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# THE CURVATURE FUNCTION AND SIMILARITY OF OPERATORS[1]

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1. Introduction. A popular and successful approach to operator theory is the introduction of a function that carries with it all information about a given operator. Examples are the characteristic determinant of a finite rank operator, the characteristic operator function in the Brodskii-Livšic study of dissipative operators and the Sz.-Nagy-Foias theory of contractions and the Carey-Pincus determining function.

Recently, Cowen and Douglas [5] introduced another function, useful for studying operators in a class they call $B_1(\Omega)$. The operators in this class are determined by their eigenvectors, corresponding to eigenvalues from the planar set $\Omega$, and the function involved is the curvature of the line bundle whose fibres are the one dimensional eigenspaces. Cowen and Douglas proved that curvature is a unitary invariant. The present paper is a survey of some recent results which attempt to relate the curvatures of two similar operators or to replace curvature with an invariant more appropriate to similarity questions.

For precise definitions, let $\Omega$ be an open planar set. A bounded operator $T$ on a Hilbert space $H$ is said to belong to $B_1(\Omega)$ if, for $\zeta \in \Omega$,

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(i) $T - \zeta I$ is surjective

(ii) $\ker(T-\zeta I)$ is one dimensional

and (iii) $\bigvee_{\zeta \in \Omega} \ker(T-\zeta I)$ is dense in $H$.

For $T \in B_1(\Omega)$, $E_T$ denotes the line bundle with fibres $\ker(T-\zeta I)$, $\zeta \in \Omega$, and the curvature of $T$ is defined to be

$$\mathcal{K}_T(\zeta) = - \frac{\partial^2}{\partial \bar{\zeta} \partial \zeta} \log \|k_\zeta\|^2 ,$$

where $k_\zeta$ is a nonvanishing holomorphic section of $E_T$ (i.e. determination of the eigenvectors of $T$).

Geometrically, $\mathcal{K}_T$ is the curvature of $E_T$. A useful analytic characterization of $\mathcal{K}_T$ is given by considering the <u>localizations</u> of $T$, i.e. the restrictions $T_\zeta$ of $T$ to $\ker(T-\zeta I)^2$, for $\zeta \in \Omega$. Let $k_\zeta$ be a section of $E_T$, as above, and choose an orthonormal basis

$$e(\zeta) = k_\zeta/\|k_\zeta\| , \quad f(\zeta)$$

of $\ker(T-\zeta I)^2$. Then, with respect to $\{e(\zeta), f(\zeta)\}$, $T_\zeta$ has the matrix

(1.1)
$$T_\zeta = \begin{bmatrix} \zeta & |\mathcal{K}_T(\zeta)|^{-\frac{1}{2}} \\ 0 & \zeta \end{bmatrix};$$

see Cowen-Douglas [5, p. 191].

2. <u>Curvature and unitarily equivalence.</u> As mentioned above, Cowen and Douglas have proved

Theorem 2.1 [5, Theorem 1.17]. *Two operators* $S,T \in B_1(\Omega)$ *are unitarily equivalent if and only if*

$$\mathcal{K}_S(\zeta) = \mathcal{K}_T(\zeta)$$

*for* $\zeta \in \Omega$.

The proof in [5] depends heavily on complex geometry; a more transparent proof is given in [6].

The theorem shows that all information about the operator $T \in B_1(\Omega)$ is contained in the real analytic function $\mathcal{K}_T(\zeta)$. The disadvantage is that the behavior of the curvature function is not easy to relate to the properties of $T$ and the function itself may not be easy to compute. To curcumvent these difficulties, Curto and Salinas [7] have introduced another invariant depending more directly upon the sections of $E_T$ themselves.

