Homogeneous Operators

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by
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Declaration

I hereby declare that the work reported in this thesis is entirely original and has been carried out by me under the supervision of Prof. Gadadhar Misra at the Department of Mathematics, Indian Institute of Science, Bangalore. I further declare that this work has not been the basis for the award of any degree, diploma, fellowship, associateship or similar title of any University or Institution.

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DEDICATED TO MY PARENTS, TEACHERS AND MY WIFE

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Abstract

A bounded operator T on a complex separable Hilbert space is said to be homogeneous if $\varphi(T)$ is unitarily equivalent to T for all φ in Möb, where Möb is the Möbius group. A complete description of all homogeneous weighted shifts was obtained by Bagchi and Misra. The first examples of irreducible bi-lateral homogeneous 2-shifts were given by Korányi. For $0 < a < b < 1$, the bi-lateral shifts $T(a,b)$ and $T(b,a)$ with weights $\sqrt{\frac{n+a}{n+b}}$ and $\sqrt{\frac{n+b}{n+a}}$, respectively, are homogeneous and the associated representation is the Complementary series $C_{\lambda,\sigma}$, where $\lambda = a+b-1$ and $\sigma = (b-a)/2$. Consequently, the operator $\begin{pmatrix} T(a,b) & \alpha(T(a,b)-T(b,a)) \\ 0 & T(b,a) \end{pmatrix}$, $\alpha > 0$, is homogeneous. It has been proved by Korányi that these are all irreducible homogeneous operators, modulo unitary equivalence, whose associated representation is a direct sum of two copies of a Complementary series representation. We describe all irreducible homogeneous 2-shifts up to unitary equivalence completing the list of homogeneous 2-shifts of Korányi.

After completing the list of all irreducible homogeneous 2-shifts, we show that every homogeneous operator whose associated representation is a direct sum of three copies of a Complementary series representation, is reducible. Moreover, we show that such an operator is either a direct sum of three bi-lateral weighted shifts, each of which is a homogeneous operator or a direct sum of a homogeneous bi-lateral weighted shift and an irreducible bi-lateral 2-shift.

It is known that the characteristic function θ_T of a homogeneous contraction T with an associated representation π is of the form

$$\theta_T(a) = \sigma_L(\phi_a)^*\theta_T(0)\sigma_R(\phi_a),$$

where σ_L and σ_R are projective representations of the Möbius group Möb with a common multiplier. We give another proof of the "product formula".

We point out that the defect operators of a homogeneous contraction in $B_2(\mathbb{D})$ are not always quasi-invertible (recall that an operator T is said to be *quasi-invertible* if T is injective and $\operatorname{ran}(T)$ is dense).

We prove that when the defect operators of a homogeneous contraction in $B_2(\mathbb{D})$ are not quasi-invertible, the projective representations σ_L and σ_R are unitarily equivalent to the

holomorphic Discrete series representations $D^+_{\lambda-1}$ and $D^+_{\lambda+3}$, respectively. Also, we prove that, when the defect operators of a homogeneous contraction in $B_2(\mathbb{D})$ are quasi-invertible, the two representations σ_L and σ_R are unitarily equivalent to certain known pairs of representations $D_{\lambda-1,\mu_2}$ and $D_{\lambda+1,\mu_1}$, respectively. These are described explicitly.

Let G be either (i) the direct product of n-copies of the bi-holomorphic automorphism group of the disc or (ii) the bi-holomorphic automorphism group of the polydisc $\mathbb{D}^n$.

A commuting tuple of bounded operators $\mathsf{T} = (T_1, T_2, \ldots, T_n)$ is said to be homogeneous with respect to G if the joint spectrum of T lies in $\bar{\mathbb{D}}^n$ and $\varphi(\mathsf{T})$, defined using the usual functional calculus, is unitarily equivalent to T for all $\varphi \in G$.

If the tuple of multiplication operators on a reproducing kernel Hilbert space is homogeneous with respect to G, then we prove that the curvature obeys a transformation rule. This transformation rule is the key to identifying the equivalence classes of homogeneous operators in $B_1(\mathbb{D}^n)$. However, the commuting tuples of homogeneous operators in $B_m(\mathbb{D}^n)$ cannot be classified using the curvature since it is not a complete invariant when $m > 1$. Nevertheless, the commuting tuples of homogeneous operators in $B_2(\mathbb{D}^n)$ have been classified here.

We show that a commuting tuple T in the Cowen-Douglas class of rank 1 is homogeneous with respect to G if and only if it is unitarily equivalent to the tuple of the multiplication operators on either the reproducing kernel Hilbert space with reproducing kernel $\prod_{i=1}^n \frac{1}{(1-z_i\bar{w}_i)^{\lambda_i}}$ or $\prod_{i=1}^n \frac{1}{(1-z_i\bar{w}_i)^{\lambda}}$, where λ, λ_i, $1 \le i \le n$, are positive real numbers, according as G is as in (i) or (ii).

Let $\mathsf{T} := (T_1, \ldots, T_{n-1})$ be an $(n-1)$-tuple of rank 1 Cowen-Douglas class operators and homogeneous with respect to G, where G is the direct product of $(n-1)$-copies of the bi-holomorphic automorphism group of the disc. Let $\hat{T}$ be an irreducible homogeneous (with respect to the bi-holomorphic group of automorphisms of the disc) operator in the Cowen-Douglas class on the disc of rank 2. We show that every irreducible homogeneous operator with respect to G, G as in (i), of rank 2 must be of the form

$$(T_1 \otimes I_{\widehat{H}}, \ldots, T_{n-1} \otimes I_{\widehat{H}}, I_H \otimes \hat{T}).$$

We also show that if G is chosen to be the group as in (ii), then there are no irreducible operators of rank 2 which is homogeneous with respect to G.

Contents

Chapter 1

Introduction

Let Möb denote the Möbius group of all biholomorphic automorphisms ϕ of the unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. These are of the form $\phi(z) = e^{i\theta}\frac{z-a}{1-\bar{a}z}$, $\theta \in \mathbb{R}$, $a \in \mathbb{D}$.

Definition 1.1. A bounded linear operator T on a complex separable Hilbert space H is said to be homogeneous if the spectrum of T is contained in $\overline{\mathbb{D}}$, the closed unit disc and $\phi(T)$ is unitarily equivalent to T for every ϕ in Möb.

These assumptions on an operator T and the Hilbert space H, namely that the operator is linear and bounded, the Hilbert space is complex and separable will be in force throughout this thesis.

The definition of a homogeneous operator while ensuring the existence of a unitary operator U_ϕ intertwining $\phi(T)$ with T does not impose any additional condition on the map $\phi \mapsto U_\phi$. To investigate some of these properties, we recall some basic notions from the representation theory of locally compact second countable (lcsc) groups, in particular, the Möbius group. Most of what follows is from [7, 9].

Definition 1.2. Let G be a locally compact second countable group, H be a Hilbert space and $\mathcal{U}(H)$ be the group of unitary operators on H. A Borel function $\pi : G \to \mathcal{U}(H)$ is said to be a projective representation of G on the Hilbert space H, if

$$\pi(1) = I,\ \pi(gh) = m(g,h)\pi(g)\pi(h);\ g,h \in G$$

where $m : G \times G \to \mathbb{T}$ is a Borel function.

The function m associated with a projective representation π is called the *multiplier* of π and satisfies the equations

$$m(g,1) = m(1,g) = 1,\ m(g_1,g_2)m(g_1g_2,g_3) = m(g_1,g_2g_3)m(g_2,g_3)$$

for all g, g_1, g_2 and g_3 in G. Two multipliers m and $\tilde{m}$ are said to be equivalent if there is a Borel function $f : G \to \mathbb{T}$ such that $m(g,h) = \frac{f(gh)}{f(g)f(h)}\tilde{m}(g,h),\ g,h \in G$.

Let π_1 and π_2 be two projective representations of G on Hilbert spaces H_1 and H_2, respectively. The representations π_1 and π_2 are called equivalent if there exists a unitary operator $U : H_1 \to H_2$ and a Borel function $f : G \to \mathbb{T}$ such that

$$\pi_1(g) = f(g)U^*\pi_2(g)U$$

holds for all g in G.

Definition 1.3. Let T be a homogeneous operator on a Hilbert space H. If there is a projective representation π of Möb on H with the property

$$\phi(T) = \pi(\phi)^* T\pi(\phi),\ \phi \in \text{Möb},$$

then π is said to be the representation associated with the operator T.

A homogeneous operator need not possess an associated representation. However, the following theorem says that for every irreducible homogeneous operator, there exists a unique (upto equivalence) projective representation associated with it.

Theorem 1.4. *[9, Theorem 2.2] If T is an irreducible homogeneous operator, then T has a unique (upto equivalence) projective representation of Möb associated with it.*

Clearly, to describe the homogeneous operators, we need a complete set of unitary invariants. For example, the spectral theorem for normal operators provides such a complete set of unitary invariants. It is possible to describe all the homogeneous operators which are normal using these invariants, see [7, Theorem 6.6].

In this thesis, we use the characteristic function for a contraction introduced by Sz.-Nagy and Foias as well as the curvature invariant introduced by Cowen and Douglas to investigate homogeneous operators which are not necessarily normal. We therefore recall these notions below and then describe our main results.

An operator T on a Hilbert space H is said to be a contraction if $\|T\| \leq 1$ and T is said to be a pure contraction if $\|Tx\| < \|x\|,\ x \in H$. Given a contraction T, the operators $D_T = (I - T^*T)^{\frac{1}{2}}$ and $D_{T^*} = (I - TT^*)^{\frac{1}{2}}$ are called the defect operators of T. The closed subspaces $\mathscr{D}_T = \overline{D_T H}$ and $\mathscr{D}_{T^*} = \overline{D_{T^*} H}$ are called the defect spaces.

Given a Hilbert space $\mathscr{K}$, let $L^2_{\mathscr{K}}$ be a Hilbert space consisting of $\mathscr{K}$ valued square integrable function on $\mathbb{T}$ and $H^2_{\mathscr{K}}$ be the vector valued Hardy space consisting of $\mathscr{K}$ valued holomorphic functions on $\mathbb{D}$. Every element of $H^2_{\mathscr{K}}$ has square integrable boundary value on $\mathbb{T}$ (cf. [32, pp. 185]). Therefore, $H^2_{\mathscr{K}}$ is naturally identified with a subspace of $L^2_{\mathscr{K}}$.

Given two Hilbert spaces $\mathscr{K}$ and $\mathscr{L}$, let $\theta : \mathbb{D} \to \mathscr{B}(\mathscr{K}, \mathscr{L})$ be a bounded holomorphic function. In [32, pp. 187], it has been proved $\theta(e^{it}) = \lim \theta(z)$ ($z \to e^{it}$ non tangentially) exists almost everywhere. Therefore θ naturally defines $\Theta : L^2_{\mathscr{K}} \to L^2_{\mathscr{L}}$ by

$$(\Theta f)(e^{it}) = \theta(e^{it}) f(e^{it}),\ f \in L^2_{\mathscr{K}}.$$

Since θ is holomorphic, it follows that Θ maps $H^2_{\mathscr{K}}$ to $H^2_{\mathscr{L}}$. The function θ is said to be pure contractive if $\theta(0)$ is pure contractive and inner if Θ is an isometry.

Two bounded holomorphic functions $\theta_1 : \mathbb{D} \to \mathscr{B}(\mathscr{K}_1, \mathscr{L}_1)$ and $\theta_2 : \mathbb{D} \to \mathscr{B}(\mathscr{K}_2, \mathscr{L}_2)$ are said to be coincide, if there exists two unitary operator $\eta : \mathscr{K}_1 \to \mathscr{K}_2$ and $\tau : \mathscr{L}_2 \to \mathscr{L}_1$ such that $\theta_1(z) = \tau\theta_2(z)\eta,\ z \in \mathbb{D}$.

Given a contraction T, define $\theta_T : \mathbb{D} \to \mathscr{B}(\mathscr{D}_T, \mathscr{D}_{T^*})$ by

$$\theta_T(a) = -T_{|\mathscr{D}_T} + aD_{T^*}(I - aT^*)^{-1} D_{T|\mathscr{D}_T},\ a \in \mathbb{D}.$$

In [32, pp. 239], it has been proved that θ_T is a purely contractive holomorphic function. Given a contraction T, the purely contractive holomorphic function θ_T is called the characteristic function of T. A contraction T on a Hilbert space H is said to be completely non-unitary (cnu) if there does not exists any non-trivial reducing subspace L of H such that $T_{|L}$ is unitary. The following theorem says that the unitary equivalence class of a cnu contraction is determined by the characteristic function.

Theorem 1.5. *[32, Theorem 3.4, Chapter VI] Let T and $\tilde{T}$ be two cnu contractions and θ_T and $\theta_{\tilde{T}}$ be the characteristic functions of T and $\tilde{T}$, respectively. The operators T and $\tilde{T}$ are unitarily equivalent if and only if θ_T and $\theta_{\tilde{T}}$ coincide.*

A contraction T on a Hilbert space H is said to $C_{.0}$, if $T^{*n}x \to 0$ as $n \to \infty$ for all x in H. In [32, Proposition 3.5, Chapter VI], it has been proved that a cnu contraction T is $C_{.0}$ if and only if the characteristic function θ_T of T is inner.

Let $\theta : \mathbb{D} \to \mathscr{B}(\mathscr{K}, \mathscr{L})$ be an inner function and $\Theta : L^2_{\mathscr{K}} \to L^2_{\mathscr{L}}$ be the operator induced by θ. Suppose M is the operator of multiplication by the coordinate function z on the Hilbert space $H^2_{\mathscr{L}}$. Let $\mathscr{M}$ be the range of the operator Θ and $T = P_{\mathscr{M}^\perp} M_{|\mathscr{M}^\perp}$, where $P_{\mathscr{M}^\perp}$ is the projection of $H^2_{\mathscr{L}}$ onto $\mathscr{M}^\perp$. From [32, Theorem 3.1, Chapter VI] and [32, Proposition 3.5, Chapter VI], it follows that the operator T is $C_{.0}$ and the characteristic function of T coincides with θ.

This completes the preliminaries needed from the Sz.-Nagy–Foias model theory for contractions for our work. Now, we recall another important class of operators, introduced by Cowen and Douglas, studied extensively over the past five decades.

Definition 1.6. Let Ω be an open and connected subset of $\mathbb{C}$. An operator T acting on a complex separable Hilbert space H is said to be in the Cowen-Douglas class of rank m, denoted by $B_m(\Omega)$, if it meets the following requirements:

1. $\operatorname{ran}(T - \bar{w}I) = H, \ \ w \in \Omega,$

2. $\dim(\ker(T - \bar{w}I)) = m, \ \ w \in \Omega$, and

3. $\bigvee_{w\in\Omega} \ker(T - \bar{w}I) = H$.

The definition of the Cowen-Douglas class ensures that, if an operator T is in $B_m(\Omega)$, then there exists a rank m Hermitian holomorphic vector bundle over $\Omega^* := \{\bar{w} : w \in \Omega\}$

$$E_T = \{(w,h) \in \Omega^* \times H : h \in \ker(T - wI)\} \text{ and } p(w,h) = w$$

which is a sub-bundle of the trivial bundle $\Omega^* \times H$. The following theorem says that the unitary equivalence class of a rank m Cowen-Douglas operator is determined by the associated Hermitian holomorphic vector bundle.

Theorem 1.7. *[12, Theorem 1.14] Let T and $\tilde{T}$ be two operators in $B_m(\Omega)$. The operators T and $\tilde{T}$ are unitarily equivalent if and only if the associated Hermitian holomorphic vector bundles E_T and $E_{\tilde{T}}$ are equivalent.*

Given an operator T on a Hilbert space H in $B_m(\Omega)$, the Hermitian structure of the bundle E_T at w in Ω^* is obtained from that of the subspace $\ker(T - wI)$ of H. It is shown in [12, Proposition 1.11] that if an operator T on a Hilbert space H is in $B_m(\Omega)$, then there exists a holomorphic map $w \in \Omega^* \to \gamma(w) := \big(\gamma_1(w), \gamma_2(w), \ldots, \gamma_m(w)\big)$ such that $\ker(T - wI) = \operatorname{span}\{\gamma_i(w) : 1 \le i \le m\}$. Let $h(w) = \big(\big(\langle \gamma_j(w), \gamma_i(w)\rangle\big)\big)$. The curvature of the K_T of the bundle E_T with respect to the frame γ is given by the following formula ([34, Proposition 1.11])

$$\mathsf{K}_T(w) = \frac{\partial}{\partial \bar{w}}\left(h(w)^{-1}\frac{\partial}{\partial w}h(w)\right) d\bar{w} \wedge dw.$$

Like in the case of pure contractions, Cowen and Douglas show that the curvature $\mathsf{K}_T(w)$ is a complete invariant for operators T in $B_1(\Omega)$.

Theorem 1.8. *[12, Theorem 1.17] Let T and $\tilde{T}$ be two operators in $B_1(\Omega)$. The operators T and $\tilde{T}$ are unitarily equivalent if and only if $\mathsf{K}_T(w) = \mathsf{K}_{\tilde{T}}(w)$ for all w in Ω^*.*

Let $K : \Omega \times \Omega \to \mathscr{M}_m$ be a positive definite kernel which is holomorphic in the first and anti-holomorphic in the second variable. The linear span of the vectors

$$\{K(\cdot, w)x : x \in \mathbb{C}^m, w \in \Omega\}$$

equipped with the inner product

$$\langle K(\cdot, w_2)x, K(\cdot, w_1)y\rangle = \langle K(w_1, w_2)x, y\rangle$$

is a pre-Hilbert space. The completion is a Hilbert space, say H_K, of holomorphic functions on Ω. For each fixed but arbitrary $w \in \Omega$, the vector $K(\cdot, w)x$, $x \in \mathbb{C}^m$, is in H_K and has the reproducing property:

$$\langle f, K(\cdot, w)x\rangle = \langle f(w), x\rangle,\ f \in H_K.$$

Given an operator T in $B_m(\Omega)$, there exists a reproducing kernel Hilbert space H_K possessing a reproducing kernel $K : \Omega \times \Omega \to \mathscr{M}_m$ such that T is unitarily equivalent to M_z^*, where M_z is the operator of multiplication by the co-ordinate function z on H_K (see [14, Theorem 4.12]).

Soon afterwards, Cowen and Douglas isolated a class of commuting tuples of bounded linear operators $B_m(\Omega)$, $\Omega \subseteq \mathbb{C}^n$, which, like in the one variable case, determines and is determined by a certain class of holomorphic Hermitian vector bundles, see [13, 14]. As in the case of one variable, these operators can be realized as the adjoint $M^* := (M_{z_1}^*, \ldots, M_{z_n}^*)$ of the commuting tuple of multiplication by the co-ordinate functions on a Hilbert space H of holomorphic functions defined on Ω possessing a reproducing kernel $K : \Omega \times \Omega \to \mathscr{M}_m$. In this case, the joint eigenspace $\mathscr{E}_w := \cap_{i=1}^n \ker(M_{z_i} - w_i)^*$, by assumption, is of constant dimension m for all w in Ω. The vector bundle is then $E_{M^*} := \{(w, h) : h \in \mathscr{E}_w, w \in \Omega\}$, where the map $s_i : w \to K(\cdot, w)e_i$, $1 \le i \le m$, $w \in \Omega$, serves as an anti-holomorphic frame on Ω. Consequently, the Hermitian structure of the vector bundle E_{M^*} is given by $K(w, w)$. Now, the curvature $\mathsf{K}(w)$ of this vector bundle is defined by the formula

$$\mathsf{K}(w) = \sum_{i,j=1}^n \partial_i \left[K(w,w)^{-1} \bar{\partial}_j K(w,w)\right] dw_i \wedge d\bar{w}_j.$$

A study of commuting tuples of operators homogeneous with respect to the bi-holomorphic automorphism group of an irreducible bounded symmetric domain was begun in [28], where a transformation rule for the curvature was established in the case: $m = 1$. This transformation rule is the key to identify the equivalence classes of homogeneous operators in $B_1(\Omega)$. The study of commuting tuples of homogeneous operators has been continued in [3, 4, 24]. However, the cases of bounded symmetric domains like the polydisc $\mathbb{D}^n$, which are reducible, were left out so far. A study of homogeneous operators with respect to the bi-holomorphic automorphism group of $\mathbb{D}^n$ is begun here. The transformation rule for the curvature of a commuting tuple of homogeneous operators in $B_1(\Omega)$ is independent of whether the bounded symmetric domain Ω is reducible or irreducible. Thus using the transformation rule, we classify all commuting tuples of homogeneous operators in $B_1(\mathbb{D}^n)$.

However, the commuting tuples of homogeneous operators in $B_m(\mathbb{D}^n)$ cannot be classified using the curvature since it is not a complete invariant when $m > 1$. Nevertheless, building on existing techniques from [22], the commuting tuples of homogeneous operators in $B_2(\mathbb{D}^n)$ have been classified.

Finally, we point out that there are some situations, where tools from the well-known theory of representations of Möb appear to be more effective than the use of unitary invariants. This situation is described later.

Definition 1.9. A bounded operator T on a Hilbert space H is said to be a shift if H admits a direct sum decomposition of the form $\oplus_{i\in I} H_i$, where each H_i is a closed subspace of H and T maps H_i into H_{i+1}, $i \in I$. The operator T is a bi-lateral, forward or backward shift according as I equals $\mathbb{Z}$, $\{n \in \mathbb{Z} : n \geq n_0\}$ or $\{n \in \mathbb{Z} : n \leq n_0\}$.

The realization of an irreducible operator as a block shift is uniquely determined, that is, there is exactly one possible decomposition of the Hilbert space on which T acts as a shift (see [9, Lemma 2.2]).

Definition 1.10. An irreducible operator T is said to be an n-shift if dim $H_i = n$, for all $i \in I$ except for finitely many of them.

All irreducible homogeneous forward (and consequently backward) 2-shifts were described by Korányi and Misra in [22]. First example of an irreducible homogeneous bilateral 2-shift was given by Korányi in [20]. In [20], a three parameter family of irreducible homogeneous bilateral 2-shifts was constructed by Korányi using the following theorem which is proved by combining [7, Theorem 5.3] and [5, Proposition 2.4.].

Theorem 1.11. *Let π be a representation of Möb and T_i, $i = 1,2$ be homogeneous operators with associated representation π. Then the operator $\begin{pmatrix} T_1 & \alpha(T_1 - T_2) \\ 0 & T_2 \end{pmatrix}$, $\alpha \in \mathbb{C}$, is homogeneous with associated representation $\pi \oplus \pi$.*

1.0.1 Main results

A complete list of the three parameter family of bi-lateral 2-shifts, discovered by Korányi, is given at the end of this introductory chapter. In chapter 2, we describe all irreducible homogeneous 2-shifts up to unitary equivalence completing the list of homogeneous 2-shifts of Korányi.

In chapter 3, we show that every homogeneous operator whose associated representation is a direct sum of three copies of a Complementary series representation, is reducible. Moreover, we show that such an operator is either a direct sum of three bi-lateral weighted shifts, each of which is a homogeneous operator or a direct sum of a homogeneous bi-lateral weighted shift and an irreducible bi-lateral 2-shift.

In chapter 4, we describe the characteristic functions of all homogeneous contractions in $B_2(\mathbb{D})$. In [2], it has been proved that the characteristic function θ_T of a homogeneous contraction T with an associated representation π is of the form

$$\theta_T(a) = \sigma_L(\phi_a)^*\theta_T(0)\sigma_R(\phi_a),$$

where σ_L and σ_R are projective representations of Möb with a common multiplier. We give another proof of the "product formula".

We point out that the defect operators of a homogeneous contraction in $B_2(\mathbb{D})$ are not always quasi-invertible (recall that an operator T is said to be *quasi-invertible* if T is injective and $\mathrm{ran}(T)$ is dense).

We prove that when the defect operators of a homogeneous contraction in $B_2(\mathbb{D})$ are not quasi-invertible, the projective representations σ_L and σ_R are unitarily equivalent to the holomorphic Discrete series representations $D^+_{\lambda-1}$ and $D^+_{\lambda+3}$, respectively. Also, we prove that when the defect operators of a homogeneous contraction in $B_2(\mathbb{D})$ are quasi-invertible, the two representations σ_L and σ_R are unitarily equivalent to certain known pair of representations $D_{\lambda-1,\mu_2}$ and $D_{\lambda+1,\mu_1}$, respectively. These are described explicitly.

In chapter 5, we show that a commuting tuple $(T_1, T_2, \ldots, T_n)$ in the Cowen-Douglas class of rank 1 is homogeneous with respect to G if and only if it is unitarily equivalent to the tuple of the multiplication operators on either the reproducing kernel Hilbert space with reproducing kernel $\prod_{i=1}^n \frac{1}{(1-z_i\bar{w}_i)^{\lambda_i}}$ or $\prod_{i=1}^n \frac{1}{(1-z_i\bar{w}_i)^{\lambda}}$, where λ, λ_i, $1 \le i \le n$, are positive real numbers, according as G is (i) the direct product of n-copies of the bi-holomorphic automorphism group of the disc, denoted by Möb^n or (ii) the bi-holomorphic automorphism group of the polydisc $\mathbb{D}^n$, denoted by $\mathrm{Aut}(\mathbb{D}^n)$. Finally, we show that a commuting tuple $(T_1, T_2, \ldots, T_n)$ in the Cowen-Douglas class of rank 2 is homogeneous with respect to Möb^n if and only if it is unitarily equivalent to the tuple of the multiplication operators on the reproducing kernel Hilbert space whose reproducing kernel is a product of $n-1$ rank one kernels and a rank two kernel. We also show that there is no irreducible tuple of operators in $B_2(\mathbb{D}^n)$, which is homogeneous with respect to the group $\mathrm{Aut}(\mathbb{D}^n)$.

1.0.2 Preliminaries

For any projective representation π of Möb, let $\pi^\#$ be the representation of Möb defined by $\pi^\#(\phi) = \pi(\phi^*)$ where $\phi^*(z) = \overline{\phi(\bar{z})}$, $z \in \mathbb{D}$, for every ϕ in Möb.

Proposition 1.12. *[9, Proposition 2.1] Suppose T is a homogeneous operator and π is an associated representation of T. Then the adjoint, T^*, is also homogeneous and $\pi^\#$ is an associated representation of T^*. If T is invertible, then T^{-1} is also homogeneous and $\pi^\#$ is an associated representation of T^{-1}. In particular, T and T^{*-1} have the same associated representation.*

To fix notation and terminology, we reproduce below a complete list of irreducible projective representations of Möb from [9].

(i) Holomorphic Discrete series representations $D_\lambda^+, \lambda > 0$: The holomorphic Discrete series representation, denoted by D_λ^+, acts on the reproducing kernel Hilbert space $H^{(\lambda)}$ determined by the positive definite kernel $\frac{1}{(1-z\bar{w})^\lambda}$, $z, w \in \mathbb{D}$, by the rule

$$\left(D_\lambda^+(\phi^{-1})f\right)(z) = \left(\phi'(z)\right)^{\frac{\lambda}{2}}(f\circ\phi)(z),\, \phi \in \text{Möb},\; f \in H^{(\lambda)},\; z \in \mathbb{D}.$$

(ii) Anti-holomorphic Discrete series representations $D_\lambda^-, \lambda > 0$: The anti-holomorphic Discrete series representation is denoted by D_λ^-. The representation space for D_λ^- is the reproducing kernel Hilbert space $H^{(\lambda)}$. The representation D_λ^- is defined by $D_\lambda^- := (D_\lambda^+)^\#$.

(iii) Principal series representations $P_{\lambda,s}$, $-1 < \lambda \le 1$, s purely imaginary: The representation space of each $P_{\lambda,s}$ is $L^2(\mathbb{T})$. The action of $P_{\lambda,s}$ on $L^2(\mathbb{T})$ is given by

$$\left(P_{\lambda,s}(\phi^{-1})f\right)(z) = \left(\phi'(z)\right)^{\frac{\lambda}{2}}\left|\phi'(z)\right|^{\mu}(f\circ\phi)(z),\; \phi \in \text{Möb},\; f \in L^2(\mathbb{T}),\; z \in \mathbb{T},$$

where $\mu = \frac{1-\lambda}{2} + s$. We point out that the representations $P_{\lambda,s}$ and $P_{\lambda,-s}$ are equivalent.

(iv) Complementary series representations $C_{\lambda,\sigma}$, $-1 < \lambda < 1$, $0 < \sigma < \frac{1}{2}(1-|\lambda|)$: The representation space of $C_{\lambda,\sigma}$ is the Hilbert space spanned by the orthogonal set of vectors $\left\{f_n : \mathbb{T} \to \mathbb{C} \mid f_n(z) = z^n\right\}_{n\in\mathbb{Z}}$ where $\|f_n\| = \frac{\Gamma(1-\mu+n)}{\Gamma(\lambda+\mu+n)}$, $n \in \mathbb{Z}$, $\mu = \frac{1-\lambda}{2} + \sigma$. The action of the representation $C_{\lambda,\sigma}$ on the Hilbert space $H^{\lambda,\sigma}$ is given by

$$\left(C_{\lambda,\sigma}(\phi^{-1})f\right)(z) = \left(\phi'(z)\right)^{\frac{\lambda}{2}}\left|\phi'(z)\right|^{\mu}(f\circ\phi)(z),\; \phi \in \text{Möb},\; f \in H^{\lambda,\sigma},\; z \in \mathbb{T}.$$

Note that the Complementary series representations and Principal series representations are together called Continuous series representations.

Remark 1.13. It is known that all the Principal series representations are irreducible except $P_{1,0}$. The representation $P_{1,0}$ is a direct sum of two irreducible representations, one of which is equivalent to the holomorphic Discrete series representation D_1^+ and the other one is equivalent to the anti-holomorphic Discrete series representation D_1^-.

Let $n : \text{Möb} \times \text{Möb} \to \mathbb{Z}$ be the measurable function defined by

$$n(\phi_1^{-1}, \phi_2^{-1}) = \frac{1}{2\pi}\left(\arg(\phi_2\phi_1)'(0) - \arg\phi_1'(0) - \arg\phi_2'(\phi_1(0))\right),\; \phi_1, \phi_2 \in \text{Möb}.$$

Using the chain rule, it is easy to check that n is an integer valued function. For any $w \in \mathbb{T}$, define $m_w : \text{Möb} \times \text{Möb} \to \mathbb{T}$ by

$$m_w(\phi_1, \phi_2) = w^{n(\phi_1,\phi_2)}. \tag{1.1}$$

Theorem 1.14. *[9, Theorem 3.2] (a) For each w in $\mathbb{T}$, the map m_w is a multiplier of Möb. Up to equivalence, these are all multipliers of Möb. Furthermore, these are mutually inequivalent multipliers and therefore $H^2(\text{Möb}, \mathbb{T})$ is naturally isomorphic to $\mathbb{T}$ via the map $w \mapsto [m_w]$.*

(b) For each representation of Möb in the above list, the associated multiplier is m_w where $w = e^{i\pi\lambda}$, except for the anti-holomorphic Discrete series representations for which $w = e^{-i\pi\lambda}$.

Corollary 1.15. *[9, Corollary 3.2] Let π and σ be two representations from the above list of irreducible representations of Möb. The multipliers of π and σ are either equal or inequivalent. If both or neither of π and σ are from the anti-holomorphic Discrete series representations, then they have same multiplier if and only if their λ parameters differ by an even integer. If exactly one of π and σ is from the anti-holomorphic Discrete series representations, then they have the same multiplier if and only if their λ parameters add to an even integer.*

A projective representation π of Möb on a Hilbert space H, containing a dense subspace $\mathscr{M}$ consisting of functions on some set X, is called a multiplier representation if

$$\left(\pi(\phi^{-1})f\right)(x) = c(\phi, x)(f \circ \phi)(x),\ \phi \in \text{Möb},\ f \in \mathscr{M},\ x \in X$$

where c is a non-vanishing measurable function on Möb $\times X$.

Theorem 1.16. *[9, Theorem 2.3] Suppose there is a multiplier representation π of Möb on a Hilbert space H, containing a dense subspace $\mathscr{M}$ consisting of functions on some set X. Suppose the operator T given on $\mathscr{M}$ by*

$$(Tf)(x) = xf(x),\ f \in \mathscr{M},\ x \in X$$

leaves $\mathscr{M}$ invariant and has a bounded extension to H. Then the extension of T is homogeneous and π is associated with T.

From the list of the irreducible projective representations of Möb, we see that every irreducible projective representation of Möb is a multiplier representation. Therefore Theorem 1.16 says that if the multiplication by the coordinate function on the representation space of an irreducible projective representation of Möb is bounded, then it must be homogeneous. Indeed this is true. The following list of homogeneous operators is given in [9, List 4.1].

(i) The Principal series example: The unweighted bilateral shift B is homogeneous. To prove this, apply Theorem 1.16 to any Principal series representation. Up to unitary equivalence, the operator B is the only weighted shift which is reducible [9, Theorem 2.1].

(ii) The Holomorphic Discrete series examples: For any positive real number $\lambda > 0$, the multiplication operator, $M^{(\lambda)}$, on the reproducing kernel Hilbert space $H^{(\lambda)}$ with reproducing kernel $\frac{1}{(1-z\bar{w})^\lambda}$, $z, w \in \mathbb{D}$, is homogeneous. Applying Theorem 1.16, we see that the holomorphic Discrete series representation D_λ^+ is associated with it. These are irreducible operators.

(iii) The anti-Holomorphic Discrete series examples: Since the adjoint of a homogeneous operator is homogeneous, it follows that $M^{(\lambda)^*}$, $\lambda > 0$, is homogeneous. Proposition 1.12 implies that the representation D_λ^- is associated with $M^{(\lambda)^*}$.

(iv) The Complementary series examples: For any two real numbers a, b in $(0,1)$, let $T(a,b)$ be the bilateral weighted shift with weight sequence $\left\{\sqrt{\frac{n+a}{n+b}} : n \in \mathbb{Z}\right\}$. If $0 < a < b < 1$, then applying Theorem 1.16 to the Complementary series representation $C_{\lambda,\sigma}$ with $\lambda = a+b-1$ and $\sigma = \frac{b-a}{2}$, it follows that $T(a,b)$ is homogeneous with associated representation $C_{\lambda,\sigma}$. Since $T(b,a) = T(a,b)^{*-1}$, using Proposition 1.12 it follows that the operator $T(b,a)$ is also a homogeneous operator with associated representation $C_{\lambda,\sigma}$. These are irreducible operators.

(v) The Constant Characteristic examples: For $x > 0$, let B_x be the bilateral weighted shift with weight sequence $\{\ldots, 1, 1, x, 1, 1, \ldots\}$, where x is the zeroth weight. In [5], it has been proved that the operator B_x is homogeneous with associated representation $P_{1,0}$. For $x \neq 1$, the operator B_x is irreducible. For $0 < x < 1$, the operator B_x is completely non unitary and the characteristic function of B_x is constant.

Remark 1.17. The operators given in the above list are not unitarily equivalent (see [9, 31]). Also note that the only contractions in the above list of homogeneous operators are $M^{(\lambda)}, \lambda \geq 1$ (consequently $M^{(\lambda)^*}$) and B_x, $0 < x \leq 1$.

The following theorem completes the description of all homogeneous shifts which was obtained by Bagchi and Misra in [9].

Theorem 1.18. *[9, Theorem 5.2] Up to unitary equivalence, the only homogeneous scalar weighted shifts with non-zero weights are the ones given in the above list of homogeneous weighted shifts.*

The first examples of irreducible bi-lateral homogeneous 2-shifts were given by Korányi in [20]. Recall from Theorem 1.11 that if $\pi(\phi)^* T_i \pi(\phi) = \phi(T_i)$, $i = 1, 2$, for some representation of Möb, then the operator $\begin{pmatrix} T_1 & \alpha(T_1 - T_2) \\ 0 & T_2 \end{pmatrix}$, $\alpha > 0$, is homogeneous. From the above list of homogeneous weighted shifts, we see that for $0 < a < b < 1$, the bi-lateral shifts $T(a,b)$ and $T(b,a)$ with weights $\sqrt{\frac{n+a}{n+b}}$ and $\sqrt{\frac{n+b}{n+a}}$, respectively, are homogeneous and the associated representation is the Complementary series $C_{\lambda,\sigma}$, where $\lambda = a+b-1$ and $\sigma = (b-a)/2$. Consequently, the operator $\left(\begin{smallmatrix} T(a,b) & \alpha(T(a,b)-T(b,a)) \\ 0 & T(b,a) \end{smallmatrix}\right)$, $\alpha \in \mathbb{C}$, is homogeneous. In [20], Korányi shows that the family

$$\mathscr{C} := \left\{ T(a,b,\alpha) = \begin{bmatrix} T(a,b) & \alpha(T(a,b)-T(b,a)) \\ 0 & T(b,a) \end{bmatrix} : 0 < a < b < 1, \alpha > 0 \right\}$$

contains all irreducible homogeneous operators, modulo unitary equivalence, whose associated representation is $C_{\lambda,\sigma} \oplus C_{\lambda,\sigma}$.

Chapter 2

Irreducible Homogeneous 2-shifts

A bounded operator T on a Hilbert space H, which is always assumed to be complex and separable, is said to be a shift if H admits a direct sum decomposition of the form $\oplus_{i\in I} H_i$, where each H_i is a closed subspace of H and T maps H_i into H_{i+1}, $i \in I$. The operator T is a bi-lateral, forward or backward shift according as I equals $\mathbb{Z}$, $\{n \in \mathbb{Z} : n \geq n_0\}$ or $\{n \in \mathbb{Z} : n \leq n_0\}$. Also, the realization of an irreducible operator as a block shift is uniquely determined, that is, there is exactly one possible decomposition of the Hilbert space on which T acts as a shift (see [9, Lemma 2.2]). An irreducible operator T is said to be an n-shift if dim $H_i = n$, for all $i \in I$ except for finitely many of them.

Let T be an irreducible homogeneous operator acting on a Hilbert space H. Then there exists a *projective unitary representation* π of Möb on H, associated with the operator T as shown in [9], that is, $\varphi(T) = \pi(\varphi)^* T\pi(\varphi)$ for all φ in the group Möb. Indeed this *associated representation* π is uniquely determined up to unitary equivalence.

Let $\mathbb{K}$ be the maximal compact subgroup consisting of those elements of Möb which fix the point 0. Recall that a subspace

$$V_n(\pi) := \{h : \pi(k)h = k^{-n}h,\ k \in \mathbb{K}\}$$

of the representation space H is said to be $\mathbb{K}$-isotypic. Setting $I(\pi) = \{n \in \mathbb{Z} : \dim\ V_n(\pi) \neq 0\}$, we note that the operator T must be a shift from $V_n(\pi)$ to $V_{n+1}(\pi)$, $n \in I(\pi)$ by virtue of [9, Theorem 5.1]. The set $I(\pi)$ is said to be connected if for any three elements a, b, c in $\mathbb{Z}$ with $a < b < c$ and $a, c \in I(\pi)$, then $b \in I(\pi)$.

Theorem 2.1. *Suppose T is an irreducible 2-shift homogeneous operator. Then the associated representation π is the direct sum of two or three or at most four irreducible representations.*

Proof. Let T be an irreducible homogeneous 2 - shift and π be the associated representation.

Since the $\mathbb{K}$-isotypic subspace of an irreducible projective representation is one dimensional (cf. [9]), it follows that π cannot be irreducible.

Thus we may assume without loss of generality that π is a direct sum of two non-trivial representations, say, $\pi_{00} \oplus \pi_{22}$. If both of them are irreducible, then we are done.

If not, one of them, say, π_{00} must be reducible. Then π_{00} is the direct sum of two non-trivial representations, namely, $\pi_{00} = \pi_{01} \oplus \pi_{21}$. Hence $\pi = \pi_{01} \oplus \pi_{21} \oplus \pi_{22}$. If all of them are irreducible, then we are done.

If not, one of them, say π_{01}, is reducible. Then π_{01} is the direct sum of two non-trivial representations, namely, $\pi_{01} = \pi_{11} \oplus \pi_{12}$. Then

$$\pi = \pi_{11} \oplus \pi_{12} \oplus \pi_{21} \oplus \pi_{22}.$$

Now, we claim that each summand in π must be irreducible. If not, then one of them, say, π_{11} is reducible. Then $\pi_{11} = \sigma \oplus \rho$, where σ and ρ are non-trivial representations. Therefore, we have the decomposition

$$\pi = \sigma \oplus \rho \oplus \pi_{12} \oplus \pi_{21} \oplus \pi_{22}.$$

Now [9, Lemma 3.2] says that connected component of each $I(\sigma)$, $I(\rho)$, $I(\pi_{12})$, $I(\pi_{21})$ and $I(\pi_{22})$ is unbounded. Therefore, each of $I(\sigma)$, $I(\rho)$, $I(\pi_{12})$, $I(\pi_{21})$ and $I(\pi_{22})$ contains a tail of $\mathbb{Z}$. This implies that one tail of $\mathbb{Z}$ must occur three times. Therefore, $\dim V_n(\pi) \geq 3$ for all those n in that tail of $\mathbb{Z}$ which occurs three times in $I(\pi)$. This contradicts the assumption that the operator T is a 2 - shift. Therefore each of $\pi_{11}, \pi_{12}, \pi_{21}$ and π_{22} must be irreducible. □

The following theorem lists the possibilities of the associated representation for an irreducible homogeneous 2-shift T.

Theorem 2.2. *If T is an irreducible homogeneous 2-shift and π is the associated representation, then π must be of the form*

> $\pi = \oplus_{i=1}^{2}\pi_i$: *In this case, the only possibilities for π_1 and π_2 are that they must be simultaneously one of the holomorphic Discrete series, anti-holomorphic Discrete series or Continuous series representations.*
>
> $\pi = \oplus_{i=1}^{3}\pi_i$: *In this case, one of the summands must be a Continuous series representation. Among the other two, one of them must be a holomorphic Discrete series and the other one an anti-holomorphic Discrete series representation.*
>
> $\pi = \oplus_{i=1}^{4}\pi_i$: *In this case, two of the summands must be holomorphic Discrete series representations while the other two summands must be anti-holomorphic Discrete series representations simultaneously.*

Proof. Now suppose π is a direct sum of two irreducible representations, say, $\pi = \pi_1 \oplus \pi_2$. If one of them is a holomorphic Discrete series representation then the other one also has to be a holomorphic Discrete series representation. Suppose not, then there is at least one tail $\mathscr{I}$ of $\mathbb{Z}$ such that the dimension of $V_n(\pi)$, $n \in \mathscr{I}$, is one. Similarly, if one of them is an anti-holomorphic Discrete series representation then the other one has to be an anti-holomorphic Discrete series representation. It follows that if one of these representations is from the Continuous series, then the other one cannot be either the holomorphic or the anti-holomorphic Discrete series representation. This completes the proof of the first case.

If π is a direct sum of three irreducible representations, then one of them must be from the Continuous series representations. If not, all the three summands are from the Discrete series representations. In consequence, the existence of a tail $\mathscr{I}$ in $\mathbb{Z}$ such that dimension of $V_n(\pi)$, $n \in \mathscr{I}$, is either one or three follows. This contradiction proves our claim. If one of the summands is a Continuous series representation, then the other two cannot be simultaneously holomorphic or anti-holomorphic Discrete series representations. If not, we find a tail $\mathscr{I}$ in $\mathbb{Z}$ for which the dimension of $V_n(\pi)$, $n \in \mathscr{I}$, is 3, which is a contradiction.

Now suppose π is the direct sum of four irreducible representations, say $\pi_1 \oplus \pi_2 \oplus \pi_3 \oplus \pi_4$. If one of them is a Continuous series representation, then there exists a tail of $\mathbb{Z}$ for which the dimension of $V_n(\pi)$ is greater than or equal to 3. So, none of the representations π_i, $1 \le i \le 4$, are from the Continuous series representations. Thus each one of the representations $\pi_1, \pi_2, \pi_3, \pi_4$ must be from the Discrete series. Now if three of them are either from the holomorphic or anti-holomorphic Discrete series representations, then the dimension of $V_n(\pi)$ must be greater than or equal to 3 for n in some tail of $\mathbb{Z}$. Therefore if π is a direct sum of four irreducible representation, then two of them have to be holomorphic Discrete series representations and the other two have to be anti-holomorphic Discrete series representations. □

Each of the three cases enumerated in Theorem 2.2 is discussed below in three different Sections.

2.1 The associated representation is the direct sum of two representations from the Continuous series

In this section, we find all the irreducible homogeneous operators for which the associated representation is a direct sum of two Continuous series representations. This naturally splits into several cases. In the paper [20], the case when the associated representation π is the direct sum $C_{\lambda,\sigma} \oplus C_{\lambda,\sigma}$, is discussed. Here we begin with the case when $\pi = \pi_1 \oplus \pi_2$, π_1, π_2 are form the Principal series.

2.1.1 $\pi = P_{\lambda,s} \oplus P_{\lambda,s}$

In this subsection, we find all the irreducible homogeneous operators for which the associated representation π is of the form $P_{\lambda,s} \oplus P_{\lambda,s}$. It is convenient to separate two cases, namely, the case of $s = 0$ and that of $s \neq 0$.

The case "s ≠ 0":

In what follows, we assume $s \neq 0$. Let $B(s)$ be the bounded linear transformation on $L^2(\mathbf{T})$ obtained by requiring that

$$B(s)\, z^n = \frac{n + \frac{1+\lambda}{2} + s}{n + \frac{1+\lambda}{2} - s} z^{n+1},\ n \in \mathbb{Z}.$$

Thus it is the weighted bilateral shift with weight sequence $\left\{ w_n = \frac{n+\frac{1+\lambda}{2}+s}{n+\frac{1+\lambda}{2}-s} \right\}$. Let B be the multiplication by the co-ordinate function on $L^2(\mathbf{T})$. The operator B is the unweighted bi-lateral shift. Both the operators $B(s)$ and B are known to be homogeneous [9, Theorem 5.2]. Each of the Principal series representations $P_{\lambda,s}$ may be taken to be the associated representation for both of these operators.

Proposition 2.3. *For all ϕ in Möb, suppose that*

$$SP_{\lambda,s}(\phi) - e^{i\theta} P_{\lambda,s}(\phi) S = \overline{a} B(s) P_{\lambda,s}(\phi) S + \overline{a} S P_{\lambda,s}(\phi) B, \tag{2.1}$$

for some operator S on $L^2(\mathbf{T})$. Then $S = \alpha\,(B(s) - B)$ for some $\alpha \in \mathbb{C}$.

Proof. Using homogeneity of $B(s)$ and B it is easy to see that $\alpha(B(s) - B)$ satisfies (2.1) for all $\alpha \in \mathbb{C}$. We show that these operators are the only solutions of the equation (2.1).

