) = 6\big(\frac{1}{C(w)} - \frac{1}{D(w)}\big) + 12|w_1|^2|w_2|^2\big(\frac{1}{C^2(w)} + \frac{1}{D^2(w)}\big),$$

$$T_{12}(w) = \bar{T}_{21}(w) = 6w_1\bar{w}_2(1 - |w_1|^2)\big(\frac{1}{C^2(w)} + \frac{1}{D^2(w)}\big),$$

$$T_{22}(w) = 3(1 - |w_1|^2)^2\big(\frac{1}{C^2(w)} + \frac{1}{D^2(w)}\big),$$

where $C(w) := (1-|w_1|^2)^2 - |w_2|^2$ and $D(w) := 3(1-|w_1|^2)^2 + |w_2|^2$. We have seen that the polarization $\boldsymbol{B}_\Omega^\lambda(z,w)$ of the function $\boldsymbol{B}_\Omega(w,w)^\lambda$ defines a Hermitian structure for $\mathcal{N}^{(\lambda)}(w)$. Specializing to $w=0$, since $-\big(\mathcal{K}(0)^{\mathrm{t}}\big)^{-1} = A(0)^{\mathrm{t}}\overline{A(0)}$, we have $a_{11}^\lambda(0) = \frac{1}{\sqrt{\lambda T_{11}(0)}}$ and $a_{22}^\lambda(0) = \frac{1}{\sqrt{\lambda T_{22}(0)}}$, where $(A^\lambda(0))^{\mathrm{t}} = \begin{pmatrix} a_{11}^\lambda(0) & 0 \\ 0 & a_{22}^\lambda(0)\end{pmatrix}$.

Proposition 4.7. *The contractivity of the homomorphism $\rho_{_{N^{(\lambda)}(0)}}$ implies $16\lambda \geq 5$.*

Proof. We have $a_{11}^\lambda(0) = \frac{1}{2\sqrt{\lambda}}, a_{12}^\lambda(0) = 0, a_{22}^\lambda(0) = \frac{3}{\sqrt{10\lambda}}$. Contractivity of homomorphism $\rho_{_{N^{(\lambda)}(0)}}$ is equivalent to $\|(A^\lambda(0))^{\mathrm{t}}\|_{\ell^2\to C_{\Omega,0}} \leq 1$. This is equivalent to $(2(a_{11}^\lambda(0))^2-1)^2 \leq (1-(a_{22}^\lambda(0))^2)$. Hence $16\lambda \geq 5$ completing our proof. □

The bi-holomorphic automorphism group of Ω is not transitive. So the contractivity of the homomorphism $\rho_{_{N^{(\lambda)}(0)}}$ does not necessarily imply the contractivity of the homomorphism $\rho_{_{N^{(\lambda)}(w)}}, w \in \Omega$. Determining which of the homomorphism $\rho_{_{N^{(\lambda)}(w)}}$ is contractive, appears to be a hard problem.

Let $P_{\mathbf{A}} : \Omega \to (\mathcal{M}_2)_1$ be the matrix valued polynomial on Ω defined by $P_{\mathbf{A}}(z) = z_1A_1 + z_2A_2$ where $A_1 = I_2$ and $A_2 = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0\end{smallmatrix}\right)$. It is natural to ask when $\rho_{_{N^{(\lambda)}(w)}}$ is completely contractive. As before, we only obtain a necessary condition using the polynomial $P_{\mathbf{A}}$.

Proposition 4.8. $\|\rho^{(2)}_{N^{(\lambda)}(0)}(P_{\mathbf{A}})\| \leq 1$ *if and only if* $\lambda \geq \frac{11}{20}$.

Proof. Suppose that $\|\rho^{(2)}_{N^{(\lambda)}(0)}(P_{\mathbf{A}})\| \leq 1$. Then we have $(a_{11}^\lambda(0))^2 + (a_{22}^\lambda(0))^2 \leq 1$. Hence $\lambda \geq \frac{11}{20}$. The converse verification is also equally easy. □

We conclude that if $\frac{5}{16} \leq \lambda < \frac{11}{20}$, the homomorphism $\rho_{N^{(\lambda)}}(0)$ is contractive but not completely contractive. An explicit description of the set

$$\{\lambda : \|\rho^{(2)}_{N^{(\lambda)}(w)}(P_{\mathbf{A}})\|_{\mathrm{op}} \leq 1, w \in \Omega\}$$

would certainly provide greater insight. However, it appears to be quite intractable, at least for now.

