# CLASSIFICATION OF QUASI-HOMOGENEOUS HOLOMORPHIC CURVES AND OPERATORS IN THE COWEN-DOUGLAS CLASS

CHUNLAN JIANG, KUI JI AND GADADHAR MISRA

ABSTRACT. In this paper we study quasi-homogeneous operators, which include the homogeneous operators, in the Cowen-Douglas class. We give two separate theorems describing canonical models (with respect to equivalence under unitary and invertible operators, respectively) for these operators using techniques from complex geometry. This considerably extends the similarity and unitary classification of homogeneous operators in the Cowen-Douglas class obtained recently by the last author and A. Korányi. In a significant generalization of the properties of the homogeneous operators, we show that quasi-homogeneous operators are irreducible and determine which of them are strongly irreducible. Applications include the equality of the topological and algebraic $K$-group of a quasi-homogeneous operator and an affirmative answer to a well-known question of Halmos.

## 1. INTRODUCTION

For a plane domain $\Omega$, in the paper [2], Cowen and Douglas introduced an important class of operators $B_n(\Omega)$. It was shown by them that for operators $T$ in $B_n(\Omega)$, the local geometry of the corresponding vector bundle $E_T$ of rank $n$ (curvature tensor and its higher derivatives) yields a complete set of unitary invariants for the operator $T$. But a tractable set of unitary (or similarity) invariants has not been found yet. The analysis of holomorphic Hermitian vector bundles in case $n > 1$ is much more complicated, see [20, Example 2.1].

In the papers [10, 11], a class $\mathcal{F}B_n(\Omega)$ of operators in the Cowen-Douglas class possessing a flag structure was isolated. A complete set of unitary invariants for this class of operators were listed. Recently, Jiang and Ji have introduced methods from $K$-theory to classify flags of holomorphic curves in the Grassmannian in order to reduce the questions involving operators in $B_n(\Omega)$ to the case of $n = 1$ (cf. [13, 12]). On the other hand, the classification of homogeneous holomorphic Hermitian vector bundles over the unit disc has been completed recently (cf. [17]) using tools from representation theory of semi-simple Lie groups. Although not complete, a similar classification over an arbitrary bounded symmetric domain is currently under way [16, 21].

The methods of $K$ - theory developed in [13, 12] together with the methods of [11] makes it possible to study a much larger class of "quasi-homogeneous" operators, where the techniques from representation theory are no longer available. These methods, applied to the class of "quasi-homogeneous" operators leads to a unitary classification. In addition the bundle maps describing the triangular decomposition of Jiang and Ji have an explicit realization in terms of the inherent harmonic analysis. A model for these operators is described explicitly, which shows, among other things, that the well-known Halmos problem for the class of "quasi-homogeneous" operators has an affirmative answer.

Prompted by these results, one might imagine that the multi-variate case (replacing the planar domain $\Omega$ by the unit ball or a bounded symmetric domain) may also be accessible to these new techniques.

2000 *Mathematics Subject Classification.* Primary 47C15, 47B37; Secondary 47B48, 47L40.

*Key words and phrases.* curvature, second fundamental form, homogeneous and quasi homogeneous vector bundles, similarity, Halmos' question, topological and algebraic $K$- groups.

The first author was Supported by National Natural Science Foundation of China (Grant No. A010602), the second author is Supported by National Natural Science Foundation of China (Grant No. 11471094), Hebei Provincial Education Department Natural Science Foundation (Grant No. 2017037) and Hebei Natural Science Foundation for Distinguished Young Scholars (Grant No. A201605219) and the third author was supported by the J C Bose National Fellowship and the UGC, SAP – CAS.

## 2. Main results

2.1. **Preliminaries.** Let $\mathcal{H}$ be a complex separable Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the algebra of bounded linear operators on $\mathcal{H}$. For an open connected subset $\Omega$ of the complex plane $\mathbb{C}$, and $n \in \mathbb{N}$, Cowen and Douglas introduced the class of operators $B_n(\Omega)$ in their very influential paper [2]. An operator $T$ acting on a Hilbert space $\mathcal{H}$ belongs to this class if each $w \in \Omega$, is an eigenvalue of the operator $T$ of constant multiplicity $n$, these eigenvectors span the Hilbert space $\mathcal{H}$ and the operator $T-w,\ w \in \Omega$, is surjective. They showed that for an operator $T$ in $B_n(\Omega)$, there exists a holomorphic choice of $n$ linearly independent eigenvectors, that is, the map $w \to \ker(T-w)$ is holomorphic. Thus $\pi : E_T \to \Omega$, where

$$E_T = \{\ker(T-w) : w \in \Omega, \pi(\ker(T-w)) = w\}$$

defines a Hermitian holomorphic vector bundle on $\Omega$.

The Grassmannian $\mathrm{Gr}(n,\mathcal{H})$, is the set of all $n$-dimensional subspaces of the Hilbert space $\mathcal{H}$. A map $t : \Omega \to \mathrm{Gr}(n,\mathcal{H})$ is said to be a holomorphic curve, if there exist $n$ (point-wise linearly independent) holomorphic functions $\gamma_1, \gamma_2, \cdots, \gamma_n$ on $\Omega$ taking values in a Hilbert space $\mathcal{H}$ such that $t(w) = \bigvee\{\gamma_1(w), \cdots, \gamma_n(w)\},\ w \in \Omega$. Any holomorphic curve $t : \Omega \to \mathrm{Gr}(n,\mathcal{H})$ gives rise to a $n$-dimensional Hermitian holomorphic vector bundle $E_t$ over $\Omega$, namely,

$$E_t = \{(x,w) \in \mathcal{H} \times \Omega \mid x \in t(w)\} \text{ and } \pi : E_t \to \Omega, \text{ where } \pi(x,w) = w.$$

Given two holomorphic curves $t, \tilde{t} : \Omega \to \mathrm{Gr}(n,\mathcal{H})$, if there exists a unitary operator $U$ on $\mathcal{H}$ such that $\tilde{t} = Ut$, that is, the restriction $U(w) := U_{|E_t(w)}$ of the unitary operator $U$ to the fiber $E_t(w)$ of $E$ at $w$ maps it to the fiber of $E_{\tilde{t}}(w)$, then $t$ and $\tilde{t}$ are said to be congruent. If $t$ and $\tilde{t}$ are congruent, then clearly the vector bundles $E_t$ and $E_{\tilde{t}}$ are equivalent via the holomorphic bundle map induced by the unitary operator $U$. Furthermore, $t$ and $\tilde{t}$ are said to be similar if there exists an invertible operator $X \in \mathcal{L}(H)$ such that $\tilde{t} = Xt$, that is, $X(w) := X_{|E_t(w)}$ is an isomorphism except that $X(w)$ is no longer an isometry. In this case, we say that the vector bundles $E_t$ and $E_{\tilde{t}}$ are similar.

An operator $T$ in the class $B_n(\Omega)$ determines a non-constant holomorphic curve $t : \Omega \to \mathrm{Gr}(n,\mathcal{H})$, namely, $t(w) = \ker(T-w), w \in \Omega$. However, if $t$ is a holomorphic curve, setting $Tt(w) = wt(w)$, defines a linear transformation on a dense subspace of the Hilbert space $\mathcal{H}$. In general, we have to impose additional conditions to ensure that the operator $T$ is bounded. Assuming that $t$ defines a bounded linear operator $T$, unitary and similarity invariants for the operator $T$ are then obtained from those of the vector bundle $E_t$.

The motivation for this work comes from three very different directions. The attempt is to describe a canonical model and obtain invariants for operators in the Cowen-Douglas class with respect to equivalence via conjugation under a unitary or invertible linear transformation. These questions have been successfully addressed using ideas from $K$-theory and representation theory of Lie groups.

First, the detailed study of the Cowen-Douglas class of operators, reported in the book [14, Theorem 1.49] provides a basic structure theorem for these operators: $T$ is an operator in the Cowen-Douglas class $B_n(\Omega)$, then there exists operators $T_0, T_1, \ldots, T_{n-1}$ in $B_1(\Omega)$ such that

$$T = \begin{pmatrix} T_0 & S_{0,1} & S_{0,2} & \cdots & S_{0,n-1} \\ 0 & T_1 & S_{1,2} & \cdots & S_{1,n-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & T_{n-2} & S_{n-2,n-1} \\ 0 & \cdots & \cdots & 0 & T_{n-1} \end{pmatrix}. \tag{2.1}$$

A slight paraphrasing of it clearly implies that if $\{\gamma_0, \gamma_1, \cdots, \gamma_{n-1}\}$ is a holomorphic frame for the vector bundle $E_t$, and $\mathcal{H} = \bigvee\{\gamma_i(w),\ w \in \Omega,\ 0 \le i \le n-1\}$, then there exists non-vanishing holomorphic curves $t_i : \Omega \to \mathrm{Gr}(1,\mathcal{H}_i)$, $0 \le i \le n-1$, such that

$$\gamma_j = \phi_{0,j}(t_0) + \cdots + \phi_{i,j}(t_i) + \cdots + \phi_{j-1,j}(t_{j-1}) + t_j,\ 0 \le j \le n-1, \tag{2.2}$$

where $\phi_{i,j}$ are certain holomorphic bundle maps. One would expect these bundle maps to reflect the properties of the operator $T$. However the tenuous relationship between the operator $T$ and the bundle maps $\phi_{i,j}$ becomes a little more transparent *only* after we impose a natural set of constraints.

Secondly, to a large extent, these constraints were anticipated in the recent paper [11, Theorem 3.6]. In that paper, a class of operators $\mathcal{F}B_n(\Omega)$ in $B_n(\Omega)$ possessing, what we called, a flag structure were isolated. The flag structure was shown to be rigid. It was then shown that the complex geometric invariants like the curvature and the second fundamental form of the vector bundle $E_T$ are unitary invariants of the operator $T$. Indeed, in that paper, a complete set of unitary invariants were found.

Finally, recall that an operators $T$ in $B_n(\mathbb{D})$ is said to be homogeneous if the unitary orbit of $T$ under the action of the Möbius group is itself, that is, $\varphi(T)$ is unitarily equivalent to $T$ for $\varphi$ in some open neighbourhood of the identity in the Möbius group (cf. [1]). A canonical element $T^{(\lambda,\boldsymbol{\mu})}$ in each unitary equivalence class of the homogeneous operators in $B_n(\mathbb{D})$ was constructed in [17]. It was then shown that two operators $T^{(\lambda,\boldsymbol{\mu})}$ and $T^{(\lambda',\boldsymbol{\mu}')}$ are similar if and only if $\lambda = \lambda'$. In particular choosing $\boldsymbol{\mu} = 0$, one verifies that a homogeneous operator in $B_n(\mathbb{D})$ is similar to the $n$-fold direct sum $T_0 \oplus \cdots \oplus T_n$, where $T_i$ is the adjoint of the multiplication operator $M^{(\lambda_i)}$ acting on the weighted Bergman space $\mathbb{A}^{(\lambda_i)}(\mathbb{D})$ determined by the positive definite kernel $\frac{1}{(1-z\bar{w})^{\lambda_i}}$ defined the unit disc $\mathbb{D}$, $0 \leq i \leq n-1$, $\lambda_i > 0$.

2.2. **Main results.** In this paper we study a class of operators, to be called quasi-homogeneous, for which we can prove results very similar to those for the homogeneous operators building on the techniques developed in [11]. This class of operators, as one may expect, contains the homogeneous operators and is characterized by the requirement that all the bundle maps of (2.2) take their values in a certain (full) jet bundle $\mathcal{J}_i(t)$ of the holomorphic curve $t$. For a detailed account of the jet bundles, we refer the reader to [23].

**Definition 2.1.** *If $t$ is a holomorphic curve in the Grassmannian of rank 1, that is, $t : \Omega \to Gr(1, \mathcal{H})$. Let $\gamma(w)$ be a non-vanishing holomorphic section for the line bundle $E_t$. The derivatives $\gamma^{(j)}$, $j \in \mathbb{N}$, taking values again in the Hilbert space $\mathcal{H}$ are holomorphic. (It can be shown that they are linearly independent.) The jet bundle $\mathcal{J}_n E_t(\gamma)$ is defined by the holomorphic frame $\{\gamma^{(0)}(:= \gamma), \gamma^{(1)}, \cdots, \gamma^{(n)}\}$. The jet bundle $\mathcal{J}_n E_t(\gamma)$ has a natural Hermitian structure obtained by taking the inner product of $\gamma^{(i)}(w)$ and $\gamma^{(j)}(w)$ in the Hilbert space $\mathcal{H}$.*

In the following definition we assume, implicitly, that the bundle map $\phi_{i,j}$ of (2.2) are from the holomorphic line bundles $E_i$ to a jet bundle $\mathcal{J}_j E_i$, where for brevity of notation and when there is no possibility of confusion, we will let $E_i$ denote the vector bundle induced by the holomorphic curve $t_i$, $0 \leq i \leq n-1$.

**Definition 2.2** ($\mathscr{J}$-holomorphic curve)**.** *Let $t$ be a holomorphic curve in the Grassmannian $Gr(n, \mathcal{H})$ of a complex separable Hilbert space $\mathcal{H}$ and $\{\gamma_0, \gamma_1, \cdots, \gamma_{n-1}\}$ be a holomorphic frame for $t$. We say that $t$ admits an atomic decomposition if there exists holomorphic curves $t_i : \Omega \to Gr(1, \mathcal{H}_i)$, to be called the atoms of $t$, corresponding to operators $T_i : \mathcal{H}_i \to \mathcal{H}_i$ in $B_1(\Omega)$ and complex numbers $\mu_{i,j} \in \mathbb{C}$, $0 \leq j \leq i \leq n-1$, such that $\mathcal{H} = \mathcal{H}_0 \oplus \cdots \oplus \mathcal{H}_{n-1}$ and*

$$\begin{aligned}
\gamma_0 &= \mu_{0,0} t_0 \\
\gamma_1 &= \mu_{0,1} t_0^{(1)} + \mu_{1,1} t_1 \\
\gamma_2 &= \mu_{0,2} t_0^{(2)} + \mu_{1,2} t_1^{(1)} + \mu_{2,2} t_2 \\
\vdots\ & \quad \vdots \\
\gamma_j &= \mu_{0,j} t_0^{(j)} + \cdots + \mu_{i,j} t_i^{(j-i)} + \cdots + \mu_{j,j} t_j \\
\vdots\ & \quad \vdots \\
\gamma_{n-1} &= \mu_{0,n-1} t_0^{(n-1)} + \cdots + \mu_{i,n-1} t_i^{(n-1-i)} + \cdots + \mu_{n-1,n-1} t_{n-1}.
\end{aligned}$$

*If $t$ admits an atomic decomposition, we call it a $\mathscr{J}$ - holomorphic curve.*

Fix $i$ in $\{0, \ldots, n-1\}$. We say that the holomorphic curve $t_i$ is homogeneous if for $w \in \mathbb{D}$, $\mathbb{C}[t_i(w)] = \ker(T_i - w)$ for some homogeneous operator $T_i$ in $B_1(\mathbb{D})$. We realize, up to unitary equivalence, such a homogeneous operator $T_i$ in $B_1(\mathbb{D})$ as the adjoint of the multiplication operator $M^{(\lambda_i)}$ on the weighted

Bergman spaces $\mathbb{A}^{(\lambda_i)}(\mathbb{D})$. Thus for a fixed $w \in \mathbb{D}$, there exists a canonical (holomorphic) choice of eigenvectors $t_i(w)$, namely, $(1 - z\bar{w})^{-\lambda_i}$.

**Definition 2.3** (quasi-homogeneous curve)**.** *We say that a $\mathscr{J}$ - holomorphic curve $t$ is quasi-homogeneous if each of the atoms $t_i$ is homogeneous, $\lambda_0 \le \lambda_1 \le \cdots \le \lambda_{n-1}$ and the difference $\lambda_{i+1} - \lambda_i$, $0 \le i \le n-2$, is a fixed positive real number $\Lambda(t)$, which is called the* valency *of $t$.*

We say that the $\mathscr{J}$ - holomorphic curve $t$ defines a bounded linear operator if the linear span of $\{\gamma_i(w) : 0 \le i \le n-1\}$, $w \in \Omega$, is dense in $\mathcal{H}$ and the linear map defined by the rule $T(\gamma_i(w)) = w\gamma_i(w)$, $0 \le i \le n-1$, extends to a bounded operator on the Hilbert space $\mathcal{H}$.

We determine conditions on the scalars $\mu_{i,j}$ and the valency $\Lambda(t)$, which ensure that the quasi-holomorphic curve $t$ defines bounded operator $T$, see Proposition 3.2

Throughout this paper, we make the standing assumption that these conditions for boundedness are fulfilled. We shall use the terms quasi-homogeneous holomorphic curve $t$, quasi-homogeneous operator $T$ and quasi-homogeneous holomorphic vector bundle $E_t$ (or, even $E_T$) interchangeably.

If $T$ is a quasi-homogeneous operator then it belong to the class $\mathcal{F}B_n(\mathbb{D})$ introduced in the paper [10, 11], see Theorem 3.3. All quasi-homogeneous operators are therefore irreducible. All the quasi-homogeneous operators that are strongly irreducible are identified in Theorem 4.5. Theorem 4.2 gives a canonical model for a quasi-homogeneous operator in the equivalence class under conjugation by an invertible transformation.

As an application of our results, in Theorem 5.5, we show that the (topological) $K^0$ group and the (algebraic) $K_0$ group of a quasi-homogeneous operator are equal, In the context of the usual $K^0$ and $K_0$ groups, this is a consequence of the well-known theorem of R. G. Swan. As a second application, we obtain an affirmative answer for the Halmos question on similarity of an operator admitting the closed unit disc as a spectral set to a contraction.

A quasi-homogeneous vector bundle $E_t$ is indeed homogeneous if $\Lambda(t) = 2$ and the constants $\mu_{i,j}$ are certain explicit functions of $\lambda$ as we point out at the end of the following section. However, a quasi-homogeneous vector bundle need not be homogeneous as the following example shows.