For the proof, let S be any operator for which (2.1) holds. Restricting the equation (2.1) to the subgroup of rotations of the form ϕ_θ, we see that S is a weighted shift operator with respect to the orthonormal basis $\{z^n\}$ in $L^2(\mathbb{T})$. Let $\{\alpha_n\}$ be the weight sequence of S. Now we find the value of α_n. Let $\phi_a(z) = -\frac{z-a}{1-\overline{a}z}$, $z \in \mathbb{D}$ be the set of involutions in the group Möb. Restricting to these in (2.1), we obtain

$$SP_{\lambda,s}(\phi_a) + P_{\lambda,s}(\phi_a) S = \overline{a} B(s) P_{\lambda,s}(\phi_a) S + \overline{a} S P_{\lambda,s}(\phi_a) B.$$

Evaluating on the vector z^m, we have

$$SP_{\lambda,s}(\phi_a) z^m + P_{\lambda,s}(\phi_a) S z^m = \overline{a} B(s) P_{\lambda,s}(\phi_a) S z^m + \overline{a} S P_{\lambda,s}(\phi_a) B z^m.$$

Therefore it follows that

$$SP_{\lambda,s}(\phi_a) z^m + \alpha_m P_{\lambda,s}(\phi_a) z^{m+1} = \overline{a} \alpha_m B(s) P_{\lambda,s}(\phi_a) z^{m+1} + \overline{a} S P_{\lambda,s}(\phi_a) z^{m+1}.$$

Consequently, we obtain

$$\langle P_{\lambda,s}(\phi_a)z^m, S^* z^n\rangle + \alpha_m \langle P_{\lambda,s}(\phi_a)z^{m+1}, z^n\rangle = \\ \overline{a}\alpha_m \langle P_{\lambda,s}(\phi_a)z^{m+1}, B(s)^* z^n\rangle + \overline{a}\langle P_{\lambda,s}(\phi_a)z^{m+1}, S^* z^n\rangle.$$

Since $S^* z^n = \bar{\alpha}_{n-1} z^{n-1}$ and $B(s)^* z^n = \bar{w}_{n-1} z^{n-1}$, it follows that

$$\alpha_{n-1}\langle P_{\lambda,s}(\phi_a)z^m, z^{n-1}\rangle + \alpha_m \langle P_{\lambda,s}(\phi_a)z^{m+1}, z^n\rangle = \overline{a}(\alpha_m w_{n-1} + \alpha_{n-1})\langle P_{\lambda,s}(\phi_a)z^{m+1}, z^{n-1}\rangle \tag{2.2}$$

The matrix coefficients of $P_{\lambda,s}(\phi_a)$ are given in [9, p. 316]:

$$\left\langle P_{\lambda,s}(\phi_a) f_m, f_n\right\rangle = c(-1)^n(\overline{a})^{n-m}\|f_n\|^2 \sum_{k\geq(m-n)^+} C_k(m,n) r^k, \tag{2.3}$$

where $f_n = z^n, r = |a|^2$, $c = \phi_a^{'}(0)^{\lambda/2}|\phi_a^{'}(0)|^{\mu}$ and $C_k(m,n) = \begin{pmatrix} -\lambda-\mu-m \\ k+n-m \end{pmatrix}\begin{pmatrix} -\mu+m \\ k \end{pmatrix}$. Using these matrix coefficients, we may rewrite the equation (2.2) in the form

$$\alpha_{n-1}\sum_{k\geq(m-n+1)^+} C_k(m,n-1)r^k - \alpha_m \sum_{k\geq(m-n+1)^+} C_k(m+1,n)r^k \\ = (\alpha_m w_{n-1} + \alpha_{n-1})\sum_{k\geq(m-n+2)^+} C_k(m+1,n-1)r^k.$$

Now putting $m = n$, we get

$$\alpha_{n-1}\sum_{k\geq 1} C_k(n,n-1)r^k - \alpha_n \sum_{k\geq 1} C_k(n+1,n)r^k = (\alpha_n w_{n-1} + \alpha_{n-1})\sum_{k\geq 2} C_k(n+1,n-1)r^k.$$

Comparing the coefficient of r, we have

$$\alpha_{n-1}C_1(n,n-1) - \alpha_n C_1(n+1,n) = 0.$$

Since $C_1(n,n-1) = (-\mu+n)$, $C_1(n+1,n) = (-\mu+n+1)$ and $\mu = \frac{1-\lambda}{2} + s$, we finally obtain

$$\alpha_{n-1}\left(-\frac{1-\lambda}{2} - s + n\right) - \alpha_n\left(-\frac{1-\lambda}{2} - s + n + 1\right) = 0,$$

which implies

$$\alpha_{n-1}(\lambda + 2n - 1 - 2s) = \alpha_n(\lambda + 2n + 1 - 2s).$$

An easy induction argument shows that $\alpha_n = \alpha\left(\frac{n+\frac{1+\lambda}{2}+s}{n+\frac{1+\lambda}{2}-s} - 1\right)$ for some $\alpha \in \mathbb{C}$. This shows that $S = \alpha(B(s) - B)$ for some $\alpha \in \mathbb{C}$.

□

Corollary 2.4. *The operator* $\begin{bmatrix} B(s) & \alpha(B(s)-B) \\ 0 & B \end{bmatrix}$ *is homogeneous with associate representation* $P_{\lambda,s} \oplus P_{\lambda,s}$.

Proof. The proof follows from Theorem 1.11. □

It is evident that $\left[\begin{smallmatrix} B(s) & \alpha(B(s)-B) \\ 0 & B \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} B(s) & \beta(B(s)-B) \\ 0 & B \end{smallmatrix}\right]$ are unitarily equivalent when $|\alpha| = |\beta|$. A particular case of what is proved in [20, Lemma 1.1] is that $\left[\begin{smallmatrix} B(s) & \alpha(B(s)-B) \\ 0 & B \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} B & \alpha(B-B(s)) \\ 0 & B(s) \end{smallmatrix}\right]$ are unitarily equivalent. We show that these are irreducible, which is very similar to the proof of [20, Theorem 1.2].

Theorem 2.5. *For a fixed but arbitrary* $\alpha > 0$, *the operator* $T := \begin{bmatrix} B(s) & \alpha(B(s)-B) \\ 0 & B \end{bmatrix}$ *is irreducible.*

Proof. Let $H(n)$ be the subspace of $L^2(\mathbb{T}) \oplus L^2(\mathbb{T})$ spanned by the orthonormal set

$$\mathscr{B}_n = \left\{ \begin{pmatrix} z^n \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ z^n \end{pmatrix} \right\}.$$

Clearly, T sends $H(n)$ to $H(n+1)$. Let $T_n := T_{|H(n)}$. The matrix representations of T_n, T_n^* with respect to $\mathscr{B}_n$ and $\mathscr{B}_{n+1}$ are of the form

$$\begin{bmatrix} w_n & \alpha(w_n-1) \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} \bar{w}_n & 0 \\ \alpha(\bar{w}_n-1) & 1 \end{bmatrix},$$

respectively. The operators $A_n = T_n^* T_n$ and $B_n = T_{n-1} T_{n-1}^*$ map $H(n)$ to $H(n)$, their matrix representation with respect to the orthonormal basis $\mathscr{B}_n$ is easy to compute, namely,

$$A_n = \begin{bmatrix} 1 & \alpha(1-\bar{w}_n) \\ \alpha(1-w_n) & 1+\alpha^2|w_n-1|^2 \end{bmatrix} \text{ and } B_n = \begin{bmatrix} 1+\alpha^2|w_{n-1}-1|^2 & \alpha(w_{n-1}-1) \\ \alpha(\bar{w}_{n-1}-1) & 1 \end{bmatrix}.$$

Since determinant of A_n is 1, and $A_n \neq I$, it follows that the eigenvalues of A_n are of the form $\lambda_n^2, \frac{1}{\lambda_n^2}$ for some real number $\lambda_n > 1$. Consequently, the trace of A_n is $\lambda_n^2 + \frac{1}{\lambda_n^2}$. Thus $\lambda_n^2 + \frac{1}{\lambda_n^2} = 2 + \alpha^2|w_n-1|^2$ and therefore $\left(\lambda_n - \frac{1}{\lambda_n}\right)^2 = \alpha^2|w_n-1|^2$.

Now suppose there exists n, m such that $|w_n-1|^2 = |w_m-1|^2$. Then putting the value of w_n and w_m, we get $|n + \frac{1+\lambda}{2} - s|^2 = |m + \frac{1+\lambda}{2} - s|^2$. Since $s = ia$, equivalently, $\left(n + \frac{1+\lambda}{2}\right)^2 + a^2 = \left(m + \frac{1+\lambda}{2}\right)^2 + a^2$ and it follows that $n = m$ or $n + m + 1 + \lambda = 0$. Consequently, if λ is not an integer, then $\lambda_n \neq \lambda_m$ for $n \neq m$. Since $-1 < \lambda \leq 1$, the possible integer values of λ are either 0 or 1. If $\lambda = 0$ then $\lambda_n = \lambda_{-n-1}$ and if $\lambda = 1$, then $\lambda_n = \lambda_{-n-2}$. Note that $\lambda_n \neq \lambda_m$ if $n \neq m$ and $n, m \geq 0$. Let $\lambda_n^{(1)} = \lambda_n^2$, and $\lambda_n^{(2)} = \frac{1}{\lambda_n^2}$. Pick an orthonormal basis $\{v_n^{(1)}, v_n^{(2)}\}$ of

$H(n)$ which makes A_n diagonal. Then $A_n v_n^{(i)} = \lambda_n^{(i)} v_n^{(i)}$. Let $u_n^{(i)} = T_{n-1} v_{n-1}^{(i)}$. Then $B_n u_n^{(i)} = T_{n-1} T_{n-1}^* T_{n-1} v_{n-1}^{(i)} = \lambda_{n-1}^{(i)} T_{n-1} v_{n-1}^{(i)} = \lambda_{n-1}^{(i)} u_n^{(i)}$. Also it is easily checked that $u_n^{(1)}$ and $u_n^{(2)}$ are orthogonal. So, $\{u_n^{(1)}, u_n^{(2)}\}$ is an orthogonal basis of $H(n)$ which makes B_n diagonal.

Suppose $u_n^{(1)} = c v_n^{(1)}$ for $c \in \mathbb{C}$, then we show that $u_n^{(2)} = d v_n^{(2)}$ for some $d \in \mathbb{C}$. Find $d_1, d_2 \in \mathbb{C}$ such that $u_n^{(2)} = d_1 v_n^{(1)} + d_2 v_n^{(2)}$. Taking inner product of $v_n^{(1)}$ with $u_n^{(2)}$, we see that $d_1 = 0$ using the equality $v_n^{(1)} = \frac{1}{c} u_n^{(1)}$ and the orthogonality of the two vectors $u_n^{(1)}$, $u_n^{(2)}$. Thus we conclude that $u_n^{(2)}$ is a scalar multiple of $v_n^{(2)}$.

Similarly, we can show that if $u_n^{(1)}$ is a scalar multiple of $v_n^{(2)}$, then $u_n^{(2)}$ is a scalar multiple of $v_n^{(1)}$. This shows that if one of $\{v_n^{(1)}, v_n^{(2)}\}$ is a scalar multiple of one of $\{u_n^{(1)}, u_n^{(2)}\}$, then the same is true of the other one. If this statement is true for all n, then we must have $A_n B_n - B_n A_n = 0$ for all n. But an easy computation shows that $A_n B_n \neq B_n A_n$ for any $n \geq 1$.

Now let $\mathscr{K}$ be a reducing subspace of T. Then $\mathscr{K}$ is an invariant subspace of both TT^* and T^*T and therefore, for $f \in \mathscr{K}$, the projections of f onto any eigenspaces of TT^* and T^*T are also in $\mathscr{K}$.

$\lambda \neq 0, 1$: Let $\mathscr{A}_{n,i}$ be the space spanned by the vector $v_n^{(i)}$. It is the eigenspace of T^*T with eigenvalue $\lambda_n^{(i)}$. Then $L^2(\mathbb{T}) \oplus L^2(\mathbb{T}) = \oplus_{n \in \mathbb{Z}, i=1,2} \mathscr{A}_{n,i}$. Let $f \in \mathscr{K}$. Then $f = \sum_{n \in \mathbb{Z}, i=1,2} \alpha_{n,i} v_n^{(i)}$.
Since f is non-zero, we can find n, i such that $\alpha_{n,i} \neq 0$. Therefore, the vector $v_n^{(i)}$ is in $\mathscr{K}$. This implies that $\mathscr{K} \cap H(n) \neq \emptyset$, for some $n \in \mathbb{Z}$.

$\lambda = 0$: Let $\mathscr{A}_{n,i}$, be the space spanned by the two vectors $v_n^{(i)}, v_{-n-1}^{(i)}$. It is the eigenspace of T^*T with eigenvalue $\lambda_n^{(i)}$. Then $L^2(\mathbb{T}) \oplus L^2(\mathbb{T}) = \oplus_{n \geq 0, i=1,2} \mathscr{A}_{n,i}$. Now suppose $f \in \mathscr{K}$. Then

$$f = \sum_{n \geq 0, i=1,2} \alpha_{n,i} h_{n,i},$$

where $h_{n,i}$ is in $\mathscr{A}_{n,i}$. Since $f \neq 0$, we can find $\alpha_{n,i} \neq 0$ for some n, i. Also there exist scalars γ, δ such that $h_{n,i} = \gamma v_{-n-1}^{(i)} + \delta v_n^{(i)}$. Since $v_{-n-1}^{(i)} \in H(-n-1)$ and $v_n^{(i)} \in H(n)$, applying T^{n+2} we see that $T^{n+2} h_{n,i} = \tilde{\gamma} h_1 + \tilde{\delta} h_{2n+2}$ for some $h_1 \in H(1)$ and $h_{2n+2} \in H(2n+2)$. Therefore, there are scalars $\gamma_1, \gamma_2, \delta_1, \delta_2$, such that $h_1 = \gamma_1 v_1^{(1)} + \gamma_2 v_1^{(2)}$ and $h_{2n+2} = \delta_1 v_{2n+2}^{(1)} + \delta_2 v_{2n+2}^{(2)}$. So,

$$T^{n+2} h_{n,i} = \tilde{\gamma}\gamma_1 v_1^{(1)} + \tilde{\gamma}\gamma_2 v_1^{(2)} + \tilde{\delta}\delta_1 v_{2n+2}^{(1)} + \tilde{\delta}\delta_2 v_{2n+2}^{(2)}.$$

Note that $v_1^{(1)} \in \mathscr{A}_{1,1}$, $v_1^{(2)} \in \mathscr{A}_{1,2}$, $v_{2n+2}^{(1)} \in \mathscr{A}_{2n+2,1}$ and $v_{2n+2}^{(2)} \in \mathscr{A}_{2n+2,2}$. Each of these correspond to distinct eigenspaces of T^*T. Since $h_{n,i}$ is non-zero, so is $T^{n+2} h_{n,i}$. Therefore one of the coefficients of this sum must be non zero. This implies that one of $v_1^{(1)}, v_1^{(2)}, v_{2n+2}^{(1)}$ or $v_{2n+2}^{(2)}$ is in $\mathscr{K}$. It follows that $H(n) \cap \mathscr{K} \neq \emptyset$ for some n.

$\lambda = 1$: A similar calculation as in the case of $\lambda = 0$ ensures the existence of some n with $\mathscr{K} \cap H(n) \neq \emptyset$.

These three cases ensure the existence of n such that $\mathcal{K} \cap H(n) \neq \emptyset$. Since each T_n is invertible, by applying T^k for sufficiently large k it follows that there exists $m > 0$ such that $\mathcal{K} \cap H(m) \neq \emptyset$. Pick a non-zero element h_m from $\mathcal{K} \cap H(m)$. Then $h_m = \alpha v_m^{(1)} + \beta v_m^{(2)} = \gamma u_m^{(1)} + \delta u_m^{(2)}$. We have already shown that $A_m B_m - B_m A_m \neq 0$, therefore either $\alpha\beta \neq 0$ or $\gamma\delta \neq 0$. If $\alpha\beta \neq 0$, then $v_m^{(1)}, v_m^{(2)} \in \mathcal{K}$ since $v_m^{(1)}, v_m^{(2)}$ are in different eigenspaces of T^*T. Similarly, $u_m^{(1)}, u_m^{(2)} \in \mathcal{K}$ if $\gamma\delta \neq 0$. We conclude that $H(m) \subseteq \mathcal{K}$. Now since T_n is invertible for all n, applying T^n and T^{*n} on $H(m)$, we find that $H(k) \subseteq \mathcal{K}$ for all k. This implies that $\mathcal{K} = L^2(\mathbb{T}) \oplus L^2(\mathbb{T})$ completing the proof. □

Let $B(\lambda, s, \alpha)$ denote the operator $\begin{bmatrix} B(s) & \alpha(B(s) - B) \\ 0 & B \end{bmatrix}$. Now we show that the unitary equivalence class of $B(\lambda, s, \alpha)$ depends only on $\lambda, |a|$, (where $s = ia$) and $|\alpha|$.

Theorem 2.6. *The operators $B(\lambda_1, s_1, \alpha_1)$ and $B(\lambda_2, s_2, \alpha_2)$ are unitarily equivalent if and only if $\lambda_1 = \lambda_2$, $a_1 = a_2$ and $\alpha_1 = \alpha_2$ for any choice of a pair of purely imaginary numbers $s_1 = ia_1, s_2 = ia_2$, $a_1, a_2 > 0$, and $\alpha_1, \alpha_2 > 0$.*

Proof. The operators $B(\lambda_i, s_i, \alpha_i)$ are homogeneous with associated representation $P_{\lambda_i, s_i} \oplus P_{\lambda_i, s_i}$ for $i = 1, 2$, see Corollary 2.4. If $\lambda_1 \neq \lambda_2$ then the multipliers of $P_{\lambda_1, s_1} \oplus P_{\lambda_1, s_1}$ and $P_{\lambda_2, s_2} \oplus P_{\lambda_2, s_2}$ are inequivalent [9, Corollary 3.2]. Therefore $P_{\lambda_1, s_1} \oplus P_{\lambda_1, s_1}$ and $P_{\lambda_2, s_2} \oplus P_{\lambda_2, s_2}$ are inequivalent. Since the representation associated with an irreducible homogeneous operator is uniquely determined, it follows that $B(\lambda_1, s_1, \alpha_1)$ and $B(\lambda_2, s_2, \alpha_2)$ cannot be inequivalent and consequently $\lambda_1 = \lambda_2$.

Now, setting $\lambda_1 = \lambda_2 = \lambda$, we show that if $B(\lambda, s_1, \alpha_1)$ and $B(\lambda, s_2, \alpha_2)$ are equivalent, then $s_1 = s_2$ and $\alpha_1 = \alpha_2$.

The set of singular values of the operators $B(\lambda_1, s_1, \alpha_1)$ and $B(\lambda_2, s_2, \alpha_2)$ are

$$\mathcal{S}_1 := \left\{ \alpha_1^2 \frac{|2s_1|^2}{|n + \frac{1+\lambda}{2} - s_1|^2} : n \in \mathbb{Z} \right\} \text{ and } \mathcal{S}_2 := \left\{ \alpha_2^2 \frac{|2s_2|^2}{|n + \frac{1+\lambda}{2} - s_2|^2} : n \in \mathbb{Z} \right\},$$

respectively. Since $B(\lambda, s_1, \alpha_1)$ and $B(\lambda, s_2, \alpha_2)$ are equivalent, it follows that the set of singular values of these two operators must be the same, that is, $\mathcal{S}_1 = \mathcal{S}_2$. To complete the proof, we discuss three cases.

$\lambda < 0$: In this case the maximum of the sets $\mathcal{S}_1$ and $\mathcal{S}_2$, which is achieved at $n = 0$ in both cases, must be equal, that is,

$$\frac{4\alpha_1^2 a_1^2}{(\frac{1+\lambda}{2})^2 + a_1^2} = \frac{4\alpha_2^2 a_2^2}{(\frac{1+\lambda}{2})^2 + a_2^2}. \tag{2.4}$$

Removing this maximum from both $\mathscr{S}_1$ and $\mathscr{S}_2$, again, the maximum in each of them is achieved at $n = -1$ and they must be equal, that is,

$$\frac{4\alpha_1^2 a_1^2}{(-1+\frac{1+\lambda}{2})^2 + a_1^2} = \frac{4\alpha_2^2 a_2^2}{(-1+\frac{1+\lambda}{2})^2 + a_2^2}. \tag{2.5}$$

Combining equations (2.4) and (2.5), we obtain the equation

$$\alpha_1^2 a_1^2 = \alpha_2^2 a_2^2.$$

Using this relationship in (2.4), we find that $a_1^2 = a_2^2$. Since both a_1 and a_2 are positive, it follows that $a_1 = a_2$. Therefore $\alpha_1 = \alpha_2$.

$\lambda = 0$: As before, in this case, the maximum and the second maximum value of $\mathscr{S}_1$ and $\mathscr{S}_2$ are achieved at $n = 0$ and $n = 1$, respectively. So, equating these two values, we get

$$\frac{4\alpha_1^2 a_1^2}{\frac{1}{4} + a_1^2} = \frac{4\alpha_2^2 a_2^2}{\frac{1}{4} + a_2^2} \text{ and } \frac{4\alpha_1^2 a_1^2}{\frac{9}{4} + a_1^2} = \frac{4\alpha_2^2 a_2^2}{\frac{9}{4} + a_2^2}.$$

A similar calculation, as in the case of $\lambda < 0$, implies that $a_1 = a_1$ and $\alpha_1 = \alpha_2$.

$\lambda > 0$: One last time, we note that the maximum and the second maximum of the two sets $\mathscr{S}_1$ and $\mathscr{S}_2$ are achieved at $n = -1$ and $n = 0$, respectively. Equating these values, we have the equations:

$$\frac{4\alpha_1^2 a_1^2}{(-1+\frac{1+\lambda}{2})^2 + a_1^2} = \frac{4\alpha_2^2 a_2^2}{(-1+\frac{1+\lambda}{2})^2 + a_2^2}$$

and

$$\frac{4\alpha_1^2 a_1^2}{(\frac{1+\lambda}{2})^2 + a_1^2} = \frac{4\alpha_2^2 a_2^2}{(\frac{1+\lambda}{2})^2 + a_2^2}.$$

But these two equations are identical to the equations we had obtained in the case of $\lambda < 0$. Therefore we conclude that $a_1 = a_2$ and $\alpha_1 = \alpha_2$.

Thus $B(\lambda_1, s_1, a_1)$ and $B(\lambda_2, s_2, a_2)$ are equivalent if and only if $\lambda_1 = \lambda_2$, $a_1 = a_2$ and $\alpha_1 = \alpha_2$. □

If $U_{\lambda,s} : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ is the operator $U_{\lambda,s} z^n = \frac{\Gamma(n+\frac{1+\lambda}{2}-s)}{\Gamma(n+\frac{1+\lambda}{2}+s)} z^n$, then from [9, p. 318], we have

$U_{\lambda,s}$ is unitary,

$U_{\lambda,-s} = U_{\lambda,s}^*$,

$P_{\lambda,-s} U_{\lambda,s} = U_{\lambda,s} P_{\lambda,s}$, and $B(s) = U^*_{\lambda,s} B U_{\lambda,s}$.

Replacing s by $-s$, we see that $B(-s) = U^*_{\lambda,-s} B U_{\lambda,-s}$. This is the same as $B(-s) = U_{\lambda,s} B U^*_{\lambda,s}$. Consequently, $U_{\lambda,s} \oplus U_{\lambda,s}$ intertwines $\left[\begin{smallmatrix} B(s) & \alpha(B(s)-B) \\ 0 & B \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} B & \alpha(B-B(-s)) \\ 0 & B(-s) \end{smallmatrix}\right]$. It follows, after conjugating with a permutation, that $\left[\begin{smallmatrix} B(s) & \alpha(B(s)-B) \\ 0 & B \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} B(-s) & \alpha(B(-s)-B) \\ 0 & B \end{smallmatrix}\right]$ are equivalent. Hence

$$\mathscr{P} := \left\{ \begin{bmatrix} B(s) & \alpha(B(s)-B) \\ 0 & B \end{bmatrix} : \lambda, s = ia, a > 0, \alpha > 0 \right\}$$

is a mutually unitarily inequivalent set of homogeneous operators with associated representation $P_{\lambda,s} \oplus P_{\lambda,s}$.

The associated representation of the family of irreducible homogeneous operators

$$\mathscr{C} := \left\{ T(a,b,\alpha) = \begin{bmatrix} T(a,b) & \alpha(T(a,b) - T(b,a)) \\ 0 & T(b,a) \end{bmatrix} : 0 < a < b < 1, \alpha > 0 \right\}$$

is the direct sum of two copies of a Complementary series representation [20].

We now show that these two sets of homogeneous operators are mutually unitarily inequivalent.

Theorem 2.7. *The homogeneous operators in the two sets $\mathscr{P}$ and $\mathscr{C}$ are mutually unitarily inequivalent.*

Proof. Let $T(a,b,\alpha)$ and $B(\lambda_1, s, \beta)$ be unitarily equivalent for some

$(a,b,\alpha) : 0 < a < b < 1,\ \alpha > 0;$

$(\lambda_1, \beta, s) : -1 < \lambda_1 \le 1,\ \beta > 0$ and $s,\ k = \mathrm{Im}(s) > 0$.

The associated representation of the operator $T(a,b,\alpha)$ is $C_{\lambda,\sigma} \oplus C_{\lambda,\sigma}$, where $\lambda = a + b - 1$, $\sigma = \frac{b-a}{2}$ (cf. Theorem 1.11) and the associated representation of $B(\lambda_1, s, \beta)$ is $P_{\lambda_1,s} \oplus P_{\lambda_1,s}$, see Corollary 2.4. Since the representation associated with an irreducible homogeneous operator is uniquely determined, it follows that $C_{\lambda,\sigma} \oplus C_{\lambda,\sigma}$, and $P_{\lambda_1,s} \oplus P_{\lambda_1,s}$ must be equivalent. This, in particular, implies that their multipliers are equivalent and, therefore, λ_1 must be equal to λ. For the remaining portion of the proof, we therefore assume that $\lambda_1 = \lambda$ without loss of generality.

We know that $T(a,b,\alpha)$ and $B(\lambda, s, \beta)$ are 2-shifts. Let $\lambda_n, \frac{1}{\lambda_n}$ be the singular values of the n-th block of $T(a,b,\alpha)$ for each $n \in \mathbb{Z}$. From [20, p. 227], we have

$$\left(\lambda_n - \frac{1}{\lambda_n}\right)^2 = \frac{(1+\alpha^2)(a-b)^2}{\left(n + \frac{1+\lambda}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2}.$$

For $n \in \mathbb{Z}$, let $\mu_n, \frac{1}{\mu_n}$ be the singular values of the n-th block of $B(\lambda, s, \beta)$. We have shown (see Theorem 2.5) that

$$\left(\mu_n - \frac{1}{\mu_n}\right)^2 = \frac{4\beta^2 k^2}{\left(n + \frac{1+\lambda}{2}\right)^2 + k^2}.$$

Since $T(a, b, \alpha)$ and $B(\lambda, s, \beta)$ are unitarily equivalent, it follows that the two sets

$$\mathscr{S}_1 = \left\{ \frac{(1+\alpha^2)(a-b)^2}{\left(n + \frac{1+\lambda}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2} : n \in \mathbb{Z} \right\} \text{ and } \mathscr{S}_2 = \left\{ \frac{4\beta^2 k^2}{\left(n + \frac{1+\lambda}{2}\right)^2 + k^2} : n \in \mathbb{Z} \right\}$$

must be equal. The proof naturally splits into three separate cases.

$\lambda < 0$: Since $\mathscr{S}_1$ and $\mathscr{S}_2$ are equal, their maximum (which is achieved at $n = 0$ in both of these sets) must be the same. Equating these, we get

$$\frac{(1+\alpha^2)(a-b)^2}{\left(\frac{1+\lambda}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2} = \frac{4\beta^2 k^2}{\left(\frac{1+\lambda}{2}\right)^2 + k^2}. \tag{2.6}$$

Removing the maximal element from $\mathscr{S}_1$ and $\mathscr{S}_2$, we must get two equal sets and again the maximum in each of them, which is achieved at $n = -1$, must be equal, that is,

$$\frac{(1+\alpha^2)(a-b)^2}{\left(1 - \frac{1+\lambda}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2} = \frac{4\beta^2 k^2}{\left(1 - \frac{1+\lambda}{2}\right)^2 + k^2}. \tag{2.7}$$

Putting $\lambda = a+b-1$ in the equations (2.6) and (2.7), after a little simplification, we obtain

$$(1+\alpha^2)(a-b)^2 = \frac{4\beta^2 k^2 ab}{\left(\frac{1+\lambda}{2}\right)^2 + k^2},$$

and

$$(1+\alpha^2)(a-b)^2 = \frac{4\beta^2 k^2 (1-a)(1-b)}{\left(\frac{1-\lambda}{2}\right)^2 + k^2}.$$

Equating these two values of $(1+\alpha^2)(a-b)^2$, we get $(a-b)^2 + 4k^2 = 0$, which is a contradiction since $a < b$.

$\lambda > 0$: As in the case of $\lambda < 0$, equating the maximum of $\mathscr{S}_1$ and $\mathscr{S}_2$, which is achieved at $n = -1$ for both the sets, we get

$$\frac{(1+\alpha^2)(a-b)^2}{\left(1 - \frac{1+\lambda}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2} = \frac{4\beta^2 k^2}{\left(1 - \frac{1+\lambda}{2}\right)^2 + k^2}.$$

Like before, removing the maximal elements from both the sets $\mathscr{S}_1$ and $\mathscr{S}_2$, the maximum of these two sets, which is achieved at $n=0$, are again equal, that is,

$$\frac{(1+\alpha^2)(a-b)^2}{\left(\frac{1+\lambda}{2}\right)^2-\left(\frac{a-b}{2}\right)^2}=\frac{4\beta^2k^2}{\left(\frac{1+\lambda}{2}\right)^2+k^2}.$$

These two equations are the same as in the case of $\lambda<0$. Thus repeating the same calculation as in that case, we arrive at a contradiction.

$\lambda=0$: For a third time, equating the maximum of the two sets $\mathscr{S}_1$ and $\mathscr{S}_2$, which is achieved at $n=0$ for both of them, we get

$$\frac{(1+\alpha^2)(a-b)^2}{\frac{1}{4}-\frac{(a-b)^2}{4}}=\frac{4\beta k^2}{\frac{1}{4}+k^2}.$$

Equate the maximum of the sets $\mathscr{S}_1$ and $\mathscr{S}_2$ after removing the maximal elements. This new maximum is achieved at $n=1$, and we obtain

$$\frac{(1+\alpha^2)(a-b)^2}{\frac{9}{4}-\frac{(a-b)^2}{4}}=\frac{4\beta^2k^2}{\frac{9}{4}+k^2}.$$

Now, equating the value of $(1+\alpha^2)(a-b)^2$, we have

$$\frac{4\beta^2k^2\left(\frac{1}{4}-\frac{(a-b)^2}{4}\right)}{\frac{1}{4}+k^2}=\frac{4\beta^2k^2\left(\frac{9}{4}-\frac{(a-b)^2}{4}\right)}{\frac{9}{4}+k^2}.$$

This is equivalent to $4k^2+(a-b)^2=0$. But this is a contradiction since $a<b$.

It follows that $T(a,b,\alpha)$ is not equivalent to $B(\lambda_1,s,\beta)$ for any choice of (a,b,α) and (λ_1,s,β). □

The case of " s = 0":

Having disposed of the case of $s\neq 0$, in what follows, we assume $s=0$ with one exception in the Proposition below.

Proposition 2.8. *Suppose S is an operator on $L^2(\mathbb{T})$ for which the equation*

$$SP_{\lambda,s}(\phi)-e^{i\theta}P_{\lambda,s}(\phi)S=\overline{a}BP_{\lambda,s}(\phi)S+\overline{a}SP_{\lambda,s}(\phi)B,\ \phi\in\text{Möb},\tag{2.8}$$

holds. Then

(*a*) *if* $s \neq 0$, *then* $S = 0$;

(*b*) *if* $s = 0$ *and* $\lambda \neq 1$, *then* S *is a weighted shift operator on* $L^2(\mathbb{T})$ *with respect to the orthonormal basis* $\{z^n\}$ *with weight sequence* $\left\{\alpha_n = \frac{1}{\lambda+2n-1}\right\}$ *and*

(*c*) *if* $s = 0$ *and* $\lambda = 1$, *then* S *is a weighted shift operator on* $L^2(\mathbb{T})$ *with respect to the orthonormal basis* $\{z^n\}$ *with weight sequence* $\{\alpha_n\}$ *where each* $\alpha_n = 0$ *except* α_{-1}.

Proof. Restricting the equation (2.8) to the subgroup of rotations of the group Möb, we see that S is a weighted shift operator with respect to the orthonormal basis $\{z^n\}$ in $L^2(\mathbb{T})$. Let $\{\alpha_n\}$ be the weight sequence of S. Let $\phi_a \in$ Möb be an involution, i.e., $\phi_a(z) = -\frac{z-a}{1-\overline{a}z}$, $z \in \mathbb{D}$. Restricting the equation (2.8) to involutions of the form ϕ_a, evaluating on the vector z^m and then taking inner product with the vector z^n, we obtain

$$\alpha_{n-1}\langle P_{\lambda,s}(\phi_a)z^m, z^{n-1}\rangle + \alpha_m\langle P_{\lambda,s}(\phi_a)z^{m+1}, z^n\rangle = \overline{a}(\alpha_m + \alpha_{n-1})\langle P_{\lambda,s}(\phi_a)z^{m+1}, z^{n-1}\rangle$$

Using the matrix coefficients of the representation $P_{\lambda,s}(\phi_a)$ from the equation (2.3), we have the equality

$$\alpha_{n-1}\sum_{k\geq(m-n+1)^+} C_k(m,n-1)r^k - \alpha_m \sum_{k\geq(m-n+1)^+} C_k(m+1,n)r^k = (\alpha_m+\alpha_{n-1})\sum_{k\geq(m-n+2)^+} C_k(m+1,n-1)r^k. \quad (2.9)$$

Proof of (*a*) : Substituting $m = n-1$ in the equation (2.9) and comparing the coefficient of r^k, $k \geq 1$, we get

$$\alpha_{n-1}C_k(n-1,n-1) - \alpha_{n-1}C_k(n,n) = 2\alpha_{n-1}C_k(n,n-1).$$

Substituting the values of $C_k(n-1,n-1), C_k(n,n)$ and $C_k(n,n-1)$, we obtain the equation

$$\alpha_{n-1}(\lambda + 2\mu - 1) = 0.$$

Since $\mu = \frac{1-\lambda}{2} + s$, it follows that

$$2s\alpha_{n-1} = 0.$$

Therefore, if $s \neq 0$, then $\alpha_{n-1} = 0$ for all $n \in \mathbb{Z}$. This completes the proof of (*a*).

Proof of (*b*) : Putting $m = n$ in the equation (2.9) and comparing the coefficient of r, we have

$$\alpha_{n-1}(\lambda + 2n - 1) = \alpha_n(\lambda + 2n + 1) \quad (2.10)$$

after substituting the values $C_1(n,n-1)$ and $C_1(n+1,n)$. If $\lambda = 1$, then putting $n = 0$ we get $\alpha_0 = 0$. This recursion makes $\alpha_n = 0$ for all n except for $n = -1$. Thus α_{-1} remains arbitrary completing the proof of part (*b*).

Proof of (c) **:** Now assume that $\lambda \neq 1$. Then $\alpha_n = \frac{1}{\lambda+2n+1}$ is a solution to the recursion (2.10).

Equating the coefficient of r^k in the equation (2.9) and substituting the values of $C_k(m, n-1)$, $C_k(m+1, n)$, $C_k(m+1, n-1)$, we obtain

$$\alpha_{n-1}(\lambda + 2n - 1) = \alpha_m(\lambda + 2m + 1).$$

Thus $\alpha_n = \frac{1}{\lambda+2n+1}$ is a solution of this recursion. This shows that S satisfies (2.8) for any involution ϕ_a, $a \in \mathbb{D}$. Also S satisfies (2.8) for the subgroup of rotations ϕ_θ. Since any elements of Möb is composition of ϕ_θ and ϕ_a for some θ and a, it follows that S satisfies (2.8) for every element of Möb. This completes the proof of part (c).

□

Let $S[\lambda]$ be the weighted shift operator on $L^2(\mathbb{T})$ with respect to the orthonormal basis $\{z^n\}$ with weight sequence $\{\frac{1}{\lambda+2n-1}\}$. Also, let

$$B(\lambda,\alpha) : L^2(\mathbb{T}) \oplus L^2(\mathbb{T}) \to L^2(\mathbb{T}) \oplus L^2(\mathbb{T}),\ B(\lambda,\alpha) := \begin{bmatrix} B & \alpha S[\lambda] \\ 0 & B \end{bmatrix},\ \alpha \in \mathbb{C}.$$

Corollary 2.9 (Theorem 1.11)**.** *The operator $B(\lambda,\alpha)$ is homogeneous with associated representation $P_{\lambda,0} \oplus P_{\lambda,0}$ with $\lambda \neq 1$.*

Theorem 2.10. *For every fixed but arbitrary λ, α with $-1 < \lambda < 1$ and $\alpha > 0$, the operator $B(\lambda,\alpha)$ is irreducible.*

Proof. Let $H(n)$ and $\mathscr{B}_n$ be as in Theorem 2.5. Clearly, T maps $H(n)$ to $H(n+1)$. Let $T_n := T_{|H(n)}$. The matrix representation of T_n is of the form

$$\begin{bmatrix} 1 & \frac{\alpha}{\lambda+2n+1} \\ 0 & 1 \end{bmatrix}.$$

Let $A_n = T_n^* T_n$ and $B_n = T_{n-1} T_{n-1}^*$. Then both A_n and B_n are operators on $H(n)$. The matrix representation of A_n with respect to the orthonormal basis $\mathscr{B}_n$ is

$$\begin{bmatrix} 1 & \frac{\alpha}{\lambda+2n+1} \\ \frac{\alpha}{\lambda+2n+1} & \frac{\alpha^2}{(\lambda+2n+1)^2} + 1 \end{bmatrix}.$$

Since determinant of A_n is 1 and $A_n \neq I$, the eigenvalues of A_n are of the form λ_n^2 and $\frac{1}{\lambda_n^2}$ for some real number $\lambda_n > 1$. Then $\lambda_n^2 + \frac{1}{\lambda_n^2} = 2 + \frac{\alpha^2}{(\lambda+2n+1)^2}$ and consequently $\left(\lambda_n - \frac{1}{\lambda_n}\right)^2 = \frac{\alpha^2}{(\lambda+2n+1)^2}$. This implies that if $\lambda \neq 0$, then λ_n are distinct and if $\lambda = 0$, then $\lambda_n = -\lambda_{-n-1}$. Also from a straight forward computation we see that $A_n B_n - B_n A_n \neq 0$, for all $n \geq 1$. Now repeating the same argument as in Theorem 2.5 we conclude that $B(\lambda,\alpha)$ must be irreducible. □

Now we have another class of irreducible homogeneous operators, $\{B(\lambda,\alpha) : -1 < \lambda < 1, \alpha\}$, whose associated representation is $P_{\lambda,0} \oplus P_{\lambda,0}$. Clearly, $B(\lambda,\alpha)$ and $B(\lambda,|\alpha|)$ are unitarily equivalent.

Proposition 2.11. *Let $\alpha_1,\alpha_2 > 0$ and $-1 < \lambda_1,\lambda_2 < 1$. The operators $B(\lambda_1,\alpha_1)$ and $B(\lambda_2,\alpha_2)$ are unitarily equivalent if and only if $\lambda_1 = \lambda_2$ and $\alpha_1 = \alpha_2$.*

Proof. Assume that the operators $B(\lambda_1,\alpha_1)$ and $B(\lambda_2,\alpha_2)$ are unitarily equivalent. We infer, from a similar argument as in Theorem 2.6, that $\lambda_1 = \lambda_2$.

Since the set of singular values of $B(\lambda,\alpha_1)$ and $B(\lambda,\alpha_2)$ must be the same, it follows that the maximal elements of the two sets $\left\{\frac{\alpha_1^2}{(\lambda+2n+1)^2} : n \in \mathbb{Z}\right\}$ and $\left\{\frac{\alpha_2^2}{(\lambda+2n+1)^2} : n \in \mathbb{Z}\right\}$, which are achieved at $n = 0$, must be the same. Therefore equating them we have $\alpha_1 = \alpha_2$. □

We have therefore shown that the set of homogeneous operators

$$\mathscr{P}_0 = \{B(\lambda,\alpha) : -1 < \lambda < 1, \alpha > 0\}$$

is irreducible and mutually unitarily inequivalent. In summary, we have proved the following theorem.

Theorem 2.12. *(a) The homogeneous operators in the two sets $\mathscr{P}$ and $\mathscr{P}_0$ are mutually unitarily inequivalent.*

(b) The homogeneous operators in the two sets $\mathscr{C}$ and $\mathscr{P}_0$ are mutually unitary inequivalent.

Proof. The proof of the statement in (a) is similar to the proof of the Theorem 2.6 and the proof of the statement in (b) is similar to the proof of the Theorem 2.7. □

2.1.2 Classification

Theorem 2.13. *Let $\pi_1 = R_{\lambda_1,\mu_1}$ and $\pi_2 = R_{\lambda_2,\mu_2}$ be two representations from the continuous series excluding $P_{1,0}$ acting on the Hilbert spaces H_1 and H_2, respectively. Assume that $(\lambda_1,\mu_1) \neq (\lambda_2,\mu_2)$. Suppose*

$$T = \begin{bmatrix} T_1 & S_1 \\ S_2 & T_2 \end{bmatrix}$$

is a homogeneous operator on $H = H_1 \oplus H_2$ with associated representation $\pi_1 \oplus \pi_2$. Then either $S_1 = 0$ or $S_2 = 0$. Furthermore, $S_1 = 0$ and $S_2 = 0$ except when $R_{\lambda_1,\mu_1} = P_{\lambda,s}$ and $R_{\lambda_2,\mu_2} = P_{\lambda,-s}$.

Proof. Since T is a homogeneous operator with associated representation $\pi_1 \oplus \pi_2$, we have

$$\phi(T) = (\pi_1(\phi)^* \oplus \pi_2(\phi)^*)\,T\,(\pi_1(\phi) \oplus \pi_2(\phi)),\ \phi \in \text{Möb}.$$

For $\phi_{\theta,a}$ in Möb, this is equivalent to the four equations listed below:

$$e^{i\theta}\pi_1(\phi_{\theta,a})(T_1-aI)=T_1\pi_1(\phi_{\theta,a})(I-\overline{a}T_1)-\overline{a}S_1\pi_2(\phi_{\theta,a})S_2 \tag{2.11}$$

$$S_1\pi_2(\phi_{\theta,a})-e^{i\theta}\pi_1(\phi_{\theta,a})S_1=\overline{a}T_1\pi_1(\phi_{\theta,a})S_1+\overline{a}S_1\pi_2(\phi_{\theta,a})T_2 \tag{2.12}$$

$$S_2\pi_1(\phi_{\theta,a})-e^{i\theta}\pi_2(\phi_{\theta,a})S_2=\overline{a}S_2\pi_1(\phi_{\theta,a})T_1+\overline{a}T_2\pi_2(\phi_{\theta,a})S_2 \tag{2.13}$$

$$e^{i\theta}\pi_2(\phi_{\theta,a})(T_2-aI)=T_2\pi_2(\phi_{\theta,a})(I-\overline{a}T_2)-\overline{a}S_2\pi_1(\phi_{\theta,a})S_1. \tag{2.14}$$

Let $\langle .,.\rangle_i$ denote the inner product of H_i. We know that $\{z^n : n\in\mathbb{Z}\}$ is an orthogonal basis of H_i. Let $\phi_\theta\in$ Möb be such that $\phi_\theta(z)=e^{i\theta}z$. Then

$$\pi_i(\phi_\theta)z^n=e^{-i\left(n+\frac{\lambda_i}{2}\right)\theta}z^n$$

for all $n\in\mathbb{Z}$ and $i=1,2$. Since $-1<\frac{\lambda_1-\lambda_2}{2}<1$, the only possible integer value for $\frac{\lambda_1-\lambda_2}{2}$ is 0, it follows that there exists a θ such that

(i) for any pair $n,m\in\mathbb{Z}$, $e^{-i\left(n+\frac{\lambda_1}{2}\right)\theta}\neq e^{-i\left(m+\frac{\lambda_2}{2}\right)\theta}$, whenever $\lambda_1\neq\lambda_2$;

(ii) $e^{-i\left(n+\frac{\lambda_i}{2}\right)\theta}\neq e^{-i\left(m+\frac{\lambda_i}{2}\right)\theta}$ for $i=1,2$, whenever $n\neq m$.

Fix a θ as above. Evaluating the equation (2.11) on z^n for ϕ_θ, we get

$$\pi_1(\phi_\theta)T_1z^n=e^{-i\left(n+1+\frac{\lambda_1}{2}\right)\theta}T_1z^n.$$

This proves that T_1 is a weighted shift operator with respect to the orthonormal basis $\left\{\frac{z^n}{\|z^n\|_1}\right\}$. Let $\{u_n\}$ be the weight sequence of T_1.

Similarly, it may be shown that T_2 is a weighted shift operator with respect to the orthonormal basis $\left\{\frac{z^n}{\|z^n\|_2}\right\}$. Let $\{v_n\}$ be the weight sequence of T_2.

Evaluating the equation (2.12) on z^n and putting $\phi=\phi_\theta$, we obtain

$$\pi_1(\phi_\theta)S_1z^n=e^{-i\left(n+1+\frac{\lambda_2}{2}\right)\theta}S_1z^n. \tag{2.15}$$

If $\lambda_1\neq\lambda_2$, then the equation (2.15) implies that $S_1z^n=0$, $n\in\mathbb{Z}$. Consequently, $S_1=0$. Similarly, it can be shown that $S_2=0$, whenever $\lambda_1\neq\lambda_2$. Therefore, the proof, in this case, is complete and we may assume, without loss of generality, that $\lambda:=\lambda_1=\lambda_2$.