The formula for the Bergman kernel for the domain

$$\Omega := \{(z_1,z_2,z_3) : |z_2|^2 \leq (1-|z_1|^2)(1-|z_3|^2), 1-|z_3|^2 \geq 0\} \subset \mathbb{C}^3.$$

is given in Lemma 3.3. From Lemma 3.3 we have $\boldsymbol{B}_\Omega^\lambda(z,0) = 1$ and $\partial_i \boldsymbol{B}_\Omega^\lambda(z,0) = 0$ for $i = 1,2,3$. Hence the desired curvature matrix is of the form

$$\big(\!\!\big((\partial_i\bar{\partial}_j \log \boldsymbol{B}_\Omega^\lambda)(0,0)\big)\!\!\big)_{i\,j=1}^m.$$

Let $T_{ij}(0) = \partial_i\bar{\partial}_j \log \boldsymbol{B}_\Omega^\lambda(0,0)$, that is, $\mathcal{K}_{\boldsymbol{B}_\Omega}(0) = \sum_{i,j=1}^3 T_{ij}(0) dw_i \wedge d\bar{w}_j$. An easy computation shows that $T_{11}(0) = 3\lambda = T_{33}(0), T_{22}(0) = \frac{9\lambda}{2}$ and $T_{ij}(0) = 0$ for $i \neq j$. As before, we have $a_{11}^\lambda(0) = \frac{1}{\sqrt{T_{11}(0)}}, a_{22}^\lambda(0) = \frac{1}{\sqrt{T_{22}(0)}}$ and $a_{33}^\lambda(0) = \frac{1}{\sqrt{T_{33}(0)}}$, where $A(0)^{\mathrm{t}} = \begin{pmatrix} a_{11}^\lambda(0) & 0 & 0 \\ 0 & a_{22}^\lambda(0) & 0 \\ 0 & 0 & a_{33}^\lambda(0)\end{pmatrix}$.

Proposition 4.9. *The contractivity of the homomorphism $\rho_{_{N^{(\lambda)}(0)}}$ implies $\lambda \geq \frac{1}{4}$.*

Proof. From Lemma (4.2) we have $a_{11}^\lambda(0) = \frac{1}{\sqrt{3\lambda}}, a_{12}^\lambda(0) = a_{13}^\lambda(0) = 0, a_{22}^\lambda(0) = \frac{\sqrt{2}}{3\sqrt{\lambda}}, a_{23}^\lambda(0) = 0$ and $a_{33}^\lambda(0) = \frac{1}{\sqrt{3\lambda}}$. The contractivity of the homomorphism $\rho_{_{N^{(\lambda)}(0)}}$ is the requirement $\|A(0)^{\mathrm{t}}\|^2_{\ell^2\to C_{\Omega,0}} \leq 1$, which is equivalent to $|a_{11}^\lambda(0)|^2(1-|a_{33}^\lambda(0)|^2) \geq (|a_{22}^\lambda(0)|^2 - |a_{33}^\lambda(0)|^2)$. Hence we have $\lambda \geq \frac{1}{4}$. □

For our final example, let $P_{\mathbf{A}} : \Omega \to (\mathcal{M}_2)_1$ be also the matrix valued polynomial on Ω defined by $P_{\mathbf{A}}(z) = z_1A_1 + z_2A_2 + z_3A_3$ where $A_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0\end{smallmatrix}\right), A_2 = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0\end{smallmatrix}\right), A_3 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1\end{smallmatrix}\right)$.

Proposition 4.10. $\|\rho^{(2)}_{N^{(\lambda)}(0)}(P_{\mathbf{A}})\| \leq 1$ *if and only if* $\lambda \geq \frac{5}{9}$.

Proof. Suppose that $\|\rho^{(2)}_{N^{(\lambda)}(0)}(P_{\mathbf{A}})\| \leq 1$. Then we have

$$\max\{(a^{\lambda}_{11}(0))^2 + (a^{\lambda}_{22}(0))^2, (a^{\lambda}_{33}(0))^2\} \leq 1.$$

Hence $\lambda \geq \frac{5}{9}$. The converse statement is easily verified. □

Thus if $\frac{1}{4} \leq \lambda < \frac{5}{9}$, the homomorphism $\rho_{_{N^{(\lambda)}(0)}}$ is contractive but not completely contractive.

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