**Example 2.4.** Let $S$ be the adjoint of the multiplication operator on arbitrary weighted Bergmann space $\mathbb{A}^{(\lambda)}(\mathbb{D})$ and let $T$ be the operator

$$T = \begin{pmatrix} S & \mu_1 I & 0 & \cdots & 0 \\ 0 & S & \mu_2 I & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & S & \mu_n I \\ 0 & \cdots & \cdots & 0 & S \end{pmatrix}, \ \mu_i \in \mathbb{C},$$

defined on the $n+1$ fold direct sum $\bigoplus \mathbb{A}^{(\lambda)}(\mathbb{D})$. Then $T$ is in $\mathcal{F}B_{n+1}(\mathbb{D})$ and therefore belongs to $B_{n+1}(\mathbb{D})$ and the corresponding holomorphic curve $t(w) = \ker(T - w)$, $w \in \mathbb{D}$, is quasi homogeneous with $\Lambda(t) = 0$. In fact, in this Example, if we replace $S$ with an arbitrary operator, say $R$, from $B_1(\mathbb{D})$, then the resulting operator $T$ while no longer quasi-homogeneous, remains a member of $\mathcal{F}B_{n+1}(\mathbb{D})$. Indeed, it has already appeared, via module tensor products, in our earlier work [11, Section 4].

The class of quasi-homogeneous operators, contrary to what might appear to be a rather small class of operators, contains apart from the homogeneous operators, many other operators. Indeed, in rank 2, for instance, it is parametrized by the multiplier algebra of two homogeneous operators. In the definition of the quasi-homogeneous operators given above, if we let the atoms occur with some multiplicity rather than being multiplicity-free, it will make it even larger. This would cause additional complications, which we are not able to resolve at this time. In another direction, we need not assume that the atoms themselves are homogeneous. Most of our results would appear to go through if we merely assume that the kernel function $K^{(\lambda)}(w,w) \sim \frac{1}{(1-|w|^2)^\lambda}$, $|w| < 1$. Deep results about such functions were obtained by Hardy and Littlewood (cf. [7]) and have already appeared in the context of similarity, see [3].

## 3. Canonical model under unitary equivalence

An operator $T$ in the Cowen and Douglas class $B_n(\Omega)$ is determined, modulo unitary equivalence, by the curvature (of the vector bundle $E_T$) together with a finite number of its partial derivatives. However, if the rank $n$ of this vector bundle is $> 1$, then the computation of the curvature and its derivatives is somewhat impractical. Here we show that if the operator is quasi-homogeneous, it is enough to restrict ourselves to the computation of the curvature of the atoms and a $n-1$ second fundamental forms of pair-wise neighbouring vector bundles. We first recall, following [2, 4], that an operator $T$ in $B_n(\Omega)$ may be realized as the adjoint of a multiplication operator on a Hilbert space of holomorphic functions on $\Omega^* := \{w : \bar{w} \in \Omega\}$ possessing a reproducing kernel.

3.1. Holomorphic Curves. For an operator $T$ in the Cowen-Douglas class $B_n(\Omega)$, acting on a Hilbert space $\mathcal{H}$, there is a holomorphic frame $\{\gamma_0, \gamma_1, \cdots, \gamma_{n-1}\}$ and atoms $t_0, \ldots, t_{n-1}$, for which we have

$$\gamma_i = \mu_{0,i} t_0^{(i)} + \cdots + \mu_{j,i} t_i^{(i-j)} + \cdots + \mu_{i,i} t_i,\ \mu_{j,i} \in \mathbb{C}.$$

At this point, assuming that the operator is quasi-homogeneous makes the atoms $T_0, T_1, \ldots, T_{n-1}$ homogeneous. Conjugating with a diagonal unitary, if necessary, we assume without loss of generality that $t_i$ is the holomorphic curve defined by

$$t_i(w) := (1 - \bar{w}z)^{-\lambda_i},\ \lambda_i = \lambda_0 + i\,\Lambda(t),\ 0 \le i \le n-1,\ \lambda_0 > 0,$$

in the weighted Bergman space $\mathbb{A}^{(\lambda_i)}(\mathbb{D})$. We assume without loss of generality that $\mu_{i,i} = 1,\ 0 \le i \le n-1$.

3.2. Atomic decomposition. Let $t$ be a quasi-homogeneous holomorphic curve in $\mathrm{Gr}(n, \mathcal{H})$. Assume that it defines a bounded linear operator $T$ on the Hilbert space $\mathcal{H}$. An appeal to the decomposition (2.1) provides, what we would *now* call an atomic decomposition for the operator $T$. This decomposition has several additional properties arising out of our assumption of quasi-homogeneity.

**Proposition 3.1.** *Let $t$ be a $\mathscr{J}$-holomorphic curve with atoms $\{t_0, \ldots, t_{n-1}\}$ and let $\{\gamma_0, \ldots, \gamma_{n-1}\}$ be a holomorphic frame for the vector bundle $E_t$. Let $\mathcal{H}$ be the closed linear span of the set of vectors $\{\gamma_0(w), \ldots, \gamma_{n-1}(w) : w \in \Omega\}$ and $\mathcal{H}_i$ be the closed linear span of the set of vectors $\{t_i(w), w \in \Omega\}$, $0 \le i \le n-1$. We have*

(1) $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_{n-1}$;
(2) *There exists an operator $T$, defined on a dense subset of vectors in $\mathcal{H}$, which is upper triangular with respect to the direct sum decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \cdots \oplus \mathcal{H}_{n-1}$ :*

$$T = \begin{pmatrix} T_0 & S_{0,1} & S_{0,2} & \cdots & S_{0,n-1} \\ 0 & T_1 & S_{1,2} & \cdots & S_{1,n-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & T_{n-2} & S_{n-2,n-1} \\ 0 & 0 & \ldots & 0 & T_{n-1} \end{pmatrix},$$

*where $S_{i,j}(t_j(w)) = m_{i,j} t_i^{(j-i-1)}(w), T_i(t_i(w))) = w\,t_i(w),\ w \in \Omega,\ i,j = 0,1,\cdots,n-1$, for some choice of complex constants $m_{i,j}$ depending on the $\mu_{i,j}$.*
(3) *The constants $m_{i,j}$ and $\mu_{i,j}$ determine each other.*

For convenience of notation, in the proof below, we set $S_{i,i} := T_i,\ 0 \le i \le n-1$, in the proof. We will adopt this practice often and call $T_0, T_1, \ldots, T_{n-1}$, the atoms of $T$. Also, $S_{i,i+1}(t_{i+1}) = \mu_{i,i+1} t_i$, with the assumption that $\mu_{i,i} = 1,\ 0 \le i \le n-2$.

*Proof.* Note that $\{\gamma_0, \gamma_1, \cdots, \gamma_{n-1}\}$ is a frame for $E_t$ and the atoms $t_i,\ 0 \le i \le n-1$ are pairwise orthogonal. From Definition 2.2, the first statement of the Proposition is included in the definition of a holomorphic quasi-homogeneous curve.

For $0 \le i \le j \le n-1$, let $S_{i,j} : \mathcal{H}_j \to \mathcal{H}$ be the linear transformation induced by bundle maps $s_{i,j} : E_{t_j} \to \mathcal{J}_{j-i-1}E_{t_i}$, namely,

$$\sum_{i\le j} s_{i,j}(\gamma_k(w)) = w\gamma_k(w),\ w \in \Omega.$$

It follows that

$$(s_{k,k} - w)(\mu_{k,k}t_k(w)) = 0, (s_{k-1,k-1} - w)(\mu_{k-1,k}t^{(1)}_{k-1}(w)) + s_{k-1,k}(\mu_{k,k}t_k(w)) = 0, \tag{3.1}$$

Thus $s_{k,k}$ induces an operator $S_{k,k}$ with $\ker(S_{k,k} - w) = \mathbb{C}[t_k(w)]$ and $s_{k-1,k}$ is a bundle map from $E_{t_k}(w)\,(:= \mathbb{C}[t_k(w)])$ to $E_{t_{k-1}}(w)\,(:= \mathbb{C}[t_{k-1}(w)])$.

For any $i \le j \le n-1$, $s_{i,j}$ is a bundle map from $E_{t_j}$ to $\mathcal{J}_{j-i-1}E_{t_i}$ and there exists $m_{i,j} \in \mathbb{C}$ such that $S_{i,j}(t_j(w)) = m_{i,j}t^{(j-i-1)}_i(w), w \in \Omega$.

Since $(s_{0,0} - w)\gamma_1(w) = (s_{0,0} - w)(\mu_{0,1}t^{(1)}_0(w)) + s_{0,1}(\mu_{1,1}t_1(w)) = 0$, we have

$$s_{0,1}(t_1(w)) = m_{0,1}t_0(w),$$

where $m_{0,1} = -\frac{\mu_{0,1}}{\mu_{1,1}}$. Thus we have

$$s_{0,2}(t_2(w)) = -\frac{2\mu_{0,2} + \mu_{1,2}m_{0,1}}{\mu_{2,2}}t^{(1)}_0(w) = m_{0,2}t^{(1)}_0(w).$$

Now assume that for any fixed $k$ and some $k < j \le n-1$, there exits $m_{k,i} \in \mathbb{C}$ such that

$$s_{k,i}(t_i(w)) = m_{k,i}t^{(i-k-1)}_k(w), i < j.$$

Then from equation (3.1), we have

$$(s_{k,k} - w)(\mu_{k,j}t^{(j-k)}_k)(w) + s_{k,k+1}(\mu_{k+1,j}t^{(j-k-1)}_{k+1}(w)) + \cdots + s_{k,j}(\mu_{j,j}t_j(w)) = 0$$

and from the induction hypothesis, we may rewrite this as

$$\mu_{k,j}(j-k)t^{(j-k-1)}_k(w) + \mu_{k+1,j}m_{k,k+1}t^{(j-k-1)}_k(w) + \cdots + \mu_{j,j}s_{k,j}(t_j(w)) = 0.$$

Thus

$$s_{k,j}(t_j(w)) = m_{k,j}t^{(j-k-1)}_k(w),$$

or, equivalently

$$m_{k,j} = -\frac{\mu_{k,j}(j-k) + \sum\limits_{l=1}^{j-k-1} \mu_{k+l,j}m_{k,k+l}}{\mu_{j,j}} \tag{3.2}$$

completing the the proof of the second statement of the Proposition.

Claim: For any operator $T$ in $B_n(\Omega)$ with atomic decomposition exactly as in the second statement of the lemma, there exists $\mu_{i,j}$ satisfying the conditions in Definition 2.2, that is, there exists a holomorphic frame for $E_T$, which is a linear combination of the non-vanishing holomorphic sections of $E_{t_i}$ and a certain number of jets.

Indeed, the proof of the second part of the Proposition already verifies this Claim for $n \le 2$. To prove the Claim by induction, let us assume that it is valid for $k \le n-2$. Note that the operator $\left(\!\left(S_{i,j}\right)\!\right)_{i,j\le n-2}$ is in $B_{n-1}(\Omega)$. By the induction hypothesis, we can find $m_{i,j}, i,j \le n-2$ verifying Claim 2 for any operator $\left(\!\left(S_{i,j}\right)\!\right)_{i,j\le n-2}$. If we consider the operator

$$\begin{pmatrix} T_{n-2} & S_{n-2,n-1} \\ 0 & T_{n-1} \end{pmatrix},$$

then we have that $S_{n-2,n-1}(t_{n-1}) = m_{n-2,n-1}t_{n-2}$. Now, setting $\mu_{n-2,n-1} = -m_{n-2,n-1}$, we can define all the coefficients $\mu_{n-k,n-1}, 2 \leq k \leq n$ recursively. In fact, if we consider

$$\begin{pmatrix} T_{n-k} & S_{n-k,n-k+1} & S_{n-k,n-k+2} & \cdots & S_{n-k,n-1} \\ & T_{n-k+1} & S_{n-k+1,n-k+2} & \cdots & S_{n-k+1,n-1} \\ & \ddots & & & \vdots \\ & & \ddots & & \vdots \\ & 0 & & T_{n-2} & S_{n-2,n-1} \\ & & & & T_{n-1} \end{pmatrix},$$

where $2 \leq k \leq n$, and set

$$\mu_{n-k,n-1} = -\frac{\sum_{i=1}^{k-2} m_{n-k,n-k+i}\mu_{n-k+i,n-1} + m_{n-k,n-1}}{k-1},$$

then $\mu_{n-k,n-1}$ is defined involving only the coefficients $\mu_{n-k+i,n-1}$ which exist by the induction hypothesis. Thus coefficients $\mu_{i,j}$ depends only on the $m_{i,j}, i, j \leq n-1$. By a direct computation, $\gamma_k = \mu_{0,k}t_0^{(k)} + \mu_{1,k}t_1^{(k-1)} + \cdots + \mu_{k,k}t_k, 0 \leq k < n-1$ together defines a frame for $E_T$. This completes the proof of the Claim and the third statement of the lemma. □

3.3. Boundedness. Having shown that a holomorphic quasi-homogeneous curve $t$ defines a linear transformation on a dense subset of $\mathcal{H}_t$, we determine when it extends to a bounded linear operator on all of $\mathcal{H}_t$. We make the following conventions here which will be in force throughout this paper.

3.3.1. Conventions. The positive definite kernel $K^{(\lambda)}(z, w)$ is the function $(1-\bar{w}z)^{-\lambda}$ defined on $\mathbb{D}\times\mathbb{D}$ and is the reproducing kernel for the weighted Bergman space $\mathbb{A}^{(\lambda)}(\mathbb{D})$. The coefficient $a_n(\lambda)$ of $\bar{w}^n z^n$ in the power series expansion for $K^{(\lambda)}$ (in powers of $z\bar{w}$) is of the form $a_n(\lambda) \sim n^{\lambda-1}$ using Stirling's formula: $\frac{\Gamma(\lambda+n)}{\Gamma(n)} \sim n^{\lambda}$. The set of vectors $e_n^{(\lambda)} := \sqrt{a_n(\lambda)}\, z^n,\ n \geq 0$, is an orthonormal basis in $\mathbb{A}^{(\lambda)}(\mathbb{D})$. The action of the multiplication operator on $\mathbb{A}^{(\lambda)}(\mathbb{D})$ is easily determined:

$$M(e_n^{(\lambda)}) \sim \Big(\frac{n}{n+1}\Big)^{\frac{\lambda-1}{2}} e_{n+1}^{(\lambda)}.$$

Often, one sets $w_n^{(\lambda)} := \frac{\sqrt{a_n(\lambda)}}{\sqrt{a_{n+1}(\lambda)}}$ and says that $M$ is a weighted shift with weights $w_n^{(\lambda)}$ since $M(e_n^{(\lambda)}) = w_n^{(\lambda)} e_{n+1}^{(\lambda)}$. The other way round, $\prod_{i=0}^{n} w_i^{(\lambda)} = \sqrt{\frac{a_0(\lambda)}{a_{n+1}(\lambda)}} \sim (n+1)^{\frac{1-\lambda}{2}}$. The adjoint of this operator is then given by the formula:

$$M^*(e_n^{(\lambda)}) = w_{n-1}^{(\lambda)} e_{n-1}^{(\lambda)} \sim \Big(\frac{n-1}{n}\Big)^{\frac{\lambda-1}{2}} e_{n-1}^{(\lambda)}.$$

The following Proposition shows that if the valency $\Lambda(t)$ is less than 2, then every possible linear combination of the atoms and their jets need not define a bounded linear transformation. However, from the proof of this Proposition, we infer that no such obstruction can occur if $\Lambda(t) \geq 2$.

**Proposition 3.2.** *Fix a natural number $n \geq 2$. Let $t$ be a quasi-homogeneous holomorphic curve with atoms $t_i,\ i = 0, 1, \ldots, n-1$. For $0 \leq i, j \leq n-1$, let $s_{i,j}(t_j(w)) = m_{i,j}t_i^{(j-i-1)}(w)$ be the bundle map from $E_{t_j}$ to $\mathcal{J}_{j-i-1}E_{t_i}$ and $S_{i,j} : \mathcal{H}_j \to \mathcal{H}_i$ be the densely defined linear transformation induced by the* *maps $s_{i,j}$. The linear transformation of the form*

$$T = \begin{pmatrix} T_0 & S_{0,1} & S_{0,2} & \cdots & S_{0,n-1} \\ 0 & T_1 & S_{1,2} & \cdots & S_{1,n-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & T_{n-2} & S_{n-2,n-1} \\ 0 & 0 & \dots & 0 & T_{n-1} \end{pmatrix}$$

*is densely defined on the Hilbert space $\mathbb{A}^{(\lambda_0)}(\mathbb{D}) \oplus \cdots \oplus \mathbb{A}^{(\lambda_{n-1})}(\mathbb{D})$. Suppose that $\Lambda(t) < 2$.*

(1) *If $\Lambda(t) \in [1 + \frac{n-3}{n-1}, 2)$, $n \geq 2$, then $T$ is bounded.*

(2) *If $\Lambda(t) \in [1 + \frac{n-k-4}{n-k-2}, 1 + \frac{n-k-3}{n-k-1})$, the operator $T$ is bounded only if we set $m_{i,j} = 0$ whenever $j - i \geq n - k - 2, n - 1 > k \geq 0$, $n \geq 4$, that is, $T$ must be of the form*

$$\begin{pmatrix} S_{0,0} & S_{0,1} & \cdots & S_{0,n-k-2} & 0 & \cdots & 0 & 0 \\ & S_{1,1} & S_{1,2} & \cdots & S_{1,n-k-1} & 0 & \cdots & 0 \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ & & & \ddots & \ddots & \ddots & \ddots & 0 \\ & \text{\huge 0} & & & S_{k+1,k+1} & S_{k+1,k+2} & \cdots & S_{k+1,n-1} \\ & & & & & \ddots & \ddots & \vdots \\ & & & & & & S_{n-2,n-2} & S_{n-2,n-1} \\ & & & & & & & S_{n-1,n-1} \end{pmatrix}$$

(3) *If $\Lambda(t) \in (0, 1)$, then the densely defined linear transformation $T$ is bounded only if we set $m_{i,j} = 0, i < j + 1, i = 0, 1, \cdots, n - 2, n \geq 3$.*

*Proof.* For $i = 0, 1, \cdots, n-1$, the operators $S_{i,i}$ are homogeneous by definition. Thus the operator $S_{i,i}$, as we have said before, is realized as the adjoint of the multiplication operator on the weighted Bergman space $\mathbb{A}^{(\lambda_i)}(\mathbb{D})$. The reproducing kernel $K^{(\lambda_i)}(z, w)$ for this Hilbert space is of the form $\frac{1}{(1-z\bar{w})^{\lambda_i}}$. Consequently,

$$\ker (S_{i,i} - w)^* = \mathbb{C}[t_i(\bar{w})] = \mathbb{C}[K^{(\lambda_i)}(z, w)], w \in \mathbb{D}.$$

Claim : If $\lambda_j - \lambda_i > 2(j - i) - 2, j > i = 0, 1, 2, \cdots, n - 2$, then each $s_{i,j}$ induces a non-zero linear bounded operator $S_{i,j}$.