The existence of a sequence $\{\alpha_n : n\in\mathbb{Z}\}$ such $S_1e_n^2=\alpha_ne_{n+1}^1$, where $e_n^i=\frac{z^n}{\|z^n\|_i}$, $i=1,2$, follows from the equation (2.15). Now evaluating equation (2.12) on the vector e_m^2, putting $\phi=\phi_a$ and then taking inner product with e_n^1, we get

$$\begin{aligned}\alpha_{n-1}\left\langle\pi_2(\phi_a)e_m^2,e_{n-1}^2\right\rangle&+\alpha_m\left\langle\pi_1(\phi_a)e_{m+1}^1,e_n^1\right\rangle\\&=\overline{a}\alpha_mu_{n-1}\left\langle\pi_1(\phi_a)e_{m+1}^1,e_{n-1}^1\right\rangle+\overline{a}v_m\alpha_{n-1}\left\langle\pi_2(\phi_a)e_{m+1}^2,e_{n-1}^2\right\rangle.\end{aligned}$$

Using the matrix coefficient of $R_{\lambda,\mu_i}(\phi_a)$ (see [9, p. 316]), we obtain

$$\alpha_{n-1}\frac{\|z^{n-1}\|_2}{\|z^m\|_2}|\phi'_a(0)|^{\mu_2}\sum_{k\geq(m-n+1)^+}C_k^2(m,n-1)r^k-\alpha_m\frac{\|z^n\|_1}{\|z^{m+1}\|_1}|\phi'_a(0)|^{\mu_1}\sum_{k\geq(m-n+1)^+}C_k^1(m+1,n)r^k$$
$$=v_m\alpha_{n-1}\frac{\|z^{n-1}\|_2}{\|z^{m+1}\|_2}|\phi'_a(0)|^{\mu_2}\sum_{k\geq(m-n+2)^+}C_k^2(m+1,n-1)r^k$$
$$+\alpha_m u_{n-1}\frac{\|z^{n-1}\|_1}{\|z^{m+1}\|_1}|\phi'_a(0)|^{\mu_1}\sum_{k\geq(m-n+2)^+}C_k^1(m+1,n-1)r^k, \quad (2.16)$$

where $C_k^i(m,n)=\begin{pmatrix}-\lambda-\mu_i-m\\ k+n-m\end{pmatrix}\begin{pmatrix}-\mu_i+m\\ k\end{pmatrix}$, $i=1,2$. Putting $m=n-1$, we have

$$\alpha_{n-1}(1-r)^{(\mu_2-\mu_1)}\sum_{k\geq 0}C_k^2(n-1,n-1)r^k-\alpha_{n-1}\sum_{k\geq 0}C_k^1(n,n)r^k$$
$$=v_{n-1}\alpha_{n-1}\frac{\|z^{n-1}\|_2}{\|z^n\|_2}(1-r)^{(\mu_2-\mu_1)}\sum_{k\geq 1}C_k^2(n,n-1)r^k$$
$$+\alpha_{n-1}u_{n-1}\frac{\|z^{n-1}\|_1}{\|z^n\|_1}\sum_{k\geq 1}C_k^1(n,n-1)r^k.$$

Differentiating this equation with respect to r and then substituting $r=0$, we get

$$v_{n-1}\alpha_{n-1}\frac{\|z^{n-1}\|_2}{\|z^n\|_2}(-\mu_2+n)+\alpha_{n-1}u_{n-1}\frac{\|z^{n-1}\|_1}{\|z^n\|_1}(-\mu_1+n)$$
$$=\alpha_{n-1}[(\mu_2-\mu_1)(\lambda+\mu_2+\mu_1-1)+(\lambda+2n-1)].$$

It follows that if $\alpha_{n-1}\neq 0$, then

$$v_{n-1}\tfrac{\|z^{n-1}\|_2}{\|z^n\|_2}(-\mu_2+n)+u_{n-1}\tfrac{\|z^{n-1}\|_1}{\|z^n\|_1}(-\mu_1+n)=(\mu_2-\mu_1)(\lambda+\mu_2+\mu_1-1)+(\lambda+2n-1), n\in\mathbb{Z}. \quad (2.17)$$

The existence of a sequence $\{\beta_n\}$ such that $S_2e_n^1=\beta_n e_{n+1}^2$, $n\in\mathbb{Z}$, follows from a similar computation. As before, for the sequence β_{n-1}, we also have

$$v_{n-1}\beta_{n-1}\frac{\|z^{n-1}\|_2}{\|z^n\|_2}(-\mu_2+n)+\beta_{n-1}u_{n-1}\frac{\|z^{n-1}\|_1}{\|z^n\|_1}(-\mu_1+n)$$
$$=\beta_{n-1}[(\mu_1-\mu_2)(\lambda+\mu_2+\mu_1-1)+(\lambda+2n-1)].$$

It follows that if $\beta_{n-1}\neq 0$, then

$$v_{n-1}\tfrac{\|z^{n-1}\|_2}{\|z^n\|_2}(-\mu_2+n)+u_{n-1}\tfrac{\|z^{n-1}\|_1}{\|z^n\|_1}(-\mu_1+n)=(\mu_1-\mu_2)(\lambda+\mu_2+\mu_1-1)+(\lambda+2n-1), n\in\mathbb{Z}. \quad (2.18)$$

Equating the right hand sides of equations (2.17) and (2.18), we get $\mu_1=\mu_2$, contradicting our hypothesis that $\mu_1\neq\mu_2$. Therefore, we can find an integer n such that either $\alpha_{n-1}=0$ or $\beta_{n-1}=0$. Assume that $\alpha_p=0$, for some integer p.

Now putting $m = n$ in the equation (2.16), we get

$$\begin{aligned}\alpha_{n-1}\frac{\|z^{n-1}\|_2}{\|z^n\|_2}(1-r)^{(\mu_2-\mu_1)}\sum_{k\geq 1}C_k^2(n,n-1)r^k - \alpha_n\frac{\|z^n\|_1}{\|z^{n+1}\|_1}\sum_{k\geq 1}C_k^1(n,n-1)r^k \\ = v_n\alpha_{n-1}\frac{\|z^{n-1}\|_2}{\|z^{n+1}\|_2}(1-r)^{(\mu_2-\mu_1)}\sum_{k\geq 2}C_k^2(n+1,n-1)r^k \\ +\alpha_n v_{n-1}\frac{\|z^{n-1}\|_1}{\|z^{n+1}\|_1}\sum_{k\geq 2}C_k^1(n+1,n-1)r^k.\end{aligned}$$

Differentiating with respect to r, then putting $r = 0$ and substituting the values of $C_1^2(n,n-1)$ and $C_1^1(n,n-1)$, we get

$$\alpha_{n-1}\frac{\|z^{n-1}\|_2}{\|z^n\|_2}(-\mu_2+n) = \alpha_n\frac{\|z^n\|_1}{\|z^{n+1}\|_1}(-\mu_1+n).$$

In this recursion, for all $n \in \mathbb{Z}$, the coefficients of α_{n-1}, α_n are non zero. Thus if $\alpha_p = 0$ for some integer p, then $\alpha_n = 0$ for all $n \in \mathbb{Z}$ and consequently, $S_1 = 0$.

Similarly, if $\beta_p = 0$ for some p, then $S_2 = 0$. This completes the proof of the first part of the Theorem.

Now assume that $S_2 = 0$. Then [5, Proposition 2.4] implies that T_i's are homogeneous operators with associated representation π_i. Since all the homogeneous shifts are known, the weights of T_1 and T_2 are therefore known.

Suppose $S_1 \neq 0$. Then one of the weights of S_1 must be non-zero. Choose, without loss of generality, $\alpha_{n-1} \neq 0$ for some $n \in \mathbb{Z}$. For this choice of α_{n-1}, we have equation (2.17).

(a) Assume that both π_1 and π_2 are from the Complementary series, that is, $\pi_i = C_{\lambda,\sigma_i}$, where $0 < \sigma_i < \frac{1}{2}(1-|\lambda|)$ and $\mu_i = \frac{1-\lambda}{2} + \sigma_i$, $i = 1,2$. In this case, we have $\frac{\|z^{n-1}\|_i^2}{\|z^n\|_i^2} = \frac{\lambda+\mu_i+n-1}{-\mu_i+n}$, $i = 1,2$. Since T_i are homogeneous operators with associated representation C_{λ,σ_i}, $T_i^{-1^*}$ is also homogeneous with the same associated representation and we have the following possibilities for the weight sequences.

(i) For $n \in \mathbb{Z}$, assume that $u_{n-1} = \frac{\|z^n\|_1}{\|z^{n-1}\|_1}$ and $v_{n-1} = \frac{\|z^n\|_2}{\|z^{n-1}\|_2}$. Substituting these values of u_{n-1} and v_{n-1} in the equation (2.17) we get

$$(\sigma_2+\sigma_1)(\sigma_2-\sigma_1+1) = 0.$$

This is a contradiction since $0 < \sigma_i < \frac{1}{2}(1-|\lambda|)$ by assumption.

(ii) For all $n \in \mathbb{Z}$, assume that $u_{n-1} = \frac{\|z^{n-1}\|_1}{\|z^n\|_1}$ and $v_{n-1} = \frac{\|z^n\|_2}{\|z^{n-1}\|_2}$. Substituting these values of u_{n-1} and v_{n-1} in the equation (2.17), we get

$$(\mu_2-\mu_1)(\sigma_1+\sigma_2+1) = 0.$$

This is a contradiction since $\mu_1 \neq \mu_2$ and $\sigma_1+\sigma_2+1 > 0$.

(iii) For all $n \in \mathbb{Z}$, assume that $u_{n-1} = \frac{\|z^n\|_1}{\|z^{n-1}\|_1}$ and $v_{n-1} = \frac{\|z^{n-1}\|_2}{\|z^n\|_2}$. Substituting these values of u_{n-1} and v_{n-1} in the equation (2.17), we get

$$(\sigma_2 - \sigma_1)(\sigma_1 + \sigma_2 - 1) = 0.$$

This is a contradiction since $\sigma_2 - \sigma_1 \neq 0$ and $\sigma_2 + \sigma_1 - 1 < 0$.

(iv) For all $n \in \mathbb{Z}$, assume that $u_{n-1} = \frac{\|z^{n-1}\|_1}{\|z^n\|_1}$ and $v_{n-1} = \frac{\|z^{n-1}\|_2}{\|z^n\|_2}$. Substituting these values of u_{n-1} and v_{n-1} in the equation (2.17), we get

$$\sigma_2 - \sigma_1 = 1.$$

This is a contradiction since $0 < \sigma_i < \frac{1}{2}(1 - |\lambda|)$.

Combining (i) – (iv), we find that there does not exists any n for which $\alpha_{n-1} \neq 0$ and we conclude that $S_1 = 0$ in this case.

(b) Let $\pi_1 = C_{\lambda,\sigma}$ for some $0 < \sigma < \frac{1}{2}(1 - |\lambda|)$ and $\pi_2 = P_{\lambda,s}$ where s is purely imaginary. Now, $\mu_1 = \frac{1-\lambda}{2} + \sigma$ and $\mu_2 = \frac{1-\lambda}{2} + s$. Since the representation space of π_2 is $L^2(\mathbb{T})$, $\|z^n\|_2 = 1$, for all $n \in \mathbb{Z}$.

Recall that there are two homogeneous operators whose associated representation is π_2, one is the unweighted bilateral shift and the other one is the weighted shift with weight sequence $\left\{v_{n-1} = \frac{-\mu_2+n+2s}{-\mu_2+n}\right\}$. As before, we consider four different possibilities that arise in this case. In each of these cases, a contradiction is obtained by noting that s is purely imaginary.

(i) For all $n \in \mathbb{Z}$, assume that $v_{n-1} = 1$ and $u_{n-1} = \frac{\|z^n\|_1}{\|z^{n-1}\|_1}$. Substituting these values of u_{n-1} and v_{n-1} in the equation (2.17) we get $s^2 - \sigma^2 + \sigma + s = 0$.

(ii) For all $n \in \mathbb{Z}$, assume that $v_{n-1} = 1$ and $u_{n-1} = \frac{\|z^{n-1}\|_1}{\|z^n\|_1}$. Substituting these values of u_{n-1} and v_{n-1} in the equation (2.17) we get $(s - \sigma)(s + \sigma + 1) = 0$.

(iii) For all $n \in \mathbb{Z}$, assume that $v_{n-1} = \frac{-\mu_2+n+2s}{-\mu_2+n}$ and $u_{n-1} = \frac{\|z^n\|_1}{\|z^{n-1}\|_1}$. Substituting these values of u_{n-1} and v_{n-1} in the equation (2.17) we get $s^2 - \sigma^2 + \sigma - s = 0$.

(iv) For all $n \in \mathbb{Z}$, assume that $v_{n-1} = \frac{-\mu_2+n+2s}{-\mu_2+n}$ and $u_{n-1} = \frac{\|z^{n-1}\|_1}{\|z^n\|_1}$. Substituting these values of u_{n-1} and v_{n-1} in the equation (2.17) we get $s^2 - \sigma^2 - \sigma - s = 0$.

Combining (i) - (iv), again in this case, we see that $S_1 = 0$.

(c) For $i = 1, 2$, assume that $\pi_i = P_{\lambda,s_i}$ are two Principal series representations. We have the following four cases to consider.

(i) For all $n \in \mathbb{Z}$, assume that $u_{n-1} = \frac{-\mu_1+n+2s_1}{-\mu_1+n}$ and $v_{n-1} = 1$. Substituting these values of u_{n-1} and v_{n-1} in the equation (2.17) we get $(s_2 - s_1)(s_2 + s_1 + 1) = 0$. This is a contradiction since $s_1 \neq s_2$.

(ii) For all $n \in \mathbb{Z}$, assume that $u_{n-1} = 1$ and $v_{n-1} = \frac{-\mu_2+n+2s_1}{-\mu_2+n}$. Substituting these values of u_{n-1} and v_{n-1} in the equation (2.17) we get $s_2^2 - s_1^2 + s_1 - s_2 = 0$. This is a contradiction since $s_1 \neq s_2$.

(iii) For all $n \in \mathbb{Z}$, assume that $u_{n-1} = \frac{-\mu_1+n+2s_1}{-\mu_1+n}$ and $v_{n-1} = \frac{-\mu_2+n+2s_1}{-\mu_2+n}$. Substituting these values of u_{n-1} and v_{n-1} in the equation (2.17) we get $s_2^2 - s_1^2 = s_2 + s_1$. Since $s_1 \neq s_2$ and both of them are purely imaginary, it follows, from the preceding equation, that $s_2 = -s_1$.

(iv) For all $n \in \mathbb{Z}$, assume that $u_{n-1} = 1$ and $v_{n-1} = 1$. Substituting these values of u_{n-1} and v_{n-1} in the equation (2.17), we get $s_2^2 - s_1^2 + s_2 + s_1 = 0$. We conclude that $s_2 = -s_1$ exactly as before.

The proof is complete by putting together the results of the three cases (a) - (c). $\square$

Proposition 2.14. *Let $P_{\lambda,s}$ be a representation from the Principal series with $s \neq 0$. If S is any operator on $L^2(\mathbb{T})$ such that*

$$SP_{\lambda,s}(\phi) - e^{i\theta} P_{\lambda,s}(\phi)S = \overline{a}B(s)P_{\lambda,s}(\phi)S + \overline{a}SP_{\lambda,s}(\phi)B(s),\ \phi \in \text{Möb}, \tag{2.19}$$

then $S = 0$.

Proof. The operator S must be an weighted shift with respect to the orthonormal basis $\{z^n : n \in \mathbb{Z}\}$ in $L^2(\mathbb{T})$ as before. Let $\{\alpha_n\}$ be the weight sequence of S. In the equation (2.19), substituting $\phi = \phi_a$, evaluating on z^m and taking inner product with z^n, we obtain

$$\alpha_{n-1}\left\langle P_{\lambda,s}(\phi_a)z^m, z^{n-1}\right\rangle + \alpha_m\left\langle P_{\lambda,s}(\phi_a)z^{m+1}, z^n\right\rangle = \overline{a}(\alpha_m w_{n-1} + w_m\alpha_{n-1})\left\langle P_{\lambda,s}(\phi_a)z^{m+1}, z^{n-1}\right\rangle.$$

Using the matrix coefficients of $P_{\lambda,s}(\phi_a)$ and putting $m = n-1$, we get

$$\alpha_{n-1}\sum_{k\geq 0} C_k(n-1,n-1)r^k - \alpha_{n-1}\sum_{k\geq 0} C_k(n,n)r^k = 2\alpha_{n-1}w_{n-1}\sum_{k\geq 1} C_k(n,n-1)r^k.$$

Now comparing the coefficient of r, we have $2\alpha_{n-1}s = 0$. Since $s \neq 0$, it follows that $\alpha_{n-1} = 0$. This implies that $S = 0$. $\square$

Corollary 2.15. *If T is a homogeneous operator with associated representation $P_{\lambda,s} \oplus P_{\lambda,s}$, where $s \neq 0$, then, upto unitary equivalence, T must be of the form*

$$\begin{bmatrix} B(s) & \alpha(B(s)-B) \\ 0 & B \end{bmatrix}, \begin{bmatrix} B(s) & 0 \\ 0 & B(s) \end{bmatrix} \text{ or } \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix}.$$

Proof. Let T be a homogeneous operator with associated representation $P_{\lambda,s} \oplus P_{\lambda,s}$. Recall that $P_{\lambda,s}$ and $P_{\lambda,-s}$ are unitarily equivalent via the unitary operator $U_{\lambda,s}$. Clearly, the operator $\left(I \oplus U_{\lambda,s}\right) T \left(I \oplus U^*_{\lambda,s}\right)$ is homogeneous with associated representation $P_{\lambda,s} \oplus P_{\lambda,-s}$. Then Theorem 2.13 implies that $\left(I \oplus U_{\lambda,s}\right) T \left(I \oplus U^*_{\lambda,s}\right)$ is of the form $\begin{bmatrix} \tilde{T}_1 & \tilde{S}_1 \\ 0 & \tilde{T}_2 \end{bmatrix}$ on $L^2(\mathbb{T}) \oplus L^2(\mathbb{T})$, therefore the operator T is also of the form $\begin{bmatrix} T_1 & S_1 \\ 0 & T_2 \end{bmatrix}$ on $L^2(\mathbb{T}) \oplus L^2(\mathbb{T})$. Now [5, Proposition 2.4] and [5, Lemma 2.5] imply that T_1 and T_2 are homogeneous operators with associated representation $P_{\lambda,s}$ and S satisfies

$$S\pi_{\lambda,s}(\phi) - e^{i\theta}\pi_{\lambda,s}(\phi)S = \overline{a}T_1\pi_{\lambda,s}(\phi)S + \overline{a}S\pi_{\lambda,s}(\phi)T_2.$$

Since $B(s)$ and B are the only homogeneous operators with associated representation $P_{\lambda,s}$, the proof is complete applying Proposition 2.3, Proposition 2.8 and Proposition 2.14. □

Now we characterize all homogeneous operators whose associated representation is $P_{\lambda,0} \oplus P_{\lambda,0}$ with $\lambda \neq 1$.

Let $\sigma = P_{\lambda,0} \oplus P_{\lambda,0}$. For all $i, j \in \mathbb{Z}$, let $\sigma_{i,j} = P_i \sigma|_{H(j)}$ where P_i is the orthogonal projection of $L^2(\mathbb{T}) \oplus L^2(\mathbb{T})$ onto $H(i)$, the $\mathbb{K}$-isotypic subspace of σ as in Theorem 2.5. Then $\sigma_{i,j}$ is a map from $H(j)$ to $H(i)$, $i, j \in \mathbb{Z}$. Let $P^{i,j}_{\lambda,0}$ be the map from the subspace of $L^2(\mathbb{T})$ spanned by the vector $\{z^j\}$ to the subspace of $L^2(\mathbb{T})$ spanned by the vector $\{z^i\}$ defined by $P^{i,j}_{\lambda,0}(z^j) = \left\langle P^{i,j}_{\lambda,0} z^j, z^i \right\rangle z^i$. Then

$$\sigma_{i,j}(\phi)\begin{pmatrix} az^j \\ bz^j \end{pmatrix} = \left\langle P_{\lambda,0}(\phi) z^j, z^i \right\rangle \begin{pmatrix} az^i \\ bz^i \end{pmatrix}, \tag{2.20}$$

for all $a, b \in \mathbb{C}$. Recall that the matrix coefficient of $P_{\lambda,0}$ is

$$\left\langle P_{\lambda,0}(\phi_a) z^m, z^n \right\rangle = c(-1)^n (\overline{a})^{n-m} \sum_{k \geq (m-n)^+} C_k(m,n) r^k \tag{2.21}$$

where $r = |a|^2$, $c = \phi_a'(0)^{\lambda/2} |\phi_a'(0)|^{\mu}$ and $C_k(m,n) = \begin{pmatrix} -\lambda - \mu - m \\ k + n - m \end{pmatrix} \begin{pmatrix} -\mu + m \\ k \end{pmatrix}$.

Definition 2.16. Let $A_{m,n} \subset (-1,1)$ be the set of all zeros of the power series $\sum_{k \geq (m-n)^+} C_k(m,n) r^k$.

Since for every $n, m \in \mathbb{Z}$, the radius of convergence of the power series $\sum_{k \geq (m-n)^+} C_k(m,n) r^k$ is 1, it follows $A_{m,n}$ is countable. Thus the set $A = \bigcup_{m,n \in \mathbb{Z}} A_{m,n}$ is also countable. Therefore, there exists $b \in (0,1) \setminus A$ such that $\left\langle P_{\lambda,0}(\phi_b) z^m, z^n \right\rangle \neq 0$, for all $n, m \in \mathbb{Z}$. In the following, we fix this ϕ_b and let e_n denote the function z^n.

Now assume that u_0, v_0 are two non-zero mutually orthogonal vectors in $H(0)$. Define $u_n = \sigma_{n,0}(\phi_b) u_0$, $v_n = \sigma_{n,0}(\phi_b) v_0$ for all $n \neq 0$. Then each of the vectors u_n, v_n are non-zero.

Lemma 2.17. *The set of vectors $\{u_n, v_n\}_{n\in\mathbb{Z}}$ is a complete orthogonal set of $L^2(\mathbb{T})\oplus L^2(\mathbb{T})$.*

Proof. As $u_n, v_n \in H(n)$ for every $n\in\mathbb{Z}$ and $H(n)$ is orthogonal to $H(m)$, so $\{u_n, v_n\}$ is orthogonal to $\{u_m, v_m\}$, if $n\neq m$. Now we will show that u_n is orthogonal to v_n, $n\in\mathbb{Z}$. From the definition of $\sigma_{n,0}(\phi_b): H(0)\to H(n)$ obtained from (2.20) and a similar one for $\sigma_{n,0}(\phi_b)^*: H(n)\to H(0)$, we have

$$\sigma_{n,0}(\phi_b)^*\sigma_{n,0}(\phi_b) = |\left\langle P_{\lambda,0}(\phi_b)e_0, e_n\right\rangle|^2 Id.$$

Consequently,

$$\begin{aligned}\langle u_n, v_n\rangle &= \left\langle \sigma_{n,0}(\phi_b)u_0, \sigma_{n,0}(\phi_b)v_0\right\rangle\\ &= \left\langle \sigma_{n,0}(\phi_b)^*\sigma_{n,0}(\phi_b)u_0, v_0\right\rangle\\ &= |\left\langle P_{\lambda,0}(\phi_b)e_0, e_n\right\rangle|^2\langle u_0, v_0\rangle = 0.\end{aligned}$$

Since $H(n)$ is spanned by $\{u_n, v_n\}$ and $L^2(\mathbb{T})\oplus L^2(\mathbb{T}) = \oplus_{n\in\mathbb{Z}}H(n)$, it follows that $\{u_n, v_n\}_{n\in\mathbb{Z}}$ is a complete orthogonal set. □

Now let H_1 be the subspace of $L^2(\mathbb{T})\oplus L^2(\mathbb{T})$ spanned by the set of vectors $\{u_n\}_{n\in\mathbb{Z}}$ and H_2 be the subspace of $L^2(\mathbb{T})\oplus L^2(\mathbb{T})$ spanned by the set of vectors $\{v_n\}_{n\in\mathbb{Z}}$.

Lemma 2.18. *The subspaces H_1 and H_2 are invariant under σ. Moreover, $\sigma_{|H_i}$ is equivalent to $P_{\lambda,0}$ for $i=1,2$.*

Proof. Let $\mathbb{K}$ be the set of all rotations in Möb. If $k\in\mathbb{K}$, then $\sigma(k)$ is a scalar multiple of the identity on each $H(n)$. Thus each H_i is invariant under $\sigma(k)$, $k\in\mathbb{K}$.

Pick a ψ in Möb which is not in $\mathbb{K}$. For all i, j, note that $\left\langle P_{\lambda,0}(\phi_b)z^j, z^i\right\rangle\neq 0$, and

$$\sigma_{i,j}(\psi) = \frac{\left\langle P_{\lambda,0}(\psi)e_j, e_i\right\rangle}{\left\langle P_{\lambda,0}(\phi_b)e_j, e_i\right\rangle}\sigma_{i,j}(\phi).$$

Since

$$\sigma_{0,j}(\phi_b)\sigma_{j,0}(\phi_b) = \left\langle P_{\lambda,0}(\phi_b)e_0, e_j\right\rangle\left\langle P_{\lambda,0}(\phi_b)e_j, e_0\right\rangle \mathrm{Id},$$

it follows that $\sigma_{0,j}(\phi_b)u_j$ is in the span of $\{u_0\}$. Therefore,

$$\sigma_{i,j}(\phi_b)u_j = \frac{\left\langle P_{\lambda,0}(\phi_b)e_j, e_i\right\rangle}{\left\langle P_{\lambda,0}(\phi_b)e_j, e_0\right\rangle\left\langle P_{\lambda,0}(\phi_b)e_0, e_i\right\rangle}\sigma_{i,0}(\phi_b)\sigma_{0,j}(\phi_b)u_j$$

is a scalar multiple of u_i. This implies that

$$\sigma_{i,j}(\psi)u_j = \frac{\left\langle P_{\lambda,0}(\psi)e_j, e_i\right\rangle}{\left\langle P_{\lambda,0}(\phi_b)e_j, e_i\right\rangle}\sigma_{i,j}(\phi_b)u_j$$

is a scalar multiple of u_i. We conclude that $\sigma(\psi)u_j = \sum_{i\in\mathbb{Z}}\sigma_{i,j}(\psi)u_j$ is in H_1, proving that H_1 is invariant under σ. A similar argument shows that H_2 is invariant under σ.

Let $t_n \in \mathbb{R}$ be such that $\langle P_{\lambda,0}(\phi_b)e_0, e_n\rangle = e^{it_n}\,|\langle P_{\lambda,0}(\phi_b)e_0, e_n\rangle|$. Now if ψ is any element in Möb, then

$$\begin{aligned}\langle\sigma(\psi)u_j, u_i\rangle = \langle\sigma_{i,j}(\psi)u_j, u_i\rangle &= \langle P_{\lambda,0}(\psi)e_j, e_i\rangle\langle P_{\lambda,0}(\phi_b)e_0, e_j\rangle\overline{\langle P_{\lambda,0}(\phi_b)e_0, e_i\rangle}\|u_0\|^2\\ &= \langle P_{\lambda,0}(\psi)e_j, e_i\rangle e^{it_j}\,|\langle P_{\lambda,0}(\phi_b)e_0, e_j\rangle|\,e^{-it_i}\,|\langle P_{\lambda,0}(\phi_b)e_0, e_i\rangle|\|u_0\|^2.\end{aligned}$$

Find $a, b \in \mathbb{C}$, such that $u_0 = \binom{ae_0}{be_0}$ and note that

$$u_n = \sigma_{n,0}(\phi_b)u_0 = \langle P_{\lambda,0}(\phi_b)e_0, e_n\rangle\begin{pmatrix}ae_n\\ be_n\end{pmatrix} \text{ and } \|u_n\| = |\langle P_{\lambda,0}(\phi_b)e_0, e_n\rangle|\|u_0\|.$$

The set of vectors $\{\hat{u}_i\}$, $\hat{u}_i = e^{-it_i}\frac{u_i}{\|u_i\|}$ is an orthonormal basis of H_1. From the preceding computation, we see that $\langle\sigma(\psi)\hat{u}_j, \hat{u}_i\rangle = \langle P_{\lambda,0}(\psi)e_j, e_i\rangle$. It is now evident that $\sigma_{|H_1}$ is equivalent to $P_{\lambda,0}$. Similarly, it can be seen that $\sigma_{|H_2}$ is equivalent to $P_{\lambda,0}$. □

Suppose T is a homogeneous operator with associated representation σ. Since $H(n)$ is a $\mathbb{K}$-isotypic subspace of σ and σ is associated with T, therefore, we have $T(H(n)) \subseteq H(n+1)$ ([9, Theorem 5.1]). Let $T_n := T_{|H(n)}$. We first prove that each T_n is invertible.

Lemma 2.19. *For every $n \in \mathbb{Z}$, the operator T_n is invertible.*

Proof. Let $\psi(z) = e^{i\theta}\frac{z-a}{1-\bar{a}z}$. The homogeneity of T implies that

$$e^{i\theta}\sigma(\psi)T - ae^{i\theta}\sigma(\psi) = T\sigma(\psi) - \overline{a}T\sigma(\psi)T.$$

From this equation, using the orthogonality of the subspaces $H(n)$, we have

$$e^{i\theta}\sigma_{i+1,n+1}(\psi)T_n - ae^{i\theta}\sigma_{i+1,n}(\psi) = T_i\sigma_{i,n}(\psi) - \overline{a}T_i\sigma_{i,n+1}(\psi)T_n, \tag{2.22}$$

for all $i, n \in \mathbb{Z}$.

For all $i, j \in \mathbb{Z}$, the operator $\sigma_{i,j}(\phi_b)$ is invertible. Substituting $i = n$ and $\psi = \phi_b$ in the equation (2.22), we get

$$b\sigma_{n+1,n}(\phi_b) + \sigma_{n+1,n+1}(\phi_b)T_n = T_n\sigma_{n,n}(\phi_b) - \overline{b}T_n\sigma_{n,n+1}(\phi_b)T_n.$$

If there exists $h_n \in H(n)$ such that $T_nh_n = 0$, then from the equation appearing above, we have

$$b\sigma_{n+1,n}(\phi_b)h_n = 0$$

and consequently, $h_n = 0$. This proves that T_n is invertible. □

Theorem 2.20. *Suppose T is a homogeneous operator with associated representation σ. Then there exists H_1 and H_2 such that $L^2(\mathbb{T}) \oplus L^2(\mathbb{T}) = H_1 \oplus H_2$, $T(H_1) \subseteq H_1$. The subspaces H_i, $i = 1,2$, are invariant under σ and $\sigma_{|H_i}$ is unitarily equivalent to $P_{\lambda,0}$.*

Proof. There exists $\lambda_0 \in \mathbb{C}$ and a pair of orthonormal vectors u_0, v_0 in $H(0)$ such that the vector u_0 is an eigenvector for the operator $\sigma_{1,0}(\phi_b)^{-1} T_0$ with eigenvalue λ_0, that is,

$$\sigma_{1,0}(\phi_b)^{-1} T_0 u_0 = \lambda_0 u_0.$$

Now, define

$$u_n = \sigma_{n,0}(\phi_b) u_0,\ v_n = \sigma_{n,0}(\phi_b) v_0$$

for all $n \in \mathbb{Z}$. Suppose H_1 and H_2 are the closed subspaces spanned by $\{u_n\}_{n\in\mathbb{Z}}$ and $\{v_n\}_{n\in\mathbb{Z}}$, respectively. Then by Lemma 2.17, $L^2(\mathbb{T}) \oplus L^2(\mathbb{T}) = H_1 \oplus H_2$ and by Lemma 2.18, each H_i is invariant under σ such that $\sigma_{|H_i}$ is equivalent to $P_{\lambda,0}$. Now we show that $T(H_1) \subseteq H_1$.

We have $T_0 u_0 = \lambda_0 \sigma_{1,0}(\phi_b) u_0$, which is a scalar multiple of the vector u_1. An inductive argument given below shows that $T_n u_n$ is a scalar multiple of the vector u_{n+1} for every $n \in \mathbb{Z}$.

Assume that $T_k u_k = \lambda_{k+1} u_{k+1}$ for some $\lambda_{k+1} \in \mathbb{C}$, $k \geq 0$. Let $A_k = \bigcup_{0\leq i,j\leq k+2} A_{i,j}$, where $A_{i,j}$ are described in Definition 2.16. Since 0 is not a limit point of any $A_{i,j}$, there exists $r_k \in (0,1)$ such that $\langle P_{\lambda,0}(\phi_a) z^j, z^i \rangle \neq 0$, $0 \leq i,j \leq k+2$, for all $a \in \mathbb{D}$ with $0 < |a| < r_k$. Combining the two equalities

$$\sigma_{k+1,0}(\phi_b) = \frac{\langle P_{\lambda,0}(\phi_b) e_0, e_{k+1} \rangle}{\langle P_{\lambda,0}(\phi_b) e_0, e_k \rangle \langle P_{\lambda,0}(\phi_b) e_k, e_{k+1} \rangle} \sigma_{k+1,k}(\phi_b) \sigma_{k,0}(\phi_b)$$

and

$$\sigma_{k+1,k}(\phi_b) = \frac{\langle P_{\lambda,0}(\phi_b) e_k, e_{k+1} \rangle}{\langle P_{\lambda,0}(\phi_a) e_k, e_{k+1} \rangle} \sigma_{k+1,k}(\phi_a),\ |a| < r_k,$$

we have $T_k u_k = \lambda_{k+1}(a) \sigma_{k+1,k}(\phi_a) u_k$, where

$$\lambda_{k+1}(a) = \lambda_{k+1} \frac{\langle P_{\lambda,0}(\phi_b) e_0, e_{k+1} \rangle \langle P_{\lambda,0}(\phi_b) e_k, e_{k+1} \rangle}{\langle P_{\lambda,0}(\phi_b) e_0, e_k \rangle \langle P_{\lambda,0}(\phi_b) e_k, e_{k+1} \rangle \langle P_{\lambda,0}(\phi_a) e_k, e_{k+1} \rangle}.$$

For every ϕ_a with $|a| < r_k$, this proves the existence of $\lambda_{k+1}(a) \in \mathbb{C}$ such that

$$T_k u_k = \lambda_{k+1}(a) \sigma_{k+1,k}(\phi_a) u_k.$$

Now, for every ϕ_a with $|a| < r_k$, substituting $n = k, i = k+1$ in the equation (2.22), and then evaluating on the vector u_k, we get

$$\begin{aligned} &a\sigma_{k+2,k}(\phi_a) u_k - \lambda_{k+1}(a) \sigma_{k+2,k+1}(\phi_a) \sigma_{k+1,k}(\phi_a) u_k \\ &\qquad = T_{k+1} \sigma_{k+1,k}(\phi_a) u_k - \overline{a} \lambda_{k+1}(a) \langle P_{\lambda,0}(\phi_a) e_{k+1}, e_{k+1} \rangle T_{k+1} \sigma_{k+1,k}(\phi_a) u_k. \end{aligned}$$

The equality below is easily verified using the definition of the $\sigma_{i,j}$:

$$\sigma_{k+2,k}(\phi_a) = \frac{\langle P_{\lambda,0}(\phi_a)e_k, e_{k+2}\rangle}{\langle P_{\lambda,0}(\phi_a)e_k, e_{k+1}\rangle\langle P_{\lambda,0}(\phi_a)e_{k+1}, e_{k+2}\rangle}\sigma_{k+2,k+1}(\phi_a)\sigma_{k+1,k}(\phi_a).$$

In consequence, we have

$$\begin{aligned}\left(\frac{a\langle P_{\lambda,0}(\phi_a)e_k, e_{k+2}\rangle}{\langle P_{\lambda,0}(\phi_a)e_k, e_{k+1}\rangle\langle P_{\lambda,0}(\phi_a)e_{k+1}, e_{k+2}\rangle} - \lambda_{k+1}(a)\right)\sigma_{k+2,k+1}(\phi)_a)\phi_{k+1,k}(\phi_a)u_k \\ = \left(1 - \overline{a}\lambda_{k+1}(a)\langle P_{\lambda,0}(\phi_a)e_{k+1}, e_{k+1}\rangle\right) T_{k+1}\sigma_{k+1,k}(\phi_a)u_k. \quad (2.23)\end{aligned}$$

Suppose

$$\left(\frac{a\langle P_{\lambda,0}(\phi_a)e_k, e_{k+2}\rangle}{\langle P_{\lambda,0}(\phi_a)e_k, e_{k+1}\rangle\langle P_{\lambda,0}(\phi_a)e_{k+1}, e_{k+2}\rangle} - \lambda_{k+1}(a)\right) \neq 0$$

and

$$\left(1 - \overline{a}\lambda_{k+1}(a)\langle P_{\lambda,0}(\phi_a)e_{k+1}, e_{k+1}\rangle\right) \neq 0$$

for all ϕ_a with $|a| < r_k$. Then we have

$$|a|^2\langle P_{\lambda,0}(\phi_a)e_k, e_{k+2}\rangle\langle P_{\lambda,0}(\phi_a)e_{k+1}, e_{k+1}\rangle = \langle P_{\lambda,0}(\phi_a)e_k, e_{k+1}\rangle\langle P_{\lambda,0}(\phi_a)e_{k+1}, e_{k+2}\rangle$$

for all $|a| < r_k$.

Now, using the matrix coefficient for $P_{\lambda,0}(\phi_a)$, we obtain

$$r\left(\sum_{n\geq 0} C_n(k, k+2)r^n\right)\left(\sum_{n\geq 0} C_n(k+1, k+1)r^n\right) = \left(\sum_{n\geq 0} C_n(k, k+1)r^n\right)\left(\sum_{n\geq 0} C_n(k+1, k+2)r^n\right)$$

for all $0 \leq r \leq r_k^2$. Putting $r = 0$ we arrive at a contradiction.

We can therefore find ϕ_a with $0 < |a| < r_k$ such that

$$\left(\frac{a\langle P_{\lambda,0}(\phi_a)e_k, e_{k+2}\rangle}{\langle P_{\lambda,0}(\phi_a)e_k, e_{k+1}\rangle\langle P_{\lambda,0}(\phi_a)e_{k+1}, e_{k+2}\rangle} - \lambda_{k+1}(a)\right) \neq 0$$

and hence

$$\left(1 - \overline{a}\lambda_{k+1}(a)\langle P_{\lambda,0}(\phi_a)e_{k+1}, e_{k+1}\rangle\right) \neq 0$$

as both $\sigma_{k+2,k+1}(\phi_a)$ and T_{k+1} are invertible. Since $0 < |a| < r_k$, it follows from (2.23) that $T_{k+1}u_{k+1}$ is a scalar multiple of the vector u_{k+2} completing half the induction argument.

A similar but slightly different proof gives the other half of the induction argument, namely, $T_{-n}^{-1}u_{-n+1}$ is a scalar multiple of $\{u_{-n}\}$ for all $n \in \mathbb{N}$. □

Corollary 2.21. *If T is a homogeneous operator with associated representation $P_{\lambda,0} \oplus P_{\lambda,0}$, then T is unitarily equivalent to one of the following operator*

$$\begin{bmatrix} B & \alpha S[\lambda] \\ 0 & B \end{bmatrix}, \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix},$$

where $S[\lambda]$ is the weighted shift on $L^2(\mathbb{T})$ with respect to the orthonormal basis $\{z^n : n \in \mathbb{Z}\}$ with weight sequence $\{\frac{1}{\lambda+2n+1} : n \in \mathbb{Z}\}$.

Proof. The proof follows form Theorem 2.20 and Proposition 2.8 □

2.2 The associated representation is the direct sum of three irreducible representations

Now, we will prove that every homogeneous operator whose associated representation is $\pi = \pi_1 \oplus \pi_2$, where π_1 is from the irreducible Continuous series representations and π_2 is the direct sum of a holomorphic and an anti-holomorphic Discrete series representation, is reducible. Let $\pi_1 = R_{\lambda,\mu}$ and H_1 be the representation space of π_1. Let $e_n^1 = \frac{z^n}{\|z^n\|_1}$. Recall that $\{e_n^1 : n \in \mathbb{Z}\}$ is an orthonormal basis of the representation space H_1. Let $\pi_2 = D^+_{\lambda_1} \oplus D^-_{\lambda_2}$ for a pair of positive real numbers λ_1, λ_2. However, the multipliers of all the three representations π_1, $D^+_{\lambda_1}$ and $D^-_{\lambda_2}$ must be the same. In consequence, $\lambda_1 + \lambda_2$ is an even integer (see Corollary 1.15), therefore $\lambda_1 = \lambda + 2m$ and $\lambda_2 = 2 - \lambda + 2k$, $-1 < \lambda \le 1$.

Let $H^{(\lambda+2m)}$ be the representation space of $D^+_{\lambda+2m}$ and $H^{(2-\lambda+2k)}$ be the representation space of $D^-_{2-\lambda+2k}$. Let $H_2 = H^{(\lambda+2m)} \oplus H^{(2-\lambda+2k)}$. Define

$$e_n^2 := \begin{pmatrix} \frac{z^n}{\|z^n\|_{\lambda+2m}} \\ 0 \end{pmatrix},\ n \ge 0 \text{ and } e_{-n}^2 := \begin{pmatrix} 0 \\ \frac{z^{n-1}}{\|z^{n-1}\|_{2-\lambda+2k}} \end{pmatrix},\ n \ge 1.$$

The set of vectors $\{e_n^2 : n \in \mathbb{Z}\}$ is an orthonormal basis of H_2. Let ϕ_θ be a rotation in Möb. Then

$$\pi_1(\phi_\theta) e_n^1 = e^{-i\left(n+\frac{\lambda}{2}\right)\theta} e_n^1,\ n \in \mathbb{Z}.$$

Also, it is easy to see that

$$\pi_2(\phi_\theta) e_n^2 = e^{-i\left(n+m+\frac{\lambda}{2}\right)\theta} e_n^2,\ n \ge 0 \text{ and } \pi_2(\phi_\theta) e_{-n}^2 = e^{i\left(n+k-\frac{\lambda}{2}\right)\theta} e_{-n}^2,\ n \ge 1.$$

Clearly, there exists a θ such that $e^{-i\left(n+m+\frac{\lambda}{2}\right)\theta} \ne e^{i\left(p+k-\frac{\lambda}{2}\right)\theta}$ for all $n \ge 0$, $p \ge 1$ and if $n_1 \ne n_2$, then $e^{-i\left(n_1+\frac{\lambda}{2}\right)\theta} \ne e^{-i\left(n_2+\frac{\lambda}{2}\right)\theta}$.

Theorem 2.22. *Suppose* $T = \begin{bmatrix} T_1 & S_1 \\ S_2 & T_2 \end{bmatrix}$ *is a homogeneous operator with associated representation* $\pi_1 \oplus \pi_2$*, where* $\pi_1 = R_{\lambda,\mu}$ *is from the Continuous series excluding* $P_{1,0}$ *and* $\pi_2 = D^+_{\lambda+2m} \oplus D^-_{2-\lambda+2k}$*. Then* $S_1 = 0$*.*

Proof. Homogeneity of T implies that the operators T_i and S_i satisfy equations (2.11), (2.12), (2.13) and (2.14). Repeating an argument similar to the one in Theorem 2.13, we find that T_1 and T_2 are weighted shifts with respect to the orthonormal basis $\{e^1_n\}$ and $\{e^2_n\}$, respectively. Let $\{u_n\}$ and $\{v_n\}$ be the weight sequences of T_1 and T_2, respectively. It is easy to see that $v_{-1} = 0$ unless either $m = 0$, $k = 0$, $\lambda > 0$ or $m = 1$, $k = -1$, $\lambda < 0$.

Substituting $\phi = \phi_\theta$ in the equation (2.12), we obtain

$$\pi_1(\phi_\theta) S_1 e^2_n = e^{-i\left(n+1+m+\frac{\lambda}{2}\right)\theta} S_1 e^2_n,\ n \geq 0;\ \pi_1(\phi_\theta) S_1 e^2_{-n} = e^{i\left(n-2+k+\frac{\lambda}{2}\right)\theta} S_1 e^2_{-n},\ n \geq 1.$$

In consequence,

(a) for $n \geq 0$, there exists $\alpha_n \in \mathbb{C}$ such that $S_1 e^2_n = \alpha_n e^1_{n+m+1}$.

(b) for $n \geq 1$, there exists $\alpha_{-n} \in \mathbb{C}$ such that $S_1 e^2_{-n} = \alpha_{-n} e^1_{-n-k+1}$.

Substituting $\phi = \phi_a$ in the equation (2.12) and then evaluating at the vector e^2_n, $n \geq 0$, we obtain

$$S_1 \pi_2(\phi_a) e^2_n + \alpha_n \pi_1(\phi_a) e^1_{n+m+1} = \overline{a} \alpha_n T_1 \pi_1(\phi_a) e^1_{n+m+1} + \overline{a} v_n S_1 \pi_2(\phi_a) e^2_{n+1}.$$

For any integer $p > 1$, since $S_1^* e^1_{-p-k+1}$ is a scalar multiple of e^2_{-p}, it follows that

$$\left\langle S_1 \pi_2(\phi_a) e^2_n, e^1_{-p-k+1} \right\rangle = \left\langle \pi_2(\phi_a) e^2_n, S_1^* e^1_{-p-k+1} \right\rangle = 0.$$

Taking inner product with e^1_{-p-k+1} and using the matrix coefficients of $\pi_1(\phi_a)$, we get

$$\alpha_n \|z^{-p-k+1}\|_1 \sum_{i \geq n+m+p+k} C^1_i(n+m+1, -p-k+1) r^i$$
$$= -\alpha_n u_{-p-k} \|z^{-p-k}\|_1 \sum_{i \geq n+m+p+k+1} C^1_i(n+m+1, -p-k) r^i.$$

Comparing the coefficient of $r^{n+m+p+k}$, we get

$$\alpha_n \|z^{-p-k+1}\|_1 C^1_{n+m+p+k}(n+m+1, -p-k+1) = 0$$

and this implies that $\alpha_n = 0$, because $\|z^{-p-k+1}\|_1 C^1_{n+m+p+k-1}(n+m+1, -p-k+1) \neq 0$. This proves that $S_1 e^2_n = 0$ for all $n \geq 0$.