Let $T \in B_1(\Omega)$, let $k_\zeta$ be a nonzero holomorphic section of $E_T$ and let $K_T(\zeta,\lambda) = (k_\zeta,k_\lambda)$, for $\zeta,\lambda \in \Omega$. For every fixed $\zeta_0 \in \Omega$, Curto and Salinas replace $k_\zeta$, in a neighborhood of $\zeta_0$, by the unique normalized section $\tilde{k}_\zeta$ such that the corresponding $\tilde{K}_T(\zeta,\lambda)$ satisfies

$$\tilde{K}_T(\zeta,\zeta_0) = 1 \tag{2.1}$$

for $\zeta$ in this neighborhood.

Theorem 2.2 [7, Theorem 4.12a]. *Two operators* $S,T \in B_1(\Omega)$ *are unitarily equivalent if and only if the corresponding normalized functions* $\tilde{K}_S$ *and* $\tilde{K}_T$ satisfy

$$\tilde{K}_T(\zeta,\lambda) = c\tilde{K}_S(\zeta,\lambda), \quad |c| = 1$$

*in a neighborhood of* $\zeta_0$.

The transition from $\mathcal{K}_T$ to $\tilde{k}_T$ is the motivation for the modification of the similarity conjecture we make in Section 5. (We are grateful to Curto for communicating his unpublished ideas on this subject to us.)

3. Curvature and similarity. As regards similarity of two operators $S,T \in B_1(\Omega)$, Cowen and Douglas made the following conjecture [5, p. 252].

Cowen-Douglas Conjecture: *If $S,T \in B_1(D)$, $D$ the unit disk, and if $\bar{D}$ is a k-spectral set for $S$ and $T$, then $S$ and $T$ are similar if and only if*

(3.1) $$\lim_{\zeta\to\partial D} \mathcal{K}_T(\zeta)/\mathcal{K}_S(\zeta) = 1 .$$

Here "$\bar{D}$ is a k-spectral set for $S$" means that for every polynomial $p$,

$$\|p(S)\| \le k\|p\|_\infty$$

where $\|\cdot\cdot\|_\infty$ is the $H^\infty$ norm on $D$.

Before discussing the extent to which the conjecture is true, it is worth noting that one might guess that $S$ and $T$ in $B_1(\Omega)$ would be similar if and only if

(3.2) $$0 < c \le \mathcal{K}_T/\mathcal{K}_S \le C$$

on $\Omega$. That this is false can be seen from the example $S$ = shift on $H^2$, $T$ = shift on the Bergman space. In this case

$$\mathcal{K}_S = -(1-|\zeta|^2)^{-2}, \quad \mathcal{K}_T = -2(1-|\zeta|^2)^{-2}$$

but $S$ and $T$ are not similar. In general, similarity does imply (3.2). In fact, if $L$ satisfies $LS = TL$, the restriction $L_\zeta$ of $L$ to $\ker(S-\zeta I)^2$

intertwines the localizations $S_\zeta$ and $T_\zeta$ and, by (1.1), must have the form

$$L_\zeta = \begin{pmatrix} A_\zeta & B_\zeta \\ 0 & C_\zeta \end{pmatrix}$$

where

(3.3) $$A_\zeta/C_\zeta = |\mathcal{K}_S(\zeta)/\mathcal{K}_T(\zeta)|^{\frac{1}{2}} .$$

Now if $L$ is bounded and invertible, $\|L^{-1}\|^{-1}$ is a lower bound and $\|L\|$ is an upper bound for both $A_\zeta$ and $C_\zeta$ and we have

$$\|L\|^{-2}\|L^{-1}\|^{-2} \leq \mathcal{K}_S(\zeta)/\mathcal{K}_T(\zeta) \leq \|L\|^2\|L^{-1}\|^2 .$$

This inequality is due to Cowen and Douglas [5, Cor. 4.30].

Although the Cowen-Douglas Conjecture is false in general, as we shall see in the next section, the "only if" part is correct in a number of cases, even for $\Omega$ more general than the disk. For example, one has

Theorem 3.1 [3]. If $S,T \in B_1(\Omega)$, $S$, $T$ have no eigenvalues on $\partial\Omega$ and $S = L^{-1}TL$, with $L = V + K$, where $V$ is unitary and $K$ is compact, then (3.1) holds.