Without loss of generality, we set $s_{i,j}(t_j) = m_{i,j} t_i^{(j-i-1)}, m_{i,j} \in \mathbb{C}, i, j = 0, 1, \cdots, n-1$ and

$$t_i(w) = \frac{1}{(1 - zw)^{\lambda_i}}, t_j(w) = \frac{1}{(1 - zw)^{\lambda_j}}.$$

Then the linear transformation $S_{i,j} : \mathcal{H}_j \to \mathcal{H}_i$ induced by $s_{i,j}$ is densely defined by the rule

$$S_{i,j}(t_j) = m_{i,j} t_i^{(j-i-1)}, i, j = 0, 1, \cdots, n - 1.$$

We have that

$$||S_{i,j}|| = |m_{i,j}| \max_{\ell} \{ \frac{\sqrt{\prod\limits_{l=0}^{\ell-(j-i)} w_l(\lambda_j)}}{\sqrt{\prod\limits_{l=0}^{\ell-1} w_l(\lambda_i)}} \ell(\ell - 1) \cdots (\ell - (j - i) + 2) \} .$$

By a direct computation,

$$\frac{\sqrt{\prod\limits_{l=0}^{\ell-(j-i-1)} w_l(\lambda_j)}}{\sqrt{\prod\limits_{l=1}^{\ell-1} w_l(\lambda_i)}} \ell(\ell - 1) \cdots (\ell - (j - i) + 2) \sim \Big( \frac{1}{\ell^{\frac{\lambda_j - \lambda_i}{2} - (j-i-1)}} \Big).$$

It follows that each $S_{i,j}$ is a non-zero bounded linear operator if and only if

$$\frac{\lambda_j - \lambda_i}{2} \geq j - i - 1, \text{ that is, } \lambda_j - \lambda_i \geq 2(j-i) - 2.$$

If $\Lambda(t) \geq 1 + \frac{n-3}{n-1}$, then

$$\lambda_{n-1} - \lambda_0 = (n-1)\Lambda(t) \geq 2(n-2).$$

By the argument given above, we obtain $S_{0,n-1}$ is non-zero and bounded. If $\Lambda(t) < 1 + \frac{n-3}{n-1}$, then we might deduce that $m_{0,n-1} = 0$ or $\mu_{0,n-1} = 0$, i.e. $S_{0,n} = 0$. Thus the proof of the first statement is complete.

For the general case, if $\Lambda(t) \in [1 + \frac{n-k-4}{n-k-2}, 1 + \frac{n-k-3}{n-k-1})$, $k \geq 0$, then we have

$$(n-k-1)\Lambda(t) < 2(n-k-1) - 2.$$

On the other hand, if $j - i \geq n-k-1$, then we obtain $\lambda_j - \lambda_i \leq 2(j-i) - 2$. By the argument above, we have $S_{i,j} = 0, j - i \geq n-k-1$, and $S$ has the following matrix form:

$$T = \begin{pmatrix} S_{0,0} & \cdots & S_{0,n-k-2} & & 0 & \cdots & 0 \\ & S_{1,1} & \cdots & S_{1,n-k-1} & & 0 \ \cdots & 0 \\ & & \ddots & & \ddots & \ddots & \vdots \\ & & & \ddots & & \ddots & 0 \\ & \text{\huge 0} & & S_{k+1,k+1} & & \cdots & S_{k+1,n-1} \\ & & & & \ddots & & \vdots \\ & & & & & \ddots & \vdots \\ & & & & & & S_{n-1,n-1} \end{pmatrix} \tag{3.3}$$

This completes the proof of the second statement.

In particular, if $0 \leq \Lambda(t) < 1$ and $j - i \geq 2$, then we have $\lambda_j - \lambda_i \leq 2(j-i) - 2$, which implies

$$T = \begin{pmatrix} S_{0,0} & S_{0,1} & 0 & \cdots & 0 \\ 0 & S_{1,1} & S_{1,2} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & S_{n-2,n-2} & S_{n-2,n-1} \\ 0 & \cdots & \cdots & 0 & S_{n-1,n-1} \end{pmatrix}, \ \Lambda(t) \in [0,1).$$

This completes the proof of the third statement. □

Having disposed off the question of boundedness of a quasi-homogeneous operator, we show that all quasi-homogeneous operators are in the class $\mathcal{F}B_n(\mathbb{D})$.

**Theorem 3.3.** *Suppose $T$ is a quasi-homogeneous operator and $(\!(S_{i,j})\!)_{n\times n}$ is its atomic decomposition. Then we have*

$$S_{i,i}S_{i,i+1} = S_{i,i+1}S_{i+1,i+1}, i = 0, 1, \cdots, n-2,$$

*or equivalently, $T$ is in $\mathcal{F}B_n(\mathbb{D})$.*

*Proof.* We have found constants $m_{i,j} \in \mathbb{C}$ such that

$$S_{i,j}(t_j) = m_{i,j} t_i^{(j-i-1)}, i < j = 0, 1, \cdots, n-1$$

in the second statement of Proposition 3.1. Since $(S_{i,i} - w)(t_i(w)) = 0, \ w \in \Omega$, it follows that

$$S_{i,i}S_{i,i+1}(t_{i+1}(w)) = S_{i,i+1}S_{i+1,i+1}(t_{i+1}(w)).$$

We have $\mathcal{H}_i = \text{Span}_{w\in\Omega}\{t_i(w)\}, i = 0, 1 \cdots, n-1$, therefore

$$S_{i,i}S_{i,i+1} = S_{i,i+1}S_{i+1,i+1}, i = 0, 1, \cdots, n-2.$$

□

3.4. The Second fundamental form. In [5, page. 2244], an explicit formula for the second fundamental form of a holomorphic Hermitian line bundle in its first order jet bundle of rank 2 was given. The second fundamental form, in a slightly different guise, was shown to be a unitary invariant for the class of operators $\tilde{\mathcal{F}}B_n(\Omega)$ in [11]. We give the computation of the second fundamental form here, yet again, keeping track of certain constants which appear in the description of the quasi-homogeneous operators. We compute the second fundamental form of the inclusion $E_0$ in $E$, where $\{\gamma_0, \gamma_1\}$ is a frame for $E$ with atoms $t_0$ and $t_1$. The line bundle defined by the atom $t_0$ is $E_0$. By necessity, we have

$$\gamma_0 = t_0 \quad \gamma_1 = \mu_{01} t_0' + t_1$$

with $t_0 \perp t_1$. As in [5, 11], setting $h = \langle \gamma_0, \gamma_0 \rangle$, the second fundamental form $\theta_{0,1}$ is seen to be of the form

$$\theta_{0,1} = -h^{1/2} \frac{\bar{\partial}(h^{-1}\langle \gamma_1, \gamma_0 \rangle)}{\big(\|\gamma_1\|^2 - \frac{|\langle \gamma_1, \gamma_0 \rangle|^2}{\|\gamma_0\|^2}\big)^{1/2}}.$$

It is important, for what follows, to express $\theta_{0,1}$ in terms of the atoms $t_0$ and $t_1$ giving the formula

$$\theta_{0,1} = \frac{\mu_{0,1}\mathcal{K}_0}{\big(\frac{\|t_1\|^2}{\|t_0\|^2} - |\mu_{0,1}|^2 \mathcal{K}_0\big)^{1/2}}, \tag{3.4}$$

where $\mathcal{K}_0$ is the curvature of the line bundle $E_{t_0}$ given by the formula $-\bar{\partial}\partial \log \|t_0\|^2$. The following lemma shows the key role of the second fundamental form in determining the unitary equivalence class of a quasi-homogeneous holomorphic curve.

**Lemma 3.4.** *Suppose that $t$ and $\tilde{t}$ are quasi-holomorphic curves with the same atoms $t_0$, $t_1$. Then the following statements are equivalent.*

(1) *The two curves $t$ and $\tilde{t}$ are unitarily equivalent;*
(2) *The second fundamental forms $\theta_{0,1}$ and $\tilde{\theta}_{0,1}$ are equal;*
(3) *The two constants $\mu_{0,1}$ and $\tilde{\mu}_{0,1}$ are equal.*

*Proof.* The equivalence of the first two statements was proved in [11, Corollary 2.8]. The equality of $\theta_{0,1}$ and $\tilde{\theta}_{0,1}$ is clearly equivalent to

$$\tilde{\mu}_{0,1}\big(\tfrac{\|t_1\|^2}{\|t_0\|^2} + |\mu_{0,1}|^2 \bar{\partial}\partial \log \|t_0\|^2\big)^{1/2} = \mu_{0,1}\big(\tfrac{\|t_1\|^2}{\|t_0\|^2} + |\tilde{\mu}_{0,1}|^2 \bar{\partial}\partial \log \|t_0\|^2\big)^{1/2}.$$

From this equality, we infer that $\arg(\mu_{0,1}) = \arg(\tilde{\mu}_{0,1})$.

Given that we have assumed, without loss of generality, $\|t_0\|^2 = (1-|w|^2)^{-\lambda_0}$ and $\|t_0\|^2 = (1-|w|^2)^{-\lambda_1}$, squaring both sides and then taking the difference of the equality displayed above, we find that

$$\bar{\partial}\partial \log \|t_0\|^2 = \lambda_0(1-|w|^2)^{-2},$$

which can be equal to $\frac{\|t_1\|^2}{\|t_0\|^2}$ if and only if $\lambda_1 - \lambda_0 = 2$. Thus except when $\Lambda(t) = 2$, we must have $\mu_{0,1}^2 - \tilde{\mu}_{0,1}^2 = 0$. Clearly, $\tilde{\mu}_{0,1} = -\mu_{0,1}$ is not an admissible solution. So, we must have $\tilde{\mu}_{0,1} = \mu_{0,1}$. In case $\lambda_1 - \lambda_0 = 2$, if we assume $\tilde{\mu}_{0,1} \neq \mu_{0,1}$, then we must have

$$\Big(\frac{1+\lambda_0|\tilde{\mu}_{0,1}|^2}{1+\lambda_0|\mu_{0,1}|^2}\Big)^{\frac{1}{2}} = \frac{|\tilde{\mu}_{0,1}|}{|\mu_{0,1}|},$$

from which it follows that $|\tilde{\mu}_{0,1}| = |\mu_{0,1}|$. The arguments of these complex numbers being equal, they must be actually equal. □

When we consider the inclusion of the line bundle $E_{t_i}$ in the vector bundle $E_{\{t_i, \frac{m_{i,j}}{j-i}t_i^{(j-i)} + t_j\}}$ of rank 2, the situation is slightly different. This is the vector bundle which corresponds to the $2 \times 2$ operator block $T_{i,j} := \begin{pmatrix} S_{i,i} & S_{i,j} \\ 0 & S_{j,j} \end{pmatrix}$.

Clearly, $\{t_i, -\frac{m_{i,j}}{j-i}t_i^{(j-i)} + t_j\}$ is the frame for $E_{T_{i,j}}$. By the formulae above, setting *temporarily* $\gamma_0 = t_i, \gamma_1 = -\frac{m_{i,j}}{j-i}t_i^{(j-i)} + t_j$, we have that

(1) $h_i = ||\gamma_0||^2 = ||t_i||^2, h_j = ||t_j||^2$;
(2) $||\gamma_1||^2 = |\frac{m_{i,j}}{j-i}|^2 \partial^{j-i}\overline{\partial}^{j-i}||t_i||^2 + ||t_j||^2 = |\frac{m_{i,j}}{j-i}|^2 \partial^{j-i}\overline{\partial}^{j-i} h_i + h_j$;
(3) $< \gamma_1, \gamma_0 >= -\frac{m_{i,j}}{j-i}\partial^{j-i}||t_i||^2 = -\frac{m_{i,j}}{j-i}\partial^{j-i}h_i$;
(4) $| < \gamma_1, \gamma_0 > |^2 = |\frac{m_{i,j}}{j-i}|^2 \partial^{j-i}h_i\overline{\partial}^{j-i}h_i$.

The second fundamental form $\theta_{i,j}$ for the inclusion $E_{t_i} \subseteq E_{\{t_i, \frac{m_{i,j}}{j-i}t_i^{(j-i)}+t_j\}}$ is given by the formula

$$\theta_{i,j} = \frac{\frac{m_{i,j}}{j-i}\overline{\partial}(h_i^{-1}\partial^{j-i}h_i)}{(\frac{h_j}{h_i} + |\frac{m_{i,j}}{j-i}|^2(\frac{h_i\partial^{j-i}\overline{\partial}^{j-i}h_i - \partial^{j-i}h_i\overline{\partial}^{j-i}h_i}{h_i^2}))^{\frac{1}{2}}}. \tag{3.5}$$

**Lemma 3.5.** *Let* $T_{i,j} := \begin{pmatrix} S_{i,i} & S_{i,j} \\ 0 & S_{j,j} \end{pmatrix}$ *and* $\widetilde{T}_{i,j} := \begin{pmatrix} S_{i,i} & \widetilde{S}_{i,j} \\ 0 & S_{j,j} \end{pmatrix}$ *with* $\widetilde{S}_{i,j}(t_j) = \widetilde{m}_{i,j}t_i^{(j-i-1)}$. *The second fundamental forms* $\theta_{i,j}$ *and* $\widetilde{\theta}_{i,j}$ *of the operators* $T_{i,j}$ *and* $\widetilde{T}_{i,j}$ *are equal, that is,* $\theta_{i,j} = \widetilde{\theta}_{i,j}$ *if and only if* $m_{i,j} = \widetilde{m}_{i,j}$.

*Proof.* Without loss of generality, we will give the proof only for the case $i = 0, j = k, j \neq 1$. In this case, $\theta_{0,k} = \widetilde{\theta}_{0,k}$ is equivalent to the equality:

$$\frac{(\frac{h_k}{h_0} + |\frac{m_{0,k}}{k}|^2(\frac{h_0\partial^k\overline{\partial}^k h_0 - \partial^k h_0\overline{\partial}^k h_0}{h_0^2}))^{\frac{1}{2}}}{(\frac{h_k}{h_0} + |\frac{\widetilde{m}_{0,k}}{k}|^2(\frac{h_0\partial^k\overline{\partial}^k h_0 - \partial^k h_0\overline{\partial}^k h_0}{h_0^2}))^{\frac{1}{2}}} = \frac{m_{0,k}}{\widetilde{m}_{0,k}}$$

For simplicity, let $g_0$ denote $(\frac{h_0\partial^k\overline{\partial}^k h_0 - \partial^k h_0\overline{\partial}^k h_0}{h_0^2})$ and let $m$, $\widetilde{m}$ denote $\frac{m_{0,k}}{k}$, $\frac{\widetilde{m}_{0,k}}{k}$ respectively. Then the equation given above may be rewritten as

$$\frac{(\frac{h_k}{h_0} + |m|^2 g_0)^{\frac{1}{2}}}{(\frac{h_k}{h_0} + |\widetilde{m}|^2 g_0)^{\frac{1}{2}}} = \frac{m}{\widetilde{m}}$$

From this equality, we infer that $\arg(m) = \arg(\widetilde{m})$. Now, squaring both sides and then taking the difference, we have

$$\frac{h_k}{h_0}(\tilde{m}^2 - m^2) - \tilde{m}^2 m^2 g_0(\bar{m}^2 - \bar{\tilde{m}}^2) = 0.$$

Having assumed, without loss of generality, $h_0 = (1-|w|^2)^{-\lambda_0}$ and $h_k = (1-|w|^2)^{-\lambda_1}$, we find that $g_0$ is a polynomial of degree $> 1$ in $(1-|w|^2)^{-1}$. Thus $g_0$ can be equal to $\frac{h_k}{h_0}$ if and only if $\lambda_1 - \lambda_0 = 2$. Therefore, except when $\Lambda(t) = 2$, we must have $m^2 - \widetilde{m}^2 = 0$. Clearly, $m = -\widetilde{m}$ is not an admissible solution. So, we must have $m = \widetilde{m}$. Hence $m_{0,k} = \widetilde{m}_{0,k}$. □

3.5. Unitary equivalence. Recall that a positive definite kernel $K : \Omega \times \Omega \to \mathbb{C}^{n\times n}$ is said to be normalized at $w_0 \in \Omega$, if $K(z, w_0) = I$, $z \in \Omega$. An operator $T$ in $B_n(\Omega)$ may be realized, up to unitary equivalence, as the adjoint of a multiplication operator on a Hilbert space possessing a normalized reproducing kernel (cf. [4]). Realized in this form, the operator is determined completely modulo multiplication by a constant unitary operator acting on $\mathbb{C}^n$. As one might expect, finding the normalized kernel if $n > 1$ is not easy. The theorem below illustrates a rigidity phenomenon in the spirit of what was proved by Curto and Salinas for operators in $B_n(\mathbb{D})$. For quasi-homogeneous operators, the atoms are homogeneous operators in $B_1(\mathbb{D})$. These are assumed to be realized in normal form. Consequently, if $T$ is a quasi-homogeneous operator, a set of $n-1$ fundamental forms determine the operator $T$ completely, that is, two of them are unitarily equivalent if and only if they are equal assuming they have the same set second fundamental forms.

**Theorem 3.6.** *Suppose that* $t$ *and* $\tilde{t}$ *are unitarily equivalent. Then if the second fundamental forms are the same, that is,* $\theta_{i,i+1} = \tilde{\theta}_{i,i+1}$, $0 \leq i \leq n-2$, *then* $t = \tilde{t}$.