To prove that $S_1 e^2_{-n} = 0$, $n \geq 1$, we repeat the previous algorithm, namely, substitute $\phi = \phi_a$ in the equation (2.12), evaluate at the vector e^2_{-n}, $n \geq 1$, take inner product with e^1_p for some positive integer p and finally, use the matrix coefficients of $\pi_1(\phi_a)$ to conclude

$$\alpha_{-n}\|z^p\|_1 \sum_{k\geq 0} C^1_k(-n-k+1, p) r^k = -\alpha_{-n} u_{p-1}\|z^{p-1}\|_1 \sum_{k\geq 0} C^1_k(-n-k+1, p-1) r^k.$$

Equating the constant term on both sides of this equation, we get

$$\alpha_{-n}\left[u_{p-1}\frac{\|z^{p-1}\|_1}{\|z^p\|_1} + \frac{(-\lambda-\mu-p+1)}{(p+n+k-1)}\right] = 0.$$

Now, suppose there exists a subsequence (n_m) such that $\alpha_{-n_m} \neq 0$. Then

$$u_{p-1}\frac{\|z^{p-1}\|_1}{\|z^p\|_1} + \frac{(-\lambda-\mu-p+1)}{(p+n_m+k-1)} = 0,$$

for all n_m. Therefore taking $m \to \infty$, we see that $u_{p-1}\frac{\|z^{p-1}\|_1}{\|z^p\|_1} = 0$. Hence $\alpha_{-n} = 0$ for all $n \geq 1$, leading to a contradiction, since we have assumed that $\alpha_{-n_m} \neq 0$ for all $m \geq 1$. Thus there is no subsequence $\{n_m\}$ such that $\alpha_{-n_m} \neq 0$, or in other words, there exists a natural number N such that $\alpha_{-n} = 0$ for all $n \geq N$. Repeating the algorithm of substituting $\phi = \phi_a$ in the equation (2.12), evaluating at the vector e^2_{-n}, $1 \leq n < N$, taking inner product with $e^1_{-n-l-k+2}$, where $l : l > N-n$, using the matrix coefficients of $\pi_1(\phi_a)$ and finally comparing coefficients of r^l, we have

$$\alpha_{-n}\|z^{-n-l-k+1}\|_1 C^1_l(-n-k+1, -n-l-k+1) = 0.$$

It follows that $\alpha_{-n} = 0$ for all $1 \leq n < N$. Therefore we have proved that $S_1 = 0$. □

Theorem 2.23. *Suppose* $T = \begin{bmatrix} T_1 & 0 \\ S_2 & T_2 \end{bmatrix}$ *is a homogeneous operator and* $\pi_1 \oplus \pi_2$ *is the associated representation. Then* S_2 *satisfies the equation* (2.13). *If* $\pi_1 = R_{\lambda,\mu}$ *is from the Continuous series representation excluding* $P_{1,0}$ *and* $\pi_2 = D^+_{\lambda+2m} \oplus D^-_{2-\lambda+2k}$, *then* $S_2 = 0$.

Proof. Substituting $\phi = \phi_\theta$ in the equation (2.13), we have

$$\pi_2(\phi_\theta) S_2 e^1_n = e^{-i\left(n+1+\frac{\lambda}{2}\right)\theta} S_2 e^1_n,\ n \in \mathbb{Z}. \tag{2.24}$$

Case I ($m \geq 1$)**:** Assume $m \geq 1$. From the equation (2.24), only the following possibilities occur.

(a) There exists $\alpha_n \in \mathbb{C}$ such that $S_2 e^1_n = \alpha_n e^2_{n+1-m}$, $n \geq m-1$ and $S_2 e^1_n = 0$, $0 \leq n < m-1$.

(b) There exists $\alpha_{-n} \in \mathbb{C}$ such that $S_2 e^1_{-n} = \alpha_{-n} e^2_{-n+k+1}$, $n > k+1$ and $S_2 e^1_{-n} = 0$, $1 \leq n \leq k+1$.

Substituting $\phi = \phi_a$ in the equation (2.13), evaluating at the vector e_0^1, taking inner product with e^2_{-n+k+1} for $n > k+1$ and lastly, using the matrix coefficient of $\pi_1(\phi_a)$, we get

$$\alpha_{-n}\|z^0\|_1^{-1}\sum_{k\geq n} C_k^1(0,-n)r^k = \alpha_{-n}u_0\|z^1\|_1^{-1}\sum_{k\geq n+1} C_k^1(1,-n)r^k.$$

Comparing the coefficients of r^n, we see that $\alpha_{-n} = 0$. Thus for every $n \geq 1$, we have $S_2 e^1_{-n} = 0$. To complete the proof, we have to show that $S_2 e^1_n = 0$, $n \geq 0$.

Substituting $\phi = \phi_a$ in the equation (2.13), evaluating at the vector e^1_{-1}, putting $S_2 e^1_{-1} = 0$, taking inner product with e^1_{n+1-m}, for $n \geq m-1$ and then using the matrix coefficient of $\pi_1(\phi_a)$, we get

$$\alpha_n\|z^{-1}\|_1^{-1}\sum_{k\geq 0} C_k^1(-1,n)r^k = \alpha_n u_{-1}\|z^0\|_1^{-1}\sum_{k\geq 0} C_k^1(0,n)r^k,$$

Comparing the constant coefficients and the coefficients of r, respectively, we get

$$\frac{\alpha_n(-\lambda-\mu+1)}{\|z^{-1}\|_1(n+1)} = \frac{\alpha_n u_{-1}}{\|z^0\|_1} \text{ and } \frac{\alpha_n(\mu+1)(-\lambda-\mu+1)}{\|z^{-1}\|_1(n+2)} = \frac{\alpha_n \mu u_{-1}}{\|z^0\|_1}.$$

These two equations together give

$$\alpha_n\left[\frac{(\mu+1)}{(n+2)} - \frac{\mu}{(n+1)}\right] = 0.$$

Since $\frac{(\mu+1)}{(n+2)} \neq \frac{\mu}{(n+1)}$, $n \geq 0$, we must have $\alpha_n = 0$ for all $n \geq 0$. This proves that $S_2 e^1_n = 0$, $n \geq 0$.

Case II ($m = 0$): Assume $m = 0$. In this case, $\lambda > 0$. From the equation (2.24), we see that

(a) there exists $\alpha_n \in \mathbb{C}$ such that $S_2 e^1_n = \alpha_n e^2_{n+1}$, $n \geq -1$;

(b) there exists $\alpha_{-n} \in \mathbb{C}$ such that $S_2 e^1_{-n} = \alpha_{-n} e^2_{-n+k+1}$, $n > k+1$ and $S_2 e^1_{-n} = 0$, $2 \leq n \leq k+1$.

Repeating a similar computation as in the case of ($m \geq 1$), we conclude that $S_2 e^1_{-n} = 0$, $n \geq 2$.

Now we prove that $\alpha_n = 0$ for all $n \geq -1$. Substituting $\phi = \phi_a$ in the equation (2.13), evaluating at the vector e^1_{-2}, putting $S_2 e^1_{-2} = 0$, taking inner product with e^2_{n+1} and then using the matrix coefficient of $\pi_1(\phi_a)$, we get

$$\alpha_n\|z^{-2}\|_1^{-1}\sum_{k\geq 0} C_k^1(-2,n)r^k = \alpha_n u_{-2}\|z^{-1}\|_1^{-1}\sum_{k\geq 0} C_k^1(-1,n)r^k.$$

Comparing the constant coefficient and the coefficient of r respectively, we obtain

$$\frac{\alpha_n(-\lambda-\mu+2)}{\|z^{-2}\|_1(n+2)} = \frac{\alpha_n u_{-2}}{\|z^{-1}\|_1} \text{ and } \frac{\alpha_n(\mu+2)(-\lambda-\mu+2)}{\|z^{-2}\|_1(n+3)} = \frac{\alpha_n(\mu+1)u_{-2}}{\|z^{-1}\|_1}.$$

These two equations together give

$$\alpha_n\left[\frac{(\mu+2)}{(n+3)} - \frac{(\mu+1)}{(n+2)}\right] = 0.$$

Since $\pi_1 \neq P_{1,0}$, it follows that μ is not in $[0,\infty)$. This implies that $\alpha_n = 0$ for all $n \geq -1$. Therefore we have proved that $S_2 = 0$ in this case. □

Remark 2.24. Suppose $T = \begin{bmatrix} T_1 & S_1 \\ S_2 & T_2 \end{bmatrix}$ is a homogeneous operator with associated representation $\pi_1 \oplus \pi_2$, where π_1 is an irreducible Continuous series representation and $\pi_2 = P_{1,0}$. Repeating the computations of Theorem 2.22 and Theorem 2.23, we obtain that $S_1 = S_2 = 0$.

2.3 The associated representation is the direct sum of four irreducible representations

In this section, we prove that every homogeneous operator with associated representation $\pi_1 \oplus \pi_2$, where $\pi_1 = D^+_{\lambda_1} \oplus D^-_{\lambda_2}$ and $\pi_2 = D^+_{\lambda_3} \oplus D^-_{\lambda_4}$, is reducible. If $D^+_{\lambda_1} \oplus D^-_{\lambda_2} \oplus D^+_{\lambda_3} \oplus D^-_{\lambda_4}$ is a representation, then the multipliers of all the four representations $D^+_{\lambda_1}$, $D^-_{\lambda_2}$, $D^+_{\lambda_3}$ and $D^-_{\lambda_4}$ must be the same. In consequence, $\lambda_1 = \lambda + 2a$, $\lambda_2 = 2 - \lambda + 2b$, $\lambda_3 = \lambda + 2m$ and $\lambda_4 = 2 - \lambda + 2p$ for some real λ with $0 < \lambda \le 2$ and some non negative integers a, b, m, p.

Let $\lambda \in (0, 2]$ and a, b, m, p be any non-negative integers. Let $\pi_1 = D^+_{\lambda+2a} \oplus D^-_{2-\lambda+2b}$ and $\pi_2 = D^+_{\lambda+2m} \oplus D^-_{2-\lambda+2p}$. Then the representation space of π_1 is $H_1 := H^{(\lambda+2a)} \oplus H^{(2-\lambda+2b)}$ and the representation space of π_2 is $H_2 := H^{(\lambda+2m)} \oplus H^{(2-\lambda+2p)}$. Define

$$e^1_n = \begin{pmatrix} \frac{z^n}{\|z^n\|_{\lambda+2a}} \\ 0 \end{pmatrix}, n \ge 0 \text{ and } e^1_{-n} = \begin{pmatrix} 0 \\ \frac{z^{n-1}}{\|z^{n-1}\|_{2-\lambda+2b}} \end{pmatrix}, n \ge 1.$$

Also, define

$$e^2_n = \begin{pmatrix} \frac{z^n}{\|z^n\|_{\lambda+2m}} \\ 0 \end{pmatrix}, n \ge 0 \text{ and } e^2_{-n} = \begin{pmatrix} 0 \\ \frac{z^{n-1}}{\|z^{n-1}\|_{2-\lambda+2p}} \end{pmatrix}, n \ge 1.$$

Then the vectors $e^i_n, n \in \mathbb{Z}$ form an orthonormal basis of H_i, $i = 1, 2$. If ϕ_θ is a rotation in Möb, then

$$\pi_1(\phi_\theta)e^1_n = e^{-i\left(n+a+\frac{\lambda}{2}\right)\theta} e^1_n, n \ge 0;\ \pi_1(\phi_\theta)e^1_{-n} = e^{i\left(n+b-\frac{\lambda}{2}\right)\theta} e^1_{-n}, n \ge 1$$

and

$$\pi_2(\phi_\theta)e^2_n = e^{-i\left(n+m+\frac{\lambda}{2}\right)\theta} e^2_n, n \ge 0;\ \pi_2(\phi_\theta)e^2_{-n} = e^{i\left(n+p-\frac{\lambda}{2}\right)\theta} e^2_{-n}, n \ge 1.$$

We can, therefore, find θ such that $\pi_1(\phi_\theta)$ and $\pi_2(\phi_\theta)$ have distinct eigenvalues with one dimensional eigenspaces described as above.

Theorem 2.25. *Let* $T = \begin{bmatrix} T_1 & S_1 \\ S_2 & T_2 \end{bmatrix}$ *be a homogeneous operator with associated representation* $\pi_1 \oplus \pi_2$, *where* $\pi_1 = D^+_{\lambda+2a} \oplus D^-_{2-\lambda+2b}$ *and* $\pi_2 = D^+_{\lambda+2m} \oplus D^-_{2-\lambda+2p}$. *Then the following holds:*

(a) *For* $n \in \mathbb{Z}$, *there exist* $u_n \in \mathbb{C}$ *such that* $T_1 e^1_n = u_n e^1_{n+1}$, *where* $u_{-1} = 0$ *unless* $a = 0$ *and* $b = 0$.

(b) *For $n \in \mathbb{Z}$, there exists $v_n \in \mathbb{C}$ such that $T_2 e_n^2 = v_n e_{n+1}^2$, where $v_{-1} = 0$ unless $m = 0$ and $p = 0$.*

(c) *For $n \ge 0$, $S_1 e_n^2$ belongs to the span closure of the set of vectors $\{e_q^1 : q \ge 0\}$ and for $n \ge 2$, $S_1 e_{-n}^2$ belongs to the span closure of the set of vectors $\{e_{-q}^1 : q \ge 1\}$. The vector $S_1 e_{-1}^2$ belongs to the span closure of the set $\{e_{-q}^1 : q \ge 1\}$ unless $p = 0$ and $a = 0$.*

(d) *For $n \ge 0$, $S_2 e_n^1$ belongs to the span closure of the set of vectors $\{e_q^2 : q \ge 0\}$ and for $n \ge 2$, $S_2 e_{-n}^1$ belongs to the span closure of the set $\{e_{-q}^2 : q \ge 1\}$. The vector $S_2 e_{-1}^1$ belongs to the span closure of the set $\{e_{-q}^2 : q \ge 1\}$ unless $b = 0$ and $m = 0$.*

Proof. Homogeneity of T implies that T_i and S_i satisfy equations (2.11), (2.12), (2.13) and (2.14). Substituting $\phi = \phi_\theta$ in the equation (2.11), we get

$$\pi_1(\phi_\theta) T_1 e_n^1 = e^{-i\left(n+1+a+\frac{\lambda}{2}\right)\theta} T_1 e_n^1, n \ge 0 \text{ and } \pi_1(\phi_\theta) T_1 e_{-n}^1 = e^{i\left(n-2+b+\frac{\lambda}{2}\right)\theta} T_1 e_{-n}^1, n \ge 1.$$

Therefore, for each $n \in \mathbb{Z}$, there exists $u_n \in \mathbb{C}$ such that $T_1 e_n^1 = u_n e_{n+1}^1$, $u_{-1} = 0$, unless $a = 0$ and $b = 0$.

Similarly, we can show that for all $n \in \mathbb{Z}$, there exists $v_n \in \mathbb{C}$ such that $T_2 e_n^2 = v_n e_{n+1}^2$, $v_{-1} = 0$, unless $m = 0$ and $p = 0$. Substituting $\phi = \phi_\theta$ in equation (2.12), we obtain

$$\pi_1(\phi_\theta) S_1 e_n^2 = e^{-i\left(n+1+m+\frac{\lambda}{2}\right)\theta} S_1 e_n^2, n \ge 0 \text{ and } \pi_1(\phi_\theta) S_1 e_{-n}^2 = e^{i\left(n-2+p+\frac{\lambda}{2}\right)\theta} S_1 e_{-n}^2, n \ge 1.$$

We, therefore, see that

1. for each $n \ge 0$, $S_1 e_n^2$ belongs to the span closure of the set of vectors $\{e_q^1 : q \ge 0\}$,
2. for each $n \ge 2$, $S_1 e_{-n}^2$ belongs to the span closure of the set of vectors $\{e_{-q}^1 : q \ge 1\}$ and
3. except when $p = 0$ and $a = 0$, $S_1 e_{-1}^2$ belongs to the span closure of the set of vectors $\{e_{-q}^1 : q \ge 1\}$.

Similarly, we can show that (i) for $n \ge 0$, $S_2 e_n^1$ belongs to the span closure of the set of vectors $\{e_q^2 : q \ge 0\}$, (ii) for $n \ge 2$, $S_2 e_{-n}^1$ belongs to the span closure of the set of vectors $\{e_{-q}^2 : q \ge 1\}$ and (iii) except when $b = 0$ and $m = 0$, $S_2 e_{-1}^1$ belongs to the span closure of the set of vectors $\{e_{-q}^2 : q \ge 1\}$. □

Theorem 2.26. *Suppose $T = \begin{bmatrix} T_1 & S_1 \\ S_2 & T_2 \end{bmatrix}$ is a homogeneous operator with associated representation $\pi_1 \oplus \pi_2$, where $\pi_1 = D_{\lambda+2a}^+ \oplus D_{2-\lambda}^-$ and $\pi_2 = D_\lambda^+ \oplus D_{2-\lambda+2p}^-$, for a pair a, p of positive integers. Then T is reducible. Furthermore, $T = \tilde{T}_1 \oplus \tilde{T}_2$ where $\tilde{T}_1$ is a homogeneous operator with associated representation $D_{\lambda+2a}^+ \oplus D_\lambda^+$ and $\tilde{T}_2$ is a homogeneous operator with associated representation $D_{2-\lambda}^- \oplus D_{2-\lambda+2p}^-$.*

Proof. Homogeneity of T implies that the operators T_i and S_i satisfy equations (2.11), (2.12), (2.13) and (2.14). Since $a \neq 0$ and $p \neq 0$, from Theorem 2.25, it follows that

(a) for $n \geq 0$, $T_i e_n^2$ is in the span closure of $\{e_q^i : q \geq 0\}$, $i = 1,2$,

(b) for $n \geq 1$, $T_i e_{-n}^i$ is in the span closure of $\{e_{-q}^i : q \geq 1\}$, $i = 1,2$,

(c) for $n \geq 0$, $S_1 e_n^2$ is in the span closure of $\{e_q^1 : q \geq 0\}$ and

(d) for $n \geq 1$, $S_1 e_{-n}^2$ is in the span closure of $\{e_{-q}^1 : q \geq 1\}$,

Substituting $\phi = \phi_\theta$ in the equation (2.13), we obtain

$$\pi_2(\phi_\theta) S_2 e_n^1 = e^{-i\left(n+1+a+\frac{\lambda}{2}\right)\theta} S_2 e_n^1, n \geq 0;\ \pi_2(\phi_\theta) S_2 e_{-n}^1 = e^{i\left(n-1-\frac{\lambda}{2}\right)\theta} S_2 e_{-n}^1, n \geq 1.$$

This implies that (i) for all $n \geq 0$, there exists $\alpha_n \in \mathbb{C}$ such that $S_2 e_n^1 = \alpha_n e_{n+1+a}^2$, (ii) for all $n \geq p+2$ there exists $\alpha_{-n} \in \mathbb{C}$ such that $S_2 e_{-n}^1 = \alpha_{-n} e_{-n+p+1}^1$, (iii) for all $2 \leq n \leq p+1$, $S_2 e_{-n}^1 = 0$ and (iv) there exists $\alpha_{-1} \in \mathbb{C}$ such that $S_2 e_{-1}^1 = \alpha_{-1} e_0^2$.

Now substituting $\phi = \phi_a$ in the equation (2.13), evaluating at the vector e_{-1}^1 and then taking inner product with e_0^2, we obtain

$$\alpha_{-1} \left\langle D_{2-\lambda}^+(\phi_a^*) z^0, z^0 \right\rangle + \alpha_{-1} \left\langle D_\lambda^+(\phi_a) z^0, z^0 \right\rangle = 0.$$

If a is real, then $\phi_a^* = \phi_a$. An easy computation shows that $\left\langle D_{2-\lambda}^+(\phi_a) z^0, z^0 \right\rangle + \left\langle D_\lambda^+(\phi_a) z^0, z^0 \right\rangle \neq 0$, $a \in (0,1)$. In consequence $\alpha_{-1} = 0$.

Let $\tilde{H}_1$ and $\tilde{H}_2$ be the closed subspaces of H spanned by the orthonormal set of vectors

$$\left\{ \begin{pmatrix} e_n^1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ e_n^2 \end{pmatrix} : n \geq 0 \right\},\ \left\{ \begin{pmatrix} e_{-n}^1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ e_{-n}^2 \end{pmatrix} : n \geq 1 \right\} \tag{2.25}$$

respectively.

We have $T = \tilde{T}_1 \oplus \tilde{T}_2$, where $\tilde{T}_i$ is an operator on $\tilde{H}_i$, $i = 1,2$. Also note that $\tilde{H}_i$ is invariant under π. So, $\tilde{T}_1$ is a homogeneous operator with associated representation $D_{\lambda+2a}^+ \oplus D_\lambda^+$ and $\tilde{T}_2$ is a homogeneous operator with associated representation $D_{2-\lambda}^- \oplus D_{2-\lambda+2p}^-$.

□

Theorem 2.27. *Suppose* $T = \begin{bmatrix} T_1 & S_1 \\ S_2 & T_2 \end{bmatrix}$ *is a homogeneous operator with associated representation* $\pi_1 \oplus \pi_2$, *where* $\pi_1 = D_\lambda^+ \oplus D_{2-\lambda}^-$ *and* $\pi_2 = D_\lambda^+ \oplus D_{2-\lambda+2p}^-$ *and* p *is some positive integer. Then* T *is reducible. Furthermore,* $T = \tilde{T}_1 \oplus \tilde{T}_2$, *where* $\tilde{T}_1$ *is a homogeneous operator with associated representation* $D_\lambda^+ \oplus D_\lambda^+$ *and* $\tilde{T}_2$ *is a homogeneous operator with associated representation* $D_{2-\lambda}^- \oplus D_{2-\lambda+2p}^-$ *or* $T = T_1 \oplus T_2$.

Proof. Homogeneity of T implies that the operators T_i and S_i satisfy equations (2.11), (2.12), (2.13) and (2.14). Recall that T_1 and T_2 are weighted shifs with respect to the orthonormal basis $\{e_n^1\}$ and $\{e_n^2\}$, respectively by virtue of Theorem 2.25. Let $\{u_n\}$ and $\{v_n\}$ be the corresponding weights for T_1 and T_2, respectively. Since $p > 0$, it follows form Theorem 2.25 that $v_{-1} = 0$.

Substituting $\phi = \phi_\theta$ in the equation (2.13), we get

$$\pi_2(\phi_\theta)S_2e_n^1 = e^{-i\left(n+1+\frac{\lambda}{2}\right)\theta}S_2e_n^1,\ n \in \mathbb{Z}.$$

This implies that (i) for all $n \geq -1$ there exist $\beta_n \in \mathbb{C}$ such that $S_2e_n^1 = \beta_n e_{n+1}^2$, (ii) for all $n \geq p+2$ there exists $\beta_{-n} \in \mathbb{C}$ such that $S_2e_{-n}^1 = \beta_{-n}e_{-n+p+1}^2$ and (iii) $S_2e_{-n}^2 = 0$, for all $1 < n < p+2$.

Substituting $\phi = \phi_a$ in the equation (2.13), evaluating at the vector e_n^1, for $n \geq -1$ and taking inner product with e_0^2, we get

$$\beta_{-1}\left\langle \pi_1(\phi_a)e_n^1, e_{-1}^1\right\rangle + \beta_n\left\langle \pi_2(\phi_a)e_{n+1}^2, e_0^2\right\rangle = 0.$$

Now, if $n \geq 0$, then from the preceding equation, we find that $\beta_n\left\langle \pi_2(\phi_a)e_{n+1}^2, e_0^2\right\rangle = 0$ and therefore $\beta_n = 0$ for all $n \geq 0$. For $n = -1$, from the same equation, we have

$$\beta_{-1}\left\langle \pi_1(\phi_a)e_{-1}^1, e_{-1}^1\right\rangle + \beta_{-1}\left\langle \pi_2(\phi_a)e_0^2, e_0^2\right\rangle = 0.$$

However, it is easily verified that $\left\langle \pi_1(\phi_a)e_{-1}^1, e_{-1}^1\right\rangle + \left\langle \pi_2(\phi_a)e_0^2, e_0^2\right\rangle \neq 0$. Therefore, $\beta_{-1} = 0$.

Substituting $\phi = \phi_a$ in the equation (2.13), evaluating at the vector e_{-1}^1 and then taking inner product with e_{-n+p+1}^2, $n \geq p+2$, we get

$$\beta_{-n}\left\langle \pi_1(\phi_a)e_{-1}^1, e_{-n}^1\right\rangle = 0.$$

Consequently, we have $\beta_{-n} = 0$, for $n \geq p+2$. This proves that $S_2e_{-n}^1 = 0$, for all $n \geq 2$ and therefore $S_2 = 0$.

Substituting $\phi = \phi_\theta$ in the equation (2.12), we obtain

$$\pi_1(\phi_\theta)S_1e_n^2 = e^{-i\left(n+1+\frac{\lambda}{2}\right)\theta}S_1e_n^2, n \geq 0;\ \pi_1(\phi_\theta)S_1e_{-n}^2 = e^{i\left(n-1+p-\frac{\lambda}{2}\right)\theta}S_1e_{-n}^2, n \geq 1.$$

Thus (i) for all $n \geq 0$, there exists $\alpha_n \in \mathbb{C}$ such that $S_1e_n^2 = \alpha_n e_{n+1}^1$ and (ii) for all $n \geq 1$, there exists $\alpha_{-n} \in \mathbb{C}$ such that $S_1e_{-n}^2 = \alpha_{-n}e_{-n-p+1}^1$.

Substituting $\phi = \phi_a$ in the equation (2.12), evaluating at the vector e_n^2, for $n \geq 0$, taking inner product with e_0^1 and using $S_1^*e_0^1 = 0$, we get

$$\alpha_n\left\langle \pi_1(\phi_a)e_{n+1}^1, e_0^1\right\rangle = 0.$$

Consequently, for all $n \geq 0$, we see that $\alpha_n = 0$. This proves that $S_1e_n^2 = 0$, $n \geq 0$.

Again, substituting $\phi = \phi_a$ in the equation (2.12), evaluating at e^2_{-n}, for $n \geq 1$, taking inner product with e^1_0 and using $S_1^* e^1_0 = 0$, we get

$$\alpha_{-n} u_{-1} \left\langle \pi_1(\phi_a) e^1_{-n-p+1}, e^1_{-1} \right\rangle = 0.$$

It follows that $\alpha_{-n} u_{-1} = 0$, $n \geq 1$. Hence if $u_{-1} \neq 0$, then for all $n \geq 1$, we see that $\alpha_{-n} = 0$ and therefore $S_1 = 0$. Putting all of these together, we infer that $T = T_1 \oplus T_2$, where T_1 is a homogeneous operator with associated representation π_1 and T_2 is a homogeneous operator with associated representation π_2.

Let $\tilde{T}_1, \tilde{T}_2$ be the operators which were constructed in Theorem 2.26. If $u_{-1} = 0$, then we have $T = \tilde{T}_1 \oplus \tilde{T}_2$. The operators $\tilde{T}_1$ and $\tilde{T}_2$ are homogeneous, and in this case, the associated representations are $D^+_\lambda \oplus D^+_\lambda$ and $D^-_{2-\lambda} \oplus D^-_{2-\lambda+2p}$, respectively. □

Theorem 2.28. *Let* $T = \begin{bmatrix} T_1 & S_1 \\ S_2 & T_2 \end{bmatrix}$ *be a homogeneous operator with associated representation* $\pi_1 \oplus \pi_2$ *where* $\pi_1 = D^+_\lambda \oplus D^-_{2-\lambda}$ *and* $\pi_2 = D^+_\lambda \oplus D^-_{2-\lambda}$. *Then* $S_1 = 0$ *and* $S_2 = 0$.

Proof. In this case $\pi_1 = \pi_2$. Denote $\pi_1 = \pi_2 = \pi$ and $e^1_n = e^2_n = e_n$. Homogeneity of T implies that the operators T_i and S_i satisfy equations (2.11), (2.12), (2.13) and (2.14). Repeating an argument similar to the one in Theorem 2.25, we find that T_1, T_2, S_1 and S_2 are weighted shifts with respect to the orthonormal basis $\{e_n\}$. Let $\{u_n\}$, $\{v_n\}$, $\{\alpha_n\}$ and $\{\beta_n\}$ be the weights for T_1, T_2, S_1 and S_2, respectively.

Now we prove that $S_1 = 0$. Substituting $\phi = \phi_a$ in the equation (2.12), evaluating at the vector e_n, for $n \geq 0$ and then taking inner product with e_0, we obtain

$$\alpha_n \left\langle \pi(\phi_a) e_{n+1}, e_0 \right\rangle = 0.$$

This implies that $\alpha_n = 0$, $n \geq 0$.

Substituting $\phi = \phi_a$ in the equation (2.12), evaluating at the vector e_{-1} and then taking inner product with e_{n+1}, for $n \leq -1$, we get

$$\alpha_n \left\langle \pi(\phi_a) e_{-1}, e_n \right\rangle + \alpha_{-1} \left\langle \pi(\phi_a) e_0, e_{n+1} \right\rangle = 0.$$

This implies that $\alpha_n = 0$, $n \leq -1$, proving that $S_1 = 0$. A similar computation shows that $S_2 = 0$. □

Theorem 2.29. *Suppose* $T = \begin{bmatrix} T_1 & S_1 \\ S_2 & T_2 \end{bmatrix}$ *is a homogeneous operator with associated representation* $\pi_1 \oplus \pi_2$ *where* $\pi_1 = D^+_{\lambda+2a} \oplus D^-_{2-\lambda+2b}$ *and* $\pi_2 = D^+_{\lambda+2m} \oplus D^-_{2-\lambda+2p}$. *Then either* $T = \tilde{T}_1 \oplus \tilde{T}_2$, *where* $\tilde{T}_1$ *is a homogeneous operator with associated representation* $D^+_{\lambda+2a} \oplus D^+_{\lambda+2m}$ *and* $\tilde{T}_2$ *is a homogeneous operator with associated representation* $D^-_{2-\lambda+2b} \oplus D^-_{2-\lambda+2p}$ *or* $T = T_1 \oplus T_2$. *In particular,* T *is reducible.*

Proof. We divide the proof into several cases and discuss each case separately. Let $\tilde{H}_1$, $\tilde{H}_2$ be as in (2.25) and $\tilde{T}_i = T_{|\tilde{H}_i}$, $i = 1,2$.

(i) Assume that none of the a,b,m,p are zero. Then from Theorem 2.25, it follows that $T = \tilde{T}_1 \oplus \tilde{T}_2$. Also note that $\tilde{H}_i$ is invariant under π. So, $\tilde{T}_1$ is a homogeneous operator with associated representation $D^+_{\lambda+2a} \oplus D^+_{\lambda+2m}$ and $\tilde{T}_2$ is a homogeneous operator with associated representation $D^-_{2-\lambda+2b} \oplus D^-_{2-\lambda+2p}$.

(ii) Assume that exactly one of a,b,m,p is non-zero. Then from Theorem 2.25, it follows that $T = \tilde{T}_1 \oplus \tilde{T}_2$.

(iii) It follows from Theorem 2.25 that $T = \tilde{T}_1 \oplus \tilde{T}_2$ if either $a = 0, b \neq 0, m = 0, p \neq 0$ or $a \neq 0, b = 0, m \neq 0, p = 0$.

(iv) The case of $a \neq 0, b = 0, p \neq 0, m = 0$ is precisely Theorem 2.26.

(v) Assume that $a = 0, b \neq 0, m \neq 0, p = 0$. Since T^* is a homogeneous operator with associated representation $\pi_1^\# \oplus \pi_2^\#$, the proof follows by applying Theorem 2.26 to T^*.

(vi) Assume that $a = 0, b = 0, m \neq 0, p \neq 0$. The associated representation of the operator T is $D^+_\lambda \oplus D^-_{2-\lambda} \oplus D^+_{\lambda+2m} \oplus D^-_{2-\lambda+2p} = \left(D^+_\lambda \oplus D^-_{2-\lambda+2p}\right) \oplus \left(D^+_{\lambda+2m} \oplus D^-_{2-\lambda}\right)$. Now, the proof follows form Theorem 2.26.

(vii) Assume that $a \neq 0, b \neq 0, m = 0, p = 0$. This is same as (vi).

(viii) The cases of $a = 0, b = 0, m = 0, p \neq 0$ and $a = 0, m = 0, p = 0, b \neq 0$ are covered in Theorem 2.27.

(ix) In case, $b = 0, m = 0, p = 0, a \neq 0$ or $a = 0, b = 0, p = 0, m \neq 0$, the proof is completed by applying the Theorem 2.27 to T^*.

(x) Assume $a = 0, b = 0, m = 0, p = 0$. This case is exactly Theorem 2.28.

This is an enumeration of all the sixteen possibilities (each of the integers a,b,m,p is either zero or positive) completing the proof. □

Now we prove that there is no irreducible homogeneous operator with associated representation $\pi := P_{1,0} \oplus D^+_{1+2m} \oplus D^-_{1+2k}$. The representation space of π is $H := L^2(\mathbb{T}) \oplus H^{(1+2m)} \oplus H^{(1+2k)}$.

Theorem 2.30. *Suppose* $T = \begin{bmatrix} T_1 & S_1 \\ S_2 & T_2 \end{bmatrix}$ *is a homogeneous operator with associated representation* $\pi_1 \oplus \pi_2$, *where* $\pi_1 = P_{1,0}$ *and* $\pi_2 = D^+_{1+2m} \oplus D^-_{1+2k}$, $m,k \geq 0$. *Then we have the following.*

(a) *The operators T_1 and T_2 are weighted shifts with respect to the orthonormal basis $\{e_n^1\}$ and $\{e_n^2\}$ with weights $\{u_n\}$ and $\{v_n\}$, respectively. Also $T_2 e_{-1}^2 = 0$ except when $m = 0$ and $k = 0$.*

(b) *If $k \geq 1$, then for all $n \geq 0$, $S_1 e_n^2 = 0$, and for all $n \geq 1$, $S_1 e_{-n}^2 = \alpha_{-n} e_{-n-k+1}^1$ such that $u_{-1}\alpha_{-n} = 0$ where $\alpha_{-n} \in \mathbb{C}$. If $k = 0$, then $S_1 e_n^2 = 0$ for all $n \neq -1$ and $S_1 e_{-1}^2 = \alpha_{-1} e_0^1$ for some $\alpha_{-1} \in \mathbb{C}$.*

(c) *If $m > 1$, then $S_2 = 0$. If $m = 1$, then for $n \leq -1$, $S_2 e_n^1 = 0$ and for $n \geq 0$, there exists $\beta_n \in \mathbb{C}$ such that $S_2 e_n^1 = \beta_n e_n^2$ and $u_{-1}\beta_n = 0$. If $m = 0$, then $S_2 e_n^1 = 0$, for all $n \neq -1$ and $S_2 e_{-1}^1 = \beta_{-1} e_0^2$ for some $\beta_{-1} \in \mathbb{C}$.*

Proof. (a) Homogeneity of T implies that the operators T_i and S_i satisfy equations (2.11), (2.12), (2.13) and (2.14). Using the equations (2.11) and (2.12), we find that T_1 and T_2 are weighted shifts with respect to the orthonormal basis $\{e_n^1\}$ and $\{e_n^2\}$, respectively. Let $\{u_n\}$ and $\{v_n\}$ be the weights of T_1 and T_2, respectively. It is easy to see that that $v_{-1} = 0$ except when $m = 0$ and $k = 0$.
(b) Restricting the equation (2.12) to the rotation group, we obtain

$$\pi_1(\phi_\theta) S_1 e_n^2 = e^{-i(n+1+m+\frac{1}{2})\theta} S_1 e_n^2, n \geq 0 \tag{2.26}$$

and

$$\pi_1(\phi_\theta) S_1 e_{-n}^2 = e^{i(n-1+k-\frac{1}{2})\theta} S_1 e_{-n}^2, n \geq 1. \tag{2.27}$$

It follows that there exists a sequence $\{\alpha_n\}$ such that

$$S_1 e_n^2 = \alpha_n e_{n+m+1}^1, n \geq 0 \text{ and } S_1 e_{-n}^2 = \alpha_{-n} e_{-n-k+1}^1, n \geq 1. \tag{2.28}$$

Substituting $\phi = \phi_a$ in the equation (2.12) and then evaluating at e_n^2, $n \geq 0$, we get

$$S_1 \pi_2(\phi_a) e_n^2 + \alpha_n \pi_1(\phi_a) e_{n+m+1}^1 = \overline{a}\alpha_n T_1 \pi_1(\phi_a) e_{n+m+1}^1 + \overline{a} v_n S_1 \pi_2(\phi_a) e_{n+1}^2.$$

The equation (2.28) implies that if $k > 0$, then $S_1^* e_0^1 = 0$ and if $k = 0$, then $S_1^* e_0^1 = \alpha_{-1} e_{-1}^2$. Therefore, taking inner product with e_0^1, we have

$$\alpha_n \langle \pi_1(\phi_a) e_{n+m+1}^1, e_0^1 \rangle = 0.$$

In consequence, $\alpha_n = 0$ for all $n \geq 0$.

$k \geq 1$: Substituting $\phi = \phi_a$ in the equation (2.12), evaluating it at e_{-n}^2 and then taking inner product with e_0^1, we have

$$\bar{a}\alpha_{-n} u_{-1} \langle \pi_1(\phi_a) e_{-n-k+1}^1, e_{-1}^1 \rangle = 0,$$

which implies that $\alpha_{-n} u_{-1} = 0$ for all $n \geq 1$.

$k=0$: Substituting $\phi=\phi_a$ in the equation (2.12), evaluating it at e^2_{-1} and then taking inner product with e^1_{-n+1}, we get

$$\alpha_{-n}\left\langle \pi_2(\phi_a)e^2_{-1}, e^2_{-n}\right\rangle + \alpha_{-1}\left\langle \pi_1(\phi_a)e^1_0, e^1_{-n+1}\right\rangle = 0.$$

This implies that $\alpha_{-n}=0$ for all $n\geq 2$.

(c) Restricting the equation (2.13) to the rotation group, we obtain

$$\pi_2(\phi_\theta)S_2 e^1_n = e^{-(n+1+\frac{1}{2})\theta} S_2 e^1_n,\ n\in\mathbb{Z}. \tag{2.29}$$

Equation (2.29) implies that

(i) for all $n, n\geq \max\{m-1,0\}$, there exist $\beta_n\in\mathbb{C}$ such that $S_2e^1_n=\beta_n e^2_{-m+n+1}$ and for all $n, 0\leq n<\max\{m-1,0\}$, $S_2e^1_n=0$,

(ii) for all $n, n\geq k+2$, there exist $\beta_{-n}\in\mathbb{C}$ such that $S_2e^1_{-n}=\beta_{-n}e^2_{-n+k+1}$ and for all $n, 2\leq n< k+2$, $S_2e^1_{-n}=0$,

(iii) there exists $\beta_{-1}\in\mathbb{C}$ such that $S_2e^1_{-1}=\beta_{-1}e^2_0$ where $\beta_{-1}=0$ if $m\neq 0$.

Substituting $\phi=\phi_a$ in the equation (2.13), evaluating at e^1_{-1} and then taking inner product with e^2_{-n+k+1}, for $n\geq k+2$, we see that $\beta_{-n}=0$ since

$$\beta_{-n}\left\langle \pi_1(\phi_a)e^1_{-1}, e^1_{-n}\right\rangle = 0.$$

Thus, we have $S_2e^1_{-n}=0$ for all $n\geq 2$.

$m>1$: Substituting $\phi=\phi_a$ in the equation (2.13), evaluating at e^1_n, $n\geq m-1$ and then taking inner product with e^2_0, we obtain

$$\beta_n\left\langle \pi_2(\phi_a)e^2_{-m+n+1}, e^2_0\right\rangle = 0.$$

Thus, for $n\geq m-1$, $\beta_n=0$. Consequently, $S_2=0$.

$m=1$: Substituting $\phi=\phi_a$ in the equation (2.13), evaluating at e^1_{-1} and then taking inner product with e^2_n, $n\geq 0$, we obtain

$$\bar{a}\beta_n u_{-1}\left\langle \pi_2(\phi_a)e^1_0, e^1_n\right\rangle = 0.$$

Thus, for $n\geq 0$, $u_{-1}\beta_n=0$.

$m=0$: Substituting $\phi=\phi_a$ in the equation (2.13), evaluating at e^1_n, $n\geq -1$, and then taking inner product with e^2_0, we obtain

$$\beta_{-1}\left\langle \pi_1(\phi_a)e^1_n, e^1_{-1}\right\rangle + \beta_n\left\langle \pi_2(\phi_a)e^2_{n+1}, e^2_0\right\rangle = 0.$$

This implies that $\beta_n=0$, $n\geq 0$.

□

Theorem 2.31. *Suppose* $T=\begin{bmatrix} T_1 & S_1\\ S_2 & T_2\end{bmatrix}$ *is a homogeneous operator with associated representation* $\pi=\pi_1\oplus\pi_2$, *where* $\pi_1=P_{1,0}$ *and* $\pi_2=D^+_{1+2m}\oplus D^-_{1+2k}$, $m,k\geq 0$. *Then* T *is reducible.*

Proof. It follows from Theorem 2.30(a) that the operators T_1 and T_2 are weighted shifts with respect to the orthonormal basis $\{e^1_n\}$ and $\{e^2_n\}$, respectively. Let $\{u_n\}$ and $\{v_n\}$ be the corresponding weights, where $v_{-1}=0$ except when $m=0$ and $k=0$.

$m>1$**,** $k\geq 1$: We have $v_{-1}=0$. From Theorem 2.30(b), we see that $S_1e^2_n=0$, $n\geq 0$, and $S_1e^2_{-n}=\alpha_{-n}e^1_{-n-k+1}$, $\alpha_{-n}\in\mathbb{C}$, $n\geq 1$. Here $u_{-1}\alpha_{-n}=0$. Thus either $u_{-1}=0$ or $\alpha_{-n}=0$ for all $n\geq 1$.

Also, Theorem 2.30(c) shows that $S_2=0$

If $u_{-1}=0$, then the closed subspace $\tilde{H}_1$ (defined in (2.25)) is a reducing subspace of T.

If $\alpha_{-n}=0$ for all $n\geq 1$, then $S_1=0$ and therefore T is reducible in this case as well.

$m=1$**,** $k\geq 1$: We have $v_{-1}=0$. From Theorem 2.30(b), we see that $S_1e^2_n=0$, $n\geq 0$, and $S_1e^2_{-n}=\alpha_{-n}e^1_{-n-k+1}$, $\alpha_{-n}\in\mathbb{C}$, $n\geq 1$. Here $u_{-1}\alpha_{-n}=0$.

Theorem 2.30(c) shows that $S_2e^1_{-n}=0$, $n\geq 1$ and $S_2e^1_n=\beta_ne^2_n$, $\beta_n\in\mathbb{C}$, $n\geq 0$. Here $u_{-1}\beta_n=0$.

Consequently, either $u_{-1}=0$ or $\alpha_{-n}=0$ for all $n\geq 1$ and $\beta_n=0$ for all $n\geq 0$.

If $u_{-1}=0$, then the subspace $\tilde{H}_1$ is a reducing subspace of T.

If $u_{-1}\neq 0$, then $S_1=0$ and $S_2=0$.

$m=0$, $k\geq 1$: We have $v_{-1}=0$. From Theorem 2.30(b), we see that $S_1e^2_n=0$, $n\geq 0$, and $S_1e^2_{-n}=\alpha_{-n}e^1_{-n-k+1}$, $\alpha_{-n}\in\mathbb{C}$, $n\geq 1$. Here $u_{-1}\alpha_{-n}=0$.

Furthermore, from Theorem 2.30(c), we see that for $n\neq -1$, $S_2e^1_n=0$ and $S_2e^1_{-1}=\beta_{-1}e^2_0$ for some $\beta_{-1}\in\mathbb{C}$.

If $u_{-1}=0$, then the closed subspace spanned by the set of vectors $\left\{\begin{pmatrix} e^1_n\\ 0\end{pmatrix} : n\geq 0\right\}$ is a reducing subspace of T.

If $\alpha_{-n} = 0$ for all $n \geq 1$, then the closed subspace of H spanned by the orthonormal set $\left\{\begin{pmatrix} 0 \\ e_n^2 \end{pmatrix} : n \leq -1\right\}$ is a reducing subspace of T.

$m > 1$**,** $k = 0$: We have $v_{-1} = 0$. From Theorem 2.30(b), we see that $S_1 e_{-1}^2 = \alpha_{-1} e_0^1$ for some $\alpha_{-1} \in \mathbb{C}$ and $S_1 e_n^2 = 0$, $n \neq -1$.

Also, Theorem 2.30(c) shows that $S_2 = 0$.

Consequently, the closed subspace spanned by the orthonormal set $\left\{\begin{pmatrix} 0 \\ e_n^2 \end{pmatrix} : n \geq 0\right\}$ is a reducing subspace of T.

$m = 1$, $k = 0$: We have $v_{-1} = 0$. From Theorem 2.30(b), we see that $S_1 e_{-1}^2 = \alpha_{-1} e_0^1$ for some $\alpha_{-1} \in \mathbb{C}$ and $S_1 e_n^2 = 0$, $n \neq -1$.

Theorem 2.30(c) shows that $S_2 e_{-n}^1 = 0$, $n \geq 1$ and $S_2 e_n^1 = \beta_n e_n^2$, $\beta_n \in \mathbb{C}$, $n \geq 0$. Here $u_{-1}\beta_n = 0$.

If $u_{-1} = 0$, then the closed subspace spanned by the orthonormal set $\left\{\begin{pmatrix} e_n^1 \\ 0 \end{pmatrix} : n \leq -1\right\}$ is a reducing subspace of T.

If $\beta_n = 0$ for all $n \geq 0$, then the closed subspace of H spanned by the orthonormal set $\left\{\begin{pmatrix} 0 \\ e_n^2 \end{pmatrix} : n \geq 0\right\}$ is a reducing subspace of T.

$m = 0$, $k = 0$: From Theorem 2.30(b), we see that for $n \neq -1$, $S_1 e_n^2 = 0$ and $S_2 e_n^1 = 0$. Clearly, $\tilde{H}_1$ is invariant under T. Let $A := T_{|\tilde{H}_1}$ and $B := PT_{|\tilde{H}_2}$, where $\tilde{H}_2$ is defined in (2.25) and P is the projection of H onto $\tilde{H}_2$. Since $\tilde{H}_1$ and $\tilde{H}_2$ are invariant under π, it follows from [5, Proposition 2.4] that A and B are homogeneous operators with associated representations $\pi_{|\tilde{H}_1}$ and $\pi_{|\tilde{H}_2}$, respectively. Since $\pi_{|\tilde{H}_1}$ is equivalent to $D_1^+ \oplus D_1^+$ and $S_1 e_n^2 = 0$, $S_2 e_n^1 = 0$ for all $n \geq 0$, it follows, using homogeneity of A, that $u_n = 1$, $v_n = 1$ for all $n \geq 0$. Similarly, it follows that $u_n = 1$, $v_n = 1$ for all $n \leq -2$. Therefore T must be reducible.