A special case of Theorem 3.1 is given by the Toeplitz operators considered in [4]. Let $F(z)$ be a rational function with no poles on $|z| = 1$, such that $t \to F(e^{it})$ is a simple closed positively oriented curve $\gamma$ and suppose $F$ is one-to-one in some annulus $s \leq |z| \leq 1$. Then $T = T_F^*$ is similar to $S = T_\tau^*$, where $\tau$ is the conformal map of $D$ onto the interior of $\gamma$. A similarity of the form $V + K$ was obtained in [4].

Another example of Theorem 3.1 is that in which $T$ is a backward weighted shift with weights $\alpha_0, \alpha_1, \ldots$ (so that $T(x_0, x_1, \ldots) = (\alpha_0 x_1, \alpha_1 x_2, \ldots))$ and

S is the unilateral backward shift $(\alpha_0 = \alpha_1 = \cdots = 1)$. If the sequence given by

$$c_n = \prod_0^n (\alpha_{j+1}/\alpha_j)$$

is nonzero and converges to a nonzero limit, then Theorem 3.1 applies.

The last example can be generalized to the case where S and T are similar and $c_n^2$ converges (c,1) to a nonzero limit $\lambda$ (i.e., $(c_0^2 + \ldots + c_n^2)/(n+1) \to \lambda$), a case not covered by the theorem. See [3, §3.1], where it is also shown that S may be replaced by a somewhat more general weighted shift.

4. Counterexamples to the Cowen-Douglas Conjecture. In this section, we describe the counterexamples to the "only if" and "if" portions of the Cowen-Douglas Conjecture given in [2] and [3].

Let $T_F$ denote the Toeplitz operator on $H^2$ with symbol

$$F(z) = z^2/(z-\beta) \qquad \tfrac{1}{2} < \beta < 1.$$

The function F maps the circle to a figure eight having winding number +1 with respect to the points inside the left loop and -1 with respect to the points inside the right loop. The loops intersect at $z = 1/\beta$. Now each point inside the left loop is an eigenvalue for $T_F^*$. We let T be the restriction of $T_F^*$ to the closed span of the corresponding eigenvectors. From [1, Theorem 1], it follows that T is similar to $S = T_\tau^*$, where $\tau$ is the Riemann mapping function from D onto the interior of the left loop.

A direct computation shows that

$$\lim_{1/\beta > \lambda \to 1/\beta} \mathcal{K}_S/\mathcal{K}_T \neq 1.$$

Replacing $S$ and $T$ by $\tau^{-1}(S)$ and $\tau^{-1}(T)$ gives a counterexample to the "only if" half of the Cowen-Douglas Conjecture.

Turning to the if half of the conjecture, let $S$ be the backward shift on $H^2$, $\{c_n\}$ the sequence

$$c_n = \left(\sum_{j=1}^{n} 1/j\right)^{\frac{1}{2}}$$

and $T$ the weighted backward shift with weights

$$\alpha_n = c_n/c_{n+1} .$$

Then $\|T\| \leq 1$ and $TL = LS$, where $L$ is the diagonal matrix (with respect to $\{e^{int}\}$) with diagonal $c_1, c_2, \ldots$

It follows from results of Sheddighi [8] that $T \in B_1(D)$ and from known results that $S$ and $T$ are not similar.

Choose $\{k_\zeta = (1-\zeta z)^{-1}\}$ as the eigenvectors of $S$, for $\zeta \in D$, and let $h_\zeta = Lk_\zeta$ be the corresponding eigenvectors of $T$. Clearly

$$\|k_\zeta\|^2 = (1-|\zeta|^2)^{-1}$$

$$\|h_\zeta\|^2 = \sum_{n=1}^{\infty} c_n^2|\zeta|^{2(n-1)} = (1-|\zeta|^2)^{-1}[1 - \log(1-|\zeta|^2)] .$$

Therefore

$$\mathcal{K}_T/\mathcal{K}_S = 1 + [1 - \log(1-|\zeta|^2)]^{-1} - |\zeta|^2[1 - \log(1-|\zeta|^2)]^{-2} \to 1$$

as $|\zeta| \to 1$.