*Proof.* If necessary, conjugating by a diagonal unitary, without loss of generality, we may assume that the atoms of the operators $T$ and $\widetilde{T}$ are the same. If there exists a unitary operator $U$ such that $TU = U\tilde{T}$, then $U$ must be diagonal with unitaries $U_0, U_1, \ldots U_{n-1}$ on its diagonal. Then we have

$$U_i S_{i,j} = \tilde{S}_{i,j} U_j,\ i,j = 0,1,\ldots,n-1.$$

In particular, $U_i$ commutes with the fixed set of atoms $T_i$, which are irreducible, therefore there exists $\beta_i \in [0, 2\pi]$ such that

$$U_i = e^{\imath\beta_i} I_{\mathcal{H}_i}, i = 0,1,\cdots,n-1.$$

Then on the one hand, we have

$$U_i S_{i,i+1}(t_{i+1}) = U_i(-\mu_{i,i+1} t_i) = -\mu_{i,i+1} e^{\imath\beta_i} t_i$$

and on the other hand, we have

$$\tilde{S}_{i,i+1} U_{i+1}(t_{i+1}) = S_{i,i+1}(e^{\imath\beta_{i+1}} t_{i+1}) = -\tilde{\mu}_{i,i+1} e^{\imath\beta_{i+1}} t_i.$$

Consequently,

$$-\mu_{i,i+1} e^{\imath\beta_i} = -\tilde{\mu}_{i,i+1} e^{\imath\beta_{i+1}},\ 0 \leq i \leq n-2.$$

The assumption that the second fundamental forms are the same for the two operators $T$ and $\widetilde{T}$ implies that $\mu_{i,i+1} = \tilde{\mu}_{i,i+1}$. Therefore, we have $\beta_i = \beta_{i+1} := \beta,\ i = 0,1,\ldots,n-2$. Since

$$U_i S_{i,j} = \tilde{S}_{i,j} U_j,\ i,j = 0,1,\ldots,n-1,$$

we have

$$U_i S_{i,j}(t_j) = e^{\imath\beta} m_{i,j} t_i^{(j-i-1)} = e^{\imath\beta} \widetilde{m}_{i,j} t_i^{(j-i-1)} = \widetilde{S}_{i,j} U_j(t_j).$$

Then $m_{i,j} = \widetilde{m}_{i,j}, i,j = 0,1,\ldots,n-1$. It follows that $S_{i,j} = \widetilde{S}_{i,j}$ and $t = \tilde{t}$. □

**Remark 3.7.** It is natural to ask which of the quasi-homogeneous operators are homogeneous. A comparison with the homogeneous operators given in [18] shows that a quasi-homogeneous operator is homogeneous if and only if

$$\mu_{i,j} = \frac{\Gamma_{i,j}(\lambda)\mu_i}{\mu_j},\ \Gamma_{i,j}(\lambda) = \binom{i}{j}\frac{1}{(2\lambda_j)_{i-j}},\ \lambda_j = \lambda - \tfrac{m}{2} + j, \tag{3.6}$$

for some choice of positive constants $\mu_0(:= 1), \mu_1, \ldots, \mu_{n-1}$. Here $(\alpha)_\ell := \alpha(\alpha+1)\cdots(\alpha+\ell-1)$ is the Pochhammer symbol. Clearly, if two homogeneous operators with $(\lambda, \boldsymbol{\mu})$ and $(\tilde{\lambda}, \tilde{\boldsymbol{\mu}})$ were unitarily equivalent, then $\lambda$ must equal $\tilde{\lambda}$. Since it is easy to see that $\mu_{i,i+1} = \tilde{\mu}_{i,i+1}$ if and only if $\mu_i = \tilde{\mu}_{i+1}$, we conclude that two of these homogeneous operators are unitarily equivalent if and only if they are equal recovering previous results of [18].

## 4. Canonical model under similarity

In this section, our main focus is on the question of reducibility and strong irreducibility of a quasi-homogeneous operator. We recall that an operator $T$ is said to be strongly irreducible if there is no idempotent in its commutant, or equivalently, there does not exist an invertible operator $L$ for which $LTL^{-1}$ is reducible. The (multiplicity-free) homogeneous operators in the Cowen-Douglas class of rank $n$ are irreducible (cf. [18]). However, they were shown (cf. [17]) to be similar to the $n$ - fold direct sum of their atoms making them strongly reducible. It is this phenomenon that we investigate here for quasi-homogeneous operators. Along the way, we determine when two quasi-homogeneous operators are similar. Our investigations show that there is dichotomy which depends on whether or not the valency $\Lambda(t)$ is less than 2 or greater or equal to 2. In what follows, we will say that a holomorphic curve $t : \mathbb{D} \to Gr(n, \mathcal{H})$ is strongly irreducible if there is no invertible operator $X$ on the Hilbert space $\mathcal{H}$ for which $Xt$ splits into orthogonal direct sum of two holomorphic curves, say $t_1$ and $t_2$, in $Gr(n_1, \mathcal{H})$ and $Gr(n_2, \mathcal{H})$, $n_1 + n_2 = n$, respectively.

Suppose $t : \mathbb{D} \to Gr(n, \mathcal{H})$ is a quasi-homogeneous holomorphic curve with atoms $t_0, t_1, \ldots, t_{n-1}$. Then $t$ is strongly reducible, $t \sim t_0 \oplus t_1 \cdots \oplus t_{n-1}$, if $\Lambda(t) \geq 2$ and strongly irreducible otherwise. The dichotomy involving the valency $\Lambda(t)$ is also clear from the main theorem on similarity Theorem 4.2 of quasi-homogeneous holomorphic curves.

The atoms of a quasi-homogeneous operator are homogeneous operators in $B_1(\mathbb{D})$ by definition. Therefore, they are uniquely determined not only up to unitary equivalence but upto similarity as well. Now, pick any two quasi-homogeneous operators. They possess an atomic decomposition by virtue of Proposition 3.1. Any invertible operator intertwining these two quasi-homogeneous operators is necessarily upper triangular:

**Lemma 4.1.** *Let $t$ and $\tilde{t}$ be two quasi-homogeneous holomorphic curves with atomic decomposition $\{t_i : i = 0, 1, \ldots, n-1\}$ and $\{\tilde{t}_i : i = 0, 1, \ldots, n-1\}$, respectively. If they are quasi-similar via the intertwining operators $X$ and $Y$, that is, $Xt = \tilde{t}$ and $Y\tilde{t} = t$, then for $i \leq n-1$, we have*

$$X\big(\bigvee\{t_0(w), t_1(w), \cdots, t_i(w) : w \in \mathbb{D}\}\big) \subseteq \bigvee\{\tilde{t}_0(w), \tilde{t}_1(w), \cdots, \tilde{t}_i(w) : w \in \mathbb{D}\},$$
$$Y\big(\bigvee\{\tilde{t}_0(w), \tilde{t}_1(w), \cdots, \tilde{t}_i(w) : w \in \mathbb{D}\}\big) \subseteq \bigvee\{t_0(w), t_1(w), \cdots, t_i(w) : w \in \mathbb{D}\}.$$

This is easily proved by modifying the proof [11, Proposition 3.3] slightly. Hence if two quasi-homogeneous operators are similar, then each of the atoms for one must be similar to the other. Consequently, to determine equivalence of quasi-homogeneous operators $T$ under an invertible linear transformation, we may assume (as before) without loss of generality that the atoms are fixed with the weight $\lambda_0$ and the valency $\Lambda(t)$. Clearly, the valency $\Lambda(t)$ is both an unitary as well as a similarity invariant of the quasi-homogeneous curve $t$.

Note that if we let $R$ be the $n \times n$ diagonal matrix with $\big(\prod_{\ell=0}^{i} \mu_{\ell,\ell+1}\big)\big(\prod_{\ell=0}^{i} \tilde{\mu}_{\ell,\ell+1}\big)^{-1}$ on its diagonal and set $\tilde{t} = R\,t\,R^{-1}$, then $\tilde{S}_{i,i+1}(t_{i+1}) = \tilde{\mu}_{i,i+1}$, $0 \leq i \leq n-2$. Thus up to similarity, we may assume that the constants $\mu_{i,i+1}$ and $\tilde{\mu}_{i,i+1}$ are the same. Or equivalently (see Lemma 3.4), we may assume that the choice of the second fundamental forms $\theta_{i,i+1}$, $0 \leq i \leq n-2$, does not change the similarity class of a quasi-homogeneous holomorphic curve. Therefore the condition in the second statement of the theorem given below is not a restriction on the similarity class of the holomorphic curves $t$ and $\tilde{t}$.

**Theorem 4.2.** *Suppose $t$ and $\tilde{t}$ are quasi-homogeneous holomorphic curves.*

(1) *If $\Lambda(t) \geq 2$, then $t$ is similar to the $n$ - fold direct sum of the atoms $t_0 \oplus t_1 \oplus \cdots \oplus t_{n-1}$.*
(2) *If $\Lambda(t) = \Lambda(\tilde{t}) < 2$ and $\theta_{i,i+1} = \tilde{\theta}_{i,i+1}$, $i = 0, 1, \cdots, n-2$, then $t$ and $\tilde{t}$ are similar if and only if they are equal.*

4.1. The Key Lemma. The following lemma is the key to determining when a bundle map that intertwines two quasi-homogeneous holomorphic vector bundles extends to an invertible bounded operator. It reveals the intrinsic structure of the intertwiners between two quasi-homogeneous bundles. We follow the conventions set up in Section 3.3.1.

**Lemma 4.3.** *Let $E_t$ be a quasi-homogeneous vector bundle and $s_{i,j}, i, j = 0, 1, \cdots, n-1$ be the induced bundle maps. There exists a bundle map $X : E_{t_{n-1}} \to \mathcal{J}_{n-1}(E_{t_0})$ with the intertwining property*

$$s_{0,0}X - Xs_{n-1,n-1} = s_{0,n-1}$$

*that extends to a bounded linear operator only if $\Lambda(t) \geq 2$.*

*Proof.* Let $T_0$ and $T_{k+1}$ be the operators induced by $s_{0,0}$ and $s_{k+1,k+1}$ as in in Proposition 3.1. These are then necessarily the operators $M^{(\lambda_0)^*}$ and $M^{(\lambda_{k+1})^*}$ acting on the weighted Bergman spaces $\mathbb{A}^{(\lambda_0)}(\mathbb{D})$ and $\mathbb{A}^{(\lambda_{k+1})}(\mathbb{D})$, respectively.

The kernel of the operator $(T_i - w)$, $w \in \mathbb{D}$, is spanned by the vector $t_i(w) := (1 - z\bar{w})^{-\lambda_i}$, $i = 0, k+1$. By hypothesis, for each fixed $w \in \mathbb{D}$, we have $S_{0,k+1}((1 - z\bar{w})^{-\lambda_{k+1}}) = \bar{\partial}^k(1 - z\bar{w})^{-\lambda_0}$. Differentiating both sides of this equation $\ell$ times and then evaluating at $w = 0$, we get $S_{0,k+1}\big((\lambda_{k+1})_\ell z^\ell\big) = (\lambda_0)_{\ell+k} z^{\ell+k}$. For $j = 0$ or $j = k-1$, the set of vectors $e_\ell^{(\lambda_j)} := \sqrt{a_\ell(\lambda_j)}\, z^\ell$, $\ell \geq 0$ is an orthonormal basis in $\mathbb{A}^{(\lambda_j)}(\mathbb{D})$. The matrix representation for the operator $S_{0,k+1} : \mathbb{A}^{(\lambda_{k+1})}(\mathbb{D}) \to \mathbb{A}^{(\lambda_0)}(\mathbb{D})$ with respect to this orthonormal basis is obtained from the computation:

$$S_{0,k+1}\big(e_\ell^{(\lambda_{k+1})}\big) = \tfrac{(\ell+k)!}{\ell!}\sqrt{\tfrac{a_{\ell+k}(\lambda_0)}{a_\ell(\lambda_{k+1})}}\, e_{\ell+k}^{(\lambda_0)}.$$

Thus $S_{0,k+1}$ is a forward shift of multiplicity $k$. We claim that if $\Lambda(t) \geq 2$, then we can find a forward shift $X$ of multiplicity $k+1$, namely, $X(e_\ell^{(\lambda_{k+1})}) = x_\ell e_{\ell+k+1}^{(\lambda_0)}$ which has the required intertwining property. Thus evaluating the equation $S_{0,0}X - XS_{n-1,k+1} = S_{0,k+1}$ on the vectors $e_\ell^{(\lambda_{k+1})}$, $\ell \geq 0$, we obtain

$$\tfrac{(\ell+k)!}{\ell!}\frac{\prod_{i=0}^{\ell-1} w_i^{(\lambda_{k+1})}}{\prod_{i=0}^{\ell+k-1} w_i^{(\lambda_0)}}\, e_{\ell+k}^{(\lambda_0)} = \big(x_\ell w_{\ell+k}^{(\lambda_0)} - x_{\ell-1} w_{\ell-1}^{(\lambda_{k+1})}\big) e_{\ell+k}^{(\lambda_0)}. \tag{4.1}$$

From this we obtain $x_\ell$ recursively:

$$w_k^{(\lambda_0)} x_0 = k!\frac{\sqrt{a_k(\lambda_0)}}{\sqrt{a_0(\lambda_{(k+1)})}}$$

and for $\ell \geq 1$,

$$x_\ell = \sqrt{\tfrac{a_{k+\ell}(\lambda_0)}{a_\ell(\lambda_{k+1})}}\sum_{i=1}^{k}(\ell)_i \sim \Big(\ell^{\frac{\lambda_0 - \lambda_{k+1}+2k+2}{2}}\Big),$$

where $(\ell)_k := \ell(\ell+1)\cdots(\ell+k-1) = \frac{\Gamma(\ell+k)}{\Gamma(k)}$ is the Pochhammer symbol as before. Here, using the Stirling approximation for the $\Gamma$ function, we infer that $\sum_{i=1}^{k}(\ell)_i \sim \ell^{k+1}$.

If $\Lambda(t) \geq 2$, then $\lambda_1 - \lambda_0 \geq 2, \lambda_2 - \lambda_1 \geq 2, \cdots, \lambda_{k+1} - \lambda_k \geq 2$. Consequently, $\lambda_{k+1} - \lambda_0 \geq 2k+2$ making the operator $X$ bounded.

It follows that if $\Lambda(t) \geq 2$, then the shift $X$ of multiplicity $n$ that we have constructed is bounded and has the desired intertwining property. To show that there is no such intertwining operator if $\Lambda(t) < 2$, assume to the contrary the existence of such an operator. Then we show that there must also exist a shift of multiplicity $k+1$ with this property leading to a contradiction. For the proof, suppose

$$X\big(e_\ell^{(\lambda_{k+1})}\big) = \sum_{i=0}^{\infty} x_{i,\ell}\, e_i^{(\lambda_0)},\ X = \big(\!\!\big(x_{i,\ell}\big)\!\!\big).$$

Then

$$\big(S_{0,0}X - XS_{k+1,k+1}\big)\big(e_\ell^{(\lambda_{k+1})}\big) = \sum_{i=0}^{\infty}\big(x_{i+1,\ell+1}w_i^{(\lambda_0)} - x_{i,\ell}w_{\ell-1}^{(\lambda_{k+1})}\big)\big(e_i^{(\lambda_0)}\big).$$

In particular, we have

$$(x_{\ell+k+1,\ell+1}\, w_{\ell+k}^{(\lambda_0)} - x_{\ell+k,\ell}\, w_{\ell-1}^{(\lambda_{k+1})})(e_{l+k}^{(\lambda_0)}) = S_{0,k+1}\big(e_\ell^{(\lambda_{k+1})}\big).$$

Repeating the proof above, we will have the conclusion $x_{l+k,l} \to \infty, l \to \infty$ which proving the claim. $\square$

Recall that if $A$ and $B$ are two operators in $\mathcal{L}(\mathcal{H})$, then the Rosenblum operator $\tau_{A,B}$ is defined to be the operator $\tau_{A,B}(X) = AX - XB$, $X \in \mathcal{L}(\mathcal{H})$. If $A = B$, then we set $\sigma_A := \tau_{A,B}$.

**Lemma 4.4.** *Let $t$ be a quasi-homogeneous holomorphic curve with atoms $t_i$, $0 \leq i \leq n-1$. Let $T := \big(\!\!\big(S_{i,j}\big)\!\!\big)$ be the atomic decomposition of the operator $T$ representing $t$ as in Proposition 3.1.*

(1) *If $\Lambda(t) \in [1 + \frac{n-3}{n-1}, 1 + \frac{n-2}{n})$, then for any $1 \leq r < n-1$, we have*

$$S_{0,r}S_{r,r+1}\cdots S_{n-2,n-1} \in \operatorname{ran}\sigma_{S_{0,0},S_{n-1,n-1}}.$$

(2) *Suppose that* $\Lambda(t) \geq 2$. *Then there exists a bounded linear operator* $X \in \mathcal{L}(\mathcal{H}_{n-1}, \mathcal{H}_{n-2})$ *such that*

$$S_{n-2,n-2}X - XS_{n-1,n-1} = S_{n-2,n-1}$$

*and*

$$S_{n-3,n-2}X \in \operatorname{ran}\sigma_{S_{n-3,n-3},S_{n-1,n-1}}.$$

*Proof.* We only prove that $S_{0,n-2}S_{n-2,n-1}$ is in $\operatorname{ran}\sigma_{S_{0,0},S_{n-1,n-1}}$. Clearly, as can be seen from the proof we present below, the proof in all the other cases are exactly the same.

Let $T_0$, $T_{n-2}$ and $T_{n-1}$ be the operators induced by $s_{0,0}$, $s_{n-2,n-2}$ and $s_{n-1}$ as in in Proposition 3.1. These are then necessarily the operators $M^{(\lambda_0)^*}$, $M^{(\lambda_{n-2})^*}$ and $M^{(\lambda_{n-1})^*}$ acting on the weighted Bergman spaces $\mathbb{A}^{(\lambda_0)}(\mathbb{D})$, $\mathbb{A}^{(\lambda_{n-2})}(\mathbb{D})$ and $\mathbb{A}^{(\lambda_{n-1})}(\mathbb{D})$, respectively.