This completes the proof since we have shown that the operator T is reducible in every possible combination of the associated representation. □

Theorem 2.32. *Suppose T is a homogeneous operator on $L^2(\mathbb{T}) \oplus L^2(\mathbb{T})$ with associated representation $\pi = P_{1,0} \oplus P_{1,0}$. Then T is reducible.*

Proof. Let H_+ and H_- be closed subspaces of the Hilbert space $L^2(\mathbb{T})$ spanned by the orthonormal sets $\{z^n : n \geq 0\}$ and $\{z^n : n < 0\}$, respectively. Suppose $H(n)$ is the subspace of

$L^2(\mathbb{T}) \oplus L^2(\mathbb{T})$ spanned by the orthonormal set $\mathscr{B}_n = \left\{\begin{pmatrix} z^n \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ z^n \end{pmatrix}\right\}$. Since $H(n)$'s are $\mathbb{K}$-isotypic subspaces of the representation $P_{1,0} \oplus P_{1,0}$, it follows, form [9, Theorem 5.1], that T maps $H(n)$ to $H(n+1)$. Consequently, $H_+ \oplus H_+$ is an invariant subspace of T. Let $\begin{bmatrix} A & S \\ 0 & B \end{bmatrix}$ be the representation of T with respect to the decomposition of $L^2(\mathbb{T}) \oplus L^2(\mathbb{T})$ as $(H_+ \oplus H_+) \oplus (H_- \oplus H_-)$. We note that S maps $H(-1)$ to $H(0)$ and is 0 elsewhere. Since $H_+ \oplus H_+$ and $H_- \oplus H_-$ are invariant under π, it follows, from [5, Proposition 2.4], that A and B are homogeneous operators with associated representations $\pi_{|H_+ \oplus H_+}$ and $\pi_{|H_- \oplus H_-}$, respectively.

Using a similar argument as in Theorem 2.30, we infer that H_+ and H_- are reducing subspaces of A and B, respectively. Since $\pi_{|H_+ \oplus H_+}$ and $\pi_{|H_- \oplus H_-}$ are equivalent to $D_1^+ \oplus D_1^+$ and $D_1^- \oplus D_1^-$, respectively and A, B are homogeneous, it follows that A is a two-fold direct sum of the forward unilateral shift and B is a two-fold direct sum of the backward unilateral shift. Therefore, T is reducible. $\square$

Chapter 3

Homogeneous 3-shifts

The first examples of irreducible (bi-lateral) homogeneous 2-shifts were given by Korányi in [20]. Recall, from Theorem 1.11, that if $\pi(\phi)^* T_i \pi(\phi) = \phi(T_i)$, $i = 1,2$, for some representation of Möb, then the operator $\begin{pmatrix} T_1 & \alpha(T_1 - T_2) \\ 0 & T_2 \end{pmatrix}$, $\alpha > 0$, is homogeneous. It was shown in [9, List 4.1(4)] that for $0 < a < b < 1$, the bi-lateral shift $T(a,b)$ with weights $\sqrt{\frac{n+a}{n+b}}$ is homogeneous and the associated representation is the Complementary series $\pi = C_{\lambda,\sigma}$, where $\lambda = a + b - 1$ and $\sigma = (b-a)/2$. It follows from [9, Proposition 2.1] that the operator $T(a,b)^{*-1} = T(b,a)$ is also homogeneous with the same associated representation, namely, $C_{\lambda,\sigma}$. Consequently, the operator $\begin{pmatrix} T(a,b) & \alpha(T(a,b) - T(b,a)) \\ 0 & T(b,a) \end{pmatrix}$ is homogeneous. In the paper [20], Korányi shows that (a) these operators are irreducible and (b) unitarily inequivalent. Also, he proved, modulo unitary equivalence, these are the only homogeneous operators for which the associated representation is $C_{\lambda,\sigma} \oplus C_{\lambda,\sigma}$.

In Chapter 2, we completed these earlier results of Korányi by describing all the irreducible homogeneous 2-shifts. In this chapter, we prove that all homogeneous operators whose associated representation is a direct sum of three copies of a Complementary series representation, is reducible. Consequently, in this case, the important question of the existence of an irreducible homogeneous 3 - shift remains unanswered.

Let $\pi = C_{\lambda,\sigma}$ be a Complementary series representation of Möb acting on the Hilbert space $H^{\lambda,\sigma}$ and $\eta = \pi \oplus \pi \oplus \pi$. The representation space of η is $H := H^{\lambda,\sigma} \oplus H^{\lambda,\sigma} \oplus H^{\lambda,\sigma}$.

Lemma 3.1. *For $i = 1,2,3$, let T_i, S_i be bounded operators on some Hilbert space H and U_i be unitary representations of Möb on the same Hilbert space H. Then the operator*

$$T = \begin{bmatrix} T_1 & S_1 & S_2 \\ 0 & T_2 & S_3 \\ 0 & 0 & T_3 \end{bmatrix}$$

is homogeneous with associated representation $U_1 \oplus U_2 \oplus U_3$ if and only if T_i is homogeneous *with associated representation* U_i, $i = 1,2,3$, *and for* $\phi_{\theta,a}$ *in Möb, the following three conditions are meet.*

(a) $S_1U_2(\phi_{\theta,a}) - e^{i\theta}U_1(\phi_{\theta,a})S_1 = \bar{a}T_1U_1(\phi_{\theta,a})S_1 + \bar{a}S_1U_2(\phi_{\theta,a})T_2$;

(b) $S_3U_3(\phi_{\theta,a}) - e^{i\theta}U_2(\phi_{\theta,a})S_3 = \bar{a}T_2U_2(\phi_{\theta,a})S_3 + \bar{a}S_3U_3(\phi_{\theta,a})T_3$ *and*

(c) $S_2U_3(\phi_{\theta,a}) - e^{i\theta}U_1(\phi_{\theta,a})S_2 = \bar{a}T_1U_1(\phi_{\theta,a})S_2 + \bar{a}S_2U_3(\phi_{\theta,a})T_3 + \bar{a}S_1U_2(\phi_{\theta,a})S_3$.

Proof. Let T be a homogeneous operator with associated representation $U_1 \oplus U_2 \oplus U_3$. From [5, Lemma 2.5], it follows that $\begin{bmatrix} T_1 & S_1 \\ 0 & T_2 \end{bmatrix}$ is a homogeneous operator with associated representation $U_1 \oplus U_2$ and T_3 is a homogeneous operator with associated representation U_3. Again [5, Lemma 2.5] implies that T_i, $i = 1,2$, is a homogeneous operator with associated representation U_i, $i = 1,2$, respectively and S_1 satisfies the following equation

$$S_1U_2(\phi_{\theta,a}) - e^{i\theta}U_1(\phi_{\theta,a})S_1 = \bar{a}T_1U_1(\phi_{\theta,a})S_1 + \bar{a}S_1U_2(\phi_{\theta,a})T_2 \tag{3.1}$$

for all $\phi_{\theta,a}$ in Möb. Once again using Proposition [5, Lemma 2.5], we obtain that $\begin{bmatrix} T_2 & S_3 \\ 0 & T_3 \end{bmatrix}$ is a homogeneous operator with associated representation $U_2 \oplus U_3$ and S_3 satisfies the following equation

$$S_3U_3(\phi_{\theta,a}) - e^{i\theta}U_2(\phi_{\theta,a})S_3 = \bar{a}T_2U_2(\phi_{\theta,a})S_3 + \bar{a}S_3U_3(\phi_{\theta,a})T_3 \tag{3.2}$$

for all $\phi_{\theta,a}$ in Möb. Now a direct computation, using homogeneity of T, gives us

$$S_2U_3(\phi_{\theta,a}) - e^{i\theta}U_1(\phi_{\theta,a})S_2 = \bar{a}T_1U_1(\phi_{\theta,a})S_2 + \bar{a}S_2U_3(\phi_{\theta,a})T_3 + \bar{a}S_1U_2(\phi_{\theta,a})S_3,\ \phi_{\theta,a} \in \text{Möb}. \tag{3.3}$$

Now we prove the converse. The given conditions imply that

$$\begin{aligned} e^{i\theta}&\left(U_1(\phi_{\theta,a}) \oplus U_2(\phi_{\theta,a}) \oplus U_3(\phi_{\theta,a})\right)(T - aI) \\ &= T(U_1(\phi_{\theta,a}) \oplus U_2(\phi_{\theta,a}) \oplus U_3(\phi_{\theta,a}))(I - \bar{a}T),\ \phi_{\theta,a} \in \text{Möb}. \end{aligned}$$

Thus there exists an open set containing the identity element of Möb for which the following equation holds:

$$\phi_{\theta,a}(T) = \left(U_1(\phi_{\theta,a}) \oplus U_2(\phi_{\theta,a}) \oplus U_2(\phi_{\theta,a})\right)^* T(U_1(\phi_{\theta,a}) \oplus U_2(\phi_{\theta,a}) \oplus U_3(\phi_{\theta,a})).$$

Now using the [5, Lemma 2.2], we conclude that T is a homogeneous operator. $\square$

Lemma 3.2. *Assume that T_1 and T_2 are homogeneous operators with the same associated representation U. Then for any pair of scalars $\alpha,\gamma \in \mathbb{C}$, the operator*

$$A(\alpha,\gamma) = \begin{bmatrix} T_1 & \alpha(T_1-T_2) & \gamma(T_1-T_2) \\ 0 & T_2 & 0 \\ 0 & 0 & T_2 \end{bmatrix}$$

is homogeneous with associated representation $U \oplus U \oplus U$.

Proof. The proof follows easily from Lemma 3.1. However, we give a different proof below which is similar to [20, Lemma 2.1]. The invertible operator

$$S = \begin{bmatrix} I & \alpha I & \gamma I \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

commutes with $U \oplus U \oplus U$ and $A(\alpha,\gamma) = S^{-1}(T_1 \oplus T_2 \oplus T_2)S$. Suppose $\phi \in$ M$\ddot{o}$b, then

$$\begin{aligned} \phi(A(\alpha,\gamma)) &= S^{-1}(\phi(T_1) \oplus \phi(T_2) \oplus \phi(T_2))S \\ &= S^{-1}(U(\phi)^* \oplus U(\phi)^* \oplus U(\phi)^*)(T_1 \oplus T_2 \oplus T_2)(U(\phi) \oplus U(\phi) \oplus U(\phi))S \\ &= (U(\phi)^* \oplus U(\phi)^* \oplus U(\phi)^*)S^{-1}(T_1 \oplus T_2 \oplus T_2)S(U(\phi) \oplus U(\phi) \oplus U(\phi)) \\ &= (U(\phi)^* \oplus U(\phi)^* \oplus U(\phi)^*)A(\alpha,\gamma)(U(\phi) \oplus U(\phi) \oplus U(\phi)). \end{aligned}$$

This proves that $A(\alpha,\gamma)$ is a homogeneous operator with associated representation $U \oplus U \oplus U$. □

Lemma 3.3. *If $|\alpha_1| = |\alpha_2|$ and $|\gamma_1| = |\gamma_2|$, then $A(\alpha_1,\gamma_1)$ and $A(\alpha_2,\gamma_2)$ are unitarily equivalent.*

Proof. We have $|\alpha_1| = |\alpha_2|$ and $|\gamma_1| = |\gamma_2|$. So there exist $t,s \in \mathbb{T}$ such that $\alpha_1 = t\alpha_2$ and $\gamma_1 = s\gamma_2$. Now, the operator

$$V = \begin{bmatrix} tI & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & t\bar{s}I \end{bmatrix}$$

is unitary. Since $A(\alpha_1,\gamma_1) = VA(\alpha_2,\gamma_2)V^*$, the proof is complete. □

Recall that $\{f_n = z^n\}_{n\in\mathbb{Z}}$ is a complete orthogonal set in the Hilbert space $H^{\lambda,\sigma}$. For $n \in \mathbb{Z}$, let $e_n = \frac{f_n}{\|f_n\|}$ and $H(n)$ be the span of $\{(e_n,0,0)^t, (0,e_n,0)^t, (0,0,e_n)^t\}$. Let $\eta_{n,m} = P_n\eta_{|_{H(m)}}$ where P_n is the orthogonal projection of H onto $H(n)$. Then $\eta_{n,m}$ is a map from $H(m)$ to $H(n)$ for all $n,m \in \mathbb{Z}$. For $\phi \in$ Möb, let $\pi_{n,m}(\phi)$ be the map defined by $\pi_{n,m}(\phi)(e_m) = \langle \pi(\phi)e_m, e_n\rangle e_n$. Then

$$\eta_{n,m}(\phi)(ae_m, be_m, ce_m)^t = \langle \pi(\phi)e_m, e_n\rangle (ae_n, be_n, ce_n)^t$$

and

$$\eta_{n,m}(\phi)^*(ae_n, be_n, ce_n)^t = \overline{\langle \pi(\phi)e_m, e_n\rangle}(ae_m, be_m, ce_m)^t,$$

for all $a, b, c \in \mathbb{C}$. It is useful to define the set $A_{m,n}$, analogous to the ones in Definition 2.16, replacing the representation $P_{\lambda,0}$ by the representation π. Thus, if the matrix coefficient of π is

$$\langle \pi(\phi_a)z^m, z^n\rangle = c(-1)^n(\bar{a})^{n-m} \sum_{k\geq(m-n)^+} C_k(m,n)r^k, \tag{3.4}$$

where $r = |a|^2$, $c = \phi_a^{'}(0)^{\lambda/2}|\phi_a^{'}(0)|^{\mu}$ and the coefficients $C_k(m,n)$ are explicitly determined as in [9, p. 316].

Definition 3.4. Let $A_{m,n} \subseteq (-1,1)$ be the set of all zeros of the power series $\sum_{k\geq(m-n)^+} C_k(m,n)r^k$ and $A = \cup_{m,n\in\mathbb{Z}} A_{m,n}$.

The sets $A_{m,n}$ are countable and therefore so is A. Take $b \in (0,1)\setminus A$. Then $\langle \pi(\phi_b)f_m, f_n\rangle \neq 0$, for all $n, m \in \mathbb{Z}$. Fix ϕ_b in Möb. Now assume that u_0, v_0, w_0 are three non-zero mutually orthogonal vectors in $H(0)$. For $n \neq 0$, define $u_n = \sigma_{n,0}(\phi_b)u_0$, $v_n = \sigma_{n,0}(\phi_b)v_0$, $w_n = \sigma_{n,0}(\phi_b)w_0$. Then each of the vectors u_n, v_n, w_n is non-zero.

Lemma 3.5. *The set of vectors $\{u_n, v_n, w_n\}_{n\in\mathbb{Z}}$ is a complete orthogonal set in H.*

Proof. The proof is similar to that of Lemma 2.17. □

Let H_1, H_2 and H_3 be the close subspaces spanned by the sets of vectors $\{u_n : n \in \mathbb{Z}\}$, $\{v_n : n \in \mathbb{Z}\}$ and $\{w_n : n \in \mathbb{Z}\}$, respectively.

Lemma 3.6. *The subspaces H_1, H_2 and H_3 are invariant under η. Moreover, $\eta_{|H_i}$ is equivalent to π for all $i = 1,2,3$.*

Proof. The proof is similar to that of Lemma 2.18. □

Suppose T is a homogeneous operator with associated representation η. Since $H(n)$ are $\mathbb{K}$-isotypic subspaces of η, it follows, from [9, Theorem 5.1], that T maps $H(n)$ to $H(n+1)$ for each $n \in \mathbb{Z}$. Let $T_n := T_{|H(n)}$.

Lemma 3.7. *For every $n \in \mathbb{Z}$, T_n is invertible.*

Proof. Homogeneity of T implies that the following equation holds for every ψ in Möb where $\psi(z) = e^{i\theta}\frac{z-a}{1-\bar{a}z}$, $z \in \mathbb{D}$:

$$e^{i\theta}\eta_{i+1,n+1}(\psi)T_n - ae^{i\theta}\eta_{i+1,n}(\psi) = T_i\eta_{i,n}(\psi) - \bar{a}T_i\eta_{i,n+1}(\psi)T_n, \tag{3.5}$$

The proof is completed following arguments similar to the ones appearing in the proof of Lemma 2.19. □

Theorem 3.8. *Suppose T is a homogeneous operator with associated representation η. Then there exist closed subspaces H_1, H_2 and H_3 such that $H = H_1 \oplus H_2 \oplus H_3$ and $T(H_1) \subseteq H_1, T(H_2) \subseteq H_1 \oplus H_2$. For each $i = 1,2,3$, the subspace H_i is invariant under η and $\eta_{|H_i}$ is equivalent to π.*

Proof. Since $\eta_{1,0}(\phi_b)^{-1}T_0$ is an operator on $H(0)$, there exist three non-zero mutually orthonormal vectors u_0, v_0, w_0 in $H(0)$ such that $\eta_{1,0}(\phi_b)^{-1}T_0 u_0 = a_0 u_0$ and $\eta_{1,0}(\phi_b)^{-1}T_0 v_0 = b_0 u_0 + c_0 v_0$ for some $a_0, b_0, c_0 \in \mathbb{C}$.

For every $n \in \mathbb{Z} \setminus \{0\}$, define $u_n = \eta_{n,0}(\phi_b)u_0$, $v_n = \eta_{n,0}(\phi_b)v_0$ and $w_n = \eta_{n,0}(\phi_b)w_0$. Let H_1, H_2 and H_3 be the close subspaces spanned by the sets of vectors $\{u_n : n \in \mathbb{Z}\}$, $\{v_n : n \in \mathbb{Z}\}$ and $\{w_n : n \in \mathbb{Z}\}$, respectively. From Lemma 3.5, we have $H = H_1 \oplus H_2 \oplus H_3$. It follows, from Lemma 3.6, that for each $i = 1,2,3$, H_i is invariant under η and $\eta_{|H_i}$ is equivalent with π. We show that $T(H_1) \subseteq H_1, T(H_2) \subseteq H_1 \oplus H_2$.

We have $T_0 u_0 = a_0 \eta_{1,0}(\phi_b)u_0$ which is a scalar multiple of the vector u_1. Using an argument similar to the one appearing in the proof of Theorem 2.20, it follows that Tu_n is a scalar multiple of u_{n+1} for every $n \in \mathbb{Z}$.

(a) We show that $T_n v_n$ is in the subspace spanned by the set of vectors $\{u_{n+1}, v_{n+1}\}$, $n \in \mathbb{N}$.

The claim is easily verified for $n = 0 : T_0 v_0 = b_0\eta_{1,0}(\phi_b)u_0 + c_0\eta_{1,0}(\phi_b)v_0$. For an inductive proof, assume that $T_k v_k$ is in the subspace spanned by the set of vectors $\{u_{k+1}, v_{k+1}\}$ for some $k \geq 0$.

Let $A_k = \bigcup_{0\leq i,j\leq k+2} A_{i,j}$, where $A_{i,j}$ are described in Definition 3.4. Since 0 is not a limit point of any $A_{i,j}$, there exists $r_k \in (0,1)$ such that for $0 \leq i, j \leq k+2$, $\langle \pi(\phi_a)e_j, e_i\rangle \neq 0$ for all $a \in \mathbb{D}$ with $0 < |a| < r_k$. Substituting $n = k, i = k+1$ in the equation (3.5), we get

$$a\eta_{k+2,k}(\phi_a) - \eta_{k+2,k+1}(\phi_a)T_k = T_{k+1}\eta_{k+1,k}(\phi_a) - \bar{a}T_{k+1}\eta_{k+1,k+1}(\phi_a)T_k \tag{3.6}$$

for every ϕ_a in Möb with $|a| < r_k$.

A similar argument to that in the proof of Theorem 2.20 implies that for every ϕ_a in Möb with $|a| < r_k$, there exist $b_{k+1}(a), c_{k+1}(a) \in \mathbb{C}$ such that

$$T_k v_k = b_{k+1}(a)\eta_{k+1,k}(\phi_a)u_k + c_{k+1}(a)\eta_{k+1,k}(\phi_a)v_{k+1} \tag{3.7}$$

holds. Since $T_{k+1}u_{k+1}$ is a scalar multiple of u_{k+2}, for every ϕ_a, there exists $\lambda_{k+2}(a) \in \mathbb{C}$ such that

$$T_{k+1}\eta_{k+1,k}(\phi_a)u_k = \lambda_{k+2}(a)\eta_{k+2,k+1}(\phi_a)\eta_{k+1,k}(\phi_a)u_k. \tag{3.8}$$

Combining the equations (3.6), (3.7) and (3.8), we get

$$\begin{aligned}\Big(\frac{a\langle\pi(\phi_a)e_k,e_{k+2}\rangle}{\langle\pi(\phi_a)e_k,e_{k+1}\rangle\langle\pi(\phi_a)e_{k+1},e_{k+2}\rangle}-c_{k+1}(a)\Big)\eta_{k+2,k+1}(\phi_a)\eta_{k+1,k}(\phi_a)v_k\\ +\left(\bar{a}\lambda_{k+2}(a)\langle\pi(\phi_a)e_{k+1},e_{k+1}\rangle-1\right)b_{k+1}(a)\eta_{k+2,k+1}(\phi_a)\eta_{k+1,k}(\phi_a)u_k\\ =\left(1-\bar{a}c_{k+1}(a)\langle\pi(\phi_a)e_{k+1},e_{k+1}\rangle\right)T_{k+1}\eta_{k+1,k}(\phi_a)v_k \quad (3.9)\end{aligned}$$

for all $0<|a|<r_k$. It is not hard to verify that $\eta_{k+2,k+1}(\phi_a)\eta_{k+1,k}(\phi_a)u_k$ is a scalar multiple of the vector u_{k+2} and $\eta_{k+2,k+1}(\phi_a)\eta_{k+1,k}(\phi_a)v_k$ is a scalar multiple of the vector v_{k+2}. Therefore, the two vectors $\eta_{k+2,k+1}(\phi_a)\eta_{k+1,k}(\phi_a)u_k$ and $\eta_{k+2,k+1}(\phi_a)\eta_{k+1,k}(\phi_a)v_k$ are linearly independent. Now, if

$$\left(1-\bar{a}c_{k+1}(a)\langle\pi(\phi_a)e_{k+1},e_{k+1}\rangle\right)=0$$

for every ϕ_a with $0<|a|<r_k$, then we have

$$\Big(\frac{a\langle\pi(\phi_a)e_k,e_{k+2}\rangle}{\langle\pi(\phi_a)e_k,e_{k+1}\rangle\langle\pi(\phi_a)e_{k+1},e_{k+2}\rangle}-c_{k+1}(a)\Big)=0,\ \left(\bar{a}\lambda_{k+2}(a)\langle\pi(\phi_a)e_{k+1},e_{k+1}\rangle-1\right)b_{k+1}(a)=0$$

for every ϕ_a with $0<|a|<r_k$. Suppose for every ϕ_a with $0<|a|<r_k$,

$$\left(1-\bar{a}c_{k+1}(a)\langle\pi(\phi_a)e_{k+1},e_{k+1}\rangle\right)=0 \text{ and } c_{k+1}(a)-\frac{a\langle\pi(\phi_a)e_k,e_{k+2}\rangle}{\langle\pi(\phi_a)e_k,e_{k+1}\rangle\langle\pi(\phi_a)e_{k+1},e_{k+2}\rangle}=0.$$

Then combining these two equations, we get

$$|a|^2\langle\pi(\phi_a)e_k,e_{k+2}\rangle\langle\pi(\phi_a)e_{k+1},e_{k+1}\rangle=\langle\pi(\phi_a)e_k,e_{k+1}\rangle\langle\pi(\phi_a)e_{k+1},e_{k+2}\rangle$$

Now, using the matrix coefficient of $\pi(\phi_a)$ and then comparing the constant coefficient, we arrive at a contradiction. So, there exists ϕ_a with $0<|a|<r_k$ such that

$$\left(1-\bar{a}c_{k+1}(a)\langle\pi(\phi_a)e_{k+1},e_{k+1}\rangle\right)\neq 0.$$

Thus from equation (3.9), we see that $T_{k+1}v_{k+1}$ is in the space spanned by the set of vectors $\{u_{k+2},v_{k+2}\}$.

(b) Now we prove that $T_{-n}^{-1}v_{-n+1}$ is in the subspace spanned by $\{u_{-n},v_{-n}\}$ for all $n\in\mathbb{N}\cup\{0\}$.

The claim for $n=0$ follows from what we have said before, namely, combine the two statements: $T_0v_0=b_0\eta_{1,0}(\phi_b)u_0+c_0\eta_{1,0}(\phi_b)v_0$ and $T_0^{-1}u_1$ is a scalar multiple of u_0.

Assume that there exists $p\in\mathbb{N}\cup\{0\}$ such that $T_{-p}^{-1}v_{-p+1}$ is in the subspace spanned by the set of vectors $\{u_{-p},v_{-p}\}$. Let $A_p=\bigcup\limits_{-p-1\le i,j\le 0}A_{i,j}$. Then, since 0 is not a limit point of A_p, there exists $r_p\in(0,1)$ such that $\langle\pi(\phi_a)e_j,e_i\rangle\neq 0$, $-p-1\le i,j\le 0$ and $0<|a|<r_p$.

Since $T_{-p}v_{-p}$ is in the subspace spanned by the set of vectors $\{u_{-p+1}, v_{-p+1}\}$, it follows that for every ϕ_a with $0<|a|<r_p$, there exists $b_{-p+1}(a), c_{-p+1}(a)$ in $\mathbb{C}$ such that

$$T_{-p}v_{-p} = b_{-p+1}(a)\eta_{-p+1,-p}(\phi_a)u_{-p} + c_{-p+1}(a)\eta_{-p+1,-p}(\phi_a)v_{-p}. \tag{3.10}$$

Substituting $\phi=\phi_a$, $i=-p-1, n=-p$ in the equation (3.5), evaluating at the vector v_{-p} and then using equation (3.10), we get

$$\begin{aligned}T_{-p-1}^{-1}&\left[a\left\langle\pi(\phi_a)e_{-p}, e_{-p}\right\rangle v_{-p} - c_{-p+1}(a)\eta_{-p,-p+1}(\phi_a)\eta_{-p+1,-p}(\phi_a)v_{-p}\right]\\ &- b_{-p+1}(a)T_{-p-1}^{-1}\left(\eta_{-p,-p+1}(\phi_a)\eta_{-p+1,-p}(\phi_a)u_{-p}\right) = \eta_{-p-1,-p}(\phi_a)v_{-p}\\ &\qquad - \bar{a}b_{-p+1}(a)\eta_{-p-1,-p+1}(\phi_a)\eta_{-p+1,-p}(\phi_a)u_{-p}\\ &\qquad\qquad - \bar{a}c_{-p+1}(a)\eta_{-p-1,-p+1}(\phi_a)\eta_{-p+1,-p}(\phi_a)v_{-p}.\end{aligned} \tag{3.11}$$

We know that the vector $b_{-p+1}(\phi_a)T_{-p-1}^{-1}\left(\eta_{-p,-p+1}(\phi_a)\eta_{-p+1,-p}(\phi_a)u_{-p}\right)$ is a scalar multiple of u_{-p-1}. Therefore, we can find $\alpha(a)\in\mathbb{C}$ such that

$$\begin{aligned}b_{-p+1}(\phi_a)&T_{-p-1}^{-1}\left(\eta_{-p,-p+1}(\phi_a)\eta_{-p+1,-p}(\phi_a)u_{-p}\right)\\ &- \bar{a}b_{-p+1}(a)\eta_{-p-1,-p+1}(\phi_a)\eta_{-p+1,-p}(\phi_a)u_{-p} = \alpha(a)u_{-p-1}.\end{aligned} \tag{3.12}$$

Using the equalities

$$\eta_{-p,-p+1}(\phi_a)\eta_{-p+1,-p}(\phi_a) = \left\langle\pi(\phi_a)e_{-p}, e_{-p+1}\right\rangle\left\langle\pi(\phi_a)e_{-p+1}, e_{-p-1}\right\rangle Id,$$

$$\eta_{-p-1,-p+1}(\phi_a)\eta_{-p+1,-p}(\phi_a) = \frac{\left\langle\pi(\phi_a)e_{-p}, e_{-p+1}\right\rangle\left\langle\pi(\phi_a)e_{-p+1}, e_{-p-1}\right\rangle}{\left\langle\pi(\phi_a)e_{-p}, e_{-p-1}\right\rangle}\eta_{-p-1,-p}(\phi_a)$$

and then combining with the equations (3.11), (3.12), we get

$$\begin{aligned}&\left[a\left\langle\pi(\phi_a)e_{-p}, e_{-p}\right\rangle - c_{-p+1}(a)\left\langle\pi(\phi_a)e_{-p}, e_{-p+1}\right\rangle\left\langle\pi(\phi_a)e_{-p+1}, e_{-p}\right\rangle\right]T_{-p-1}^{-1}v_{-p}\\ &\quad = \alpha(a)u_{-p-1} + \left[1-\bar{a}c_{-p+1}(a)\frac{\left\langle\pi(\phi_a)e_{-p}, e_{-p+1}\right\rangle\left\langle\pi(\phi_a)e_{-p+1}, e_{-p-1}\right\rangle}{\left\langle\pi(\phi_a)e_{-p}, e_{-p-1}\right\rangle}\right]\eta_{-p-1,-p}(\phi_a)v_{-p}.\end{aligned} \tag{3.13}$$

The two vectors u_{-p-1} and $\eta_{-p-1,-p}(\phi_a)v_{-p}$ are orthogonal. Therefore, if

$$a\left\langle\pi(\phi_a)e_{-p}, e_{-p}\right\rangle - c_{-p+1}(a)\left\langle\pi(\phi_a)e_{-p}, e_{-p+1}\right\rangle\left\langle\pi(\phi_a)e_{-p+1}, e_{-p}\right\rangle = 0$$

for all ϕ_a with $0<|a|<r_p$, then

$$\alpha(a)=0 \text{ and } 1-\bar{a}c_{-p+1}(a)\frac{\left\langle\pi(\phi_a)e_{-p}, e_{-p+1}\right\rangle\left\langle\pi(\phi_a)e_{-p+1}, e_{-p-1}\right\rangle}{\left\langle\pi(\phi_a)e_{-p}, e_{-p-1}\right\rangle} = 0$$

for every ϕ_a with $0 < |a| < r_p$. This gives us

$$|a|^2 \langle \pi(\phi_a)e_{-p}, e_{-p}\rangle \langle \pi(\phi_a)e_{-p+1}, e_{-p-1}\rangle = \langle \pi(\phi_a)e_{-p+1}, e_{-p}\rangle \langle \pi(\phi_a)e_{-p}, e_{-p-1}\rangle$$

for all ϕ_a with $0 < |a| < r_p$. Now, using the matrix coefficient of $\pi(\phi_a)$ and then comparing the coefficients of r^3, we arrive at a contradiction. So, there exists ϕ_a with $0 < |a| < r_p$ such that

$$a\langle \pi(\phi_a)e_{-p}, e_{-p}\rangle - c_{-p+1}(a)\langle \pi(\phi_a)e_{-p}, e_{-p+1}\rangle \langle \pi(\phi_a)e_{-p+1}, e_{-p}\rangle \neq 0.$$

Thus, from equation (3.13), we conclude that $T^{-1}_{-p-1} v_{-p}$ is in the subspace spanned by the set of vectors $\{u_{-p-1}, v_{-p-1}\}$. □

Let $0 < a < b < 1$ and $T(a,b)$, $T(b,a)$ be the weighted shifts defined by $T(a,b)e_n = t_n e_{n+1}$ and $T(b,a)e_n = \frac{1}{t_n} e_{n+1}$ where $t_n = \frac{\|f_{n+1}\|}{\|f_n\|} = \sqrt{\frac{n+a}{n+b}}$. Recall that $T(a,b)$ and $T(b,a)$ are the only homogeneous operators whose associated representation is $\pi = C_{\lambda,\sigma}$, where $\lambda = a+b-1$ and $\sigma = \frac{b-a}{2}$.

Lemma 3.9. *Let T_1 and T_2 be two homogeneous operators with associated representation π and S be an operator on $H^{\lambda,\sigma}$ such that S satisfies the following equation*

$$S\pi(\phi) - e^{i\theta}\pi(\phi)S = \bar{a}T_1\pi(\phi)S + \bar{a}S\pi(\phi)T_2 \tag{3.14}$$

for all $\phi = \phi_{\theta,a}$ in Möb. Then $S = \alpha(T_1 - T_2)$, for some $\alpha \in \mathbb{C}$.

Proof. First assume that $T_1 = T(a,b)$, $T_2 = T(b,a)$. It follows from [7, Theorem 5.3] and [9, Lemma 2.5] that $\alpha(T_1 - T_2)$ satisfies equation (3.14) for all $\alpha \in \mathbb{C}$.

Restricting the equation (3.14) to the group of rotations, we see that S is a weighted shift with respect to the orthonormal basis $\{e_n\}$ of $H^{\lambda,\sigma}$. Let $\{\alpha_n\}$ be the weight sequence of S. Substituting $\phi = \phi_a$ in the equation (3.14), evaluating at e_m, taking inner product with e_n and then using the matrix coefficient of $\pi(\phi_a)$, we obtain

$$\begin{aligned}\alpha_{n-1} t_m \sum_{k\geq(m-n+1)^+} C_k(m, n-1)r^k - \alpha_m t_{n-1} \sum_{k\geq(m-n+1)^+} C_k(m+1, n)r^k \\ = \left(\alpha_m t_{n-1} + \frac{\alpha_{n-1}}{t_m}\right) \sum_{k\geq(m-n+2)^+} C_k(m+1, n-1)r^k.\end{aligned}$$

Taking $m = n$ and comparing the coefficient of r, we get

$$\alpha_n t_{n-1}(n+a) = \alpha_{n-1} t_n (n-1+a).$$

Now applying induction, we find that for $n \in \mathbb{Z}$, $\alpha_n = \alpha\left(t_n - \frac{1}{t_n}\right)$ for some $\alpha \in \mathbb{C}$. This shows that if $T_1 = T(a,b)$ and $T_2 = T(b,a)$, then the solution of the equation (3.14) is $\alpha(T_1 - T_2)$, for some $\alpha \in \mathbb{C}$.

Similarly, we can show that if $T_1 = T(b,a)$ and $T_2 = T(a,b)$, then also the solution of equation the (3.14) is $\alpha(T_1 - T_2)$, for some $\alpha \in \mathbb{C}$.

Now assume that $T_1 = T(a,b)$ and $T_2 = T(a,b)$. In this case, we show that $S = 0$. Again restricting the equation (3.14) to the group of rotations, we see that S is a weighted shift with respect to the orthonormal basis $\{e_n\}$ of $H^{\lambda,\sigma}$. Let $\{\alpha_n\}$ be the weights of S.

Substituting $\phi = \phi_a$ in the equation (3.14), evaluating at e_m, taking inner product with e_n and then using the matrix coefficient of $\pi(\phi_a)$, we obtain

$$\alpha_{n-1} t_m \sum_{k\geq(m-n+1)^+} C_k(m,n-1)r^k - \alpha_m t_{n-1} \sum_{k\geq(m-n+1)^+} C_k(m+1,n)r^k \\ = (\alpha_m t_{n-1} + \alpha_{n-1} t_m) \sum_{k\geq(m-n+2)^+} C_k(m+1,n-1)r^k$$

Putting $m = n-1$ and equating the coefficient of r, we get

$$\alpha_{n-1} t_{n-1}(\lambda + 2\mu - 1) = 0.$$

Since $\lambda = a+b-1$ and $\mu = 1-a$, so $\lambda + 2\mu - 1 = b - a$, which is different from 0. Also we know that $t_{n-1} \neq 0$, for all $n \in \mathbb{Z}$. This implies that $\alpha_n = 0$ for all $n \in \mathbb{Z}$. This shows that if $T_1 = T(a,b)$, $T_2 = T(a,b)$ and S satisfies equation (3.14), then S must be 0.

Similarly, we can prove that if $T_1 = T(b,a)$, $T_2 = T(b,a)$ and S satisfies equation (3.14), then also $S = 0$. $\square$

Now, we describe all homogeneous operators whose associated representation is η. Denote

$$T(a,b,\alpha,\beta) := \begin{bmatrix} T(a,b) & \alpha(T(a,b)-T(b,a)) & \beta(T(a,b)-T(b,a)) \\ 0 & T(b,a) & 0 \\ 0 & 0 & T(b,a) \end{bmatrix}$$

and

$$T(b,a,\alpha,\beta) := \begin{bmatrix} T(b,a) & \alpha(T(a,b)-T(b,a)) & \beta(T(a,b)-T(b,a)) \\ 0 & T(a,b) & 0 \\ 0 & 0 & T(a,b) \end{bmatrix}.$$

Theorem 3.10. *Up to unitary equivalence, $T(a,b,\alpha,\beta)$, $T(b,a,\alpha,\beta)$, $T(a,b) \oplus T(a,b) \oplus T(a,b)$ and $T(b,a) \oplus T(b,a) \oplus T(b,a)$ are the only homogeneous operators with associated representation η.*

Proof. Let T be a homogeneous operator with associated representation η. In view of Theorem 3.8, we may assume, without loss of generality, that

$$T = \begin{bmatrix} T_1 & S_1 & S_2 \\ 0 & T_2 & S_3 \\ 0 & 0 & T_3 \end{bmatrix}.$$

Now, Lemma 3.1 implies that T_i, $i = 1,2,3$, are homogeneous with associated representation π and S_i, $i = 1,2,3$, satisfy equations (3.1), (3.2) and (3.3). We divide the proof into several cases and give the proof in each of these cases separately.

(i) First assume that $T_1 = T(a,b)$ and $T_2 = T_3 = T(b,a)$. The operators S_1 and S_3 satisfy the equations (3.1) and (3.2), respectively. Lemma 3.9 shows that $S_1 = \alpha(T_1 - T_2)$ for some $\alpha \in \mathbb{C}$ and $S_3 = 0$. Substituting $S_3 = 0$ in the equation (3.3), we obtain

$$S_2\pi(\phi_{\theta,a}) - e^{i\theta}\pi(\phi_{\theta,a})S_2 = \bar{a}T_1\pi(\phi_{\theta,a})S_2 + \bar{a}S_2\pi(\phi_{\theta,a})T_2,\ \phi_{\theta,a} \in \text{Möb}.$$

Again, Lemma 3.9 implies that $S_2 = \beta(T_1 - T_2)$ for some $\beta \in \mathbb{C}$ and therefore $T = T(a,b,\alpha,\beta)$.

(ii) Assume $T_1 = T(b,a)$ and $T_2 = T_3 = T(a,b)$. Repeating the argument given in (i), it follows that $T = T(b,a,\alpha,\beta)$.

(iii) Assume $T_2 = T(b,a)$ and $T_1 = T_3 = T(a,b)$. Since S_1 satisfies equation (3.1), Lemma 3.9 applies and we see that $S_1 = \alpha(T_1 - T_2)$ for some $\alpha \in \mathbb{C}$. Using arguments similar to the ones in the proof of Lemma 3.3, we can take $\alpha > 0$. Consider the unitary operator

$$U_\alpha := \frac{1}{\sqrt{\alpha^2+1}}\begin{bmatrix} -\alpha I & I \\ I & \alpha I \end{bmatrix}.$$

Clearly,

$$U\begin{bmatrix} T_1 & \alpha(T_1 - T_2) \\ 0 & T_2 \end{bmatrix}U^* = \begin{bmatrix} T_2 & \alpha(T_2 - T_1) \\ 0 & T_1 \end{bmatrix}.$$

Let $V = U_\alpha \oplus I$ where I is the identity operator on $H^{\lambda,\sigma}$. Then V is a unitary operator and

$$VTV^* = \begin{bmatrix} T_2 & \alpha(T_2 - T_1) & \widehat{S}_2 \\ 0 & T_1 & \widehat{S}_3 \\ 0 & 0 & T_3 \end{bmatrix}.$$

Now repeating the same argument as in the proof of (i), we see that $VTV^* = T(b,a,\alpha,\beta)$.

(iv) Assume that $T_2 = T(a,b)$ and $T_1 = T_3 = T(b,a)$. In this case, T is unitarily equivalent to $T(a,b,\alpha,\beta)$ for some $\alpha,\beta \in \mathbb{C}$. The proof is similar to that of (iii).

(v) If $T_1 = T_2 = T_3$, then applying Lemma 3.3 repeatedly, we find that $S_1 = S_2 = S_3 = 0$. □

Theorem 3.11. *Every homogeneous operator with associated representation η is reducible.*

Proof. Theorem 3.10 provides a list of all the homogeneous operators with associated representation η. All of these are evidently reducible except $T(a,b,\alpha,\beta)$ and $T(b,a,\alpha,\beta)$. Therefore, it is enough to show that these are reducible.

Let $H(n)$ be the subspace spanned by the set of vectors $\{(e_n,0,0)^t, (0,e_n,0)^t, (0,0,e_n)^t\}$, $n \in \mathbb{Z}$. Since $T(a,b,\alpha,\beta)$ and $T(b,a,\alpha,\beta)$ are homogeneous operators, it follows that both $T(a,b,\alpha,\beta)$ and $T(b,a,\alpha,\beta)$ map $H(n)$ to $H(n+1)$, $n \in \mathbb{Z}$. Let us define

$$u_n = (e_n,0,0)^t,\ v_n = (0,\beta e_n,-\alpha e_n)^t \text{ and } w_n = (0,\alpha e_n,\beta e_n)^t.$$

Then u_n, v_n and w_n are three mutually orthogonal elements of $H(n)$. It is easy to verify that

$$T(a,b,\alpha,\beta)u_n = t_n u_{n+1},$$

$$T(a,b,\alpha,\beta)v_n = \tfrac{1}{t_n} v_{n+1},$$

$$T(a,b,\alpha,\beta)w_n = (\alpha^2+\beta^2)\left(t_n - \tfrac{1}{t_n}\right) u_{n+1} + \tfrac{1}{t_n} w_{n+1}$$

and

$$T(b,a,\alpha,\beta)u_n = \tfrac{1}{t_n} u_{n+1},$$

$$T(b,a,\alpha,\beta)v_n = t_n v_{n+1},$$

$$T(b,a,\alpha,\beta)w_n = (\alpha^2+\beta^2)\left(t_n - \tfrac{1}{t_n}\right) u_{n+1} + t_n w_{n+1}.$$

This shows that $T(a,b,\alpha,\beta)$ and $T(b,a,\alpha,\beta)$ are reducible operators. In fact, this shows that the operator $T(a,b,\alpha,\beta)$ is unitarily equivalent to the operator

$$\begin{bmatrix} T(b,a) & 0 & 0 \\ 0 & T(a,b) & (T(a,b)-T(b,a)) \\ 0 & 0 & T(b,a) \end{bmatrix}$$

and similarly, the operator $T(b,a,\alpha,\beta)$ is unitarily equivalent to the operator

$$\begin{bmatrix} T(a,b) & 0 & 0 \\ 0 & T(b,a) & (T(a,b)-T(b,a)) \\ 0 & 0 & T(a,b) \end{bmatrix}.$$

$\square$

Chapter 4

Characteristic function of homogeneous contractions

A bounded operator T is said to be a contraction if $\|T\| \leq 1$. A very successful model theory for such operators were developed by Sz.-Nagy and Foias. In particular, the model theory provides a complete unitary invariant, namely, the characteristic function θ of the operator T. To define θ, let us recall that $D_T = (I - T^*T)^{\frac{1}{2}}$ and $D_{T^*} = (I - TT^*)^{\frac{1}{2}}$ are the defect operators of T and $\mathscr{D}_T = \text{closran} D_T$ and $\mathscr{D}_{T^*} = \text{closran} D_{T^*}$. Then $\mathscr{D}_T$ and $\mathscr{D}_{T^*}$ are the defect spaces. Now, define the characteristic function $\theta_T : \mathbb{D} \to \mathscr{B}(\mathscr{D}_T, \mathscr{D}_{T^*})$ to be the holomorphic function:

$$\theta_T(a) = -T_{|\mathscr{D}_T} + aD_{T^*}(I - aT^*)^{-1}D_{T|\mathscr{D}_T},\ a \in \mathbb{D}.$$

A very deep theorem, due to Sz.-Nagy and Foias, says that two pure contractions T and $\tilde{T}$ are unitarily equivalent if and only if $\theta_T(a) = \tau\theta_{\tilde{T}}(a)\eta$, $a \in \mathbb{D}$, for some pair of unitaries $\tau : \mathscr{D}_{\tilde{T}^*} \to \mathscr{D}_{T^*}$ and $\eta : \mathscr{D}_T \to \mathscr{D}_{\tilde{T}}$.

In the following, we will let ϕ_a, $\phi_a(w) := \frac{a-w}{1-\bar{a}w}$, $a \in \mathbb{D}$, denote an involutive automorphism of the unit disc $\mathbb{D}$. It has been proved by Bagchi and Misra (see [2]) that the characteristic function θ_T of a homogeneous contraction T with an associated representation π is of the form

$$\theta_T(a) = \sigma_L(\phi_a)^*\theta_T(0)\sigma_R(\phi_a).$$

Also, σ_L and σ_R are projective representations of Möb with common multiplier, which are explicitly determined from π. In the first section of this chapter, we give another proof of the "product formula". In [8], it has been proved that

(a) the defect operators of $M^{(\lambda)}$, $\lambda > 1$, the multiplication operator on the reproducing kernel Hilbert space $H^{(\lambda)}$ with reproducing kernel $\frac{1}{(1-z\bar{w})^\lambda}$, $z, w \in \mathbb{D}$, are quasi-invertible and

(b) the representations σ_L and σ_R for the operator $M^{(\lambda)}$, $\lambda > 1$, are equivalent to the representations $D^+_{\lambda-1}$ and $D^+_{\lambda+1}$, respectively.

Since the operators $M^{(\lambda)}$ are in the Cowen-Douglas class $B_1(\mathbb{D})$, it is natural to ask what happens to homogeneous contractions in $B_2(\mathbb{D})$. In this chapter, we show that the defect operators of an irreducible homogeneous contraction in $B_2(\mathbb{D})$ need not be quasi-invertible. We also identify the representations σ_L and σ_R for an irreducible homogeneous contraction in $B_2(\mathbb{D})$.

4.1 Product formula

Let T be a homogeneous contraction with an associated representation π and Γ_T be the minimal unitary dilation of the operator T. The original proof of the product formula of the characteristic function θ_T was obtained by first extending the representation π to the dilation space of Γ_T, say $\hat{\pi}$, and then verifying that

$$\hat{\pi}(\phi)^*\Gamma_T\hat{\pi}(\phi) = \phi(\Gamma_T),\ \phi \in \text{Möb}.$$

Thus the minimal unitary dilation Γ_T of the operator T is homogeneous whenever T is homogeneous with an associated representation π. The restriction of $\hat{\pi}$ to the subspaces $\mathscr{D}_T$ and $\mathscr{D}_{T^*}$ defines the representations σ_R and σ_L, respectively. The proof of the product formula given below is obtained without going to the dilation space.