This counterexample to the if half of the Cowen-Douglas Conjecture has been found independently by Badri Matooq.

5. Toward a revised similarity conjecture. A second look at the second example in the last section shows that the ratio

$$a_\zeta = \|h_\zeta\|^2/\|k_\zeta\|^2 \tag{5.1}$$

tends to $\infty$. Thus the quantity $a_\zeta$, as well as being easier to compute than the ratio of the curvatures, can detect a pair of nonsimilar operators S and T in a case where $\mathcal{K}_S/\mathcal{K}_T$ cannot.

However $a_\zeta$ is not a good invariant, since it depends heavily upon the choice of $h_\zeta$ and $k_\zeta$. Indeed, for some (analytic, nonzero) choice of the eigenvectors, $a_\zeta$ can have almost any prescribed behavior at the boundary. The Curto-Salinas results mentioned in §2 strongly suggest some normalization of the eigenvectors, such as

$$(k_\zeta, k_{\zeta_0}) = 1 \tag{5.2}$$

for some fixed point $\zeta_0 \in \Omega$. But whereas the normalization (2.1) is only required to hold locally it is evident that for similarity we need (5.2) to hold for all $\zeta$ in $\Omega$. The following revised conjecture has been arrived at in correspondence with Curto.

Conjecture. If there exist analytic sections of $E_T$ and $E_S$, normalized in some appropriate way, such as (5.2), and if $a_\zeta$ is defined by (5.1), then S and T are similar if and only if $a_\zeta$ is bounded and bounded from 0 in $\Omega$. For the rest of this section, we discuss positive evidence for the conjecture, in the next section, we discuss problems with it.

Suppose T is a weighted backward shift (with spectrum $\overline{D}$). Automatically the function $1 \in H^2$ (or equivalently the vector $(1,0,0,\ldots)$) is an eigenvector, with eigenvalue 0. Furthermore an analytic section of $E_T$, normalized to satisfy (5.2), exists by merely choosing $k_\zeta(0) = 1$

(or $k_\zeta = (1, a_1(\zeta), a_2(\zeta), \ldots)$). For weighted shifts, half of the above conjecture now follows.

Theorem 5.1 [3]. *If $S$ and $T$ are backward weighted shifts with eigenvectors normalized to satisfy (5.2). Then $S$ and $T$ are similar only if $a_\zeta$ is bounded and bounded from 0 in $D$.*

In this generality,

$$(5.3) \qquad S = L^{-1}TL\,, \quad L = V + K\,, \quad V \text{ unitary}, \quad K \text{ compact}$$

implies that $a_\zeta$ converges to a nonzero limit as $|\zeta| \to 1$. We do not know if the ratio of the curvatures tends to 1 in Theorem 5.1.

Now we replace $S$ with the backward weighted shift $S_\alpha$, with weights

$$\alpha = [(n+1)/(n+2)]^{\alpha/2} \qquad n = 0,1,\ldots,$$

$\alpha \geq 0$. We have

Theorem 5.2 [3]. a. *If $T$ is a backward weighted shift with $\|T\| \leq 1$ and if $S = S_\alpha$ ($\alpha \geq 0$), then $T$ is similar to $S$ if and only if $a_\zeta$ is bounded and bounded from 0.*

b. *If $S = S_0$ and $T$ is a power bounded backward weighted shift, then $S$ and $T$ are similar if and only if $a_\zeta$ does not tend to $\infty$ as $|\zeta| \to 1$.*

In part a of the theorem, (5.3) holds if and only if $a_\zeta$ tends to a nonzero limit as $|\zeta| \to 1$. In part b, $a_\zeta$ converging to a nonzero limit is equivalent to $S = L^{-1}TL$ with $L$ diagonal and the square of its diagonal entries converging (c,1) to a nonzero limit.