As in the proof of Lemma 4.3, equations (4.1), we have that

$$S_{0,n-2}\big(e_\ell^{(\lambda_{n-2})}\big) = \tfrac{(\ell+n-3)!}{\ell!}\sqrt{\tfrac{a_{\ell+n-3}(\lambda_0)}{a_\ell(\lambda_{n-2})}}\, e_{\ell+n-3}^{(\lambda_0)},$$

$$S_{n-2,n-1}(e_\ell^{(\lambda_{n-1})}) = \frac{\sqrt{a_\ell(\lambda_{n-2})}}{\sqrt{a_\ell(\lambda_{n-1})}} e_\ell^{(\lambda_{n-2})}$$

and

$$S_{0,n-2}S_{n-2,n-1}(e_\ell^{(\lambda_{n-1})}) = \tfrac{(\ell+n-3)!}{\ell!}\sqrt{\tfrac{a_{\ell+n-3}(\lambda_0)}{a_\ell(\lambda_{n-1})}}\, e_{\ell+n-3}^{(\lambda_0)}.$$

Thus $S_{0,n-2}S_{n-2,n-1}$ is a forward shift of multiplicity $n-3$. We claim that if $\Lambda(t) \geq 1 + \frac{n-3}{n-1}$, then we can find a forward shift $X$ of multiplicity $n-2$, namely, $X(e_\ell^{(\lambda_{n-1})}) = x_\ell e_{\ell+n-2}^{(\lambda_0)}$ which has the required intertwining property. Thus evaluating the equation $S_{0,0}X - XS_{n-1,n-1} = S_{0,n-1}$ on the vectors $e_\ell^{(\lambda_{n-1})}$, $\ell \geq 0$, we obtain

$$w_{n-3}^{(\lambda_0)} x_0 = (n-3)!\frac{\sqrt{a_{n-3}(\lambda_0)}}{\sqrt{a_0(\lambda_{(n-1)})}}$$

and for $\ell \geq 1$, we have that

$$w_{l+n-3}^{(\lambda_0)} x_\ell - x_{\ell-1} w_l^{(\lambda_n - 1)} = \tfrac{(\ell+n-3)!}{\ell!}\,\frac{\sqrt{a_{\ell+n-3}(\lambda_0)}}{\sqrt{a_\ell(\lambda_{(n-1)})}}.$$

It follows that

$$x_\ell = \frac{\sqrt{a_{\ell+n-3}(\lambda_0)}}{\sqrt{a_\ell(\lambda_{n-1})}}\sum_{i=1}^{n-3}(\ell)_i \sim \Big(\ell^{\frac{\lambda_0 - \lambda_{n-1} + 2n-4}{2}}\Big).$$

Note that when $\Lambda(t) > 1 + \frac{n-3}{n-1}$, we obtain

$$\lambda_{n-1} - \lambda_0 = (n-1)\Lambda(t) > (n-1)\frac{2n-4}{n-1} = 2n-4$$

making $X$ bounded. This completes the proof of the first statement.

For the proof of the second statement, note that by virtue of Lemma 4.3, we have $S_{n-2,n-1} \in \operatorname{Ran}\sigma_{S_{n-2,n-1}}$. So there exists a bounded operator $X$ such that

$$S_{n-2,n-2}X - XS_{n-1,n-1} = S_{n-2,n-1}.$$

Repeating the proof for the first part, we conclude

$$S_{n-3,n-2}X \in \operatorname{ran}\sigma_{S_{n-3,n-3},S_{n-1,n-1}}.$$

$\square$

4.2. **Strong irreducibility**. We now show that a quasi-homogeneous holomorphic curve $t$ is strongly irreducible or strongly reducible according as $\Lambda(t)$ is less than 2 or greater equal to 2. We recall that homogeneous operators (in this case, $\Lambda(t)=2$) were shown to be irreducible but strongly reducible in [17]

**Theorem 4.5.** *Fix a quasi-homogeneous holomorphic curve $t$ with atoms $t_i$ and let $T=(\!(S_{i,j})\!)$ be its atomic decomposition.*

(1) *If $\Lambda(t)\geq 2$, then $T$ is strongly reducible, indeed $T$ is similar to the direct sum of its atoms, namely, $\bigoplus_{i=0}^{n-1} T_i$ and*
(2) *if $\Lambda(t)<2$, then $T$ is strongly irreducible.*

*Proof.* If $\Lambda(t)\geq 2$, then we claim that the operator $T$ is similar to $T_0\oplus T_1\oplus\cdots\oplus T_{n-1}$.

When $n=2$, Let $T=\begin{pmatrix} S_{0,0} & S_{0,1}\\ 0 & S_{1,1}\end{pmatrix}$. By Lemma 4.3, there exists $X_{0,1}$ such that

$$S_{0,0}X_{0,1}-X_{0,1}S_{1,1}=S_{0,1}.$$

Set $Y_{0,1}=\begin{pmatrix} I & X_{0,1}\\ 0 & I\end{pmatrix}$, then we have that

$$Y_{0,1}TY_{0,1}^{-1}=\begin{pmatrix} S_{0,0} & 0\\ 0 & S_{1,1}\end{pmatrix}$$

Notice that $Y_{0,1}$ is invertible, we have that $T\sim S_{0,0}\oplus S_{1,1}$.

In this case, using Lemma 4.3, we find an invertible bounded linear operator $X_{0,n-1}$ such that

$$S_{0,0}X_{0,n-1}-X_{0,n-1}S_{n-1,n-1}=S_{0,n-1}.$$

For any $i<j$, applying Lemma 4.3 to the operators

$$\begin{pmatrix} S_{i,i} & S_{i,i+1} & S_{i,i+2} & \cdots & S_{i,j}\\ 0 & S_{i+1,i+1} & S_{i+1,i+2} & \cdots & S_{i+1,j}\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ 0 & \cdots & 0 & S_{j-1,j-1} & S_{j-1,j}\\ 0 & 0 & \cdots & 0 & S_{j,j}\end{pmatrix},$$

we find an invertible bounded linear operator $X_{i,j}$ such that $S_{i,i}X_{i,j}-X_{i,j}S_{j,j}=S_{i,j}$. Set $Y_{n-2,n-1}:=\left(\begin{array}{c|cc} I^{(n-2)} & \multicolumn{2}{c}{0}\\ \hline 0 & I & X_{n-2,n-1}\\ & 0 & I\end{array}\right)$ and note that $Y_{n-2,n-1}^{-1}=\left(\begin{array}{c|cc} I^{(n-2)} & \multicolumn{2}{c}{0}\\ \hline 0 & I & -X_{n-2,n-1}\\ & 0 & I\end{array}\right)$. Now, we have

$$\left(\begin{array}{c|cc} I^{(n-2)} & \multicolumn{2}{c}{0}\\ \hline 0 & I & X_{n-2,n-1}\\ & 0 & I\end{array}\right)\begin{pmatrix} S_{0,0} & S_{0,1} & S_{0,2} & \cdots & S_{0,n-1}\\ 0 & S_{1,1} & S_{1,2} & \cdots & S_{1,n-1}\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ 0 & \cdots & 0 & S_{n-2,n-2} & S_{n-2,n-1}\\ 0 & 0 & \cdots & 0 & S_{n-1,n-1}\end{pmatrix}\left(\begin{array}{c|cc} I^{(n-2)} & \multicolumn{2}{c}{0}\\ \hline 0 & I & -X_{n-2,n-1}\\ & 0 & I\end{array}\right)$$

$$=\begin{pmatrix} S_{0,0} & S_{0,1} & S_{0,2} & \cdots & S_{0,n-1}-S_{0,n-2}X_{n-2,n-1}\\ 0 & \ddots & \ddots & \ddots & \vdots\\ \vdots & \ddots & S_{n-3,n-3} & S_{n-3,n-2} & S_{n-3,n-1}-S_{n-3,n-2}X_{n-2,n-1}\\ 0 & \cdots & 0 & S_{n-2,n-2} & 0\\ 0 & \cdots & \cdots & 0 & S_{n-1,n-1}\end{pmatrix}.$$

By Lemma 4.4, we have

$$S_{n-3,n-2}X_{n-2,n-1}\in \operatorname{ran}\sigma_{S_{n-1,n-1},S_{n-3,n-3}}.$$

Therefore, there exists an invertible bounded linear operator $\widetilde{X}$ such that

$$S_{n-3,n-3}\widetilde{X}-\widetilde{X}S_{n-1,n-1}=S_{n-3,n-1}-S_{n-3,n-2}X_{n-2,n-1}.$$

Let $X_{n-3,n-1} := \widetilde{X}$ and $Y_{n-3,n-1} = \left(\begin{array}{c|ccc} I^{(n-3)} & & 0 & \\ \hline & I & 0 & X_{n-3,n-2} \\ 0 & 0 & I & 0 \\ & 0 & 0 & I \end{array}\right)$. Now, we have

$$Y_{n-3,n-1}\begin{pmatrix} S_{0,0} & S_{0,1} & S_{0,2} & \cdots & S_{0,n-1}-S_{0,n-2}X_{n-2,n-1} \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & S_{n-3,n-3} & S_{n-3,n-2} & S_{n-3,n-1}-S_{n-3,n-2}X_{n-2,n-1} \\ 0 & \cdots & 0 & S_{n-2,n-2} & 0 \\ 0 & \cdots & \cdots & 0 & S_{n-1,n-1} \end{pmatrix} Y^{-1}_{n-3,n-1}$$

$$= \begin{pmatrix} S_{0,0} & S_{0,1} & S_{0,2} & \cdots & S_{0,n-1}-S_{0,n-2}X_{n-2,n-1}-S_{0,n-3}X_{n-3,n-1} \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & S_{n-3,n-3} & \cdots & 0 \\ 0 & \cdots & 0 & S_{n-2,n-2} & 0 \\ 0 & \cdots & \cdots & 0 & S_{n-1,n-1} \end{pmatrix}.$$

Continuing in this manner, we clearly have

$$\begin{pmatrix} S_{0,0} & S_{0,1} & S_{0,2} & \cdots & S_{0,n-1} \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & S_{n-3,n-3} & S_{n-3,n-2} & S_{n-3,n-1} \\ 0 & \cdots & 0 & S_{n-2,n-2} & S_{n-2,n-1} \\ 0 & \cdots & \cdots & 0 & S_{n-1,n-1} \end{pmatrix} \sim \begin{pmatrix} S_{0,0} & S_{0,1} & \cdots & S_{0,n-2} & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & S_{n-3,n-3} & S_{n-3,n-2} & 0 \\ 0 & \cdots & 0 & S_{n-2,n-2} & 0 \\ 0 & \cdots & \cdots & 0 & S_{n-1,n-1} \end{pmatrix}.$$

This completes the proof of the induction step. We have therefore proved the first statement.

To prove the second statement, assuming that $\Lambda(t) < 2$, we must show that $E_t$ is strongly irreducible. First, we prove that $E_t$ is irreducible. By Lemma 4.1, any projection $P = \big(\!\big(P_{i,j}\big)\!\big)_{n\times n}$ in $\mathcal{A}'(E_t)$ is diagonal. Thus

$$P^2_{i,i} = P_{i,i} \in \mathcal{A}'(E_{t_i}).$$

It follows that for any $0 \le i \le n-1$, $P_{i,i} = 0$ or $P_{i,i} = I$. Since $PS = SP$, we have

$$P_{i,i}S_{i,i+1} = S_{i,i+1}P_{i+1,i+1}.$$

Therefore

$$P_{i,i} = P_{j,j}, i, j = 0, 1, \cdots, n-1.$$

Consequently, $P = 0$ or $P = I$ and $E_t$ is irreducible.

We first prove that $E_t$ is also strongly irreducible for $n = 2$. By Lemma 4.1, we have

$$S_{0,1} \notin \text{ran } \sigma_{S_{0,0}S_{1,1}}.$$

Let $P \in \mathcal{A}'(E_t)$ be an idempotent. By Lemma 3.3, $P$ has the following form

$$P = \begin{pmatrix} P_{0,0} & P_{0,1} \\ 0 & P_{1,1} \end{pmatrix}.$$

Since $PS = SP$, we have

$$P_{0,0}S_{0,0} = S_{0,0}P_{0,0}, P_{1,1}S_{1,1} = S_{1,1}P_{1,1}$$

and

$$P_{00}S_{0,1} - S_{0,1}P_{11} = S_{0,0}P_{0,1} - P_{0,1}S_{1,1}.$$

Since $P_{i,i} \in \{S_{i,i}\}'$, for $0 \le i \le 1$, so $P_{i,i}$ can be either $I$ or 0. If either $P_{1,1} = I$, $P_{0,0} = 0$ or $P_{0,0} = 0$, $P_{1,1} = I$, then $S_{0,1} \in \text{Ran } \sigma_{S_{0,0},S_{1,1}}$ which is a contradiction to our conclusion that $S \notin \text{ran } \sigma_{S_{0,0},S_{1,1}}$. Thus the form of P will be

$$\begin{pmatrix} I & P_{0,1} \\ 0 & I \end{pmatrix} \text{ or } \begin{pmatrix} 0 & P_{0,1} \\ 0 & 0 \end{pmatrix}.$$

Since $P$ is an idempotent operator, so we have $P_{0,1} = 0$. Hence $E_t$ is strongly irreducible.

To complete the proof of the second statement by induction, suppose that it is valid for any $n \leq k-1$. For $n = k$, let $P \in \mathcal{A}'(E_t)$ be an idempotent operator. By Lemma 4.1, $P$ has the following form:

$$P = \begin{pmatrix} P_{0,0} & P_{0,1} & P_{0,2} & \cdots & P_{0,k} \\ 0 & P_{1,1} & P_{1,2} & \cdots & P_{1,k} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & P_{k-1,k-1} & P_{k-1,k} \\ 0 & \dots & \dots & 0 & P_{k,k} \end{pmatrix},$$

and $P\big(\!\big(S_{i,j}\big)\!\big)_{k\times k} = \big(\!\big(S_{i,j}\big)\!\big)_{k\times k}P$. It follows that

$$\big(\!\big(P_{i,j}\big)\!\big)\big(\!\big(S_{i,j}\big)\!\big) = \big(\!\big(S_{i,j}\big)\!\big)\big(\!\big(P_{i,j}\big)\!\big),\ 0 \leq i,j \leq k-1,\ \big(\!\big(P_{i,j}\big)\!\big)\big(\!\big(S_{i,j}\big)\!\big) = \big(\!\big(S_{i,j}\big)\!\big)\big(\!\big(P_{i,j}\big)\!\big),\ 1 \leq i,j \leq k.$$

Both $\big(\!\big(P_{ij}\big)\!\big)_{i,j=0}^{k-1}$ and $\big(\!\big(P_{i,j}\big)\!\big)_{i,j=1}^{k}$ are idempotents. Since $\Lambda(t) < 2$, we have

$$S_{r,s} \notin \operatorname{ran}\sigma_{S_{r,r},S_{s,s}},\ r,s \leq n.$$

By the induction hypothesis, we have

$$P_{i,j} = 0, i \neq j \leq k-1,$$

and

$$P_{0,0} = P_{1,1} = \cdots = P_{k,k} = 0,\ \text{or } P_{0,0} = P_{1,1} = \cdots = P_{k,k} = I.$$

Thus $P$ has the following form:

$$P = \begin{pmatrix} I & 0 & 0 & \cdots & P_{0,k} \\ 0 & I & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & I & 0 \\ 0 & 0 & \dots & 0 & I \end{pmatrix} \text{ or } P = \begin{pmatrix} 0 & 0 & 0 & \cdots & P_{0,k} \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & 0 & \cdots & 0 & 0 \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}.$$

Since $P$ is an idempotent, it follows that $P_{0,k} = 0$. $\square$

By Lemma 4.1, an intertwining operator between two quasi-homogeneous operators with respect to any atomic decomposition must be upper triangular. Thus any operator $X$ in the commutant of such an operator, say $T$, must also be upper-triangular. In particular, $X_{i,i}$ belongs to the commutant of $S_{i,i}$, $0 \leq i \leq n-1$. Since $S_{i,i}$ is a homogeneous operator in $B_1(\mathbb{D})$, it follows that the commutant of $S_{i,i}$ is isomorphic to $\mathcal{H}^\infty(\mathbb{D})$, the space of bounded analytic functions on the unit disc $\mathbb{D}$. Consequently, for any $\phi \in \mathcal{H}^\infty(\mathbb{D})$, the operator $\phi(S_{i,i})$ is in the commutant $\mathcal{A}'(S_{i,i})$. In the following lemma, we give a description of the commutant of $T$. We will construct an operator $X$ in the commutant of $T$, where the diagonal elements are induced by the same holomorphic function $\phi \in \mathcal{H}^\infty(\mathbb{D})$, that is, $\phi(S_{i,i}) = X_{i,i}$.

**Lemma 4.6.** *Let $t$ be a quasi-homogeneous holomorphic curve with atoms $t_i, 0 \leq i \leq 1$. Let $T = \big(\!\big(S_{i,j}\big)\!\big)_{i,j\leq 1}$ be its atomic decomposition. Suppose that $X = \big(\!\big(X_{i,j}\big)\!\big)_{i,j\leq 1}$ is in $\mathcal{A}'(T)$. Then there exists $\phi \in \mathcal{H}^\infty(\mathbb{D})$ such that $X_{i,i} = \phi(S_{i,i}), i = 0,1$ and we also have that*

$$S_{0,0}X_{0,1} - X_{0,1}S_{1,1} = X_{0,0}S_{0,1} - S_{0,1}X_{1,1} = 0.$$

*In particular, $X_{0,1}$ can be chosen as zero.*

*Proof.* Set $X = \big(\!\big(X_{i,j}\big)\!\big)_{i,j\leq 1} \in \mathcal{A}'(T)$, we have the following equation

$$\begin{pmatrix} S_{0,0} & S_{0,1} \\ 0 & S_{1,1} \end{pmatrix}\begin{pmatrix} X_{0,0} & X_{0,1} \\ X_{1,0} & X_{1,1} \end{pmatrix} = \begin{pmatrix} X_{0,0} & X_{0,1} \\ X_{1,0} & X_{1,1} \end{pmatrix}\begin{pmatrix} S_{0,0} & S_{0,1} \\ 0 & S_{1,1} \end{pmatrix}.$$

As in [11, Proposition 3.4], we have $X_{1,0} = 0$. Then

$$S_{0,0}X_{0,1} + S_{0,1}X_{1,1} = X_{0,0}S_{0,1} + X_{0,1}S_{1,1},$$

and

$$S_{0,0}X_{0,1} - X_{0,1}S_{1,1} = X_{0,0}S_{0,1} - S_{0,1}X_{1,1}.$$

Note that there exist holomorphic functions $\phi_{0,0}$ and $\phi_{1,1}$ such that

$$X_{0,0}(t_0) = \phi_{0,0}t_0, X_{1,1}(t_1) = \phi_{1,1}t_1,$$

and by the definition of $S_{0,1}$, there exist constant function $\phi_{0,1}$ such that

$$S_{0,1}(t_1) = \phi_{0,1}t_0.$$

Then

$$X_{0,0}S_{0,1}(t_1) - S_{0,1}X_{1,1}(t_1) = (\phi_{0,0}\phi_{0,1} - \phi_{1,1}\phi_{0,1})t_0.$$

and $X_{0,0}S_{0,1} - S_{0,1}X_{1,1}$ also intertwines $S_{0,0}$ and $S_{1,1}$. Taking $X_{0,0}S_{0,1} - S_{0,1}X_{1,1}$ the place of $S_{0,1}$ and using the proof of Lemma 3.2, we might deduce that

$$S_{0,0}X_{0,1} - X_{0,1}S_{1,1} = X_{0,0}S_{0,1} - S_{0,1}X_{1,1} = 0, \phi_{0,0} = \phi_{1,1}.$$

Thus we can choose $X_{0,1} = 0$ and there exists a holomorphic function $\phi = \phi_{0,0} = \phi_{1,1} \in \mathcal{H}^\infty(\mathbb{D})$ such that $X = \begin{pmatrix} X_{0,0} & 0 \\ 0 & X_{1,1} \end{pmatrix}$ where $X_{i,i} = \phi(S_{i,i})$ satisfies that

$$\begin{pmatrix} S_{0,0} & S_{0,1} \\ 0 & S_{1,1} \end{pmatrix} \begin{pmatrix} X_{0,0} & 0 \\ 0 & X_{1,1} \end{pmatrix} = \begin{pmatrix} X_{0,0} & 0 \\ 0 & X_{1,1} \end{pmatrix} \begin{pmatrix} S_{0,0} & S_{0,1} \\ 0 & S_{1,1} \end{pmatrix}.$$

□

**Lemma 4.7.** *Let $t$ be a quasi-homogeneous holomorphic curve with atoms $t_i, 0 \leq i \leq n-1$. Let $T = (\!(S_{i,j})\!)$ be its atomic decomposition. Let $\phi \in \mathcal{H}^\infty(\mathbb{D})$ be a holomorphic function. If $\Lambda(t) < 2$, then there exists a bounded linear operator $X \in \mathcal{A}'(T)$ such that $X_{i,i} = \phi(S_{i,i}), i = 0, 1, \cdots, n-1$.*

*Proof.* Firstly, by Lemma 4.6, the lemma is true for the case of $n = 2$.