Lemma 4.1. *Let T be a homogeneous contraction with an associated representation π. Let $\phi \in$ Möb. Then $D_{\phi(T)} = \pi(\phi)^* D_T \pi(\phi)$ and $D_{\phi(T)^*} = \pi(\phi)^* D_{T^*}\pi(\phi)$. In particular $\pi(\phi)$ maps $\mathscr{D}_{\phi(T)}$ and $\mathscr{D}_{\phi(T)^*}$ to $\mathscr{D}_T$ and $\mathscr{D}_{T^*}$, respectively.*

Proof. Let $\phi \in$ Möb. Homogeneity of T implies that $\phi(T) = \pi(\phi)^* T\pi(\phi)$. Using this relation it is easy to see that $D^2_{\phi(T)} = \pi(\phi)^* D^2_T \pi(\phi)$ and $D^2_{\phi(T)^*} = \pi(\phi)^* D^2_{T^*}\pi(\phi)$. This implies that $D_{\phi(T)} = \pi(\phi)^* D_T \pi(\phi)$ and $D_{\phi(T)^*} = \pi(\phi)^* D_{T^*}\pi(\phi)$. It follows that $\pi(\phi)$ maps $\mathscr{D}_{\phi(T)}$ and $\mathscr{D}_{\phi(T)^*}$ to $\mathscr{D}_T$ and $\mathscr{D}_{T^*}$, respectively. □

Let $\phi_\alpha(z) = e^{i\alpha}z$, $z \in \mathbb{D}$ and let T_α denote the operator $\phi_\alpha(T) = e^{i\alpha}T$. It is easy to see that $D_{T_\alpha} = D_T$ and $D_{T^*_\alpha} = D_{T^*}$. Lemma 4.1 implies that $\mathscr{D}_T$ and $\mathscr{D}_{T^*}$ are invariant under $\pi(\phi_\alpha)$.

Lemma 4.2. *Let θ_{T_α} and θ_T be the characteristic function of T_α and T, respectively. Then $\theta_{T_\alpha}(z) = \pi(\phi_\alpha)^*_{|\mathscr{D}_{T^*}}\theta_T(z)\pi(\phi_\alpha)_{|\mathscr{D}_T}$ and $\theta_{T_\alpha}(z) = e^{i\alpha}\theta_T(e^{-i\alpha}z)$, $z \in \mathbb{D}$.*

Proof. Using homogeneity of the operator T, we have

$$\begin{aligned}\theta_{T_\alpha}(z) &= -T_{\alpha|\mathscr{D}_T} + zD_{T_\alpha^*}(I - zT_\alpha^*)^{-1}D_{T_\alpha|\mathscr{D}_T}\\ &= -\pi(\phi_\alpha)^* T\pi(\phi_\alpha)_{|\mathscr{D}_T} + z\pi(\phi_\alpha)^* D_{T^*}\pi(\phi_\alpha)(I - z\pi(\phi_\alpha)^* T^*\pi(\phi_\alpha))^{-1}\pi(\phi_\alpha)^* D_T\pi(\phi_\alpha)_{|\mathscr{D}_T}\\ &= \pi(\phi_\alpha)^*{}_{|\mathscr{D}_{T^*}}\theta_T(z)\pi(\phi_\alpha)_{|\mathscr{D}_T}.\end{aligned}$$

This proves the first part. Also,

$$\begin{aligned}\theta_{T_\alpha}(z) &= -T_{\alpha|\mathscr{D}_T} + zD_{T_\alpha^*}(I - zT_\alpha^*)^{-1}D_{T_\alpha|\mathscr{D}_T}\\ &= -e^{i\alpha}T_{|\mathscr{D}_T} + zD_{T^*}(I - e^{-i\alpha}zT^*)^{-1}D_{T|\mathscr{D}_T}\\ &= e^{i\alpha}\theta_T(e^{-i\alpha}z).\end{aligned}$$

This completes the proof of the lemma. □

Let $\rho_a(z) = \frac{z-a}{1-\bar{a}z}$, for all $z \in \mathbb{D}$. Let T_a denote the operator $\rho_a(T)$. Following Sz.-Nagy and Foias [32, p. 240], define $Z(a) : \mathscr{D}_{T_a} \to \mathscr{D}_T$ and $Z_*(a) : \mathscr{D}_{T_a^*} \to \mathscr{D}_{T^*}$ by $Z(a)D_{T_a}h = D_T S_a h$ and $Z_*(a)D_{T_a^*}h = D_{T^*}S_a^* h$, respectively, where $S_a = (1-|a|^2)^{\frac{1}{2}}(I - \bar{a}T)^{-1}$. Then $Z(a)$ and $Z_*(a)$ are unitary operators. Also, from the definition of $Z(a)$ and $Z_*(a)$, we have

$$Z(a)D_{T_a} = D_T S_a \text{ and } Z_*(a)D_{T_a^*} = D_{T^*}S_a^*.$$

Lemma 4.3. *Let T be a homogeneous contraction with associated representation π. For ϕ in Möb of the form $\phi(z) = e^{i\alpha}\frac{z-a}{1-\bar{a}z}$, $z \in \mathbb{D}$, we have*

$$(\theta_T \circ \phi^{-1})(z) = (e^{-i\frac{\alpha}{2}}Z_*(a)\pi(\phi)^*_{|\mathscr{D}_{T^*}})\theta_T(z)(e^{-i\frac{\alpha}{2}}\pi(\phi)_{|\mathscr{D}_{\phi(T)}}Z(a)^*),\ z \in \mathbb{D}.$$

Proof. Clearly, $\phi(T) = e^{i\alpha}T_a$. From [32, p. 241], it follows that

$$\theta_T(\rho_a^{-1}(z)) = Z_*(a)\theta_{T_a}(z)Z(a)^*. \tag{4.1}$$

Substituting $e^{-i\alpha}z$ for z in the equation (4.1), we obtain

$$\theta_{T_a}(e^{-i\alpha}z) = Z_*(a)^*\theta_T(\rho_a^{-1}(e^{-i\alpha}z))Z(a). \tag{4.2}$$

Now, Lemma 4.2 implies that

$$\theta_{\phi(T)}(z) = e^{i\alpha}\theta_{T_a}(e^{-i\alpha}z),\ z \in \mathbb{D}. \tag{4.3}$$

Homogeneity of T and Lemma 4.1 gives us

$$\begin{aligned}\theta_{\phi(T)}(z) &= -\phi(T)_{|\mathscr{D}_{\phi(T)}} + zD_{\phi(T)^*}(I - a\phi(T)^*)^{-1}D_{\phi(T)|\mathscr{D}_{\phi(T)}}\\ &= -\pi(\phi)^*{}_{|\mathscr{D}_{T^*}}T\pi(\phi)_{|\mathscr{D}_{\phi(T)}}\\ &\quad + z\pi(\phi)^*{}_{|\mathscr{D}_{T^*}}D_{T^*}\pi(\phi)\left(I - z\pi(\phi)^*T^*\pi(\phi)\right)^{-1}\pi(\phi)^*D_T\pi(\phi)_{|\mathscr{D}_{\phi(T)}}\\ &= \pi(\phi)^*_{|\mathscr{D}_{T^*}}\theta_T(z)\pi(\phi)_{|\mathscr{D}_{\phi(T)}}.\end{aligned} \tag{4.4}$$

Combining equation (4.2), equation (4.3) and equation (4.4), we get

$$(\theta_T \circ \phi^{-1})(z) = (e^{-i\frac{\alpha}{2}} Z_*(a)\pi(\phi)^*_{|\mathscr{D}_{T^*}})\theta_T(z)(e^{-i\frac{\alpha}{2}}\pi(\phi)_{|\mathscr{D}_{\phi(T)}} Z(a)^*)$$

completing the proof. □

Theorem 4.4. *Let T be a homogeneous contraction with associated representation π. For ϕ in Möb of the form $\phi(z) = e^{i\alpha}\frac{z-a}{1-\bar{a}z}$, $z \in \mathbb{D}$, define $\sigma_L(\phi) = e^{i\frac{\alpha}{2}}\pi(\phi)_{|\mathscr{D}_{\phi(T)^*}} Z_*(a)^*$ and $\sigma_R(\phi) = e^{-i\frac{\alpha}{2}}\pi(\phi)_{|\mathscr{D}_{\phi(T)}} Z(a)^*$. Suppose m_1 and m are the multipliers of D_1^+ and π, respectively. Then σ_L and σ_R are projective representations of Möb with common multiplier $m_1 m$. Also, we have the following relationships:*

$$\sigma_L(\phi)D_{T^*} = D_{T^*}\pi(\phi)((\phi')^{-\frac{1}{2}}(T))^*,\ \sigma_R(\phi)D_T = D_T\pi(\phi)(\phi')^{-\frac{1}{2}}(T),\ \phi \in \text{Möb}.$$

Proof. First, we prove the second part of the Theorem using the formulas $Z(a)D_{T_a} = D_T S_a$, $Z_*(a)^* D_{T^*} = D_{T_a^*}(S_a^*)^{-1}$ from [32, p. 240] and Lemma 4.1. Let $\phi \in$ Möb be such that $\phi(z) = e^{i\alpha}\frac{z-a}{1-\bar{a}z}$, $z \in \mathbb{D}$. Then

$$\begin{aligned}
\sigma_L(\phi)D_{T^*} &= e^{i\frac{\alpha}{2}}\pi(\phi)_{|\mathscr{D}_{T_a^*}} Z_*(a)^* D_{T^*} \\
&= e^{i\frac{\alpha}{2}}\pi(\phi)_{|\mathscr{D}_{T_a^*}} D_{T_a^*}(S_a^*)^{-1} \\
&= e^{i\frac{\alpha}{2}} D_{T^*}\pi(\phi)(S_a^*)^{-1} \\
&= D_{T^*}\pi(\phi)((\phi')^{-\frac{1}{2}}(T))^*
\end{aligned}$$

and

$$\begin{aligned}
\sigma_R(\phi)D_T &= e^{-i\frac{\alpha}{2}}\pi(\phi)_{|\mathscr{D}_{T_a}} Z(a)^* D_T \\
&= e^{-i\frac{\alpha}{2}}\pi(\phi)_{|\mathscr{D}_{T_a}} D_{T_a} S_a^{-1} \\
&= e^{-i\frac{\alpha}{2}} D_T\pi(\phi) S_a^{-1} \\
&= D_T\pi(\phi)(\phi')^{-\frac{1}{2}}(T).
\end{aligned}$$

Now, we prove that σ_L and σ_R are projective representations. Clearly, σ_L and σ_R are Borel maps. Let ϕ_1 and ϕ_2 be any two elements in Möb. Then

$$\begin{aligned}
\sigma_L(\phi_1)\sigma_L(\phi_2)D_{T^*} &= \sigma_L(\phi_1)D_{T^*}\pi(\phi_2)((\phi_2')^{-\frac{1}{2}}(T))^* \\
&= D_{T^*}\pi(\phi_1)((\phi_1')^{-\frac{1}{2}}(T))^*\pi(\phi_2)((\phi_2')^{-\frac{1}{2}}(T))^* \\
&= D_{T^*}\pi(\phi_1)\pi(\phi_2)((\phi_1')^{-\frac{1}{2}}(\phi_2(T)))^*((\phi_2')^{-\frac{1}{2}}(T))^*.
\end{aligned}$$

Also, we have

$$\begin{aligned}
\sigma_L(\phi_1\phi_2)D_{T^*} &= D_{T^*}\pi(\phi_1\phi_2)(((\phi_1\phi_2)')^{-\frac{1}{2}}(T))^* \\
&= m_1(\phi_1,\phi_2)m(\phi_1,\phi_2)D_{T^*}\pi(\phi_1)\pi(\phi_2)((\phi_1')^{-\frac{1}{2}}(\phi_2(T)))^*((\phi_2')^{-\frac{1}{2}}(T))^*.
\end{aligned}$$

This proves that

$$\sigma_L(\phi_1\phi_2) = m_1(\phi_1,\phi_2)m(\phi_1,\phi_2)\sigma_L(\phi_1)\sigma_L(\phi_2).$$

Now,

$$\begin{aligned}\sigma_R(\phi_1)\sigma_R(\phi_2)D_T &= \sigma_R(\phi_1)D_T\pi(\phi_2)(\phi_2')^{-\frac{1}{2}}(T)\\ &= D_T\pi(\phi_1)(\phi_1')^{-\frac{1}{2}}(T)\pi(\phi_2)(\phi_2')^{-\frac{1}{2}}(T)\\ &= D_T\pi(\phi_1)\pi(\phi_2)(\phi_1')^{-\frac{1}{2}}(\phi_2(T))(\phi_2')^{-\frac{1}{2}}(T).\end{aligned}$$

Also, we have

$$\begin{aligned}\sigma_R(\phi_1\phi_2)D_T &= D_T\pi(\phi_1\phi_2)((\phi_1\phi_2)')^{-\frac{1}{2}}(T)\\ &= m_1(\phi_1,\phi_2)m(\phi_1,\phi_2)D_T\pi(\phi_1)\pi(\phi_2)(\phi_1')^{-\frac{1}{2}}(\phi_2(T))(\phi_2')^{-\frac{1}{2}}(T).\end{aligned}$$

Thus

$$\sigma_R(\phi_1\phi_2) = m_1(\phi_1,\phi_2)m(\phi_1,\phi_2)\sigma_R(\phi_1)\sigma_R(\phi_2).$$

This shows that σ_L and σ_R are projective representations of Möb and the multiplier of σ_L and σ_R is $m_1 m$. □

Combining the Theorem we have just proved and Lemma 4.3, we obtain the following Theorem which gives the product formula for the characteristic function of a homogeneous contraction with an associated representation.

Theorem 4.5. *Let ϕ_a in Möb be of the form $\phi_a(z) = -\frac{z-a}{1-\bar{a}z}$, $z \in \mathbb{D}$. The characteristic function of a homogeneous contraction T with an associated representation π, is given by*

$$\theta_T(a) = \sigma_L(\phi_a)^*\theta_T(0)\sigma_R(\phi_a),\ a \in \mathbb{D},$$

where σ_L and σ_R are as in Theorem 4.4.

4.2 Characteristic function of an irreducible homogeneous contraction of rank 2

In this section, we explicitly compute the characteristic function of an irreducible homogeneous contraction in the Cowen-Douglas class of rank 2. This, however, naturally splits into two cases which we discuss separately.

For the rest of this chapter, we denote the defect operators D_T and D_{T^*} of T by D and D_*, respectively. Similarly, we let $\mathscr{D}$ and $\mathscr{D}_*$ denote the defect spaces $\mathscr{D}_T$ and $\mathscr{D}_{T^*}$, respectively.

Let $M^{(\lambda)}$ be the multiplication operator on $H^{(\lambda)}$ and i be the inclusion map of $H^{(\lambda)}$ to $H^{(\lambda+2)}$. Let $f_n(z) = z^n$ and $e_n^\lambda = \frac{f_n}{\|f_n\|_\lambda}$. Recall that $\{e_n^\lambda\}_{n\geq 0}$ is a complete orthonormal set of $H^{(\lambda)}$.

For $\mu > 0$, following [21], define $\Gamma_{\lambda,\mu} : H^{(\lambda)} \oplus H^{(\lambda+2)} \to \mathrm{Hol}(\mathbb{D}, \mathbb{C}^2)$ by

$$\Gamma_{\lambda,\mu}(f,g)^t = \left(f, \frac{1}{\lambda} f' + \mu g\right)^t. \tag{4.5}$$

Clearly, $\Gamma_{\lambda,\mu}$ is an injective linear map. Let $H^{(\lambda,\mu)}$ be the range of $\Gamma_{\lambda,\mu}$. Define an inner product on $H^{(\lambda,\mu)}$ so as to make $\Gamma_{\lambda,\mu}$ unitary. It is known (cf. [21]) that $H^{(\lambda,\mu)}$ is a reproducing kernel Hilbert space with reproducing kernel

$$\begin{bmatrix} \frac{1}{(1-z\overline{w})^\lambda} & \frac{z}{(1-z\overline{w})^{\lambda+1}} \\ \frac{\overline{w}}{(1-z\overline{w})^{\lambda+1}} & \frac{\frac{1}{\lambda}+\mu^2+z\overline{w}}{(1-z\overline{w})^{\lambda+2}} \end{bmatrix}.$$

Also, it has been proved ([21, Theorem 3.1]) that the representation $D_{\lambda,\mu} = \Gamma_{\lambda,\mu}(D_\lambda^+ \oplus D_{\lambda+2}^+)\Gamma_{\lambda,\mu}^*$ is a multiplier representation on $H^{(\lambda,\mu)}$ with multiplier

$$J^{(\lambda)}(\phi, z) = \begin{bmatrix} (\phi'(z))^{\frac{\lambda}{2}} & 0 \\ -c_\phi(\phi'(z))^{\frac{\lambda+1}{2}} & (\phi'(z))^{\frac{\lambda}{2}+1} \end{bmatrix}, \tag{4.6}$$

where c_ϕ is a scalar depending on ϕ such that $\phi''(z) = -c_\phi(\phi'(z))^{3/2}$. Let M_z be the multiplication by z operator on $H^{(\lambda,\mu)}$. From Theorem 1.16, it follows that M_z is a homogeneous operator with associated representation $D_{\lambda,\mu}$. It is easy to see that the off-diagonal entry of the operator $\Gamma_{\lambda,\mu}^* M_z \Gamma_{\lambda,\mu}$, given in [15, p. 2255], is a scalar multiple of the inclusion map i from $H^{(\lambda)}$ into $H^{(\lambda+2)}$. Therefore, it follows that

$$\Gamma_{\lambda,\mu}^* M_z \Gamma_{\lambda,\mu} = \begin{bmatrix} M^{(\lambda)} & 0 \\ -\frac{1}{\lambda\mu} i & M^{(\lambda+2)} \end{bmatrix}.$$

It is shown in [22] that these are the only irreducible homogeneous operators in $B_2(\mathbb{D})$. The associated representation of $\begin{pmatrix} M^{(\lambda)} & 0 \\ -\frac{1}{\lambda\mu} i & M^{(\lambda+2)} \end{pmatrix}$ is $D_\lambda^+ \oplus D_{\lambda+2}^+$. Let $M(\eta)$ denote the operator $\begin{pmatrix} M^{(\lambda)} & 0 \\ \eta i & M^{(\lambda+2)} \end{pmatrix}$. It can be easily proved that if $|\eta_1| = |\eta_2|$, then $M(\eta_1)$ and $M(\eta_2)$ are unitarily equivalent. So, we consider $M(\eta)$ with $\eta \geq 0$.

Lemma 4.6. *The operator $M(\eta)$ is a contraction if and only if $\lambda \geq 1$ and $0 \leq \eta^2 \leq \frac{\lambda-1}{\lambda}$. Defect operators of $M(\eta)$ are quasi-invertible if and only if $0 \leq \eta^2 < \frac{\lambda-1}{\lambda}$.*

Proof. Since $M^{(\lambda)}$ is a contraction only when $\lambda \geq 1$, we infer that if $M(\eta)$ is a contraction, then $\lambda \geq 1$. Let $H(0)$ be the subspace spanned by the vector $(e_0^\lambda, 0)^t$ and for $n \geq 1$, let $H(n)$ be the subspace spanned by the set of vectors $\{(e_n^\lambda, 0)^t, (0, e_{n-1}^{\lambda+2})^t\}$. The operator $M(\eta)$ maps $H(n)$ to $H(n+1)$.

It is easy to see that $\|M(\eta)(e_n^\lambda, 0)^t\|^2 \leq 1$ for all $n \geq 0$ if and only if $\eta^2 \leq \frac{\lambda-1}{\lambda}$. Let $T_n := T_{|H(n)}$, for $n \geq 0$. Then $T_n^* T_n$ maps $H(n)$ to $H(n)$. Suppose $[T_n^* T_n]$ is the matrix representation of $T_n^* T_n$ with respect to the basis $\{(e_n^\lambda, 0)^t, (0, e_{n-1}^{\lambda+2})^t\}$ of $H(n)$, $n \geq 1$. Then we have

$$[T_n^* T_n] = \begin{bmatrix} \frac{n+1}{\lambda+n} + \frac{\eta^2(\lambda+1)\lambda}{(\lambda+n+1)(\lambda+n)} & \frac{\eta\sqrt{n\lambda(\lambda+1)}}{(\lambda+n+1)\sqrt{(\lambda+n)}} \\ \frac{\eta\sqrt{n\lambda(\lambda+1)}}{(\lambda+n+1)\sqrt{(\lambda+n)}} & \frac{n}{(\lambda+n+1)} \end{bmatrix}.$$

A straight forward computation shows that the eigenvalues of $[T_n^* T_n]$ are less than or equal to 1 if and only if $0 \leq \eta^2 \leq \frac{\lambda-1}{\lambda}$. From this, it follows that T is a contraction if and only if $\lambda \geq 1$ and $0 \leq \eta^2 \leq \frac{\lambda-1}{\lambda}$.

Also, it is easy to see that if one of the eigenvalues of $[T_n^* T_n]$ is 1, then $\eta^2 = \frac{\lambda-1}{\lambda}$. This shows that D is quasi-invertible if and only if $0 \leq \eta^2 < \frac{\lambda-1}{\lambda}$. For $\eta^2 = \frac{\lambda-1}{\lambda}$, $(I_n - T_n^* T_n)$ has one dimensional kernel for all $n \geq 1$, where I_n is the identity operator on $H(n)$.

A similar computation shows that D_* is quasi-invertible if and only if $0 \leq \eta^2 < \frac{\lambda-1}{\lambda}$. For $\eta^2 = \frac{\lambda-1}{\lambda}$, $(I_n - T_n T_n^*)$ has one dimensional kernel for all $n \geq 1$. □

4.2.1 The defect operators are not quasi-invertible

Given any two Hilbert spaces $\mathscr{K}, \mathscr{L}$ and a contractive holomorphic function $\theta : \mathbb{D} \to \mathscr{B}(\mathscr{K}, \mathscr{L})$, the operator $\Theta : H^2_{\mathscr{K}} \to H^2_{\mathscr{L}}$, defined by the formula $\Theta f(z) = \theta(z) f(z)$, $z \in \mathbb{D}$, $f \in H^2_{\mathscr{K}}$, is a contraction. The two Hilbert spaces $H^2_{\mathscr{K}}$ and $H^2_{\mathscr{L}}$ can be naturally identified with $H^{(1)} \otimes \mathscr{K}$ and $H^{(1)} \otimes \mathscr{L}$, respectively. If $\mathscr{K}$ is a Hilbert space consisting of holomorphic functions defined on $\mathbb{D}$, then there is a natural realization of the Hilbert space $H^{(1)} \otimes \mathscr{K}$ as a space of holomorphic functions in two variables. One way to achieve this is to take $f \otimes g \in H^{(1)} \otimes \mathscr{K}$ to the function $(z, w) \mapsto f(z) g(w)$, $z, w \in \mathbb{D}$. Similarly, we realize the Hilbert space $H^{(1)} \otimes \mathscr{L}$ as a space of holomorphic functions on $\mathbb{D}^2$, whenever $\mathscr{L}$ is a Hilbert space consisting of holomorphic functions on $\mathbb{D}$.

Recall (see [8, Theorem 3.1]) that the characteristic function of the operator $M^{(\lambda)}$, $\lambda > 1$, coincides with the purely contractive holomorphic function $\theta_\lambda : \mathbb{D} \longrightarrow \mathscr{B}(H^{(\lambda+1)}, H^{(\lambda-1)})$, where

$$\theta_\lambda(z) = \frac{1}{\sqrt{\lambda(\lambda-1)}} D^+_{\lambda-1}(\phi_z)^* \partial^* D^+_{\lambda+1}(\phi_z).$$

Here, $\partial : H^{(\lambda-1)} \to H^{(\lambda+1)}$ is the map defined by $\partial f = f'$. The characteristic function θ_λ determines, as above, an operator $\Theta_\lambda : H^{(1)} \otimes H^{(\lambda+1)} \to H^{(1)} \otimes H^{(\lambda-1)}$. The formula given below for Θ_λ^* is from [8]:

$$\left(\Theta_\lambda^* f\right)(z, w) = \frac{1}{\sqrt{\lambda(\lambda-1)}} \frac{\partial}{\partial w} f(z, w) - \sqrt{\frac{\lambda-1}{\lambda}} \frac{f(z, w) - f(w, w)}{z - w}, \; z, w \in \mathbb{D}, \; f \in H^{(1)} \otimes H^{(\lambda-1)}. \tag{4.7}$$

For $\lambda > 2$, let $\theta : \mathbb{D} \longrightarrow \mathscr{B}(H^{(\lambda+2)}, H^{(\lambda-2)})$ be the map defined by $\theta(z) = \theta_{\lambda-1}(z)\theta_{\lambda+1}(z)$, $z \in \mathbb{D}$. The map θ induces an operator Θ from $H^{(1)} \otimes H^{(\lambda+2)}$ to $H^{(1)} \otimes H^{(\lambda-2)}$. From the definition of θ, it follows that $\Theta = \Theta_{\lambda-1}\Theta_{\lambda+1}$.

Let $\Delta := \{(z,z) : z \in \mathbb{D}\}$ be the diagonal subset of the bi-disc $\mathbb{D}^2$. For $\lambda > 0$, let $s_\lambda(1) := \{f \in H^{(1)} \otimes H^{(\lambda)} : f_{|\Delta} = 0\}$ and $s_\lambda(2) := \{f \in H^{(1)} \otimes H^{(\lambda)} : f_{|\Delta} = 0, \partial_z f_{|\Delta} = 0\}$. Both $s_\lambda(1)$, $s_\lambda(2)$ are closed subspaces of $H^{(1)} \otimes H^{(\lambda)}$ with $s_\lambda(2) \subseteq s_\lambda(1)$.

Recall that $\left\{\frac{w^n}{\|w^n\|_\lambda} : n \geq 0\right\}$, $\|w^n\|_\lambda^2 = \binom{n+\lambda-1}{n}^{-1}$ is an orthonormal basis of $H^{(\lambda)}$. Suppose $\mathrm{Hom}_\lambda(p)$ is the space of all homogeneous polynomial of degree p in $H^{(1)} \otimes H^{(\lambda)}$. Then we have $H^{(1)} \otimes H^{(\lambda)} = \oplus_{p\geq 0}\mathrm{Hom}_\lambda(p)$. Let

$$f_{p,1}^{(\lambda)}(z,w) = \sum_{l=0}^{p} \frac{z^{p-l}w^l}{\|w^l\|_\lambda^2} \text{ and } f_{p,2}^{(\lambda)}(z,w) = \sum_{l=0}^{p} \frac{(b_p^{(\lambda)} + l a_p^{(\lambda)})}{\|w^l\|_\lambda^2} z^{p-l} w^l, \tag{4.8}$$

where $a_p^{(\lambda)} = -\sum_{l=0}^{p} \frac{1}{\|w^l\|_\lambda^2}$ and $b_p^{(\lambda)} = \sum_{l=0}^{p} \frac{l}{\|w^l\|_\lambda^2}$. From [15], we know that $f_{p,1}^{(\lambda)}$ is in $s_{\lambda-2}(1)^\perp$ and $f_{p,2}^{(\lambda)}$ is in $s_\lambda(1) \cap s_\lambda(2)^\perp$. Therefore the set of vectors $\left\{f_{p,1}^{(\lambda)}, f_{p,2}^{(\lambda)}\right\}$ is an orthogonal basis of $s_\lambda(2)^\perp \cap \mathrm{Hom}_\lambda(p)$.

Lemma 4.7. *For $\lambda > 2$, the operator $\Theta_{\lambda-1}^*$ maps $s_{\lambda-2}(2)$ to $s_\lambda(1)$ and $s_{\lambda-2}(2)^\perp$ to $s_\lambda(1)^\perp$.*

Proof. From the definition of $\Theta_{\lambda-1}^*$, it is easy to see that $\Theta_{\lambda-1}^*(s_{\lambda-2}(2))$ is contained in $s_\lambda(1)$. It takes a little more work to show that $\Theta_{\lambda-1}^*\left(s_{\lambda-2}(2)^\perp\right)$ is contained in $s_\lambda(1)^\perp$. First, using the formula (see [8, Equation 4.3])

$$\sum_{j=0}^{l} \begin{pmatrix} j+\lambda-3 \\ j \end{pmatrix} = \begin{pmatrix} l+\lambda-2 \\ l \end{pmatrix},$$

it is easy to prove that

$$a_p^{(\lambda-2)} = -\begin{pmatrix} p+\lambda-2 \\ p \end{pmatrix} \text{ and } b_p^{(\lambda-2)} = (\lambda-2)\begin{pmatrix} p+\lambda-2 \\ p-1 \end{pmatrix}.$$

It then follows that

$$b_p^{(\lambda-2)} = -p\left(\frac{\lambda-2}{\lambda-1}\right) a_p^{(\lambda-2)}. \tag{4.9}$$

Since ${f_{p,2}^{(\lambda-2)}}_{|\Delta} = 0$, it follows that $f_{p,2}^{(\lambda-2)}(z,w) = (z-w)\sum_{l=0}^{p-1} \alpha_l z^{p-1-l} w^l$, where $\alpha_l \in \mathbb{C}$, $0 \leq l \leq p-1$. Comparing the coefficients of $z^{p-l}w^l$ from the two sides of this equality and then using the equation (4.9), we get

$$\alpha_l = \sum_{k=0}^{l} \frac{b_p^{(\lambda-2)} + k a_p^{(\lambda-2)}}{\|w^k\|_{\lambda-2}^2} = \frac{(\lambda-2)}{(\lambda-1)}(p-l) a_p^{(\lambda-2)} a_l^{(\lambda-2)}. \tag{4.10}$$

Now, from the formula (4.7) given for Θ^*_λ, we have

$$\begin{aligned}\sqrt{(\lambda-1)(\lambda-2)}\Theta^*_{\lambda-1} f^{(\lambda-2)}_{p,2}(z,w) &= \frac{\partial}{\partial w} f^{(\lambda-2)}_{p,2}(z,w) - (\lambda-2)\frac{f^{(\lambda-2)}_{p,2}(z,w)}{z-w} \\ &= \sum_{l=1}^{p}\left(\frac{b^{(\lambda-2)}_p + l a^{(\lambda-2)}_p}{\|w^l\|^2_{\lambda-2}} - (\lambda-2)\alpha_{l-1}\right) z^{p-l} w^{l-1}.\end{aligned}$$

Using the value of α_{l-1} from the equation (4.10) and the identity $\binom{\lambda+l-3}{l} = \frac{(\lambda-2)}{l}\binom{\lambda+l-3}{l-1}$, we obtain

$$\begin{aligned}\frac{(b^{(\lambda-2)}_p + l a^{(\lambda-2)}_p) l}{\|w^l\|^2_{\lambda-2}} - (\lambda-2)\alpha_{l-1} &= (\lambda-2)\begin{pmatrix}\lambda+l-3\\ l-1\end{pmatrix}\Big[-p\left(\frac{\lambda-2}{\lambda-1}\right)a^{(\lambda-2)}_p \\ &\qquad + l a^{(\lambda-2)}_p + \frac{(\lambda-2)}{(\lambda-1)} a^{(\lambda-2)}_p (p-l+1)\Big] \\ &= (\lambda-2)\begin{pmatrix}\lambda+l-3\\ l-1\end{pmatrix} a^{(\lambda-2)}_p \frac{(\lambda+l-2)}{(\lambda-1)} \\ &= (\lambda-2) a^{(\lambda-2)}_p \begin{pmatrix}\lambda+l-2\\ l-1\end{pmatrix} \\ &= \frac{(\lambda-2) a^{(\lambda-2)}_p}{\|w^{l-1}\|^2_\lambda}.\end{aligned}$$

For the third equality, we have used the identity $\frac{(\lambda+l-2)}{(\lambda-1)}\binom{\lambda+l-3}{l-1} = \binom{\lambda+l-2}{l-1}$. Therefore, it follows that

$$\sqrt{(\lambda-1)(\lambda-2)}\left(\Theta^*_{\lambda-1} f^{(\lambda-2)}_{p,2}\right)(z,w) = (\lambda-2) a^{(\lambda-2)}_p \sum_{l=0}^{p-1}\frac{z^{p-1-l} w^l}{\|w^l\|^2_\lambda} = (\lambda-2) a^{(\lambda-2)}_p f^{(\lambda)}_{p-1,1}(z,w).$$

Consequently, the vector $\Theta^*_{\lambda-1} f^{(\lambda-2)}_{p,2}$ is in $s_\lambda(1)^\perp \cap \mathrm{Hom}_\lambda(p-1)$ showing that $\Theta^*_{\lambda-1}$ maps $s_{\lambda-2}(2)^\perp$ into $s_\lambda(1)^\perp$. □

Theorem 4.8. *Let $\theta : \mathbb{D} \longrightarrow \mathscr{B}(H^{(\lambda+2)}, H^{(\lambda-2)})$ be defined by $\theta(z) = \theta_{\lambda-1}(z)\theta_{\lambda+1}(z)$. Then θ coincides with the characteristic function of* $\begin{bmatrix} M^{(\lambda-1)} & 0 \\ \sqrt{\frac{\lambda-2}{\lambda-1}}\, i & M^{(\lambda+1)}\end{bmatrix}$.

Proof. Since $\theta_{\lambda-1}$ and $\theta_{\lambda+1}$ are purely contractive, it follows that $\theta = \theta_{\lambda-1}\theta_{\lambda+1}$ is also purely contractive. Let $\Theta : H^{(1)} \otimes H^{(\lambda+2)} \longrightarrow H^{(1)} \otimes H^{(\lambda-2)}$ be the operator induced by θ. Then $\Theta = \Theta_{\lambda-1}\Theta_{\lambda+1}$ and therefore θ is an inner function.

Let $\mathscr{M}$ be the range of Θ and $T = P_{\mathscr{M}^\perp} M^{(1)} \otimes I_{|\mathscr{M}^\perp}$, where $M^{(1)}$ is the multiplication operator on $H^{(1)}$ and $P_{\mathscr{M}^\perp}$ is the projection of $H^{(1)} \otimes H^{(\lambda-2)}$ onto $\mathscr{M}^\perp$. It follows, from [32], that the characteristic function of T coincides with θ. Since $\mathscr{M} = \mathrm{ran}\,\Theta$, so $\mathscr{M}^\perp = \ker\,\Theta^*$.

Now [8, Theorem 3.2] implies that ker $\Theta^*_{\lambda+1} = s_\lambda(1)^\perp$ and therefore, by Lemma 4.7, we have ker $\Theta^* = s_{\lambda-2}(2)^\perp$.

From [15], it follows that T is unitarily equivalent to the operator $\begin{bmatrix} M^{(\lambda-1)} & 0 \\ \sqrt{\frac{\lambda-2}{\lambda-1}}\,i & M^{(\lambda+1)} \end{bmatrix}$ and therefore the characteristic function of $\begin{bmatrix} M^{(\lambda-1)} & 0 \\ \sqrt{\frac{\lambda-2}{\lambda-1}}\,i & M^{(\lambda+1)} \end{bmatrix}$ coincides with $\theta(z)$. □

4.2.2 The defect operators are quasi-invertible

Lemma 4.6 implies that the defect operators of M_z defined on $H^{(\lambda,\mu)}$ are quasi-invertible if and only if $\lambda > 1$ and $\mu^2 > \frac{1}{\lambda(\lambda-1)}$. Now we describe the characteristic function of M_z in this case.

Let $H(0)$ be the subspace of $H^{(\lambda,\mu)}$ spanned by the vector $(1,0)^t$ and for $n \geq 1$, $H(n)$ be the subspace of $H^{(\lambda,\mu)}$ spanned by the set of vectors $\{(z^n, \frac{n}{\lambda}z^{n-1})^t, (0, \mu z^{n-1})^t\}$. Then M_z maps $H(n)$ to $H(n+1)$ for all $n \geq 0$. Note that $H^{(\lambda,\mu)}$ is densely contained in $H^{(\lambda+1,\mu')}$ for all $\mu' > 0$, because $H^{(\lambda_1)}$ is densely contained in $H^{(\lambda_2)}$ if $\lambda_1 < \lambda_2$.

Lemma 4.9. *There exist $c > 0$ and $\mu_1 > 0$ such that $\langle Df, Dg\rangle = c\langle f, g\rangle_{\lambda+1,\mu_1}$ for all $f, g \in H^{(\lambda,\mu)}$, where $\langle .,.\rangle_{\lambda+1,\mu_1}$ is the inner product on $H^{(\lambda+1,\mu_1)}$ and D is the defect operator $(I - M_z^* M_z)^{1/2}$.*

Proof. Since M_z maps $H(n)$ to $H(n+1)$, each $H(n)$ is invariant under D. Because $H(n)$'s are orthogonal in $H^{(\lambda,\mu)}$, so it is enough to prove the above equality on each $H(n)$ with some c and μ_1 which independent of n.

For $n \geq 0$, we have

$$\|D(0,\mu z^n)^t\|^2 = \|z^n\|^2_{\lambda+2} - \|z^{n+1}\|^2_{\lambda+2} = \frac{n!(\lambda+1)}{(\lambda+n+2)\cdots(\lambda+2)}.$$

Suppose there exist $c > 0$ and $\mu_1 > 0$ such that $\|D(0,\mu z^n)^t\|^2 = c\|(0,\mu z^n)^t\|^2_{\lambda+1,\mu_1}$, then we obtain

$$\frac{n!(\lambda+1)}{(\lambda+n+2)\cdots(\lambda+2)} = c\|\Gamma_{\lambda+1,\mu_1}(0, \frac{\mu}{\mu_1}z^n)^t\|^2_{\lambda+1,\mu_1} = c\|\frac{\mu}{\mu_1}z^n\|^2_{\lambda+3} = c\frac{\mu^2}{\mu_1^2}\begin{pmatrix} \lambda+n+2 \\ n \end{pmatrix}^{-1},$$

which gives us

$$\frac{\lambda+1}{\lambda+2} = c\frac{\mu^2}{\mu_1^2}. \tag{4.11}$$

Suppose that the statement of the Theorem is valid with this c and μ_1. Then we have

$$\|D(z^n, \frac{n}{\lambda}z^{n-1})^t\|^2 = c\|(z^n, \frac{n}{\lambda}z^{n-1})^t\|^2_{\lambda+1,\mu_1},\ n \geq 1. \tag{4.12}$$

But

$$\|D(z^n, \frac{n}{\lambda}z^{n-1})^t\|^2 = \|z^n\|^2_\lambda - \|(z^{n+1}, \frac{n}{\lambda}z^n)^t\|^2. \tag{4.13}$$

Since

$$\left(z^{n+1},\frac{n}{\lambda}z^n\right)^t=\left(z^{n+1},\frac{n+1}{\lambda}z^n\right)^t-\frac{1}{\lambda\mu}(0,\mu z^n)^t,$$

it follows that

$$\|(z^{n+1},\frac{n}{\lambda}z^n)^t\|^2=\|z^{n+1}\|^2_\lambda+\frac{1}{\lambda^2\mu^2}\|z^n\|^2_{\lambda+2}=\begin{pmatrix}\lambda+n\\ n+1\end{pmatrix}^{-1}+\frac{1}{\lambda^2\mu^2}\begin{pmatrix}\lambda+n+1\\ n\end{pmatrix}^{-1}.$$

Substituting the value of $\|(z^{n+1},\frac{n}{\lambda}z^n)^t\|^2$ in the equation (4.13), we obtain

$$\|D(z^n,\frac{n}{\lambda}z^{n-1})^t\|^2=\frac{n!}{(\lambda+n)(\lambda+n+1)\cdots(\lambda+2)}\left[\frac{\lambda-1}{(\lambda+1)\lambda}-\frac{1}{\lambda^2\mu^2(\lambda+n+1)}\right].$$

Now let us calculate $\|(z^n,\frac{n}{\lambda}z^{n-1})^t\|^2_{\lambda+1,\mu_1}$. Since

$$(z^n,\frac{n}{\lambda}z^{n-1})^t=(z^n,\frac{n}{\lambda+1}z^{n-1})^t+\frac{n}{\mu_1\lambda(\lambda+1)}(0,\mu_1z^{n-1})^t,$$

it follows that

$$\begin{aligned}\|(z^n,\frac{n}{\lambda}z^{n-1})^t\|^2_{\lambda+1,\mu_1}&=\|(z^n,\frac{n}{\lambda+1}z^{n-1})^t\|^2_{\lambda+1,\mu_1}+\frac{n^2}{(\lambda+1)^2\lambda^2\mu_1^2}\|(0,\mu_1z^{n-1})^t\|^2_{\lambda+1,\mu_1}\\ &=\|z^n\|^2_{\lambda+1}+\frac{n^2}{(\lambda+1)^2\lambda^2\mu_1^2}\|z^{n-1}\|^2_{\lambda+3}\\ &=\frac{n!}{(\lambda+n)\cdots(\lambda+3)}\left[\frac{1}{(\lambda+2)(\lambda+1)}+\frac{n}{\lambda^2\mu_1^2(\lambda+1)^2(\lambda+n+1)}\right].\end{aligned}$$

Now substituting the values of $\|(z^n,\frac{n}{\lambda}z^{n-1})^t\|^2_{\lambda+1,\mu_1}$ and $\|D(z^n,\frac{n}{\lambda}z^{n-1})^t\|^2$ in the equation (4.12), we get

$$c=\frac{\lambda-1}{\lambda}-\frac{1}{\lambda^2\mu^2}. \tag{4.14}$$

Quasi-invertibility of the defect operators of M_z imply that $\frac{\lambda-1}{\lambda}-\frac{1}{\lambda^2\mu^2}>0$ and therefore $c>0$. Putting the value of c in the equation (4.11), we obtain μ_1.

Now to check if this choice of c and μ_1 works, we need to verify the equality

$$\left\langle D(z^n,\frac{n}{\lambda}z^{n-1})^t,D(0,\mu z^{n-1})^t\right\rangle=c\left\langle(z^n,\frac{n}{\lambda}z^{n-1})^t,(0,\mu z^{n-1})\right\rangle_{\lambda+1,\mu_1},\ n\geq1.$$

For $n \geq 1$,

$$\begin{aligned}\left\langle D(z^n, \frac{n}{\lambda} z^{n-1})^t, D(0, \mu z^{n-1})^t \right\rangle &= \left\langle (I - M_z^* M_z)(z^n, \frac{n}{\lambda} z^{n-1})^t, (0, \mu z^{n-1})^t \right\rangle \\ &= \left\langle (z^n, \frac{n}{\lambda} z^{n-1})^t, (0, \mu z^{n-1})^t \right\rangle - \left\langle (z^{n+1}, \frac{n}{\lambda} z^n)^t, (0, \mu z^n)^t \right\rangle \\ &= -\left\langle (z^{n+1}, \frac{n}{\lambda} z^n)^t, (0, \mu z^n)^t \right\rangle \\ &= -\left\langle (z^{n+1}, \frac{n+1}{\lambda} z^n)^t - \frac{1}{\lambda\mu}(0, \mu z^n)^t, ((0, \mu z^n)^t) \right\rangle \\ &= \frac{1}{\lambda\mu} \left\langle (0, \mu z^n)^t, (0, \mu z^n)^t \right\rangle \\ &= \frac{1}{\lambda\mu} \| z^n \|_{\lambda+2}^2 = \frac{1}{\lambda\mu} \begin{pmatrix} \lambda + n + 1 \\ n \end{pmatrix}^{-1}.\end{aligned}$$

Again, we have

$$\begin{aligned}&\left\langle (z^n, \frac{n}{\lambda} z^{n-1})^t, (0, \mu z^{n-1})^t \right\rangle_{\lambda+1,\mu_1} \\ &= \left\langle (z^n, \frac{n}{\lambda+1} z^{n-1})^t + \frac{n}{\mu_1 \lambda(\lambda+1)} (0, \mu_1 z^{n-1})^t, \frac{\mu}{\mu_1} (0, \mu_1 z^{n-1})^t \right\rangle_{\lambda+1,\mu_1} \\ &= \frac{n\mu}{\mu_1^2 \lambda(\lambda+1)} \left\langle (0, \mu_1 z^{n-1})^t, (0, \mu_1 z^{n-1})^t \right\rangle_{\lambda+1,\mu_1} \\ &= \frac{n\mu}{\mu_1^2 \lambda(\lambda+1)} \| z^{n-1} \|_{\lambda+3}^2 = \frac{n\mu}{\mu_1^2 \lambda(\lambda+1)} \begin{pmatrix} \lambda + n + 1 \\ n - 1 \end{pmatrix}^{-1}.\end{aligned}$$

Now, using the equation (4.11), it is easy to see that

$$\left\langle D(z^n, \frac{n}{\lambda} z^{n-1})^t, D(0, \mu z^{n-1})^t \right\rangle = c \left\langle (z^n, \frac{n}{\lambda} z^{n-1})^t, (0, \mu z^{n-1})^t \right\rangle_{\lambda+1,\mu_1}.$$

Also, it is easy to prove that $\langle D(1,0)^t, D(1,0)^t \rangle = c \langle (1,0)^t, (1,0)^t \rangle_{\lambda+1,\mu_1}$. Since we have c and μ_1, independent of n, it follows that $\langle Df, Dg \rangle = c \langle f, g \rangle_{\lambda+1,\mu_1}$ for all $f, g \in H^{(\lambda,\mu)}$. □

From Theorem 4.5, it follows that the characteristic function of M_z on $H^{(\lambda,\mu)}$ is $\theta(a) = \sigma_L(\phi_a)^* \theta(0) \sigma_R(\phi_a)$, $a \in \mathbb{D}$, where $\phi_a(z) = -\frac{z-a}{1-\bar{a}z}$ and σ_L, σ_R are representations of Möb with common multiplier on the Hilbert space $H^{(\lambda,\mu)}$. Also, Theorem 4.4 implies that

$$\sigma_R(\phi) D = D D_{\lambda,\mu}(\phi)(\phi')^{-1/2}(T) \text{ and } \sigma_L(\phi)^* D_* = D_* (\phi')^{1/2}(T)^* D_{\lambda,\mu}(\phi)^*, \ \phi \in \text{Möb}.$$

The following Lemma along with its proof is an adaptation of a similar result from the unpublished manuscript [2].