It is important to view Theorems 5.1 and 5.2 as progress on the revised similarity conjecture, not as new criteria for similarity of weighted shifts.

In fact, the known similarity criteria for weighted shifts are generally simpler and easier to apply than considerations of $a_\zeta$.

6. Further counterexamples. Unfortunately, the conjecture in Section 5 has problems, even with the normalization (5.2).

In [3, §2.5], we gave an example of two contractive backward weighted shifts $S$ and $T$ which were not similar and yet $a_\zeta \to 1$ as $|\zeta| \to 1$. Thus we can only hope for the conjecture of the last section in case, for example, $S$ is of the form $S_\alpha$ and $T$ is contractive but not necessarily a weighted shift; or in case some different normalization than (5.2) is used.

For non-weighted shifts there can be a problem with the global existence of the normalization (5.2). The following examples show that such a normalization may exist for some $\zeta_0 \in \Omega$ and not others and that the conjecture of the last section may only hold for some $\zeta_0$.

Let

$$L = \begin{pmatrix} 1 & 1 & 0 & 0 & \\ 1 & a & 0 & 0 & \\ 0 & 0 & 1 & 0 & \\ 0 & 0 & 0 & 1 & \\ & & & & \ddots \end{pmatrix}, \quad a \neq 1,$$

$S_0$ the unilateral shift and $T = LS_0L^{-1}$. Certainly $T \in B_1(D)$ and the eigenvectors of $T$ are

$$h_\zeta = Lk_\zeta = (1+\zeta, 1+a\zeta, \zeta^2, \zeta^3, \ldots).$$

Thus

$$\text{(6.1)} \qquad \begin{aligned}(h_\zeta, h_\lambda) &= (1+\zeta)(1+\bar\lambda) + (1+a\zeta)(1+\bar a\bar\lambda) + \sum_2^\infty (\zeta\bar\lambda)^n \\ &= (1+a)\zeta + (1+\bar a)\bar\lambda + 1 + |a|^2\zeta\bar\lambda + (1-\zeta\bar\lambda)^{-1}.\end{aligned}$$

If $\lambda = 0$,

$$(h_\zeta, h_0) = (1+a)\zeta + 2 = 0 \quad \text{if} \quad \zeta = -2/(1+a) .$$

Choosing $a = -3$, we see that

$$\hat{h}_\zeta = 2^{-\frac{1}{2}}(1-\zeta)^{-1} h_\zeta$$

gives a normalized holomorphic section of $E_T$ which satisfies $\|k_\zeta\|^2/\|\hat{h}_\zeta\|^2 \to 0$, even though $S_0$ and $T$ are similar.

Choosing $a = 6/5$, we see that $(h_\zeta, h_0) = 0$ if $\zeta = -10/11$ and since $(h_\zeta, h_\zeta) \neq 0$ for this $\zeta$, there is no holomorphic section in all of $D$ satisfying (5.2) with $\zeta_0 = 0$. However, as we will show, there are holomorphic sections satisfying (5.2) with other values of $\zeta_0$.

Temporarily set $\lambda = 1$ in (6.1). Then the right side is zero if and only if

$$-91\zeta^2 + 11\zeta + 105 = 0 .$$

This equation has two (real) roots, one $< -184/182$ and the other $> 206/182$; i.e., both outside $\overline{D}$. Therefore, for $\lambda$ in a neighborhood of $1$, (6.1) is non-vanishing for $\zeta \in \overline{D}$, and therefore $E_T$ has a section, holomorphic and satisfying (5.2) in $D$. Clearly $a_\zeta = \|h_\zeta\|^2/\|k_\zeta\|^2 \to 1$ as $\zeta \to \tilde{\zeta} \in \partial D$ for this section.

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Douglas N. Clark,
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KRATAK SADRŽAJ. Ovaj rad je pregled nekih novijih rezultata koji pokušavaju da uspostave vezu izmedju krivina dvaju sličnih operatora ili da zamene krivinu nekom invarijantom koja bolje odgovara problemima sličnosti.

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