For $n = 3$, let $X = \begin{pmatrix} X_{0,0} & X_{0,1} & X_{0,2} \\ 0 & X_{1,1} & X_{1,2} \\ 0 & 0 & X_{2,2} \end{pmatrix} \in \mathcal{A}'(E_t)$. Then we have

$$\begin{pmatrix} S_{0,0} & S_{0,1} & S_{0,2} \\ 0 & S_{1,1} & S_{1,2} \\ 0 & 0 & S_{2,2} \end{pmatrix} \begin{pmatrix} X_{0,0} & X_{0,1} & X_{0,2} \\ 0 & X_{1,1} & X_{1,2} \\ 0 & 0 & X_{2,2} \end{pmatrix} = \begin{pmatrix} X_{0,0} & X_{0,1} & X_{0,2} \\ 0 & X_{1,1} & X_{1,2} \\ 0 & 0 & X_{2,2} \end{pmatrix} \begin{pmatrix} S_{0,0} & S_{0,1} & S_{0,2} \\ 0 & S_{1,1} & S_{1,2} \\ 0 & 0 & S_{2,2} \end{pmatrix}$$

and it follows that

(1) $S_{0,0}X_{0,1} + S_{0,1}X_{1,1} = X_{0,0}S_{0,1} + X_{0,1}S_{1,1}$, that is, $S_{0,0}X_{0,1} - X_{0,1}S_{1,1} = X_{0,0}S_{0,1} - S_{0,1}X_{1,1}$;
(2) $S_{1,1}X_{1,2} + S_{1,2}X_{2,2} = X_{1,1}S_{1,2} + X_{1,2}S_{2,2}$, that is, $S_{1,1}X_{1,2} - X_{1,2}S_{2,2} = X_{1,1}S_{1,2} - S_{1,2}X_{2,2}$.

By Lemma 4.6, we may choose, without loss of generality, $X_{0,1} = 0$ and $X_{1,2} = 0$. And there exists $\phi \in \mathcal{H}^\infty(\mathbb{D})$ such that $X_{i,i} = \phi(S_{i,i}), i = 0, 1, 2$. It is therefore enough to find an operator $X_{0,2}$ satisfying

$$S_{0,0}X_{0,2} - X_{0,2}S_{2,2} = X_{0,0}S_{0,2} - S_{0,2}X_{2,2}.$$

Clearly, we have

$$\begin{aligned}(X_{0,0}S_{0,2} - S_{0,2}X_{2,2})(t_2(w)) &= X_{0,0}(m_{0,2}t_0^{(1)}(w)) - S_{0,2}(\phi(w)t_2(w)) \\ &= m_{0,2}(\phi(w)t_0(w))^{(1)} - m_{0,2}\phi(w)t^{(1)}(w) \\ &= m_{0,2}\phi^{(1)}(w)t_0(w).\end{aligned}$$

We therefore set $X_{0,2}$ be the operator: $X_{0,2}(t_2(w)) = m_{0,2}\phi^{(1)}(w)t_0^{(1)}(w)$.

To complete the proof by induction, we assume that we have the validity of the conclusion for $n = k-1$. Thus we assume the existence of a bounded linear operator $X = (\!(X_{i,j})\!)$ such that $(\!(S_{i,j})\!)(\!(X_{i,j})\!) = (\!(X_{i,j})\!)(\!(S_{i,j})\!)$ where $X_{i,i} = \phi(S_{i,i})$ and $X_{i,i+1} = 0$. And there exists $l_{i,j}^r$ such that $X_{i,j}(t_j) = \sum\limits_{r=1}^{j-i-1} l_{i,j}^r \phi^{(j-k)} t_i^{(k)}$. To complete the inductive step, we only need to find the operator $X_{0,k}$ satisfying the following equation:

$$S_{0,0}X_{0,k} - X_{0,k}S_{k,k} = X_{0,0}S_{0,k} - S_{0,k}X_{k,k} + (\sum_{i=2}^{k-1} X_{0,i}S_{i,k} - \sum_{i=1}^{k-2} S_{0,i}X_{i,k}) \tag{4.2}$$

Note that the induction hypothesis ensures the existence of constants $c_{0,k}^s$ (depending on $m_{i,j}$) such that

$$(X_{0,0}S_{0,k} - S_{0,k}X_{k,k} + \sum_{i=2}^{k-1} X_{0,i}S_{i,k} - \sum_{i=1}^{k-2} S_{0,i}X_{i,k})(t_k) = \sum_{s=1}^{k-1} c_{0,k}^s \phi^{(s)} t_0^{(k-s-1)}.$$

Now, suppose that $X_{0,k}(t_k) = \sum_{s=1}^{k-1} l_{0,k}^s \phi^{(s)} t_0^{(k-s)}$, where the constants $l_{0,k}^s$ are to be found. Then we must have

$$(S_{0,0}X_{0,k} - X_{0,k}S_{k,k})(t_k(w)) = \sum_{s=1}^{k-1} c_{0,k}^s \phi^{(s)} t_0^{(k-1-s)}(w)$$

It follows that if we choose $l_{0,k}^s = \frac{c_{0,k}^s}{k-s}$, then $X_{0,k}$ with this choice of the constants validates equation (4.2). This completes the induction step.

In particular, when $\mu_{i,j}$ are all chosen to be 1, then $m_{i,j} = -1$, that is, $S_{i,j}(t_j) = -t_j^{(j-i-1)}$. In this case, $X_{0,k}(t_0) = -\sum_{s=1}^{k-1} \phi^{(s)} t_0^{(k-s)}$. Now, if $m_{i,j} = -1$, $i, j = 0, 1, \cdots, n-1$, then by a similar argument, we have

$$X_{i,j}(t_j) = -\sum_{s=1}^{j-i-1} \phi^{(s)} t_i^{(j-i-s)}, i, j = 0, 1, \cdots, n-1. \tag{4.3}$$

□

### 4.3. Proof of the main theorem.

*Proof of Theorem 4.2.* First, if "$\Lambda(t) \geq 2$", then the first conclusion of the theorem follows from Theorem 4.5. So, it remains for us to verify the second statement of the theorem, where $\Lambda(t) < 2$.

Let $T$ and $\widetilde{T}$ be the operators representing $t$ and $\widetilde{t}$ respectively. Recall from Proposition 3.1 that $S_{i,j}(t_j) = m_{i,j} t_i^{(j-i-1)}$, $\widetilde{S}_{i,j}(t_j) = \widetilde{m}_{i,j} t_i^{(j-i-1)}$. Up to similarity, we can assume that $m_{i,i+1} = \widetilde{m}_{i,i+1}$. Then $T$ and $\widetilde{T}$ have the following atomic decomposition:

$$T = \begin{pmatrix} S_{0,0} & S_{0,1} & S_{0,2} & \cdots & S_{0,n-1} \\ 0 & S_{1,1} & S_{1,2} & \cdots & S_{1,n-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & S_{n-2,n-2} & S_{n-2,n-1} \\ 0 & 0 & \cdots & 0 & S_{n-1,n-1} \end{pmatrix} \text{ and } \widetilde{T} = \begin{pmatrix} S_{0,0} & S_{0,1} & c_{0,2}S_{0,2} & \cdots & c_{0,n-1}S_{0,n-1} \\ 0 & S_{1,1} & S_{1,2} & \cdots & c_{1,n-1}S_{1,n-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & S_{n-2,n-2} & c_{n-2,n-1}S_{n-2,n-1} \\ 0 & 0 & \cdots & 0 & S_{n-1,n-1} \end{pmatrix}$$

Set $c_{i,j} = \frac{\widetilde{m}_{i,j}}{m_{i,j}}$. Now it is enough to prove the Claim stated below.

Claim: If $T \sim \widetilde{T}$, then $c_{i,j} = 1, i, j = 0, 1, \cdots, n$.

Consider the following possibilities:

(1) $\Lambda(t) \in [0, 1)$
(2) $n = 3$, $\Lambda(t) \in [1, 2)$; $n > 3$, $\Lambda(t) \in [1, \frac{4}{3})$
(3) $n = 4$, $\Lambda(t) \in [\frac{4}{3}, 2)$; $n > 4$, $\Lambda(t) \in [\frac{4}{3}, \frac{3}{2})$
(4) $n = 5$, $\Lambda(t) \in [\frac{3}{2}, 2)$; $n > 5$, $\Lambda(t) \in [\frac{3}{2}, \frac{8}{5})$

The method of the proof below combined with Lemma 4.7 and equation (4.3) completes the proof in the remaining cases.

In what follows, without loss of generality, we will always choose $m_{i,j} = -1, i, j = 0, 1, \cdots, n-1$.

Case (1): By Proposition 3.2, we have

$$T = \widetilde{T} = \begin{pmatrix} S_{0,0} & S_{0,1} & 0 & \cdots & 0 \\ & S_{1,1} & S_{1,2} & \cdots & 0 \\ & & \ddots & \ddots & \vdots \\ & 0 & & S_{n-2,n-2} & S_{n-1,n} \\ & & & & S_{n-1,n-1} \end{pmatrix}.$$

In this case, we clearly have $K_{t_i} = K_{s_i}$ and $\theta_{i,i+1} = \widetilde{\theta}_{i,i+1}, i = 0, 1, \cdots, n-1$.

Case (2): By Proposition 3.2, we have

$$T = \begin{pmatrix} S_{0,0} & S_{0,1} & S_{0,2} & \cdots & 0 & 0 \\ & S_{1,1} & S_{1,2} & S_{1,3} & \cdots & 0 \\ & & \ddots & \ddots & \ddots & \vdots \\ & \huge 0 & & & S_{n-2,n-2} & S_{n-1,n} \\ & & & & & S_{n-1,n-1} \end{pmatrix},$$

and

$$\widetilde{T} = \begin{pmatrix} S_{0,0} & S_{0,1} & c_{0,2}S_{0,2} & 0 & \cdots & 0 \\ & S_{1,1} & S_{1,2} & c_{1,3}S_{1,3} & \cdots & 0 \\ & & \ddots & \ddots & \ddots & \vdots \\ & & & S_{n-2,n-2} & S_{n-2,n-1} & c_{n-2,n}S_{n-2,n} \\ & \huge 0 & & & S_{n-1,n-1} & S_{n-1,n} \\ & & & & & S_{n,n} \end{pmatrix}.$$

In this case, by Proposition 3.2, we first assume that $n = 3$. Then we have

$$\begin{pmatrix} S_{0,0} & S_{0,1} & S_{0,2} \\ 0 & S_{1,1} & S_{1,2} \\ 0 & 0 & S_{2,2} \end{pmatrix} \begin{pmatrix} X_{0,0} & X_{0,1} & X_{0,2} \\ 0 & X_{1,1} & X_{1,2} \\ 0 & 0 & X_{2,2} \end{pmatrix} = \begin{pmatrix} X_{0,0} & X_{0,1} & X_{0,2} \\ 0 & X_{1,1} & X_{1,2} \\ 0 & 0 & X_{2,2} \end{pmatrix} \begin{pmatrix} S_{0,0} & S_{0,1} & c_{0,2}S_{0,2} \\ 0 & S_{1,1} & S_{1,2} \\ 0 & 0 & S_{2,2} \end{pmatrix} \tag{4.4}$$

By Lemma 4.6, $X_{0,1}$ and $X_{1,2}$ may be chosen to be zero. Therefore we have the equalities:

$$S_{i,i+1}X_{i+1,i+1} = X_{i,i}S_{i,i+1}, i = 0, 1, \text{ and } S_{0,0}X_{0,2} + S_{0,2}X_{2,2} = c_{0,2}X_{0,0}S_{0,2} + X_{0,2}S_{2,2}.$$

Note that $\mathcal{A}'(S_{i,i}) \cong \mathcal{H}^\infty(\mathbb{D})$, by Lemma 4.6, we can find a holomorphic function $\phi \in \mathcal{H}^\infty(\mathbb{D})$ such that $X_{i,i}t_i = \phi t_i$. Since $X_{i,i}$ is invertible, $\phi(S_{i,i})$ is also invertible. Note that

$$\begin{aligned} (c_{0,2}X_{0,0}S_{0,2} - S_{0,2}X_{2,2})(t_2) &= c_{0,2}X_{0,0}(-t_0^{(1)}) - S_{0,2}(\phi t_2) \\ &= (c_{0,2}-1)S_{0,2}\phi(S_{2,2})(t_2) - c_{0,2}S_{0,1}S_{1,2}\phi^{(1)}(S_{2,2})(t_2). \end{aligned} \tag{4.5}$$

By Lemma 4.4, we have $c_{0,2}S_{0,1}S_{1,2}\phi^{(1)}(S_{2,2}) \in \operatorname{ran}\sigma_{S_{0,0},S_{2,2}}$. From (4.5), it follows that

$$(c_{0,2}-1)S_{0,2}\phi(S_{2,2}) \in \operatorname{ran}\sigma_{S_{0,0},S_{2,2}}.$$

By Lemma 3.2, $S_{0,2} \notin \operatorname{ran}\sigma_{S_{0,0},S_{2,2}}$. Since $\phi(S_{2,2})$ is invertible and $\phi(S_{2,2}) \in \mathcal{A}'(S_{2,2})$, we have

$$S_{0,2}\phi(S_{2,2}) \notin \operatorname{ran}\sigma_{S_{0,0},S_{2,2}}$$

it follows from Theorem 4.5. This shows that $c_{0,2} = 1$.

In the following, we will prove the general case. Now suppose that we have proved Claim 1 for $n = k-1$. Pick $X = \begin{pmatrix} X_{0,0} & 0 & \cdots & X_{0,k} \\ 0 & X_{1,1} & \cdots & X_{1,k} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & X_{k,k} \end{pmatrix}$ such that $X\widetilde{T} = TX$. Then it follows that

$$X_0((\widetilde{S}_{i,j})_{i,j=0}^{k-1}) = ((S_{i,j})_{i,j=0}^{k-1})X_0, X_1((\widetilde{S}_{i,j})_{i,j=1}^{k}) = ((S_{i,j})_{i,j=1}^{k})X_1,$$

where

$$X_0 = \begin{pmatrix} X_{0,0} & 0 & \cdots & X_{0,k-1} \\ 0 & X_{1,1} & \cdots & X_{1,k-1} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & X_{k-1,k-1} \end{pmatrix}, \quad X_1 = \begin{pmatrix} X_{1,1} & 0 & \cdots & X_{1,k} \\ 0 & X_{2,2} & \cdots & X_{2,k} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & X_{k,k} \end{pmatrix}.$$

Since $X$ is invertible, $X_0$ and $X_1$ are both invertible. By the induction hypothesis $c_{i,i+2} = 1, i = 0, 1, \cdots, n-3$.