Lemma 4.10. *Suppose that the defect operators D and D_* of M_z on $H^{(\lambda,\mu)}$ are quasi-invertible. Then there exist two positive real numbers μ_1 and μ_2 such that σ_R and σ_L are unitarily equivalent to $D_{\lambda+1,\mu_1}$ and $D_{\lambda-1,\mu_2}$, respectively.*

Proof. Recall that the reproducing kernel K of the Hilbert space $H^{(\lambda,\mu)}$ is of the form

$$K(z,w) = \begin{bmatrix} \frac{1}{(1-z\bar{w})^{\lambda}} & \frac{z}{(1-z\bar{w})^{\lambda+1}} \\ \frac{\bar{w}}{(1-z\bar{w})^{\lambda+1}} & \frac{\frac{1}{\lambda}+\mu^2+z\bar{w}}{(1-z\bar{w})^{\lambda+2}} \end{bmatrix},\ z,w \in \mathbb{D}.$$

Let $\mu_2^2 = \mu^2 + \frac{1}{\lambda} - \frac{1}{\lambda-1}$. Since the defect operators of M_z are quasi-invertible, using Lemma 4.6, we have $\mu_2^2 > 0$. Define

$$K^+(z,w) = (1-z\bar{w})K(z,w) = \begin{bmatrix} \frac{1}{(1-z\bar{w})^{\lambda-1}} & \frac{z}{(1-z\bar{w})^{\lambda}} \\ \frac{\bar{w}}{(1-z\bar{w})^{\lambda}} & \frac{\frac{1}{\lambda-1}+\mu_2^2+z\bar{w}}{(1-z\bar{w})^{\lambda+1}} \end{bmatrix}.$$

Then K^+ is the kernel of the Hilbert space $H^{(\lambda-1,\mu_2)}$. Note that $H^{(\lambda-1,\mu_2)}$ is densely contained in $H^{(\lambda,\mu)}$. Let i^+ be the inclusion map of $H^{(\lambda-1,\mu_2)}$ into $H^{(\lambda,\mu)}$. Then $(i^+)^*$ is a map from $H^{(\lambda,\mu)}$ to $H^{(\lambda-1,\mu_2)}$ such that

$$(i^+)^* K(.,w)\xi = K^+(.,w)\xi,\ w \in \mathbb{D}, \xi \in \mathbb{C}^2. \tag{4.15}$$

Also, it is easy to see that

$$\langle D_* K(.,w_1)\xi_1, D_* K(.,w_2)\xi_2\rangle = \big\langle (i^+)^* K(.,w_1)\xi_1, (i^+)^* K(.,w_2)\xi_2 \big\rangle_{\lambda-1,\mu_2}$$

for all $w_1, w_2 \in \mathbb{C}$ and $\xi_1, \xi_2 \in \mathbb{C}^2$. This implies that

$$\big\langle D_* f, D_* g\big\rangle = \big\langle (i^+)^* f, (i^+)^* g\big\rangle_{\lambda-1,\mu_2} \tag{4.16}$$

for all $f, g \in H^{(\lambda,\mu)}$. Define $V : H^{(\lambda,\mu)} \to H^{(\lambda-1,\mu_2)}$ by $VD_* f = (i^+)^* f$ for all $f \in H^{(\lambda,\mu)}$. Since D_* is quasi-invertible, equation (4.16) implies that V is an isometry. Also, by definition of V, we have $VD_* = (i^+)^*$. Taking adjoint both sides of this equation, we obtain $D_* V^* = i^+$ which implies that $\ker V^* = \{0\}$. This proves that V is an unitary operator. We have

$$D_* \sigma_L(\phi) = D_{\lambda,\mu}(\phi)(\phi')^{1/2}(T) D_*.$$

Using Theorem 4.4, we obtain

$$D_* \sigma_L(\phi) f(z) = J^{(\lambda-1)}(\phi^{-1}, z) D_* f(\phi^{-1}(z)),\ f \in H^{(\lambda,\mu)}.$$

Therefore for g in $H^{(\lambda-1,\mu_2)}$, we have

$$\begin{aligned} i^+ V \sigma_L(\phi) V^* g(z) &= D_* \sigma_L(\sigma) V^* g(z) = J^{(\lambda-1)}(\phi^{-1}, z)(D_* V^* g)(\phi^{-1}(z)) \\ &= J^{(\lambda-1)}(\phi^{-1}, z)(i^+ g)(\phi^{-1}(z)) = J^{(\lambda-1)}(\phi^{-1}, z) g(\phi^{-1}(z)). \end{aligned}$$

This implies that

$$V \sigma_L(\phi) V^* g(z) = J^{(\lambda-1)}(\phi^{-1}, z) g(\phi^{-1}(z)),\ g \in H^{(\lambda-1,\mu_2)}.$$

Therefore, $V\sigma_L(\phi)V^* = D_{\lambda-1,\mu_2}(\phi)$, ϕ in Möb.

From Theorem 4.4, we know that $\sigma_R(\phi)D = DD_{\lambda,\mu}(\phi)(\phi')^{-1/2}(T)$. Lemma 4.9 implies that there exist two positive real numbers c and μ_1 such that

$$\langle Df, Dg\rangle = c\langle f, g\rangle_{\lambda+1,\mu_1},\ f, g \in H^{(\lambda,\mu)},$$

where $\langle .,.\rangle_{\lambda+1,\mu_1}$ is the inner product of $H^{(\lambda+1,\mu_1)}$. Let i^- be the inclusion map form $H^{(\lambda,\mu)}$ into $H^{(\lambda+1,\mu_1)}$. Define $U : H^{(\lambda,\mu)} \to H^{(\lambda+1,\mu_1)}$ by $UDf = \sqrt{c}i^- f,\ f \in H^{(\lambda,\mu)}$. Quasi-invertibility of D implies that U is a unitary operator. Theorem 4.4 shows that

$$\sigma_R(\phi)Df(z) = DJ^{(\lambda+1)}(\phi^{-1}, z)f(\phi^{-1}(z)),\ f \in H^{(\lambda,\mu)},\ \phi \in \text{Möb}.$$

Thus for f in $H^{(\lambda,\mu)}$, we have

$$\begin{aligned} U\sigma_R(\phi)U^* i^- f(z) &= \frac{1}{\sqrt{c}} U\sigma_R(\phi)Df(z) \\ &= \frac{1}{\sqrt{c}} UDJ^{(\lambda+1)}(\phi^{-1}, z)f(\phi^{-1}(z)) \\ &= i^- J^{(\lambda+1)}(\phi^{-1}, z)f(\phi^{-1}(z)), \end{aligned}$$

which implies that

$$U\sigma_R(\phi)U^* f(z) = J^{(\lambda+1)}(\phi^{-1}, z)f(\phi^{-1}(z)) = D_{\lambda+1,\mu}(\phi)f(z),\ f \in H^{(\lambda,\mu)}. \tag{4.17}$$

Since $H^{(\lambda,\mu)}$ is densely contained in $H^{(\lambda+1,\mu_1)}$, from equation (4.17) it follows that $U\sigma_R(\phi)U^* = D_{\lambda+1,\mu}(\phi)$ for all ϕ in Möb. □

From Lemma 4.10, it follows that the characteristic function of M_z on $H^{(\lambda,\mu)}$ coincides with $D_{\lambda-1,\mu_2}(\phi_a)^* V\theta(0)U^* D_{\lambda+1,\mu_1}(\phi_a)$, where U and V are the two unitary operators defined as above, whenever $\mu > \frac{1}{\sqrt{\lambda(\lambda+1)}}$. To concretely realize the characteristic function, set $T := UM_z^* V^*$ and compute $\Gamma^*_{\lambda+1,\mu_1} T\Gamma_{\lambda-1,\mu_2}$ explicitly. Note that for $\lambda > 0$, the map $\partial : H^{(\lambda)} \to H^{(\lambda+2)}$, defined by $\partial f = f'$, is a bounded operator.

Lemma 4.11. *The operator* $\Gamma^*_{\lambda+1,\mu_1} T\Gamma_{\lambda-1,\mu_2} : H^{(\lambda-1)} \oplus H^{(\lambda+1)} \to H^{(\lambda+1)} \oplus H^{(\lambda+3)}$ *is given by the formula*

$$\Gamma^*_{\lambda+1,\mu_1} T\Gamma_{\lambda-1,\mu_2} = \begin{bmatrix} \frac{\sqrt{c}}{\lambda-1}\partial & -\frac{\sqrt{c}}{\mu_2(\lambda-1)} I \\ \frac{\sqrt{c}}{\mu_1\lambda(\lambda^2-1)}\partial^2 & \frac{\sqrt{c}\mu_2}{\mu_1(\lambda+1)}\partial \end{bmatrix}.$$

Proof. Using the equation (4.15), the relations $V^*(i^+)^* = D_*$, $M_z D = D_* M_z$ and $UD = \sqrt{c}\ i^-$, we obtain

$$\begin{aligned} TK^+(., w)\xi &= UM_z^* V^* K^+(., w)\xi \\ &= UM_z^* V^*(i^+)^* K(., w)\xi = UM_z^* D_* K(., w)\xi \\ &= UDM_z^* K(., w)\xi = \sqrt{c}\ i^- M_z^* K(., w)\xi \\ &= \sqrt{c}\ \overline{w} K(., w)\xi, \end{aligned} \tag{4.18}$$

for any $w \in \mathbb{C}$ and $\xi \in \mathbb{C}^2$. Differentiating k times both sides of the equation (4.18) with respect to $\bar{w}$ at 0, we get

$$T(\bar{\partial}^k K^+(.,0)\xi) = \sqrt{c}\; k\bar{\partial}^{k-1}K(.,0)\xi. \tag{4.19}$$

A direct computation shows that

$$\bar{\partial}^k K^+(z,0) = \begin{bmatrix} \begin{pmatrix} \lambda+k-2 \\ k \end{pmatrix} k!z^k & \begin{pmatrix} \lambda+k-1 \\ k \end{pmatrix} k!z^{k+1} \\ \begin{pmatrix} \lambda+k-2 \\ k-1 \end{pmatrix} k!z^{k-1} & (\frac{1}{\lambda}+\mu^2)\begin{pmatrix} \lambda+k \\ k \end{pmatrix} k!z^k + \begin{pmatrix} \lambda+k-1 \\ k-1 \end{pmatrix} k!z^k \end{bmatrix}$$

and

$$\bar{\partial}^k K(z,0) = \begin{bmatrix} \begin{pmatrix} \lambda+k-1 \\ k \end{pmatrix} k!z^k & \begin{pmatrix} \lambda+k \\ k \end{pmatrix} k!z^{k+1} \\ \begin{pmatrix} \lambda+k-1 \\ k-1 \end{pmatrix} k!z^{k-1} & (\frac{1}{\lambda}+\mu^2)\begin{pmatrix} \lambda+k+1 \\ k \end{pmatrix} k!z^k + \begin{pmatrix} \lambda+k \\ k-1 \end{pmatrix} k!z^k \end{bmatrix}.$$

Evaluating the equation (4.19) at $\xi = (1,0)^t$, we obtain

$$T\left(z^k, \frac{k}{\lambda-1}z^{k-1}\right)^t = \frac{\sqrt{c}}{(\lambda-1)}\left(kz^{k-1}, \frac{k(k-1)}{\lambda}z^{k-2}\right)^t.$$

and consequently,

$$T\Gamma_{\lambda-1,\mu_2}(f,0)^t = T\left(f, \frac{1}{\lambda-1}f'\right)^t = \frac{\sqrt{c}}{(\lambda-1)}\left(f', \frac{1}{\lambda}f''\right)^t,\ f \in H^{(\lambda-1)}. \tag{4.20}$$

Now, evaluating (4.19) at $\xi = (0,1)^t$, we get

$$T\left(z^{k+1}, \left(\frac{1}{\lambda}+\mu^2\right)\frac{(\lambda+k)}{\lambda}z^k + \frac{k}{\lambda}z^k\right)^t = \frac{\sqrt{c}}{\lambda}\left(kz^k, \left(\frac{1}{\lambda}+\mu^2\right)\frac{(\lambda+k)}{(\lambda+1)}kz^{k-1} + \frac{k(k-1)}{(\lambda+1)}z^{k-1}\right)^t. \tag{4.21}$$

Using $\mu_2^2 = \mu^2 + \frac{1}{\lambda} - \frac{1}{\lambda-1}$, it is easy to see that

$$\left(z^{k+1}, \left(\frac{1}{\lambda}+\mu^2\right)\frac{(\lambda+k)}{\lambda}z^k + \frac{k}{\lambda}z^k\right)^t = \left(z^{k+1}, \frac{k+1}{\lambda-1}z^k\right)^t + \left(\frac{k}{\lambda}+1\right)\left(0, \mu_2^2 z^k\right)^t.$$

Equation (4.20) and (4.21) together imply that

$$\frac{1}{\sqrt{c}}\left(\frac{k}{\lambda}+1\right)T\left(0,\mu_2^2 z^k\right)^t = \left(\left(\frac{k}{\lambda}-\frac{k+1}{\lambda-1}\right)z^k, \left\{(\frac{1}{\lambda}+\mu^2)\frac{k(\lambda+k)}{\lambda(\lambda+1)} + \frac{k(k-1)}{\lambda(\lambda+1)} - \frac{k(k+1)}{\lambda(\lambda-1)}\right\}z^{k-1}\right)^t. \tag{4.22}$$

If $c, d \in \mathbb{C}$ such that

$$\left(\left(\frac{k}{\lambda}-\frac{k+1}{\lambda-1}\right)z^k, \left\{(\frac{1}{\lambda}+\mu^2)\frac{k(\lambda+k)}{\lambda(\lambda+1)} + \frac{k(k-1)}{\lambda(\lambda+1)} - \frac{k(k+1)}{\lambda(\lambda-1)}\right\}z^{k-1}\right)^t = c\left(z^k, \frac{k}{\lambda}z^{k-1}\right)^t + d\left(0, \mu^2 z^{k-1}\right)^t,$$

then we get $c = \frac{k}{\lambda} - \frac{k+1}{\lambda-1} = -\frac{(\lambda+k)}{\lambda(\lambda-1)}$ and $d = \frac{(\lambda+k)k}{\lambda(\lambda+1)}$. Consequently, equation (4.22) gives us

$$T(0,\mu_2^2 z^k)^t = \frac{\sqrt{c}}{(\lambda+1)}(0,\mu^2 k z^{k-1})^t - \frac{\sqrt{c}}{(\lambda-1)}(z^k, \frac{k}{\lambda} z^{k-1})^t,$$

which implies that

$$T\Gamma_{\lambda-1,\mu_2}(0,g)^t = \frac{\sqrt{c}}{\mu_2(\lambda+1)}(0,\mu^2 g')^t - \frac{\sqrt{c}}{\mu_2(\lambda-1)}(g,\frac{1}{\lambda}g')^t,\ g \in H^{(\lambda+1)}. \tag{4.23}$$

Therefore, combining equations (4.20) and (4.23), the proof is complete. □

The following theorem gives the formula for the characteristic function of a homogeneous operator in this case (defect operators are quasi-invertible) explicitly.

Theorem 4.12. *The characteristic function of M_z, assuming that the defect operators are quasi-invertible, on $H^{(\lambda,\mu)}$ coincides with*

$$-\sqrt{c}\begin{bmatrix} \sqrt{\frac{\lambda}{\lambda-1}}\theta_\lambda(a) & \frac{\theta_\lambda(a)\theta_{\lambda+2}(a)}{\sqrt{(\lambda-1)\left(\mu^2(\lambda-1)-\frac{1}{\lambda}\right)}} \\ -\frac{I}{\sqrt{(\lambda-1)\left(\mu^2(\lambda-1)-\frac{1}{\lambda}\right)}} & \sqrt{\frac{\lambda}{\lambda-1}}\theta_{\lambda+2}(a) \end{bmatrix},$$

where $\theta_\lambda(a) = \frac{1}{\sqrt{(\lambda-1)\lambda}} D^+_{\lambda-1}(\phi_a)^*\partial^* D^+_{\lambda+1}(\phi_a)$ *and* $\theta_{\lambda+2}(a) = \frac{1}{\sqrt{(\lambda+1)(\lambda+2)}} D^+_{\lambda+1}(\phi_a)^*\partial^* D^+_{\lambda+3}(\phi_a)$.

Proof. Theorem 4.5 shows that the characteristic function θ of M_z is of the form

$$\theta(a) = \sigma_L(\phi_a)^*\theta(0)\sigma_R(\phi_a),\ a \in \mathbb{D}.$$

Since the defect operators of M_z on $H^{(\lambda,\mu)}$ are quasi-invertible, it follows, from Lemma 4.10, that

$$V\theta(a)U^* = -D_{\lambda-1,\mu_2}(\phi_a)^* T^* D_{\lambda+1,\mu_1}(\phi_a),$$

where U and V are the two unitary operators defined in Lemma 4.10. So

$$\begin{aligned}
&\Gamma^*_{\lambda-1,\mu_2} V\theta(a)U^*\Gamma_{\lambda+1,\mu_1} \\
&= -\left(\Gamma^*_{\lambda-1,\mu_2} D_{\lambda-1,\mu_2}(\phi_a)\Gamma_{\lambda-1,\mu_2}\right)^*\left(\Gamma^*_{\lambda+1,\mu_1} T\Gamma_{\lambda-1,\mu_2}\right)^*\left(\Gamma^*_{\lambda+1,\mu_1} D_{\lambda+1,\mu_1}(\phi_a)\Gamma_{\lambda+1,\mu_1}\right) \\
&= -\begin{bmatrix} D^+_{\lambda-1}(\phi_a)^* & 0 \\ 0 & D^+_{\lambda+1}(\phi_a)^* \end{bmatrix}\begin{bmatrix} \frac{\sqrt{c}}{\lambda-1}\partial^* & \frac{\sqrt{c}}{\mu_1\lambda(\lambda^2-1)}(\partial^*)^2 \\ -\frac{\sqrt{c}}{\mu_2(\lambda-1)}I & \frac{\sqrt{c}\mu_2}{\mu_1(\lambda+1)}\partial^* \end{bmatrix}\begin{bmatrix} D^+_{\lambda+1}(\phi_a) & 0 \\ 0 & D^+_{\lambda+3}(\phi_a) \end{bmatrix} \\
&= -\sqrt{c}\begin{bmatrix} \sqrt{\frac{\lambda}{\lambda-1}}\theta_\lambda(a) & \frac{\sqrt{\lambda+2}}{\mu_1\sqrt{\lambda(\lambda^2-1)}}\theta_\lambda(a)\theta_{\lambda+2}(a) \\ -\frac{1}{\mu_2(\lambda-1)}I & \frac{\mu_2\sqrt{\lambda+2}}{\mu_1\sqrt{\lambda+1}}\theta_{\lambda+2}(a) \end{bmatrix}.
\end{aligned}$$

Now, using the equations (4.11), (4.14) and $\mu_2^2 = \mu^2 - \frac{1}{\lambda(\lambda-1)}$, we obtain

$$\Gamma^*_{\lambda-1,\mu_2} V\theta(a)U^*\Gamma_{\lambda+1,\mu_1} = -\sqrt{c}\begin{bmatrix} \sqrt{\frac{\lambda}{\lambda-1}}\theta_\lambda(a) & \frac{\theta_\lambda(a)\theta_{\lambda+2}(a)}{\sqrt{(\lambda-1)(\mu^2(\lambda-1)-\frac{1}{\lambda})}} \\ -\frac{I}{\sqrt{(\lambda-1)(\mu^2(\lambda-1)-\frac{1}{\lambda})}} & \sqrt{\frac{\lambda}{\lambda-1}}\theta_{\lambda+2}(a) \end{bmatrix}.$$

This completes the proof of the theorem. □

Since the characteristic function θ coincides with $\Gamma^*_{\lambda-1,\mu_2} V\theta(a)U^*\Gamma_{\lambda+1,\mu_1}$, we use the same symbol θ to denote this operator and call it the characteristic function of the operator M_z.

We conclude this chapter by describing the invariant subspace determined by the characteristic operator Θ :

$$\Theta = -\sqrt{c}\begin{bmatrix} \sqrt{\frac{\lambda}{\lambda-1}}\Theta_\lambda & \frac{\Theta_\lambda\Theta_{\lambda+2}}{\sqrt{(\lambda-1)(\mu^2(\lambda-1)-\frac{1}{\lambda})}} \\ -\frac{I}{\sqrt{(\lambda-1)(\mu^2(\lambda-1)-\frac{1}{\lambda})}} & \sqrt{\frac{\lambda}{\lambda-1}}\Theta_{\lambda+2} \end{bmatrix},$$

which is just the range of the operator Θ. We have

$$\Theta^* = -\sqrt{c}\begin{bmatrix} \sqrt{\frac{\lambda}{\lambda-1}}\Theta^*_\lambda & -\frac{I}{\sqrt{(\lambda-1)(\mu^2(\lambda-1)-\frac{1}{\lambda})}} \\ \frac{\Theta^*_{\lambda+2}\Theta^*_\lambda}{\sqrt{(\lambda-1)(\mu^2(\lambda-1)-\frac{1}{\lambda})}} & \sqrt{\frac{\lambda}{\lambda-1}}\Theta^*_{\lambda+2} \end{bmatrix}.$$

It follows, from Lemma 4.7, that $\ker\,\Theta^* = \left\{\left(f, \frac{a}{b}\Theta^*_\lambda f\right) : f \in s_{\lambda-1}(2)^\perp\right\}$, where $a = -\sqrt{\frac{c\lambda}{\lambda-1}}$ and $b = \frac{-\sqrt{c}}{\sqrt{(\lambda-1)(\mu^2(\lambda-1)-\frac{1}{\lambda})}}$.

By the model theory of Sz.-Nagy and Foias, the operator $P_{\ker\Theta^*}\left(M^{(1)} \otimes (I_{\lambda-1} \oplus I_{\lambda+1})\right)_{|\ker\Theta^*}$, where $I_{\lambda\pm1}$ is the identity operator on $H^{(\lambda\pm1)}$, is unitarily equivalent to the operator M_z on $H^{(\lambda,\mu)}$. Recall that $M^{(1)}$ is the multiplication by the coordinate function on the Hardy space $H^{(1)}$. The operator $M^{(1)} \otimes (I_{\lambda-1} \oplus I_{\lambda+1})$ is clearly a homogeneous operator with the associated representation $D_1^+ \otimes \left(D^+_{\lambda-1} \oplus D^+_{\lambda+1}\right)$.

We show that the subspace $\ker\,\Theta^*$ is also left invariant by the representation $(D_1^+ \otimes D^+_{\lambda-1}) \oplus (D_1^+ \otimes D^+_{\lambda+1})$. This would give another proof that M_z on $H^{(\lambda,\mu)}$, $\mu > \frac{1}{\sqrt{\lambda(\lambda-1)}}$, is homogeneous.

The following Lemma is the first step in proving that $\ker\,\Theta^*$ is also left invariant by the representation $D_1^+ \otimes \left(D^+_{\lambda-1} \oplus D^+_{\lambda+1}\right) = (D_1^+ \otimes D^+_{\lambda-1}) \oplus (D_1^+ \otimes D^+_{\lambda+1})$.

Lemma 4.13. *For* $\phi \in$ *Möb, we have* $\Theta^*_\lambda D_1^+(\phi) \otimes D^+_{\lambda-1}(\phi) = D_1^+(\phi) \otimes D^+_{\lambda+1}(\phi)\Theta^*_\lambda$.

Proof. Let $\tilde{\theta}(z) = \frac{1}{\sqrt{\lambda(\lambda+1)}} D^+_{\lambda+1}(\phi_z)^* \partial D^+_{\lambda-1}(\phi_z)$. Then $\tilde{\theta}(z)$ induces an operator $\tilde{\Theta}$ from $L^2(\mathbb{T}) \otimes H^{(\lambda-1)}$ to $L^2(\mathbb{T}) \otimes H^{(\lambda+1)}$. Here, we view the elements of $L^2(\mathbb{T}) \otimes H^{(\lambda-1)}$ and $L^2(\mathbb{T}) \otimes H^{(\lambda+1)}$ as complex valued functions defined on $\mathbb{T} \times \mathbb{D}$. Then for every $f \in L^2(\mathbb{T}) \otimes H^{(\lambda-1)}$, from [8, Theorem 3.2], we have

$$(\tilde{\Theta} f)(z,w) = \frac{1}{\sqrt{\lambda(\lambda-1)}} \left[\frac{\partial}{\partial w} f(z,w) - (\lambda-1) \frac{\bar{z}}{1-\bar{z}w} f(z,w) \right]$$

for all $z \in \mathbb{T}$ and $w \in \mathbb{D}$.

Let ρ_b be an element of Möb such that $\rho_b(z) = \frac{z-b}{1-\bar{b}z}$. We know that $\{a^n w^m\}_{n\in\mathbb{Z}, m\geq 0}$ is an orthogonal basis of $L^2(\mathbb{T}) \otimes H^{(\lambda-1)}$. Let $f(a,w) = a^n w^m$ where $a \in \mathbb{T}$ and $w \in \mathbb{D}$. Then

$$\begin{aligned}
&((D^+_1(\rho_b^{-1}) \otimes D^+_{\lambda+1}(\rho_b^{-1}))\tilde{\Theta} f)(a,w) \\
&= \frac{1}{\sqrt{\lambda(\lambda-1)}} D^+_1(\rho_b^{-1}) \otimes D^+_{\lambda+1}(\rho_b^{-1}) \left[\frac{\partial}{\partial w} a^n w^m - (\lambda-1) \frac{\bar{a}}{1-\bar{a}w} a^n w^m \right] \\
&= \frac{1}{\sqrt{\lambda(\lambda-1)}} D^+_1(\rho_b^{-1}) \otimes D^+_{\lambda+1}(\rho_b^{-1}) \left[m a^n w^{m-1} - (\lambda-1) a^{n-1} w^m \sum_{k=0}^{\infty} \bar{a}^k w^k \right] \\
&= \frac{1}{\sqrt{\lambda(\lambda-1)}} D^+_1(\rho_b^{-1}) \otimes D^+_{\lambda+1}(\rho_b^{-1}) \left[m a^n w^{m-1} - (\lambda-1) \sum_{k=0}^{\infty} a^{n-k-1} w^{m+k} \right] \\
&= \frac{(\rho_b'(a))^{1/2} (\rho_b'(w))^{\frac{\lambda+1}{2}}}{\sqrt{\lambda(\lambda-1)}} \left[m(\rho_b(a))^n (\rho_b(w))^{m-1} - (\lambda-1) \sum_{k=0}^{\infty} (\rho_b(a))^{n-k-1} (\rho_b(w))^{m+k} \right] \\
&= \frac{(\rho_b'(a))^{1/2} (\rho_b'(w))^{\frac{\lambda+1}{2}} (\rho_b(a))^n (\rho_b(w))^{m-1}}{\sqrt{\lambda(\lambda-1)}} \left[m - (\lambda-1) \frac{(\rho_b(w))}{(\rho_b(a))} \frac{1}{(1-(\overline{\rho_b(a)})(\rho_b(w)))} \right] \\
&= \frac{(\rho_b'(a))^{1/2} (\rho_b'(w))^{\frac{\lambda+1}{2}} (\rho_b(a))^n (\rho_b(w))^{m-1}}{\sqrt{\lambda(\lambda-1)}} \left[m - (\lambda-1) \frac{\rho_b(w)}{\rho_b(a) - \rho_b(w)} \right] \\
&= \frac{(\rho_b'(a))^{1/2} (\rho_b'(w))^{\frac{\lambda+1}{2}} (\rho_b(a))^n (\rho_b(w))^{m-1}}{\sqrt{\lambda(\lambda-1)}} \left[m - (\lambda-1) \frac{(w-b)(1-\bar{b}a)}{(a-w)(1-|b|^2)} \right]
\end{aligned}$$

The fifth and the sixth equalities are obtained by noting that if $a \in \mathbb{T}$, then $\rho_b(a) \in \mathbb{T}$. Now,

$$\begin{aligned}
&(\tilde{\Theta}\, D^+_1(\rho_b^{-1}) \otimes D^+_{\lambda-1}(\rho_b^{-1}) f)(a,w) \\
&= \tilde{\Theta}\, D^+_1(\rho_b^{-1}) \otimes D^+_{\lambda-1}(\rho_b^{-1}) a^n w^m \\
&= \tilde{\Theta}((\rho_b'(a))^{1/2} (\rho_b(a))^n (\rho_b'(w))^{\frac{\lambda-1}{2}} (\rho_b(w))^m) \\
&= \frac{(\rho_b'(a))^{1/2} (\rho_b(a))^n}{\sqrt{\lambda(\lambda-1)}} \left[\frac{\partial}{\partial w} (\rho_b'(w))^{\frac{\lambda-1}{2}} (\rho_b(w))^m - (\lambda-1) \frac{\bar{a}}{(1-\bar{a}w)} (\rho_b'(w))^{\frac{\lambda-1}{2}} (\rho_b(w))^m \right]
\end{aligned}$$

From this equation, using $\rho_b(w) = \frac{w-b}{1-\bar{b}w}, \rho_b'(w) = \frac{1-|b|^2}{(1-\bar{b}w)^2}$ and $\rho_b''(w) = \frac{2\bar{b}(1-|b|^2)}{(1-\bar{b}w)^3}$, we see that

$$\begin{aligned}(\tilde{\Theta}\ (D_1^+(\rho_b^{-1}) \otimes D_{\lambda-1}^+(\rho_b^{-1}))f)(a,w)& \\ &= \frac{(\rho_b'(a))^{1/2}(\rho_b'(w))^{\frac{\lambda+1}{2}}(\rho_b(a))^n(\rho_b(w))^{m-1}}{\sqrt{\lambda(\lambda-1)}}\left[m-(\lambda-1)\frac{(w-b)(1-\bar{b}a)}{(a-w)(1-|b|^2)}\right].\end{aligned}$$

This shows that

$$(D_1^+(\rho_b^{-1}) \otimes D_{\lambda+1}^+(\rho_b^{-1}))\ \tilde{\Theta} = \tilde{\Theta}\ (D_1^+(\rho_b^{-1}) \otimes D_{\lambda-1}^+(\rho_b^{-1})).$$

Since $\rho_b^{-1} = \rho_{-b}$ for all $b \in \mathbb{D}$, it follows that

$$(D_1^+(\rho_b) \otimes D_{\lambda+1}^+(\rho_b))\ \tilde{\Theta} = \tilde{\Theta}\ (D_1^+(\rho_b) \otimes D_{\lambda-1}^+(\rho_b)).$$

Let ϕ_θ be an element of Möb such that $\phi_\theta(z) = e^{i\theta}z$. Then it is easy to see that

$$(D_1^+(\phi_\theta) \otimes D_{\lambda+1}^+(\phi_\theta))\ \tilde{\Theta} a^n w^m = \tilde{\Theta}\ (D_1^+(\phi_\theta) \otimes D_{\lambda-1}^+(\phi_\theta)) a^n w^m,\ n \in \mathbb{Z}\ m \geq 0.$$

Since $\{a^n w^m\}_{n\in\mathbb{Z}, m\geq 0}$ is an orthonormal basis of $L^2(\mathbb{T}) \otimes H^{(\lambda-1)}$, it follows that

$$(D_1^+(\phi_\theta) \otimes D_{\lambda+1}^+(\phi_\theta))\ \tilde{\Theta} = \tilde{\Theta}\ (D_1^+(\phi_\theta) \otimes D_{\lambda-1}^+(\phi_\theta)).$$

The multipliers of the two representations $D_1^+ \otimes D_{\lambda-1}^+$ and $D_1^+ \otimes D_{\lambda+1}^+$ are same. Hence we have

$$(D_1^+(\phi) \otimes D_{\lambda+1}^+(\phi))\ \tilde{\Theta} = \tilde{\Theta}\ (D_1^+(\phi) \otimes D_{\lambda-1}^+(\phi)),\ \phi \in \text{Möb}.$$

Let P be the projection of $L^2(\mathbb{T}) \otimes H^{(\lambda+1)}$ onto $H^{(1)} \otimes H^{(\lambda+1)}$. Since $H^{(1)} \otimes H^{(\lambda+1)}$ is an invariant subspace of the representation $D_1^+ \otimes D_{\lambda+1}^+$, so P commutes with $D_1^+(\phi) \otimes D_{\lambda+1}^+(\phi)$ for all ϕ in Möb. As $\Theta_\lambda^* = P\tilde{\Theta}_{|H^{(1)}\otimes H^{(\lambda-1)}}$, so we get that

$$(D_1^+(\phi) \otimes D_{\lambda+1}^+(\phi))\ \Theta_\lambda^* = \Theta_\lambda^*\ (D_1^+(\phi) \otimes D_{\lambda-1}^+(\phi))$$

for all ϕ in Möb. □

Theorem 4.14. *For any $t \in \mathbb{C}$, the subspace $\mathscr{M}_t^\perp = \{(f, t\,\Theta_\lambda^* f) : f \in s_{\lambda-1}(2)^\perp\}$ is invariant under the representation $(D_1^+ \otimes D_{\lambda-1}^+) \oplus (D_1^+ \otimes D_{\lambda+1}^+)$. Consequently, the subspace* $\ker\Theta^*$ *is invariant under $(D_1^+ \otimes D_{\lambda-1}^+) \oplus (D_1^+ \otimes D_{\lambda+1}^+)$ as well.*

Proof. The subspace $s_{\lambda-1}(2)^\perp$ is invariant under $D_1^+ \otimes D_{\lambda-1}^+$. Thus, for f in $s_{\lambda-1}(2)^\perp$ and ϕ in Möb, $(D_1^+(\phi) \otimes D_{\lambda-1}^+(\phi))f$ is in $s_{\lambda-1}(2)^\perp$. Now, using Lemma 4.13, we get

$$\begin{aligned}(D_1^+(\phi) \otimes D_{\lambda-1}^+(\phi)) \oplus (D_1^+(\phi) \otimes D_{\lambda+1}^+(\phi))(f, t\,\Theta_\lambda^* f)& \\ &= \big((D_1^+(\phi) \otimes D_{\lambda-1}^+(\phi))f, t(D_1^+(\phi) \otimes D_{\lambda+1}^+(\phi))\Theta_\lambda^* f\big) \\ &= \big((D_1^+(\phi) \otimes D_{\lambda-1}^+(\phi))f, t\,\Theta_\lambda^*(D_1^+(\phi) \otimes D_{\lambda-1}^+(\phi))f\big)\end{aligned}$$

Since $(D_1^+(\phi) \otimes D_{\lambda-1}^+(\phi))f$ is in $s_{\lambda-1}(2)^\perp$, it follows that

$$(D_1^+(\phi) \otimes D_{\lambda-1}^+(\phi)) \oplus (D_1^+(\phi) \otimes D_{\lambda+1}^+(\phi))(f, t\,\Theta_\lambda^* f) \in \mathscr{M}_t^\perp.$$

This shows that $\mathscr{M}_t^\perp$ is invariant under $(D_1^+ \otimes D_{\lambda-1}^+) \oplus (D_1^+ \otimes D_{\lambda+1}^+)$.

Clearly, $\ker\,\Theta^* = \mathscr{M}_t^\perp$ for some $t \in \mathbb{C}$, therefore $\ker\,\Theta^*$ is invariant under the representation $(D_1^+ \otimes D_{\lambda-1}^+) \oplus (D_1^+ \otimes D_{\lambda+1}^+)$. $\square$

Chapter 5

Homogeneous tuples in the Cowen-Douglas class of the polydisk

The notion of a homogeneous operator has a natural generalization to commuting tuple of operators. In this chapter, we consider the case of a commuting tuple which is homogeneous with respect to either the automorphism group $\mathrm{Aut}(\mathbb{D}^n)$ of the polydisc $\mathbb{D}^n$ or the subgroup $\text{Möb}^n := \mathrm{Aut}(\mathbb{D}) \times \cdots \times \mathrm{Aut}(\mathbb{D}) \subseteq \mathrm{Aut}(\mathbb{D}^n)$. We note that $\mathrm{Aut}(\mathbb{D}^n) = \text{Möb}^n \ltimes S_n$, where S_n is the permutation group on a set of n elements. Throughout this chapter, we let G denote one of these two groups.

Definition 5.1. A commuting tuple of operators $(T_1, T_2, \ldots, T_n)$ is said to be homogeneous with respect to the group G, if the joint spectrum of $(T_1, T_2, \ldots, T_n)$ lies in $\overline{\mathbb{D}}^n$ and $\varphi(T_1, T_2, \ldots, T_n)$ is unitarily equivalent to $(T_1, T_2, \ldots, T_n)$ for all $\varphi \in G$.

Combining the change of variable formula with the transformation rule of the curvature forced by homogeneity, we write down the curvature $(1,1)$ form explicitly for a homogeneous operator. Since the curvature is a complete unitary invariant for an n-tuple of operators in the Cowen-Douglas class of the polydisc $B_1(\mathbb{D}^n)$, we obtain a list of unitarily inequivalent homogeneous tuples from the curvature. Determining the class of homogeneous tuples in $B_m(\mathbb{D}^n)$ is much more challenging when the rank $m > 1$. We have succeeded in obtaining a complete list of inequivalent homogeneous tuples only in the case of $m = 2$.

Let $K : \mathbb{D}^n \times \mathbb{D}^n \to \mathcal{M}_m$ be a positive definite kernel which is holomorphic in the first and anti-holomorphic in the second variable. The linear span of the set of vectors

$$\{K(z, w)x : x \in \mathbb{C}^m, w \in \mathbb{D}^n\}$$

equipped with the inner product

$$\langle K(z, w_2)x, K(z, w_1)y\rangle = \langle K(w_1, w_2)x, y\rangle$$

is a pre-Hilbert space. The completion is a Hilbert space, say H_K, of holomorphic functions on $\mathbb{D}^n$. For each fixed but arbitrary $w \in \mathbb{D}^n$, the vector $K(\cdot, w)x$, $x \in \mathbb{C}^m$, is in H_K and has the reproducing property:

$$\langle f, K(\cdot, w)x\rangle = \langle f(w), x\rangle,\ f \in H_K.$$

Let M_{z_i} be the operator of multiplication by the coordinate function z_i, $1 \leq i \leq n$. Given any commuting tuple of operators $(T_1, T_2, \ldots, T_n)$ in $B_m(\mathbb{D}^n)$, there exists a reproducing kernel Hilbert space H_K such that $(T_1, T_2, \ldots, T_n)$ is unitarily equivalent to $(M^*_{z_1}, M^*_{z_2}, \ldots, M^*_{z_n})$, (see [14, Theorem 4.12]). Thus we will assume without loss of generality that a commuting tuple of operators in $B_m(\mathbb{D}^n)$ has been realized as $(M^*_{z_1}, M^*_{z_2}, \ldots, M^*_{z_n})$ on some reproducing kernel Hilbert space H_K. Conversely, with mild assumptions on the kernel K, one may assume the commuting tuple $(M^*_{z_1}, M^*_{z_2}, \ldots, M^*_{z_n})$ is in $B_m(\mathbb{D}^n)$ (cf. [14]). Throughout this chapter, we mandate that these assumptions are in force.

Definition 5.2. For $g \in G$, let $J_g : \mathbb{D}^n \to GL(m, \mathbb{C})$ be holomorphic. A kernel $K : \mathbb{D}^n \times \mathbb{D}^n \to \mathcal{M}_m$ is said to be quasi-invariant with respect to J if for all $g \in G$ and $z, w \in \mathbb{D}^n$, the kernel K transforms as follows:

$$K(z, w) = J_g(z) K(gz, gw) J_g(w)^*.$$

In practice, the factor J is assumed to be a cocycle, that is,

$$J(gh, z) = J(h, z) J(g, hz),\ g, h \in G;\ z \in \mathbb{D}^n.$$

Here $J(g, z) := J_g(z)$, $g \in G$, $z \in \mathbb{D}^n$.

Suppose H_K is a reproducing kernel Hilbert space and $J_g : \mathbb{D}^n \to GL(m, \mathbb{C})$, $g \in G$, is holomorphic. Then $U : G \to \mathrm{Hol}(\mathbb{D}^n, \mathbb{C}^m)$, defined by

$$(U_g f)(z) = J(g^{-1}, z) f(g^{-1} z);\ f \in \mathrm{Hol}(\mathbb{D}^n, \mathbb{C}^m),\ g \in G,$$

is a unitary representation of G on H_K if and only if K is quasi-invariant and J is a cocycle.

Let $e_1, \ldots, e_m$ be the standard unit vectors in $\mathbb{C}^m$. For $1 \leq i \leq m$, define $s_i : \mathbb{D}^n \to H_K$ to be the anti-holomorphic map: $s_i(w) := K(\cdot, w) e_i$, $w \in \mathbb{D}^n$. Clearly, $(s_1, \ldots, s_m)$ defines a trivial anti-holomorphic Hermitian vector bundle E of rank m on $\mathbb{D}^n$. The fiber of E at w is the m - dimensional subspace $\{K(\cdot, w)x : x \in \mathbb{C}^m\}$ and the Hermitian structure at w is given by the positive definite matrix $K(w, w)$. Thus the the curvature K of the vector bundle E is a $(1,1)$ form given by the formula:

$$\mathsf{K}(w) = \sum_{i,j=1}^{n} \partial_i \left[K(w, w)^{-1} \bar{\partial}_j K(w, w) \right] dw_i \wedge d\bar{w}_j.$$

Although, not very common, we will let

$$\mathscr{K}(w) = (\!(\mathscr{K}^{ij}(w))\!), \; w \in \mathbb{D}^n,$$

where $\mathscr{K}^{ij}(w) := \partial_i \left[K(w,w)^{-1}\bar{\partial}_j K(w,w)\right]$ is the co-efficient of $dw_i \wedge d\bar{w}_j$ in K. We obtain a transformation rule for the curvature whenever the kernel K is quasi-invariant.

Proposition 5.3. *Let $J_\varphi : \mathbb{D}^n \to GL(m,\mathbb{C})$, $\varphi \in G$, be holomorphic and $K : \mathbb{D}^n \times \mathbb{D}^n \to \mathscr{M}_m$ be a kernel. If K is quasi-invariant with respect to J, then we have*

$$\mathscr{K}(z) = \left(D\varphi(z)^t \otimes (J(\varphi,z)^*)^{-1}\right)\mathscr{K}(\varphi(z))\left(\overline{D\varphi(z)} \otimes J(\varphi,z)^*\right)$$

for $\varphi \in G$ and $z, w \in \mathbb{D}^n$.

Proof. Let $\varphi \in G$ and $K_\varphi : \mathbb{D}^n \times \mathbb{D}^n \to \mathscr{M}_m$ be the kernel $K_\varphi(z,w) = K(\varphi z, \varphi w)$. Since K is quasi-invariant with respect to J, we have

$$K(z,w) = J(\varphi,z)K(\varphi z,\varphi w)J(\varphi,w)^*$$

for all $\varphi \in G$ and $z, w \in \mathbb{D}^n$. Now,

$$\begin{aligned}
\mathscr{K}_\varphi^{ij}(z) &= \partial_i \left[K_\varphi(z,z)^{-1}\bar{\partial}_j K_\varphi(z,z)\right] \\
&= \partial_i \left[\left\{J(\varphi,z)^{-1}K(z,z)\left(J(\varphi,z)^*\right)^{-1}\right\}^{-1} \bar{\partial}_j \left\{J(\varphi,z)^{-1}K(z,z)\left(J(\varphi,z)^*\right)^{-1}\right\}\right] \\
&= \partial_i \Big[J(\varphi,z)^* K(z,z)^{-1} J(\phi,z)\Big\{J(\varphi,z)^{-1}\bar{\partial}_j K(z,z)\left(J(\varphi,z)^*\right)^{-1} \\
&\quad + J(\varphi,z)^{-1}K(z,z)\bar{\partial}_j \left(J(\varphi,z)^*\right)^{-1}\Big\}\Big] \\
&= \partial_i \left[J(\varphi,z)^* K(z,z)^{-1}\bar{\partial}_j K(z,z)\left(J(\varphi,z)^*\right)^{-1} + J(\varphi,z)^*\bar{\partial}_j \left(J(\varphi,z)^*\right)^{-1}\right] \\
&= J(\varphi,z)^*\partial_i \left[K(z,z)^{-1}\bar{\partial}_j K(z,z)\right]\left(J(\varphi,z)^*\right)^{-1} \\
&= J(\varphi,z)^*\mathscr{K}^{ij}(z)\left(J(\varphi,z)^*\right)^{-1}.
\end{aligned}$$

This gives us

$$\mathscr{K}_\varphi(z) = \left(I \otimes J(\varphi,z)^*\right)\mathscr{K}(z)\left(I \otimes (J(\varphi,z)^*)^{-1}\right). \tag{5.1}$$

Also using the chain rule, we obtain

$$\mathscr{K}_\varphi(z) = \left(D\varphi(z)^t \otimes I\right)\mathscr{K}(\varphi(z))\left(\overline{D\varphi(z)} \otimes I\right). \tag{5.2}$$

Combining (5.1) and (5.2), we have

$$\mathscr{K}(z) = \left(D\varphi(z)^t \otimes (J(\varphi,z)^*)^{-1}\right)\mathscr{K}(\varphi(z))\left(\overline{D\varphi(z)} \otimes J(\varphi,z)^*\right).$$

verifying the transformation rule for the curvature $\mathscr{K}$. □

Since G acts transitively on $\mathbb{D}^n$, there is a φ_z in G with $\varphi_z(z) = 0$. Substituting φ_z in the transformation rule for the curvature obtained in Proposition 5.3, we see that the curvature at any $z \in \mathbb{D}^n$, is determined from its value at 0.

Corollary 5.4. *With notations and assumptions as in Proposition 5.3, we have*

$$\mathcal{K}(z) = \left(D\varphi_z(z)^t \otimes (J(\varphi_z, z)^*)^{-1}\right)\mathcal{K}(0)\left(\overline{D\varphi_z(z)} \otimes J(\varphi_z, z)^*\right)$$

where φ_z *is in* G *with* $\varphi_z(z) = 0$.

5.1 Homogeneous tuples in $B_1(\mathbb{D}^n)$

First we prove a lemma for a kernel K which is quasi-invariant with respect to $J_g : \mathbb{D}^n \to \mathbb{C}$, which is assumed to be holomorphic and $g \to J_g$ is assumed to be Borel. We show that J must be a co-cycle if K is assumed to be quasi-invariant with respect to J.

Lemma 5.5. *Suppose* $\varphi \to J_\varphi$ *is Borel and for each* $\varphi \in G$, J_φ *is holomorphic. If*

$$K(z, w) = J(\varphi, z)K(\varphi z, \varphi w)\overline{J(\varphi, w)};\ \varphi \in G,\ z, w \in \mathbb{D}^n,$$

then J *is a* projective *cocycle.*

Proof. Let $\varphi, \psi \in G$. By the given condition, we have

$$K(z, w) = J(\varphi\psi, z)K(\varphi\psi z, \varphi\psi w)\overline{J(\varphi\psi, w)}.$$

Again using the given condition repeatedly, we have

$$\begin{aligned} K(z, w) &= J(\psi, z)K(\psi z, \psi w)\overline{J(\psi, w)} \\ &= J(\psi, z)J(\varphi, \psi z)K(\varphi\psi z, \varphi\psi w)\overline{J(\varphi, \psi w)J(\psi, w)}. \end{aligned}$$

Equating these two values of $K(z, w)$ and then cancelling $K(\varphi\psi z, \varphi\psi w)$ from the both side, we get

$$J(\varphi\psi, z)\overline{J(\varphi\psi, w)} = J(\psi, z)J(\varphi, \psi z)\overline{J(\varphi, \psi w)J(\psi, w)}$$

Since for each φ in G, $J(\varphi, \cdot)$ is holomorphic, it follows that

$$J(\varphi\psi, z) = m(\varphi, \psi)J(\varphi, \psi z)J(\psi, z) \tag{5.3}$$

where $m(\varphi, \psi)$ is a constant of modulus 1. Since J is Borel in first variable, so m is also a Borel map on $G \times G$. Now, we show that m is a multiplier.