Case (3) and Case (4): By Proposition 3.2, $\widetilde{T} = (\!(\widetilde{S}_{i,j})\!), \widetilde{S}_{i,j} = 0, j-i \geq 4$ and $\widetilde{T} = (\!(\widetilde{S}_{i,j})\!), \widetilde{S}_{i,j} = 0, j-i \geq 5$. Following the proof given above, by Proposition 3.2, we only need to consider the case of $n = 4$ and $n = 5$. For case 3, we only consider $n = 4$ and the other cases would follow by induction. In this case, we have

$$\begin{pmatrix} S_{0,0} & S_{0,1} & S_{0,2} & S_{0,3} \\ 0 & S_{1,1} & S_{1,2} & S_{1,3} \\ 0 & 0 & S_{2,2} & S_{2,3} \\ 0 & 0 & 0 & S_{3,3} \end{pmatrix} \begin{pmatrix} X_{0,0} & 0 & X_{0,2} & X_{0,3} \\ 0 & X_{1,1} & 0 & X_{1,3} \\ 0 & 0 & X_{2,2} & 0 \\ 0 & 0 & 0 & X_{3,3} \end{pmatrix} = \begin{pmatrix} X_{0,0} & 0 & X_{0,2} & X_{0,3} \\ 0 & X_{1,1} & 0 & X_{1,3} \\ 0 & 0 & X_{2,2} & 0 \\ 0 & 0 & 0 & X_{3,3} \end{pmatrix} \begin{pmatrix} S_{0,0} & S_{0,1} & S_{0,2} & c_{0,3}S_{0,3} \\ 0 & S_{1,1} & S_{1,2} & S_{1,3} \\ 0 & 0 & S_{2,2} & S_{2,3} \\ 0 & 0 & 0 & S_{3,3} \end{pmatrix}.$$

It follows that $\begin{pmatrix} X_{0,0} & 0 & X_{0,2} \\ 0 & X_{1,1} & 0 \\ 0 & 0 & X_{2,2} \end{pmatrix}$ commutes with $\begin{pmatrix} S_{0,0} & S_{0,1} & S_{0,2} \\ 0 & S_{1,1} & S_{1,2} \\ 0 & 0 & S_{2,2} \end{pmatrix}$ and $\begin{pmatrix} X_{1,1} & 0 & X_{1,3} \\ 0 & X_{2,2} & 0 \\ 0 & 0 & X_{3,3} \end{pmatrix}$ commutes with $\begin{pmatrix} S_{1,1} & S_{1,2} & S_{1,3} \\ 0 & S_{2,2} & S_{2,3} \\ 0 & 0 & S_{3,3} \end{pmatrix}$. By equation (4.3), we see that $X_{0,2}$ and $X_{1,3}$ can be chosen to be $S_{0,2}\phi^{(1)}(S_{2,2})$ and $S_{1,3}\phi^{(1)}(S_{3,3})$. Note that

$$S_{0,0}X_{0,3} + S_{0,1}X_{1,3} + S_{0,3}X_{3,3} = c_{0,3}X_{0,0}S_{0,3} + X_{0,2}S_{2,3} + X_{0,3}S_{3,3}. \tag{4.6}$$

Then

$$\begin{aligned} X_{0,2}S_{2,3} - S_{0,1}X_{1,3} &= S_{0,2}\phi^{(1)}(S_{2,2})S_{2,3} - S_{0,1}S_{1,3}\phi^{(1)}(S_{3,3}) \\ &= (S_{0,2}S_{2,3} - S_{0,1}S_{1,3})\phi^{(1)}(S_{3,3}) = 0 \end{aligned}$$

So we only need to consider

$$S_{0,0}X_{0,3} - X_{0,3}S_{3,3} = c_{0,3}X_{0,0}S_{0,3} - S_{0,3}X_{3,3}.$$

Since

$$(c_{0,3}X_{0,0}S_{0,3} - S_{0,3}X_{3,3})(t_3) = (1-c_{0,3})\phi t_0^{(2)} - 2c_{0,3}\phi^{(1)}t_0^{(1)} - c_{0,3}\phi^{(2)}t_0,$$

we obtain

$$c_{0,3}X_{0,0}S_{0,3} - S_{0,3}X_{3,3} = (c_{0,3}-1)S_{0,3}\phi(S_{3,3}) + 2c_{0,3}S_{0,1}S_{1,3}\phi^{(1)}(S_{3,3}) + c_{0,3}S_{0,1}S_{1,2}S_{2,3}\phi^{(2)}(S_{3,3}).$$

By Lemma 4.4 and equation (4.6), we have

$$2c_{0,3}S_{0,1}S_{1,3}\phi^{(1)}(S_{3,3}) + c_{0,3}S_{0,1}S_{1,2}S_{2,3}\phi^{(2)}(S_{3,3}) \in \mathrm{Ran}\sigma_{S_{0,0},S_{3,3}}.$$

Since $\phi(S_{3,3})$ is invertible, we deduce that

$$(c_{0,3}-1)S_{0,3} \in \operatorname{ran}\sigma_{S_{0,0},S_{3,3}}.$$

Note that $S_{0,3} \notin \operatorname{ran}\sigma_{S_{0,0},S_{3,3}}$, we have $c_{0,3}=1$. For case 4 with $n=5$, we have

$$\begin{pmatrix} S_{0,0} & S_{0,1} & S_{0,2} & S_{0,3} & S_{0,4} \\ & S_{1,1} & S_{1,2} & S_{1,3} & S_{1,4} \\ & & S_{2,2} & S_{2,3} & S_{2,4} \\ & 0 & & S_{3,3} & S_{3,4} \\ & & & & S_{4,4} \end{pmatrix} \begin{pmatrix} X_{0,0} & 0 & X_{0,2} & X_{0,3} & X_{0,4} \\ & X_{1,1} & 0 & X_{1,3} & X_{1,4} \\ & & X_{2,2} & 0 & X_{2,4} \\ & 0 & & X_{3,3} & 0 \\ & & & & X_{4,4} \end{pmatrix} = \begin{pmatrix} X_{0,0} & 0 & X_{0,2} & X_{0,3} & X_{0,4} \\ & X_{1,1} & 0 & X_{1,3} & X_{1,4} \\ & & X_{2,2} & 0 & X_{2,4} \\ & 0 & & X_{3,3} & 0 \\ & & & & X_{4,4} \end{pmatrix} \begin{pmatrix} S_{0,0} & S_{0,1} & S_{0,2} & S_{0,3} & c_{0,4}S_{0,4} \\ & S_{1,1} & S_{1,2} & S_{1,3} & S_{1,4} \\ & & S_{2,2} & S_{2,3} & S_{2,4} \\ & 0 & & S_{3,3} & S_{3,4} \\ & & & & S_{4,4} \end{pmatrix}$$

Therefore $\big(X_{ij}\big)_{4\times 4}$ commutes with $\big(S_{i,j}\big)_{4\times 4}$ for $i,j=0,1,2,3$ and $\big(X_{ij}\big)_{4\times 4}$ commutes with $\big(S_{i,j}\big)_{4\times 4}$ for $i,j=1,2,3,4$. Then from Lemma 4.7, we find that $X_{i,j},(i,j)\neq(0,4)$. We also have

$$S_{0,0}X_{0,4} - X_{0,4}S_{4,4} = (c_{0,4}X_{0,0}S_{0,4} - S_{0,4}X_{4,4}) + (X_{0,2}S_{2,4} + X_{0,3}S_{3,4}) - (S_{0,1}X_{1,4} + S_{0,2}X_{2,4}). \tag{4.7}$$

By Lemma 4.7, we have

$$\begin{aligned} X_{0,2}S_{2,4} - S_{0,2}X_{2,4} &= S_{0,2}\phi^{(1)}(S_{2,2})S_{2,4} - S_{0,2}S_{2,4}\phi^{(1)}(S_{4,4}) \\ &= S_{0,2}S_{2,3}S_{3,4}\phi^{(2)}(S_{4,4}). \end{aligned}$$

Lemma 4.7 together with the equation (4.3) gives

$$X_{0,3} = S_{0,2}S_{2,3}\phi^{(2)}(S_{3,3}) + S_{0,3}\phi^{(1)}(S_{3,3}),$$

$$X_{1,4} = S_{1,3}S_{3,4}\phi^{(2)}(S_{4,4}) + S_{1,4}\phi^{(1)}(S_{4,4}).$$

Note that $S_{0,2}S_{2,3} = S_{0,1}S_{1,3}$ and $S_{0,3}S_{3,4} = S_{0,1}S_{1,4}$, we also have

$$\begin{aligned} &X_{0,3}S_{3,4} - S_{0,1}X_{1,4} \\ &= (S_{0,2}S_{2,3}\phi^{(2)}(S_{3,3}) + S_{0,3}\phi^{(1)}(S_{3,3}))S_{3,4} - S_{0,1}(S_{1,3}S_{3,4}\phi^{(2)}(S_{4,4}) + S_{1,4}\phi^{(1)}(S_{4,4})) = 0. \end{aligned}$$

Since

$$\begin{aligned} (c_{0,4}X_{0,0}S_{0,4} - S_{0,4}X_{4,4})(t_4) &= c_{0,4}X_{0,0}S_{0,4}(t_4) - S_{0,4}(\phi t_4) \\ &= (1-c_{0,4})\phi t_0^{(3)} - 3c_{0,4}\phi^{(2)}t_0^{(1)} - 3c_{0,4}\phi^{(1)}t_0^{(2)} - c_{0,4}\phi^{(3)}t_0, \end{aligned}$$

we also have

$$c_{0,4}X_{0,0}S_{0,4} - S_{0,4}X_{4,4} = (c_{0,4}-1)S_{0,4}\phi(S_{4,4}) + 3c_{0,4}S_{0,1}S_{1,3}\phi^{(1)}(S_{3,3}) \\ +3c_{0,4}S_{0,1}S_{1,2}S_{2,3}\phi^{(2)}(S_{3,3}) + c_{0,4}S_{0,1}S_{1,2}S_{2,3}\phi^{(3)}(S_{3,3}).$$

Combining Lemma 4.4 with the equation (4.7), we obtain

$$3c_{0,4}S_{0,1}S_{1,3}\phi^{(1)}(S_{3,3}) + 3c_{0,4}S_{0,1}S_{1,2}S_{2,3}\phi^{(2)}(S_{3,3}) + c_{0,4}S_{0,1}S_{1,2}S_{2,3}\phi^{(3)}(S_{3,3}) \in \operatorname{ran}\sigma_{S_{0,0},S_{4,4}},$$

$$S_{0,2}S_{2,3}S_{3,4}\phi^{(2)}(S_{4,4}) \in \operatorname{ran}\sigma_{S_{0,0},S_{4,4}}.$$

Then it follows that

$$(c_{0,4}-1)S_{0,4}\phi(S_{4,4}) \in \operatorname{ran}\sigma_{S_{0,0},S_{4,4}}.$$

Note that $\phi(S_{4,4})$ is invertible, therefore

$$(c_{0,4}-1)S_{0,4} \in \operatorname{ran}\sigma_{S_{0,0},S_{4,4}}.$$

Since $S_{0,4} \notin \operatorname{ran}\sigma_{S_{0,0},S_{4,4}}$, it follows that $c_{0,4}=1$.

□

## 5. Applications

We give two different applications of our results. First of these shows that the topological and algebraic $K$-groups defined in our context must coincide. Second, we show that the Halmos' question on similarity has an affirmative answer for quasi-homogeneous operators. We begin with some preliminaries on $K$- groups.

5.1. Preliminaries. Let $t:\Omega \to Gr(n,\mathcal{H})$ be a holomorphic curve. Recall that the commutant $\mathcal{A}'(E_t)$ of such a holomorphic curve $t$ is defined to be

$$\mathcal{A}'(E_t) = \{A \in \mathcal{L}(\mathcal{H}) : A\,t(w) \subseteq t(w),\ w \in \Omega.\}$$

**Definition 5.1.** *For a holomorphic curve $t:\Omega\to Gr(n,\mathcal{H})$, the Jocaboson radical Rad $\mathcal{A}'(E_t)$ of $\mathcal{A}'(E_t)$ is defined to be*

$$\{S \in \mathcal{A}'(E_t) | \sigma_{\mathcal{A}'(E_t)}(SA) = 0, A \in \mathcal{A}'(E_t)\},$$

*where $\sigma_{\mathcal{A}'(E_t)}(SA)$ denotes the spectrum of $SA$ in the algebra $\mathcal{A}'(E_t)$.*

The discussion below follows closely the paper [12] of the first two authors.

**Definition 5.2.** *A holomorphic curve $t:\Omega\to Gr(n,\mathcal{H})$ is said to be have a finite decomposition if it meets one of the equivalent conditions given in* [12, Theorem 1.3]*).*

*Suppose $\{P_1, P_2, \cdots, P_m\}$ and $\{Q_1, Q_2, \cdots, Q_n\}$ are two distinct decompositions of $t$. If $m = n$, there exists a permutation $\Pi \in S_n$ such that $XQ_{\Pi(i)}X^{-1} = P_i$ for some invertible operator $X$ in $\mathcal{A}'(E_t)$, $1 \le i \le n$, then we say that $t$ (or $E_t$) has a unique decomposition up to similarity.*

For a holomorphic curve, $f:\Omega\to Gr(n,\mathcal{H})$, let $M_k(\mathcal{A}'(E_t))$ be the collection of $k\times k$ matrices with entries from $\mathcal{A}'(E_t)$. Let

$$M_\infty(\mathcal{A}'(E_t)) = \bigcup_{k=1}^{\infty} M_k(\mathcal{A}'(E_t)),$$

and $\operatorname{Proj}(M_k(\mathcal{A}'(E_t)))$ be the algebraic equivalence classes of idempotents in $M_\infty(\mathcal{A}'(E_t))$. If $p,q$ are idempotents in $\operatorname{Proj}(\mathcal{A}'(E_t))$, then say that $p{\sim_{st}}q$ if $p\oplus r {\sim_a} q\oplus r$ for some idempotent $r$ in $\operatorname{Proj}(\mathcal{A}'(E_t))$. The relation $\sim_{st}$ is known as stable equivalence.

Let $X$ be a compact Hausdorff space, and $\xi = (E,\pi,X)$ be a (topological) vector bundle. A well-known theorem due to R. G. Swan says that a vector bundle $\xi = (E,\pi,X)$ is a direct summand of the trivial bundle, that is,

$$\xi \oplus \eta \cong (X \times \mathbb{C}^n, \pi, X)$$

for some vector bundle $\eta = (F,\rho,X)$.

5.2. Unique decomposition. None of what we have said so far applies to holomorphic vector bundles over an open subset of $\mathbb{C}$ since they are already trivial by Graut's theorem. However, the study of holomorphic vector bundles over an open subset of $\mathbb{C}$ is central to operator theory. In the context of operator theory, as shown in the foundational paper of Cowen and Douglas [2], the vector bundles of interest are equipped with a Hermitian structure inherited from a fixed inner product of some Hilbert space $\mathcal{H}$. This makes it possible to ask questions about their equivalence under a unitary or an invertible linear transformation of $\mathcal{H}$. In the paper [2], questions regarding unitary equivalence were dealt with quite successfully while equivalence under an invertible linear transformation remains somewhat of a mystery to date. However, we can ask if the uniqueness of the summand, which was a consequence of Swan's theorem, remains valid in the context of Cowen-Douglas operators.

**Question.** Let $t : \Omega \to Gr(m, \mathcal{H})$ be a Hermitian holomorphic curve and the vector bundle $E_r$ be a direct summand of $E_t$ for some other holomorphic curve $r : \Omega \to Gr(n, \mathcal{H})$. Does there exist a unique sub-bundle of $E_t$, up to similarity, such that $E_r \oplus E_s = E_t$? Here the uniqueness is meant to be in the sense of Definition 5.2

It was shown in [13] that an operator in the Cowen-Douglas class $B_n(\Omega)$ admits a unique decomposition. So, the answer to the question raised above is affirmative. However, here we give a different proof for quasi-homogeneous operators which is much more transparent. For our proof, we will need the following lemma.

**Lemma 5.3.** *Let $E_t$ be a quasi-homogeneous bundle. Then $\mathcal{A}'(E_t)/\mathrm{Rad}(\mathcal{A}'(E_t))$ is commutative.*

*Proof.* Let
$$\mathcal{S} = \{Y : \sigma(Y) = 0, Y \in \mathcal{A}'(E_t)\}.$$

Claim 1: $\mathcal{S}$ is an ideal of the algebra $\mathcal{A}'(E_t)$.

By Lemma 4.1, $Y$ is upper-triangular if $Y \in \mathcal{S}$. Since the spectrum $\sigma(Y)$ of $Y$ is $\{0\}$, the operator $Y$ must be of the form
$$Y = \begin{pmatrix} 0 & Y_{0,1} & Y_{0,2} & \cdots & Y_{0,n-1} \\ 0 & 0 & Y_{1,2} & \cdots & Y_{1,n-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & Y_{n-2,n-1} \\ 0 & \cdots & \cdots & 0 & 0 \end{pmatrix},$$
and it follows that each quasi-nilpotent element in the commutant of the holomorphic curve $t$ of rank one is zero. Using Lemma 4.1 again, each element $X \in \mathcal{A}'(E_t)$ is upper-triangular. Thus $\sigma(XY) = \sigma(YX) = 0$. This completes the proof of Claim 1 and $\mathcal{S} = \mathrm{Rad}(\mathcal{A}'(E_t))$.

Claim 2: $\mathcal{A}'(E_t)/\mathrm{Rad}(\mathcal{A}'(E_t))$ is commutative.

Note that if $X \in \mathcal{A}'(E_t)$ is (block) nilpotent, then $X \in \mathcal{S}$. A simple computation shows that $\mathcal{A}'(E_t)/\mathrm{Rad}(\mathcal{A}'(E_t))$ is commutative. □

**Theorem 5.4.** *For any quasi-homogeneous holomorphic curve $t$ with atoms $t_i$, $0 \le i \le n-1$, we have that*

(1) *$E_t$ has no non-trivial sub-bundle whenever $\Lambda(t) < 2$, and*
(2) *if $\Lambda(t) \ge 2$, then for any sub-bundle $E_r$ of $E_t$, there exists a unique sub-bundle $E_s$, up to equivalence under an invertible map, such that $E_r \oplus E_s$ is similar to $E_t$.*

For any holomorphic curve $t$, we let $t^n$ denote the $n$ - fold direct sum of $t$. For any two natural numbers $n$ and $m$, let $E_r$ and $E_s$ be the sub-bundles of $E_{t^n}$ and $E_{t^m}$, respectively. If $m > n$, then both $E_r$ and $E_s$ can be regarded as a sub-bundle of $E_{t^m}$.

Two holomorphic Hermitian vector bundles $E_r$ and $E_s$ are said to be similar if there exist an invertible operator $X \in \mathcal{A}'(E_r)$ such that $XE_r = E_s$. Analogous to the definition of Vect(X), we let $\mathrm{Vect}^0(E_t)$ be the set of equivalence classes $\overline{E_s}$ of the sub-bundles $E_s$ of $E_{t^n}$, $n = 1, 2, \cdots$ . An addition on $\mathrm{Vect}^0(E_t)$ is defined as follows, namely,
$$\overline{E_r} + \overline{E_s} = \overline{E_r \oplus E_s},$$

where $E_r$ and $E_s$ are both sub-bundles of $E_t$. Now, the group $K^0(E_t)$ is the Grothendieck group of $(\mathrm{Vect}^0(E_t), +)$. In this notation, we have the following theorem.

**Theorem 5.5.** $K^0(E_t) \cong K_0(\mathcal{A}'(E_t))$.

The proof of this theorem is split into a number of lemmas which are stated and proved below.