Let $\varphi, \psi, \eta \in G$. Then equation (5.3) implies that

$$\begin{aligned} J(\varphi\psi\eta, z) &= m(\varphi, \psi\eta) J(\varphi, \psi\eta z) J(\psi\eta, z) \\ &= m(\varphi, \psi\eta) m(\psi, \eta) J(\varphi, \psi\eta z) J(\psi, \eta z) J(\eta, z). \end{aligned}$$

Again equation (5.3) implies that

$$\begin{aligned} J(\varphi\psi\eta, z) &= m(\varphi\psi, \eta) J(\varphi\psi, \eta z) J(\eta, z) \\ &= m(\varphi\psi, \eta) m(\varphi, \psi) J(\varphi, \psi\eta z) J(\psi, \eta z) J(\eta, z). \end{aligned}$$

Equating these two values of $J(\varphi\psi\eta, z)$, we obtain

$$m(\varphi, \psi\eta) m(\psi, \eta) = m(\varphi\psi, \eta) m(\varphi, \psi).$$

This shows that m is a multiplier and therefore J is a projective cocycle. □

To describe the homogeneous tuples in $B_1(\mathbb{D}^n)$, realized as a tuple of adjoint of multiplication operators on a reproducing kernel Hilbert space, we first prove a useful Lemma showing that the reproducing kernel in this case must be quasi-invariant. The proof of the Lemma is given for $G = \mathrm{Aut}(\mathbb{D}^n)$. The proof in the case $G = \text{Möb}^n$ then follows.

Lemma 5.6. *Let $K : \mathbb{D}^n \times \mathbb{D}^n \to \mathscr{M}_m$ be a reproducing kernel. Assume that the tuple of multiplication operators is in $B_m(\mathbb{D}^n)$ and homogeneous with respect to G. Then for each $\varphi \in G$, there exists a holomorphic map $J_\varphi : \mathbb{D}^n \to GL(m, \mathbb{C})$ such that K is quasi-invariant with respect to J.*

Proof. If $g_\sigma \in \mathrm{Aut}(\mathbb{D}^n)$, then $g_\sigma(z_1, z_2, \ldots, z_n) = (g_1(z_{\sigma_1}), g_2(z_{\sigma_2}), \ldots, g_n(z_{\sigma_n}))$, $g_i \in$ Möb and $\sigma \in S_n$. Since $(M_{z_1}, M_{z_2}, \ldots, M_{z_n})$, where M_{z_i} denotes the multiplication by the coordinate function z_i on the reproducing kernel Hilbert space H_K, is homogeneous, it follows that

$$g_\sigma(M_{z_1}, M_{z_2}, \ldots, M_{z_n}) = \left(M_{g_1(z_{\sigma_1})}, M_{g_2(z_{\sigma_2})} \ldots, M_{g_n(z_{\sigma_n})}\right)$$

is unitarily equivalent to $(M_{z_1}, M_{z_2}, \ldots, M_{z_n})$.

Let $K_{g_\sigma} : \mathbb{D}^n \times \mathbb{D}^n \to \mathscr{M}_m$ be the kernel $K_{g_\sigma}(z, w) = K(g_\sigma^{-1} z, g_\sigma^{-1} w)$. It is easy to check that $U : H_K \to H_{K_{g_\sigma}}$ defined by $UK(\cdot, w)\xi = K_{g_\sigma}(\cdot, g_\sigma w)\xi$, $w \in \mathbb{D}^n$ and $\xi \in \mathbb{C}^m$, is unitary. Let $\tilde{M}_{z_i}$ denote the multiplication by z_i on the Hilbert space $H_{K_{g_\sigma}}$. The following computation shows that $K(\cdot, w)\xi$ is an eigenvector for $(U^* \tilde{M}_{z_1} U, \ldots, U^* \tilde{M}_{z_n} U)$ with the joint eigenvalue $(\overline{g_1(w_{\sigma_1})}, \ldots, \overline{g_n(w_{\sigma_n})})$:

$$\begin{aligned} U^* \tilde{M}_{z_i} U K(\cdot, w)\xi &= U^* \tilde{M}_{z_i} K_{g_\sigma}(\cdot, g_\sigma w)\xi \\ &= \overline{g_i(w_{\sigma_i})} U^* K_{g_\sigma}(\cdot, g_\sigma w)\xi \\ &= \overline{g_i(w_{\sigma_i})} K(\cdot, w)\xi. \end{aligned}$$

Since the linear span of the vectors $K(\cdot,w)\xi$, $w\in\mathbb{D}^n$, $\xi\in\mathbb{C}^m$, are dense in H_K, we conclude that

$$U^*(\tilde{M}_1,\tilde{M}_2,\ldots,\tilde{M}_n)U = g_\sigma(M_{z_1},M_{z_2},\ldots,M_{z_n}).$$

This, together with the homogeneity assumption, proves that the tuples $(\tilde{M}_{z_1},\tilde{M}_2,\ldots,\tilde{M}_{z_n})$ and $(M_{z_1},M_{z_2},\ldots,M_{z_n})$ are unitarily equivalent. Since $(M_{z_1},M_{z_2},\ldots,M_{z_n})$ is in $B_m(\mathbb{D}^n)$, it follows, from [14, Theorem 3.7], that there exists a holomorphic map $J_{g_\sigma^{-1}}:\mathbb{D}^n\to GL(m,\mathbb{C})$ such that

$$K(z,w) = J_{g_\sigma^{-1}}(z)K_{g_\sigma}(z,w)J_{g_\sigma^{-1}}(w)^*.$$

This shows that there exists a holomorphic map $J_{g_\sigma}:\mathbb{D}^n\to GL(m,\mathbb{C})$, $g_\sigma\in\mathrm{Aut}(\mathbb{D}^n)$, such that K is quasi-invariant with respect to J. □

The following theorem describes all the tuples in $B_1(\mathbb{D}^n)$ which are homogeneous with respect to (a) the group Möb^n and (b) the full automorphism group $\mathrm{Aut}(\mathbb{D}^n)$.

Theorem 5.7. *Assume that the n - tuple of multiplication operators $(M_{z_1},M_{z_2},\ldots,M_{z_n})$, defined on a reproducing kernel Hilbert space H_K, is in the Cowen-Douglas class $B_1(\mathbb{D}^n)$. Then*
(a) the n - tuple $(M_{z_1},M_{z_2},\ldots,M_{z_n})$ is homogeneous with respect to Möb^n if and only if

$$K(z,w) = h(z)\Big(\prod_{i=1}^n \frac{1}{(1-z_i\bar{w}_i)^{\lambda_i}}\Big)\overline{h(w)},\ z,w\in\mathbb{D}^n,\ \lambda_i>0,$$

for some holomorphic function $h:\mathbb{D}^n\to\mathbb{C}$;
(b) the n - tuple $(M_{z_1},M_{z_2},\ldots,M_{z_n})$ is homogeneous with respect to $\mathrm{Aut}(\mathbb{D}^n)$ *if and only if*

$$K(z,w) = h(z)\Big(\prod_{i=1}^n \frac{1}{(1-z_i\bar{w}_i)^{\lambda}}\Big)\overline{h(w)},\ z,w\in\mathbb{D}^n,\ \lambda>0,$$

for some holomorphic function $h:\mathbb{D}^n\to\mathbb{C}$.

Proof. (a) It is well-known that the n-tuple $(M_{z_1}^*,\ldots,M_{z_n}^*)$ on the Hilbert space H_K, $K(z,w)=\prod_{i=1}^n \frac{1}{(1-z_i\bar{w}_i)^{\lambda_i}}$ is in $B_1(\mathbb{D}^n)$. It is also easy to verify that these are homogeneous with respect to Möb^n. This is the proof in one direction.

For the proof in the other direction, note that the existence of a holomorphic map J_φ, $\varphi\in$ Möb^n, such that

$$K(z,w) = J(\varphi,z)K(\varphi z,\varphi w)\overline{J(\varphi,w)},\ z,w\in\mathbb{D}^n,\ \varphi\in\text{Möb}^n$$

follows from Lemma 5.6.

Since J is scalar valued, it follows from Proposition 5.3 that

$$\mathscr{K}(z) = D\varphi(z)^t\mathscr{K}(\varphi(z))\overline{D\varphi(z)} \tag{5.4}$$

where $\mathcal{K}$ is the curvature of K.

Now, let $k \in$ Möb^n be such that $k(z_1, z_2, \ldots, z_n) = (k_1 z_1, k_2 z_2, \ldots, k_n z_n)$ for $(z_1, z_2, \ldots, z_n)$ in $\mathbb{D}^n$ where each k_i is a constant of modulus 1. Then $Dk(0) = \text{diag}(k_1, k_2, ..., k_n)$. Let a_{ij} be the (i, j)-th entry of $\mathcal{K}(0)$. Evaluating the equation (5.4) for $\varphi = k$ and $z = 0$, we see that $a_{ij} = 0$ if $i \neq j$. This shows that $\mathcal{K}(0) = \text{diag}(a_{11}, a_{22}, \ldots, a_{nn})$. Now Corrolary 5.4 gives

$$\mathcal{K}(z) = D\varphi_z(z)^t \mathcal{K}(0)\overline{D\varphi_z(z)} = \text{diag}\left(\frac{a_{11}}{(1-|z_1|^2)^2}, \frac{a_{22}}{(1-|z_2|^2)^2}, \ldots, \frac{a_{nn}}{(1-|z_n|^2)^2}\right).$$

Let $\lambda_i = a_{ii}$, $1 \leq i \leq n$. Recalling that $\mathcal{K}_1 = \mathcal{K}_2$ if and only if $K_2 = hK_1\bar{h}$ for some holomorphic function h, we conclude that $K(z, w) = h(z)\left(\prod_{i=1}^n \frac{1}{(1-z_i\bar{w}_i)^{\lambda_i}}\right)\overline{h(w)}$, h is holomorphic on $\mathbb{D}^n$. Since K is a positive definite kernel, it follows that $\lambda_i > 0$, $1 \leq i \leq n$.

(b) The proof in the forward direction follows from the proof in the same direction of part (a).

For the other direction, note that the existence of a holomorphic map J_φ, $\varphi \in \text{Aut}(\mathbb{D}^n)$, such that

$$K(z, w) = J(\varphi, z)K(\varphi z, \varphi w)\overline{J(\varphi, w)},\ z, w \in \mathbb{D}^n,\ \varphi \in \text{Aut}(\mathbb{D}^n)$$

follows from Lemma 5.6. On the other hand, Proposition 5.3 gives

$$\mathcal{K}(z) = D\varphi(z)^t \mathcal{K}(\varphi(z))\overline{D\varphi(z)},\ z \in \mathbb{D}^n\ \varphi \in \text{Aut}(\mathbb{D}^n). \tag{5.5}$$

Since Möb^n is a subgroup of $\text{Aut}(\mathbb{D}^n)$, it follows that $(M_{z_1}, M_{z_2}, \ldots, M_{z_n})$ is homogeneous with respect to the group Möb^n. Therefore, $\mathcal{K}(0) = \text{diag}(a_{11}, a_{22}, \ldots, a_{nn})$ where $a_{ii} > 0$, $1 \leq i \leq n$.

Let $\sigma_k \in \text{Aut}(\mathbb{D}^n)$ be such that $\sigma_k(z_1, z_2, \ldots, z_n) = (k_2 z_2, k_1 z_1, \ldots, k_n z_n)$ for $(z_1, z_2, \ldots, z_n)$ in $\mathbb{D}^n$, where each k_i is a constant of modulus 1. Then

$$D\sigma_k(0) = \begin{bmatrix} 0 & k_2 & 0 & \ldots & 0 \\ k_1 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & k_n \end{bmatrix}.$$

Evaluating the equation (5.5) for $\varphi = \sigma_k$ and $z = 0$ and equating $(1, 2)$ entry, we get $a_{11} = a_{22}$. Similarly, $a_{ii} = a_{11}$ for all i. Putting $\lambda = a_{11}$, we have $\mathcal{K}(0) = \text{diag}(\lambda, \lambda, \ldots, \lambda)$. Now, Corollary 5.4 gives

$$\mathcal{K}(z) = D\varphi_z(z)^t \mathcal{K}(0)\overline{D\varphi_z(z)} = \text{diag}\left(\frac{\lambda}{(1-|z_1|^2)^2}, \frac{\lambda}{(1-|z_2|^2)^2}, \ldots, \frac{\lambda}{(1-|z_n|^2)^2}\right),\ z \in \mathbb{D}^n,$$

which implies that $K(z, w) = h(z)\left(\prod_{i=1}^n \frac{1}{(1-z_i\bar{w}_i)^{\lambda}}\right)\overline{h(w)}$ for some holomorphic function h on $\mathbb{D}^n$. □

This completes the description of homogeneous operators in Cowen-Douglas class $B_1(\mathbb{D}^n)$.

5.2 Irreducible Homogeneous tuples in $B_2(\mathbb{D}^n)$

In this section, we describe all irreducible homogeneous tuples in $B_2(\mathbb{D}^n)$ with respect to the group G, which is taken to be either Möb^n or Aut$(\mathbb{D}^n)$ as before. First, we describe all irreducible tuples in $B_2(\mathbb{D}^n)$ which are homogeneous with respect to Möb^n, then we show that there are no irreducible tuples in $B_2(\mathbb{D}^n)$, which are homogeneous with respect to Aut$(\mathbb{D}^n)$.

Definition 5.8. Let $(T_1, T_2, \ldots, T_n)$ be a commuting tuple of bounded operators. If there does not exist any non-trivial projection which commutes with each T_i, then $(T_1, T_2, \ldots, T_n)$ is said to be irreducible.

Let $H^{(\alpha)}$ denote the reproducing kernel Hilbert space, consisting of holomorphic functions on the unit disc $\mathbb{D}$, determined by the kernel $K^{(\alpha)}(z,w) = \frac{1}{(1-z\bar{w})^\alpha}$ defined on $\mathbb{D}$. Also, let $M^{(\alpha)}$ denote the operator of multiplication by the coordinate function z on $H^{(\alpha)}$. Finally, let $H^{(\lambda,\mu)}$ be the reproducing kernel Hilbert space determined by the kernel

$$K^{(\lambda,\mu)}(z,w) = \begin{bmatrix} \frac{1}{(1-z\bar{w})^\lambda} & \frac{z}{(1-z\bar{w})^{\lambda+1}} \\ \frac{\bar{w}}{(1-z\bar{w})^{\lambda+1}} & \frac{\frac{1}{\lambda}+\mu+z\bar{w}}{(1-z\bar{w})^{\lambda+1}} \end{bmatrix}$$

defined on $\mathbb{D}$. The operator $M^{(\lambda,\mu)}$ is the multiplication by the coordinate function z on $H^{(\lambda,\mu)}$. An homogeneous operator in $B_1(\mathbb{D})$ must be unitarily equivalent to $M^{(\alpha)*}$ for some $\alpha > 0$ (see [26]) and every irreducible homogeneous operators in $B_2(\mathbb{D})$ must be unitarily equivalent to $M^{(\lambda,\mu)*}$ for some $\lambda, \mu > 0$ (see [22, 35]).

We prove that the tuple $(M_{z_1}, M_{z_2}, \ldots, M_{z_n})$ of multiplication by the coordinate functions acting on the Hilbert space $H^{(\alpha_1)} \otimes \cdots \otimes H^{(\alpha_{n-1})} \otimes H^{(\lambda,\mu)} \subseteq \mathrm{Hol}(\mathbb{D}^n, \mathbb{C}^2)$ is irreducible. First, we prove a useful lemma.

Lemma 5.9. *Let H_1 and H_2 be two Hilbert spaces and T_i be an irreducible operator on H_i for $i = 1, 2$. Suppose P is a projection defined on $H_1 \otimes H_2$.*
(a) If P commutes with $I \otimes T_2$, then there exists a projection P_1, defined on H_1, such that $P = P_1 \otimes I$.
(b) If P commutes with $T_1 \otimes I$, then there exists a projection P_2, defined on H_2, such that $P = I \otimes P_2$.

Proof. (a) Assume that $\dim H_1 = N$, where N can be ∞. Let $\{e_i : 1 \le i \le N\}$ be an orthonormal basis of H_1. Define $U : H_1 \otimes H_2 \to \bigoplus_{i=1}^N H_2$ by $U(e_i \otimes y) = (0, 0, \ldots, y, 0, \ldots, 0)$, $y \in H_2$, where y is in the i-th position. Then U is a unitary operator and $U(I \otimes T_2)U^* = \bigoplus_{i=1}^N T_2$.

Let $\tilde{P} = UPU^*$. Suppose $((\tilde{P}_{ij}))$ is the matrix representation of $\tilde{P}$ as an operator on the Hilbert space $\oplus_{i=1}^N H_2$, where $\tilde{P}_{ij}$ is an operator on H_2. Since $\tilde{P}$ is a projection, it follows that $\tilde{P}_{ij}^* = \tilde{P}_{ji}$ for all i, j.

Since P and $I \otimes T_2$ commutes, the operators $\tilde{P}$ and $\oplus_{i=1}^N T_2$ also commute. This implies that $\tilde{P}_{ij}$ commutes with T_2 for each i, j. Since T_2 is irreducible and $\tilde{P}_{ij}^* = \tilde{P}_{ji}$, $\tilde{P}_{ij}$ both commute with T_2, it follows that $\tilde{P}_{ij} = \alpha_{ij} I$, for some $\alpha_{ij} \in \mathbb{C}$.

Thus, we have $\tilde{P} = \big((\alpha_{ij} I)\big)$. Let P_1 be the operator on H_1, whose matrix representation with respect to the orthonormal basis $\{e_i : 1 \le i \le N\}$ is $\big((\alpha_{ij})\big)$. Since $\tilde{P} = \big((\alpha_{ij} I)\big)$ is a projection, it follows that P_1 is also a projection and $P = P_1 \otimes I$.

(b) Let $V : H_1 \otimes H_2 \to H_2 \otimes H_1$ be the unitary operator, defined by $V(h_1 \otimes h_2) = h_2 \otimes h_1$, $h_1 \in H_1$, $h_2 \in H_2$. Conjugating P, $T_1 \otimes I$ by V and applying (a), the proof of (b) follows. □

Theorem 5.10. *The tuple $(M_{z_1}, M_{z_2}, \ldots, M_{z_n})$ of multiplication by the coordinate functions, acting on the Hilbert space $H^{(\alpha_1)} \otimes \cdots \otimes H^{(\alpha_{n-1})} \otimes H^{(\lambda,\mu)} \subseteq Hol(\mathbb{D}^n, \mathbb{C}^2)$ is irreducible.*

Proof. Evidently, the tuple $(M_{z_1}, M_{z_2}, \ldots, M_{z_n})$ is simultaneously unitarily equivalent to the tuple

$$(M^{(\alpha_1)} \otimes \cdots \otimes I \otimes I, \ldots, I \otimes \cdots \otimes M^{(\alpha_{n-1})} \otimes I, I \otimes \cdots \otimes I \otimes M^{(\lambda,\mu)})$$

acting on $H^{(\alpha_1)} \otimes \cdots \otimes H^{(\alpha_{n-1})} \otimes H^{(\lambda,\mu)}$.

Let P be a projection which commutes with $M^{(\alpha_1)} \otimes \cdots \otimes I \otimes I$. Then there exists a projection P_2, defined on $H^{(\alpha_2)} \otimes \cdots \otimes H^{(\alpha_{n-1})} \otimes H^{(\lambda,\mu)}$, such that $P = I \otimes P_2$, by virtue of Lemma 5.9. Now, P commutes with $I \otimes M^{(\alpha_2)} \otimes \cdots \otimes I$. This implies that P_2 commutes with $M^{(\alpha_2)} \otimes I \otimes \cdots \otimes I$. Again applying Lemma 5.9, we obtain a projection P_3 such that $P_2 = I \otimes P_3$.

Continuing in this manner, we see that $P = I \otimes I \otimes \cdots \otimes P_n$, where P_n is a projection defined on $H^{(\lambda,\mu)}$ and it commutes with $M^{(\lambda,\mu)}$. Since $M^{(\lambda,\mu)}$ is irreducible, it follows that P_n must be either 0 or I. This proves that the given tuple is irreducible. □

Recall that D_α^+ is the holomorphic Discrete series representation of Möb on $H^{(\alpha)}$ and $D_{\lambda,\mu}$ is the multiplier representation of Möb on $H^{(\lambda,\mu)}$ given by the cocycle

$$J(\varphi, z) = \begin{bmatrix} (\varphi'(z))^{\frac{\lambda}{2}} & 0 \\ \frac{\varphi''(0)}{2(\varphi'(0))^{\frac{3}{2}}} (\varphi'(z))^{\frac{\lambda+1}{2}} & (\varphi'(z))^{\frac{\lambda}{2}+1} \end{bmatrix}$$

It is easy to see that the tuple of multiplication by the coordinate functions $(M_{z_1}, M_{z_2}, \ldots, M_{z_n})$ acting on the Hilbert space $H^{(\alpha_1)} \otimes \cdots \otimes H^{(\alpha_{n-1})} \otimes H^{(\lambda,\mu)}$ is homogeneous under the action of Möb^n with associated representation $D_{\alpha_1}^+ \otimes D_{\alpha_2}^+ \otimes \cdots \otimes D_{\lambda,\mu}$ of Möb^n.

Lemma 5.11. *Let $J_\varphi : \mathbb{D}^n \to GL(2, \mathbb{C})$, $\varphi \in$ Möb^n, be holomorphic and $K : \mathbb{D}^n \times \mathbb{D}^n \to \mathcal{M}_2$ be a kernel. If K is quasi-invariant with respect to J, then $\mathcal{K}^{ij}(0) = 0$ whenever $i \neq j$.*

Proof. Since K is quasi-invariant with respect to J, it follows from Lemma 5.3 that

$$\mathcal{K}(z) = \left(D\varphi(z)^t \otimes (J(\varphi, z)^*)^{-1}\right) \mathcal{K}(\varphi(z)) \left(\overline{D\varphi(z)} \otimes J(\varphi, z)^*\right)$$

for all φ in Möb^n and z in $\mathbb{D}^n$. Let $k \in$ Möb^n be such that $k(z_1, z_2, \dots, z_n) = (k_1 z_1, k_2 z_2, \dots, k_n z_n)$ for $(z_1, z_2, \dots, z_n)$ in $\mathbb{D}^n$ where modulus of each k_i is 1. Then $Dk(0) = \mathrm{diag}(k_1, k_2, \dots, k_n)$. Now replacing φ by k and z by 0 in the equation appearing above, we get

$$\mathcal{K}(0) = \left(Dk(0)^t \otimes (J(k,0)^*)^{-1}\right)\mathcal{K}(0)\left(\overline{Dk(0)} \otimes J(k,0)^*\right),$$

which is equivalent to the equation:

$$\left(\overline{Dk(0)} \otimes J(k,0)^*\right)\mathcal{K}(0) = \mathcal{K}(0)\left(\overline{Dk(0)} \otimes J(k,0)^*\right).$$

Now, equating the (i, j)-th block from both sides, we get

$$\bar{k}_i J(k,0)^* \mathcal{K}^{ij}(0) = \bar{k}_j \mathcal{K}^{ij}(0) J(k,0)^*.$$

Thus if $i \neq j$, $\mathcal{K}^{ij}(0)$ is similar to $k_i \bar{k}_j \mathcal{K}^{ij}(0)$ for all k_i, k_j in the unit circle. This means that the set of eigenvalues of $\mathcal{K}^{ij}(0)$ must be invariant under the circle action. This is not possible unless $\mathcal{K}^{ij}(0) = 0$. □

Lemma 5.12. *Let $J_\varphi : \mathbb{D}^n \to GL(2, \mathbb{C})$, $\varphi \in \mathrm{Aut}(\mathbb{D}^n)$, be holomorphic and $K : \mathbb{D}^n \times \mathbb{D}^n \to \mathcal{M}_2$ be a kernel. If K is quasi-invariant with respect to J, then $\mathcal{K}^{ij}(0) = 0$ if $i \neq j$ and $\mathcal{K}^{ii}(0)$, $\mathcal{K}^{jj}(0)$ are similar for all i, j, $1 \le i, j \le n$.*

Proof. Since K is quasi-invariant with respect to J, therefore K is also quasi-invariant with respect to $J|_{\text{Möb}^n \times \mathbb{D}^n}$. It then follows, from Lemma 5.11, that $\mathcal{K}^{ij}(0) = 0$ if $i \neq j$.

Let $\sigma_k \in Aut(\mathbb{D}^n)$ be the automorphism such that $\sigma_k(z_1, z_2, \dots, z_n) = (k_2 z_2, k_1 z_1, \dots, k_n z_n)$ for $(z_1, z_2, \dots, z_n)$ in $\mathbb{D}^n$, where each k_i is in the unit circle. Then

$$D\sigma_k(0) = \begin{bmatrix} 0 & k_2 & 0 & \dots & 0 \\ k_1 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & k_n \end{bmatrix}.$$

Replacing φ by σ_k and z by 0 in Lemma 5.3, we obtain

$$\overline{D\sigma_k(0)} \otimes J(\sigma_k, 0)^* \mathcal{K}(0) = \mathcal{K}(0) \overline{D\sigma_k(0)} \otimes J(\sigma_k, 0)^*.$$

Equating the $(1,2)$ block from both sides of the equation, we get

$$J(\sigma_k, 0)^* \mathcal{K}^{22}(0) = \mathcal{K}^{11}(0) J(\sigma_k, 0)^*.$$

Since $J(\sigma_k, 0)^*$ is invertible, it follows that $\mathcal{K}^{11}(0)$ and $\mathcal{K}^{22}(0)$ are similar. Similar reasoning shows that $\mathcal{K}^{ii}(0)$ and $\mathcal{K}^{i+1\, i+1}(0)$ are similar for all i. □

Theorem 5.13. *Let λ, μ, α_i be positive real numbers where $i = 1, 2, \ldots, n-1$. The tuple of multiplication by the coordinate functions $(M_{z_1}, M_{z_2}, \ldots, M_{z_n})$, acting on the Hilbert space $H^{(\alpha_1)} \otimes \cdots \otimes H^{(\alpha_{n-1})} \otimes H^{(\lambda,\mu)}$, is not homogeneous under the action of* $\mathrm{Aut}(\mathbb{D}^n)$.

Proof. The reproducing kernel of the Hilbert space $H^{(\alpha_1)} \otimes \cdots \otimes H^{(\alpha_{n-1})} \otimes H^{(\lambda,\mu)}$ is

$$K^{(\alpha,\lambda,\mu)}(z,w) = \left(\prod_{i=1}^{n-1} \frac{1}{(1-z_i\bar{w}_i)^{\alpha_i}}\right)\begin{bmatrix} \frac{1}{(1-z_n\bar{w}_n)^{\lambda}} & \frac{z_n}{(1-z_n\bar{w}_n)^{\lambda+1}} \\ \frac{\bar{w}_n}{(1-z_n\bar{w}_n)^{\lambda+1}} & \frac{\frac{1}{\lambda}+\mu+z_n\bar{w}_n}{(1-z_n\bar{w}_n)^{\lambda+1}} \end{bmatrix},$$

where α represents the vector $(\alpha_1, \alpha_2, \ldots, \alpha_{n-1})$. Since the tuple is homogeneous, it follows, from Lemma 5.6, that for each $\varphi \in \mathrm{Aut}(\mathbb{D}^n)$ there exists a holomorphic map $J_\varphi : \mathbb{D}^n \to GL(2,\mathbb{C})$ such that $K^{(\alpha,\lambda,\mu)}$ is quasi-invariant with respect to J. Then Lemma 5.12 implies that $\mathscr{K}^{11}(0)$ and $\mathscr{K}^{nn}(0)$ are similar. But it is easy to see that

$$\mathscr{K}^{11}(0) = \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_1 \end{bmatrix} \text{ and } \mathscr{K}^{nn}(0) = \begin{bmatrix} \lambda - \left(\frac{1}{\lambda} - \mu^2\right)^{-1} & 0 \\ 0 & \lambda + 2 + \left(\frac{1}{\lambda} - \mu^2\right)^{-1} \end{bmatrix}.$$

This implies that $\mathscr{K}^{11}(0)$ and $\mathscr{K}^{nn}(0)$ can not be similar. This proves that the given tuple is not homogeneous under the action of $\mathrm{Aut}(\mathbb{D}^n)$. □

Now, we obtain all irreducible homogeneous tuples in $B_2(\mathbb{D}^n)$ under the action of Möb^n. If an irreducible tuple in $B_2(\mathbb{D}^n)$ is homogeneous under the action of Möb^n, then it is associated with an irreducible rank two Hermitian holomorphic vector bundle which admits an action of the universal covering group $\tilde{G}^n$ of Möb^n, where $\tilde{G}$ is the universal covering group of Möb. Indeed, even if we drop the assumption of irreducibility, the proof from [23, Theorem 2.1] goes through.

Also, recall that if $(T_1, T_2, \ldots, T_n)$ is in $B_2(\mathbb{D}^n)$, then it is unitarily equivalent to the tuple $(M_{z_1}^*, M_{z_2}^*, \ldots, M_{z_n}^*)$ acting on a Hilbert space consisting of holomorphic functions taking values in $\mathbb{C}^2$, defined on $\mathbb{D}^n$, possessing a reproducing kernel K. Next, if the tuple $(M_{z_1}^*, M_{z_2}^* \ldots, M_{z_n}^*)$ is homogeneous with respect to Möb^n, then K must be quasi-invariant relative to some family of holomophic functions J_φ, $\varphi \in$ Möb^n. Finally, if we assume that the tuple $(M_{z_1}^*, M_{z_2}^* \ldots, M_{z_n}^*)$ is irreducible, then the map $J : \text{Möb}^n \times \mathbb{D}^n \to GL(2,\mathbb{C})$, $J(\varphi, z) := J_\varphi(z)$, is Borel and satisfies the projective cocycle property:

> $J(\varphi\psi, z) = m(\varphi,\psi) J(\psi, z) J(\varphi, \psi(z))$, where $m : \text{Möb}^n \times \text{Möb}^n \to \mathbb{T}$ is a multiplier, that is, m is a Borel map satisfying the multiplier identities
>
> 1. $m(e,\varphi) = m(\varphi, e) = 1$, φ in Möb^n and e is the identity element of Möb^n.
> 2. $m(\varphi_1,\varphi_2) m(\varphi_1\varphi_2, \varphi_3) = m(\varphi_1, \varphi_2\varphi_3) m(\varphi_2,\varphi_3)$ holds for all $\varphi_1, \varphi_2, \varphi_3$ in Möb^n.

Therefore to find all the irreducible homogeneous n-tuples in $B_2(\mathbb{D}^n)$, it is enough to find all the cocycles $\tilde{J} : \tilde{G}^n \times \mathbb{D}^n \to GL(2,\mathbb{C})$ and positive diagonal matrix $K(0,0)$ such that it commutes with $\tilde{J}(\tilde{k},0)$ for each $\tilde{k}$ fixing 0 and so that the polarization of

$$K(z,z) := \tilde{J}(\tilde{g}_z,z)K(0,0)\tilde{J}(\tilde{g}_z,z)^*, \tag{5.6}$$

where $\tilde{g}_z$ maps z to 0, is positive definite. Once we find these, we show that K is quasi-invariant with respect to $\tilde{J}$ and the adjoint of the tuple of multiplication operators on H_K is in $B_2(\mathbb{D}^n)$.

Now, we shall compute all the two dimensional cocycles on $\tilde{G}^n \times \mathbb{D}^n$. Recall that $SU(1,1)$ is the 2-fold covering group of Möb and let $\tilde{G}$ be the universal covering group of $SU(1,1)$. The unit disc $\mathbb{D}$ admits an action of the group $SU(1,1)$ by the rule,

$$g(z) = \tfrac{az+b}{\bar{b}z+\bar{a}},\ g = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \in SU(1,1), z \in \mathbb{D}.$$

Therefore the direct product of n copies of the group $SU(1,1)$, denoted by $SU(1,1)^n$, acts on $\mathbb{D}^n$. Consequently, composing with the covering map, an action of the universal covering group of Möb^n on $\mathbb{D}^n$ is evident.

In the discussion below, we follow the notation of [23]. The Lie algebra $\mathfrak{g}$ of $SU(1,1)$ is spanned by

$$X_1 = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, X_0 = \frac{1}{2}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \text{ and } Y = \frac{1}{2}\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}.$$

Let $\mathfrak{g}^{\mathbb{C}}$ be the complexification of $\mathfrak{g}$. Then $\mathfrak{g}^{\mathbb{C}}$ is the Lie algebra of the complexification of the group $SU(1,1)$, which is $SL(2,\mathbb{C})$. The Lie algebra $\mathfrak{g}^{\mathbb{C}}$ is spanned by

$$h = -iX_0 = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, x = X_1 + iY = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } y = X_1 - iY = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

The subgroups $\mathbb{K}^{\mathbb{C}} = \left\{ \begin{bmatrix} z & 0 \\ 0 & \frac{1}{z} \end{bmatrix} : z \in \mathbb{C} \setminus \{0\} \right\}$, $P^+ = \left\{ \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix} : z \in \mathbb{C} \right\}$ and $P^- = \left\{ \begin{bmatrix} 1 & 0 \\ z & 1 \end{bmatrix} : z \in \mathbb{C} \right\}$ of $SL(2,\mathbb{C})$ have corresponding Lie algebras $\mathfrak{t}^{\mathbb{C}} = \left\{ \begin{bmatrix} c & 0 \\ 0 & -c \end{bmatrix} : c \in \mathbb{C} \right\}$, $\mathfrak{p}^+ = \left\{ \begin{bmatrix} 0 & c \\ 0 & 0 \end{bmatrix} : c \in \mathbb{C} \right\}$ and $\mathfrak{p}^- = \left\{ \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix} : c \in \mathbb{C} \right\}$, respectively, where $\mathbb{K}$ is the subgroup of $SU(1,1)$ which stabilzes 0. Let $\mathfrak{b}$ denote the Lie algebra spanned by $\{h, y\}$. Then $\mathfrak{b}$ is the Lie algebra of the group $\mathbb{K}^{\mathbb{C}}P^-$, which is a closed subgroup of $SL(2,\mathbb{C})$. Now, every rank two cocycles on $\tilde{G}^n \times \mathbb{D}^n$ is obtained from a two dimensional indecomposable representation of $\mathfrak{b} \oplus \mathfrak{b} \oplus \cdots \oplus \mathfrak{b}$ (see [23]).

Let $\tilde{\mathbb{K}}$ be the subgroup of $\tilde{G}$ which stabilize 0. Then $K(0,0)$ is invariant under the action of $\tilde{\mathbb{K}}^n$. Thus, we have to find all the cocycles of rank 2, which are obtained from two dimensional indecomposable representations ρ of $\mathfrak{b} \oplus \mathfrak{b} \oplus \cdots \oplus \mathfrak{b}$ such that ρ is diagonalizable on the sub-algebra spanned by the set $\{(h,0,\dots,0),(0,h,\dots,0),\dots,(0,0,\dots,h)\}$. Let $\mathfrak{b}_i$ denote the sub-algebra $0\oplus\cdots\oplus\mathfrak{b}\oplus\cdots\oplus 0$, where $\mathfrak{b}$ is in the i-th position. Similarly, set $h_i := (0,\dots,h,\dots,0)$ and $y_i := (0,\dots,y,\dots,0)$, where h and y are in the i-th position in the tuple.

Theorem 5.14. *Suppose $\rho : \mathfrak{b}\oplus\mathfrak{b}\oplus\cdots\oplus\mathfrak{b}\to\mathscr{M}_2$ is a two dimensional indecomposable representation such that $\rho(h_i)$ is diagonalizable for all i. Then there exists k such that $\rho_{|\mathfrak{b}_k}$ is indecomposable. Furthermore, $\rho(h_j)=\alpha_j I_2$ and $\rho(y_j)=0$ for all $j\neq k$ where $\alpha_j\in\mathbb{C}$.*

Proof. Since h_i and h_j commute, it follows that $\rho(h_i)$ and $\rho(h_j)$ also commute for all i,j. Therefore $\rho(h_1)$, $\rho(h_2),\dots,\rho(h_n)$ are simultaneously diagonalizable. Let $\{v_1,v_2\}$ be a basis of $\mathbb{C}^2$ such that

$$\rho(h_i)v_j=\lambda_i^j v_j$$

for all $i=1,2,\dots,n$ and $j=1,2$. The relation

$$\left[\rho(h_i),\rho(y_i)\right]=-\rho(y_i)$$

implies that

$$\rho(h_i)\rho(y_i)v_j=(\lambda_i^j-1)\rho(y_i)v_j. \tag{5.7}$$

Suppose $\lambda_i^1\neq\lambda_i^2\pm 1$ for all i. Then equation (5.7) implies that $\rho(y_i)=0$ for all i. But this is a contradiction, since ρ is indecomposable. Thus, there must exists k such that either $\lambda_k^1=\lambda_k^2-1$ or $\lambda_k^2=\lambda_k^1-1$.

Without loss of generality, we can assume that $\lambda_k^2=\lambda_k^1-1$. Then (5.7) implies that $\rho(y_k)v_1=\alpha v_2$ for some $\alpha\in\mathbb{C}$ and $\rho(y_k)v_2=0$.

Now, we claim that $\alpha\neq 0$.

Suppose $\alpha=0$. Then $\rho(y_k)$ must be 0. Since ρ is indecomposable, there must exists some i with $1\le i\le n$, such that $\rho(y_i)\neq 0$. Again, equation (5.7) implies that either $\lambda_i^1=\lambda_i^2-1$ or $\lambda_i^2=\lambda_i^1-1$. If we assume that $\lambda_i^2=\lambda_i^1-1$, then Equation (5.7) implies that $\rho(y_i)v_1=\alpha_i v_2$ for some $\alpha_i\neq 0$ and $\rho(y_i)v_2=0$. Since $\rho(y_i)$ and $\rho(h_k)$ commute, in this case, it follows that $\lambda_k^1=\lambda_k^2$, which contradicts the assumption that $\lambda_k^2=\lambda_k^1-1$. If we assume $\lambda_i^1=\lambda_i^2-1$, then we arrive at a similar contradiction.

Now, since $\rho(h_i)$ and $\rho(y_k)$ commutes for all $i\neq k$, we must have $\lambda_i^1=\lambda_i^2$ and then Equation (5.7) gives us $\rho(y_i)=0$ for all $i\neq k$.

This shows that $\rho_{|\mathfrak{b}_k}$ is indecomposable, $\rho(h_i)=\lambda_i^1 I_2$ and $\rho(y_i)=0$ for all $i\neq k$. □

Now we describe all the cocycles.

Theorem 5.15. *Let $J : SU(1,1)^n \times \mathbb{D}^n \to GL(2,\mathbb{C})$ be a projective cocycle such that $J(k,0)$ is diagonal for all $k \in \mathbb{K}^n$. Then there exists α_i, $i = 1,2,\ldots,n-1$ and $\lambda \neq 0$ such that*

$$J(g,z) = \prod_{i=1}^{n-1} g_i^{'}(z_i)^{\alpha_i} \begin{bmatrix} \left(g_n^{'}(z_n)\right)^{\lambda} & 0 \\ \frac{g_n^{''}(0)}{2\left(g_n^{'}(0)\right)^{\frac{3}{2}}} \left(g_n^{'}(z_n)\right)^{\lambda+\frac{1}{2}} & \left(g_n^{'}(z_n)\right)^{\lambda+1} \end{bmatrix},$$

where $g = (g_1, g_2, \ldots, g_n) \in SU(1,1)^n$ and $z = (z_1, z_2, \ldots, z_n) \in \mathbb{D}^n$.

Proof. If $J : SU(1,1)^n \times \mathbb{D}^n \to GL(2,\mathbb{C})$ be a projective cocycle, then there exists a two dimensional indecomposable representation ρ of $\mathfrak{b} \oplus \mathfrak{b} \oplus \cdots \oplus \mathfrak{b}$ such that

$$J(g,z) = \rho(s(z)^{-1} g^{-1} s(g \cdot z)),$$

where $s : \mathbb{D}^n \to SL(2,\mathbb{C})^n$ is a holomorphic section. If $J(k,0)$ is also diagonal for all $k \in \mathbb{K}^n$, then ρ is diagonalizable on the sub-algebra spanned by $\{h_1, h_2, \ldots, h_n\}$. Let $s : \mathbb{D}^n \to SL(2,\mathbb{C})^n$ be a holomorphic section, defined by

$$s(z) = \left(\begin{pmatrix} 1 & z_1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & z_2 \\ 0 & 1 \end{pmatrix}, \ldots, \begin{pmatrix} 1 & z_n \\ 0 & 1 \end{pmatrix} \right).$$

Suppose ρ is a two dimensional indecomposable representation of $\mathfrak{b} \oplus \mathfrak{b} \oplus \cdots \oplus \mathfrak{b}$. Applying Theorem 5.14, we assume that there exists α_i, $i = 1,2,\ldots,n-1$ and $\lambda \neq 0$ such that

$$\rho(h_i) = \alpha_i I_2,\ \rho(y_i) = 0,$$

for $i = 1,2,\ldots,n-1$ and

$$\rho(h_n) = \begin{bmatrix} -\lambda & 0 \\ 0 & -\lambda-1 \end{bmatrix},\ \rho(y_n) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Let $g = \left(\begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \right)_{i=1}^{n} \in SU(1,1)^n$. Then

$$\begin{aligned} J(g,z) &= \rho(s(z)^{-1} g^{-1} s(g \cdot z)) \\ &= \rho\Big(\exp\left(\left(\frac{-c_1}{c_1 z_1 + d_1} \right) y_1 \right) \exp\left(\left(2\log(c_1 z_1 + d_1) \right) h_1 \right) \exp\left(\left(\frac{-c_2}{c_2 z_2 + d_2} \right) y_2 \right) \\ &\quad \exp\left(\left(2\log(c_2 z_2 + d_2) \right) h_2 \right) \cdots \exp\left(\left(\frac{-c_n}{c_n z_n + d_n} \right) y_n \right) \exp\left(\left(2\log(c_n z_n + d_n) \right) h_n \right) \Big) \\ &= \exp\left(\left(\frac{-c_1}{c_1 z_1 + d_1} \right) \rho(y_1) \right) \exp\left(\left(2\log(c_1 z_1 + d_1) \right) \rho(h_1) \right) \exp\left(\left(\frac{-c_2}{c_2 z_2 + d_2} \right) \rho(y_2) \right) \\ &\quad \exp\left(\left(2\log(c_2 z_2 + d_2) \right) \rho(h_2) \right) \cdots \exp\left(\left(\frac{-c_n}{c_n z_n + d_n} \right) \rho(y_n) \right) \exp\left(\left(2\log(c_n z_n + d_n) \right) \rho(h_n) \right). \end{aligned}$$

Now, substituting the values of $\rho(h_i)$ and $\rho(y_i)$, we get that

$$J(g,z) = \prod_{i=1}^{n-1} g_i^{'}(z_i)^{\alpha_i} \begin{bmatrix} \left(g_n^{'}(z_n)\right)^{\lambda} & 0 \\ \frac{g_n^{''}(0)}{2\left(g_n^{'}(0)\right)^{\frac{3}{2}}}\left(g_n^{'}(z_n)\right)^{\lambda+\frac{1}{2}} & \left(g_n^{'}(z_n)\right)^{\lambda+1} \end{bmatrix}.$$

This completes the proof. □

We denote the (projective) cocycle

$$\prod_{i=1}^{n-1} g_i^{'}(z_i)^{\alpha_i} \begin{bmatrix} \left(g_n^{'}(z_n)\right)^{\lambda} & 0 \\ \frac{g_n^{''}(0)}{2\left(g_n^{'}(0)\right)^{\frac{3}{2}}}\left(g_n^{'}(z_n)\right)^{\lambda+\frac{1}{2}} & \left(g_n^{'}(z_n)\right)^{\lambda+1} \end{bmatrix}$$

by $J_{\alpha,\lambda}$ where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{n-1})$. Now, we find possible values of α_i and λ for which there exists a diagonal matrix $K(0,0)$ such that

(a) the polarization of $K(z,z)$, defined by the equation (5.6) is a quasi-invariant kernel with respect to $J_{\alpha,\lambda}$ and

(b) the tuple of multiplication operators is in $B_2(\mathbb{D}^n)$.

Suppose there exists a positive diagonal matrix $K(0,0)$ such that the polarization of $K(z,z)$, defined by the equation (5.6), is a quasi-invariant kernel with respect to $J_{\alpha,\lambda}$ under the action of the group Möb^n. Then the function $\frac{1}{(1-z_i\bar{w}_i)^{\alpha_i}}$ must define a positive definite kernel on $\mathbb{D}$, for each $i = 1, 2, \ldots, n-1$. This implies that α_i must be positive for each i. Also, it is easy to see that the polarization of

$$J_{\alpha,\lambda}((0,0,\ldots,g_z);(0,0,\ldots,z)K(0,0)J_{\alpha,\lambda}((0,0,\ldots,g_z);(0,0,\ldots,z))^*$$

is a positive definite kernel on $\mathbb{D}$, where g_z maps z to 0. It has been shown in [22] that the polarization of

$$J_{\alpha,\lambda}((0,0,\ldots,g_z);(0,0,\ldots,z)K(0,0)J_{\alpha,\lambda}((0,0,\ldots,g_z);(0,0,\ldots,z))^*$$

is a positive definite kernel on $\mathbb{D}$, only when λ is positive and

$$K(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\lambda}+\mu \end{bmatrix},$$

where μ is any positive real number. This implies that

$$K^{(\alpha,\lambda,\mu)}(z,w) = \left(\prod_{i=1}^{n-1} \frac{1}{(1-z_i\bar{w}_i)^{\alpha_i}}\right) \begin{bmatrix} \frac{1}{(1-z_n\bar{w}_n)^{\lambda}} & \frac{z_n}{(1-z_n\bar{w}_n)^{\lambda+1}} \\ \frac{\bar{w}_n}{(1-z_n\bar{w}_n)^{\lambda+1}} & \frac{\frac{1}{\lambda}+\mu+z_n\bar{w}_n}{(1-z_n\bar{w}_n)^{\lambda+1}} \end{bmatrix}$$

is the only kernel on $\mathbb{D}^n$ such that the tuple of multiplication operators is in $B_2(\mathbb{D}^n)$ and homogeneous under the action of Möb^n, where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{n-1})$ is a tuple of positive real numbers and $\lambda, \mu > 0$. Now, Theorem 5.13 implies that there are no irreducible tuple of operators in $B_2(\mathbb{D}^n)$ which are homogeneous with respect to the group $\mathrm{Aut}(\mathbb{D}^n)$.

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