**Lemma 5.6.** *Let $E_t$ be a quasi-homogeneous bundle. Then*

$$\mathrm{Vect}(\mathcal{A}'(E_t)) \cong \mathrm{Vect}(\mathcal{A}'(E_t)/Rad\mathcal{A}'(E_t)).$$

*Proof.* Note that $M_n(\mathcal{A}'(E_t)) \cong \mathcal{A}'(\bigoplus^n E_t)$. Let $p \in M_n(\mathcal{A}'(E_t))$ be an idempotent. Define a map $\sigma : \mathrm{Vect}(\mathcal{A}'(E_t)) \to \mathrm{Vect}(\mathcal{A}'(E_t)/\mathrm{Rad}\mathcal{A}'(E_t))$ as the following:

$$\sigma[P] = [\pi(P)],$$

where $\pi : \mathcal{A}'(E_t) \to \mathrm{Vect}(\mathcal{A}'(E_t)/\mathrm{Rad}\mathcal{A}'(E_t))$.

**Claim** $\sigma$ is well defined and it is an isomorphism.

If $[p] = [q]$, where $p \in M_n(\mathcal{A}'(E_t))$ and $q \in M_m(\mathcal{A}'(E_t))$ are both idempotents, then there exists $k \geq \max\{m, n\}$ and an invertible element $u \in M_k(\mathcal{A}'(E_t))$ such that

$$u(p \oplus 0^{k-n})u^{-1} = q \oplus 0^{k-m}.$$

Thus we have

$$\pi(u)\pi(p \oplus 0^{(k-n)})\pi(u)^{-1} = \pi(u(p \oplus 0^{k-n})u^{-1}) = \pi(q \oplus 0^{k-m}).$$

That means $[\pi(p)] = [\pi(q)]$, and $\sigma$ is well defined.

Now, we would prove that $\sigma$ is injective. In fact, if $p \in M_n(\mathcal{A}'(E_t))$ and $q \in M_m(\mathcal{A}'(E_t))$ are idempotents with

$$\sigma[p] = [\pi(p)] = [\pi(q)] = \sigma[q],$$

then we can find $k \geq \max\{m, n\}$ and an invertible element $\pi(u) \in M_k(\mathcal{A}'(E_t))/\mathrm{Rad}(M_k(\mathcal{A}'(E_t)))$ such that

$$\pi(u)(\pi(p \oplus 0^{k-n}))\pi(u)^{-1} = \pi(q \oplus 0^{k-m}).$$

Since $\pi(u)$ is invertible, there exists $\pi(s) \in \mathrm{Rad}(M_k(\mathcal{A}'(E_t)))$ such that $\pi(u)^{-1} = \pi(s)$. Then we have

$$us = I - R_1, su = I - R_2,$$

where $R_1, R_2 \in \mathrm{Rad}(M_k(\mathcal{A}'(E_t)))$. Since $\sigma(R_1) = \sigma(R_2) = \{0\}$, then $us, su$ are both invertible. Therefore, $u$ is invertible and thus

$$\pi(u(p \oplus 0^{(k-n)})u^{-1}) = \pi(u)(\pi(p \oplus 0^{k-n}))\pi(u)^{-1} = \pi(q \oplus 0^{k-m}).$$

Thus

$$u(p \oplus 0^{(k-n)})u^{-1} = q \oplus 0^{k-m} + R$$

for some $R \in \mathrm{Rad}(M_k(\mathcal{A}'(E_t)))$. Let $W_1 = 2(q \oplus 0^{(k-m)}) - I$. Since $\sigma(Q \oplus 0^{(k-m)}) \subseteq \{0, 1\}$, then $W_1$ is invertible. Since we have $R \in \mathrm{Rad}(M_k(\mathcal{A}'(E_t)))$ and $W_1^{-1} \in M_k(\mathcal{A}'(E_t))$, then $RW_1^{-1} \in \mathrm{Rad}(M_k(\mathcal{A}'(E_t)))$, so $I + RW_1^{-1}$ is invertible. Set

$$W = 2(q \oplus 0^{(k-m)}) - I + R = W_1 + R = (I + RW_1^{-1})W_1.$$

and $W$ is invertible. Since $p \oplus 0^{(k-n)}$ is an idempotent, it follows that $u(p \oplus 0^{(k-n)})u^{-1}$ and hence $(q \oplus 0^{(k-m)}) + R$ is an idempotent as well. Thus

$$(q \oplus 0^{(k-m)})^2 + (q \oplus 0^{(k-m)})R = R(q \oplus 0^{(k-m)}) + R^2 = (q \oplus 0^{(k-m)}) + R.$$

Similarly, $q \oplus 0^{(k-m)}$ is an idempotent, therefore

$$(q \oplus 0^{(k-m)})R + R(q \oplus 0^{(k-m)}) + R^2 = R.$$

So we have

$$\begin{aligned} W((q\oplus 0^{(k-m)}+R) &= (q\oplus 0^{(k-m)})+R(q\oplus 0^{(k-m)})+2(q\oplus 0^{(k-m)})R-R+R^2\\ &= (q\oplus 0^{(k-m)})+(q\oplus 0^{(k-m)})R\\ &= (q\oplus 0^{(k-m)})W \end{aligned}$$

and

$$u(p\oplus 0^{(k-n)})u^{-1}=(q\oplus 0^{(k-m)})+R=W^{-1}(q\oplus 0^{(k-m)})W.$$

It follows that $p\sim_a q$, and $\sigma$ is injective. Finally, we show that $\sigma$ is surjective. For each $[\pi(p)]\in \text{Vect}(\mathcal{A}'(E_t)/\text{Rad}\mathcal{A}'(E_t))$ with $\pi(p)\in M_n(\mathcal{A}'(E_t))/\text{Rad}(M_n(\mathcal{A}'(E_t)))$, $p\in M_n(\mathcal{A}'(E_t))$ and $\pi^2(p)=\pi(p)$, we have

$$p^2-p=R_0, R_0\in \text{Rad}(M_n(\mathcal{A}'(E_t))).$$

Note that $p=B+R$, where $B\in M_n(\mathcal{A}'(E_t))$ is a block-diagonal matrix over $\mathbb{C}$ and $R$ is in $\text{Rad}(M_n(\mathcal{A}'(E_t)))$. Then $\pi(p)=\pi(B)$ and

$$R_0=p^2-p=(B+R)^2-(B+R)=B^2-B+(BR+RB+R^2-R).$$

Since $\text{Rad}(M_n(\mathcal{A}'(E_t)))$ is an ideal of $M_n(\mathcal{A}'(E_t))$, then we have

$$B^2-B\in \text{Rad}(M_n(\mathcal{A}'(E_t))).$$

Since $B$ is a block-diagonal matrix, then we have $B$ is also an idempotent. Then we have

$$\sigma([B])=[\pi(p)].$$

That means $\sigma$ is also a surjective. And we also can see that $\sigma$ is homomorphism. Then $\sigma$ is an isomorphism and

$$\text{Vect}(\mathcal{A}'(E_t))\cong \text{Vect}(\mathcal{A}'(E_t)/\text{Rad}\mathcal{A}'(E_t)).$$

□

**Proposition 5.7.** *Let $E_t$ and $E_{\widetilde{t}}$ be two quasi-homogeneous bundles with matchable bundles $\{E_{t_i}\}_{i=0}^{n-1}$ and $\{E_{s_i}\}_{i=0}^{n-1}$ respectively. If $\Lambda(t)<2$, then $E_t$ and $E_{\widetilde{t}}$ are similarity equivalent if and only if*

$$K_0(\mathcal{A}'(E_t\oplus E_{\widetilde{t}}))\cong \mathbb{Z}.$$

*If $\Lambda(t)\geq 2$, then $E_t$ and $E_{\widetilde{t}}$ are similarity equivalent if and only if*

$$K_0(\mathcal{A}'(E_t\oplus E_{\widetilde{t}}))\cong \mathbb{Z}^n.$$

*Proof.* Suppose that $\Lambda(t)<2$. Let

$$T=\begin{pmatrix} S_{0,0} & S_{0,1} & S_{0,2} & \cdots & S_{0,n-1}\\ & S_{1,1} & S_{1,2} & \cdots & S_{1,n-1}\\ & & \ddots & \ddots & \vdots\\ & & & S_{n-1,n-1} & S_{n-1,n}\\ & & & & S_{n,n}\end{pmatrix} \text{ and } X=\begin{pmatrix} X_{0,0} & X_{0,1} & X_{0,2} & \cdots & X_{0,n-1}\\ & X_{1,1} & X_{1,2} & \cdots & X_{1,n-1}\\ & & \ddots & \ddots & \vdots\\ & & & X_{n-1,n-1} & X_{n-1,n}\\ & & & & X_{n,n}\end{pmatrix}.$$

Claim 1: If $XT=TX$,then we have $X_{i,i}=X_{j,j}$, for any $i\neq j$.

In fact, for any $i=0,1,\cdots,n-1$, we have

$$S_{i,i}X_{i,i+1}+S_{i,i+1}X_{i+1,i+1}=X_{i,i}S_{i,i+1}+X_{i,i+1}S_{i+1,i+1},$$

and

$$S_{i,i}X_{i,i+1}-X_{i,i+1}S_{i+1,i+1}=X_{i,i}S_{i,i+1}-S_{i,i+1}X_{i+1,i+1}=0.$$

Since $X_{i,i}\in \mathcal{A}'(E_{t_i})$ and each $E_{t_i}$ induces a Hilbert functional space $\mathcal{H}_i$ with reproducing kernel $\frac{1}{(1-z\overline{w})^{\lambda_i}}$, then we have $\mathcal{A}'(E_{t_i})\cong \mathcal{H}^\infty(\mathbb{D})$. Then there exists $\phi_{i,i}\in \mathcal{H}^\infty(\mathbb{D})$ such that

$$X_{i,i}=\phi_{i,i}(S_{i,i}), i=0,1,\cdots,n-1.$$

Thus we have

$$\phi_{i,i}(S_{i,i})S_{i,i+1}-S_{i,i+1}\phi_{i+1,i+1}(S_{i+1,i+1})=0.$$

Since $S_{i,i}S_{i,i+1}=S_{i,i+1}S_{i+1,i+1}$, then

$$S_{i,i+1}(\phi_{i,i}-\phi_{i+1,i+1})(S_{i+1,i+1})=0.$$

Note that $S_{i,i+1}$ has a dense range, then we can set

$$\phi_{i,i} = \phi, i = 0, 1, \cdots, n-1.$$

Claim 2: $\mathcal{A}'(E_t)/\mathrm{Rad}\mathcal{A}'(E_t) \cong \mathcal{H}^\infty(\mathbb{D})$.

Recall that $\mathrm{Rad}\mathcal{A}'(E_t) = \{S \in \mathcal{A}'(E_t) | \sigma_{\mathcal{A}'(E_t)}(SS') = 0,\ S' \in \mathcal{A}'(E_t)\}$. Any $X \in \mathcal{A}'(E_t)$ is upper triangular by Lemma 4.1 and $\mathcal{A}'(E_t)/\mathrm{Rad}\,\mathcal{A}'(E_t)$ is commutative by Lemma 5.3. Therefore if $Y$ is in $\mathrm{Rad}\,\mathcal{A}'(E_t)$, then we have

$$Y = \begin{pmatrix} 0 & Y_{0,1} & Y_{0,2} & \cdots & Y_{0,n-1} \\ & 0 & Y_{1,2} & \cdots & Y_{1,n-1} \\ & & \ddots & \ddots & \vdots \\ & \text{\huge 0} & & 0 & Y_{n-1,n} \\ & & & & 0 \end{pmatrix}.$$

Define a map $\Gamma : \mathcal{A}'(E_t)/\mathrm{Rad}\mathcal{A}'(E_t) \to \mathcal{H}^\infty(\mathbb{D})$ by the rule:

$$\Gamma([X]) = \phi, \text{ where } X = ((X_{i,j}))_{n\times n}, X_{i,i} = \phi(S_{i,i}).$$

Obviously, $\Gamma$ is well defined and if $\Gamma([X]) = 0$, then $\phi = 0$. Then $X_{i,i} = 0$, it follows that $X \in \mathrm{Rad}\mathcal{A}'(E_t)$ and $[X] = 0$. So $\Gamma$ is injective.

For any $\phi \in \mathcal{H}^\infty(\mathbb{D})$, set $X_{i,i} = \phi(S_{i,i}), i = 0, 1, 2, \cdots, n-1$. By Lemma 3.6, we can construct the operators $X_{i,j}, j \neq i$ such that $X := (\!(X_{i,j})\!)_{n\times n} \in \mathcal{A}'(E_t)$. That means $\Gamma$ is surjective. Then $\Gamma$ is an isomorphism and

$$\mathcal{A}'(E_t)/\mathrm{Rad}\mathcal{A}'(E_t) \cong \mathcal{H}^\infty(\mathbb{D}).$$

By [12, Lemma 2.10] and [12, Lemma 2.14]), we have

$$\mathrm{Vect}(\mathcal{A}'(E_t)) \cong \mathbb{N}, K_0(\mathcal{A}'(E_t)) \cong \mathbb{Z}.$$

By [12, Lemma 2.10], we have $E_t$ has a unique finite decomposition up to similarity. Similarly, $E_{\widetilde{t}}$ also has a unique finite decomposition up to similarity.

If $E_t \sim E_{\widetilde{t}}$, then $(t \oplus \widetilde{t}) \sim t^{(2)}$. So we have

$$\mathrm{Vect}(\mathcal{A}'(t \oplus \widetilde{t})) \cong \mathrm{Vect}(\mathcal{A}'(t^{(2)})) \cong \mathrm{Vect} M_2(\mathcal{A}'(t))) \cong \mathbb{N}$$

and

$$K_0(\mathcal{A}'(t \oplus \widetilde{t})) \cong \mathbb{Z}.$$

On the other hand, Note that $t$ and $\widetilde{t}$ are both strongly irreducible. If $K_0(\mathcal{A}'(t \oplus \widetilde{t})) \cong \mathbb{Z}$ and $\mathrm{Vect}(\mathcal{A}'(t \oplus \widetilde{t})) \cong \mathbb{N}$, then by [12, Lemma 2.10], we have $t \sim \widetilde{t}$, otherwise we will have

$$\mathrm{Vect}(\mathcal{A}'(t \oplus \widetilde{t})) \cong \mathbb{N}^2.$$

This is a contradiction. □

*Proof of Theorem 5.4.* When $\Lambda(t) < 2$, by Lemma 4.3, we have $E_t$ is strongly irreducible. So there exists no non-trivial idempotent in $\mathcal{A}'(E_t)$, which is the same as saying that the vector bundle $E_t$ has no non-trivial sub-bundle.

When $\Lambda(t) \geq 2$, by Lemma 4.3, we have

$$E_r \sim E_{t_0} \oplus E_{t_1} \oplus \cdots \oplus E_{t_{n-1}}.$$

Since $\mathcal{A}'(E_{t_i}) \cong \mathcal{H}^\infty(\mathbb{D})$, we have

$$\mathcal{A}'(E_r) \cong \mathcal{H}^\infty(\mathbb{D})^{(n)},$$

and by [12, Lemma 2.10],

$$\mathrm{Vect}(\mathcal{A}'(E_r)) \cong \mathbb{N}^{(n)}, K_0(\mathcal{A}'(E_r)) \cong \mathbb{Z}^{(n)}.$$

Then by [12, Lemma 2.10], we have $E_t$ has a unique finite decomposition up to similarity. Then for any non-trivial reducible sub-bundle of $E_r$ denoted by $E_r$, with

$$\mathcal{H}_r = \mathrm{Span}_{w\in\Omega}\{E_r(w)\}.$$

Let $P_t$ be the projection from $\mathcal{H}$ to $\mathcal{H}_t$. Then

$$E_t \sim E_r \oplus (E_t \ominus E_r) = P_r E_t \oplus (I - P_r)E_t.$$

Let

$$P_{t_i} : \mathcal{H} \to \mathcal{H}_i := \mathrm{Span}_{\lambda\in\Omega}\{E_{t_i}(w)\}, i = 0, 1, \cdots, n-1$$

be projections in $\mathcal{A}'(E_r)$. Then there exists an invertible operator $X$ such that $E_r = X(\oplus_{i=0}^{s} E_{t_{k_i}})$. Suppose that

$$\oplus_{i=0}^{n-1} E_{t_i} = (\oplus_{i=0}^{s} E_{t_{k_i}}) \oplus (\oplus_{i=0}^{n-s} E_{t_{l_i}}).$$

Set $E_s = X(\oplus_{i=0}^{n-s} E_{t_{l_i}})$, then we have

$$E_r \oplus E_s \sim E_t.$$

If there exists another bundle $E_{s'}$ such that

$$E_r \oplus E_{s'} \sim E_t.$$

Since $E_r$ has a unique finite decomposition up to similarity, then we have

$$E_{s'} \sim \oplus_{i=0}^{n-s} E_{t_{l_i}} \sim E_s.$$

$\square$

*Proof of Theorem 5.5.* Let $P \in P_n(\mathcal{A}'(E_t)) = P(\mathcal{A}'(E_{t^n}))$ be an idempotent. Then we have $PE_{t^n}$ be a sub-bundle of $E_{t^n}$. Define map

$$\Gamma : V(\mathcal{A}'(E_t))) \to V^0(E_t)$$

with $\Gamma([p]_0) = \overline{PE_{t^n}}$.

First, we prove that $\Gamma$ is well defined. In fact, for any $P \sim Q \in [P]_0$, there exists positive integer $n$ such that $P, Q \in \mathcal{A}'(E_{t^n})$. Since $Q = XPX^{-1}, X \in \mathcal{A}'(E_{t^n})$, then we have

$$QE_{t^n} = XPX^{-1}E_{t^n} \sim PX^{-1}E_{t^n}.$$

And Note that $X, X^{-1} \in \mathcal{A}'(E_{t^n})$, then we have

$$X^{-1}t^n(w) = t^n(w), \text{for any } w \in \Omega.$$

Thus

$$QE_{t^n} \sim PXE_{t^n},$$

and $\overline{QE_{t^n}} = \overline{PE_{t^n}}$. So $\Gamma$ is well defined.

Second, we prove that $\Gamma$ is surjective. Suppose that $E_r$ is a sub-bundle of $E_{t^n}$ with dimension $K$, where $n$ is positive integer. Suppose that

$$\mathcal{H}_r := \bigvee_{w\in\Omega} \{\gamma_1(w), \gamma_2(w), \cdots, \gamma_K(w)\},$$

where $K \in \mathbb{N}$ and $P_r$ is the projection from $\mathcal{H}$ to $\mathcal{H}_r$, then we have $P_r \in \mathcal{A}'(E_{t^n})$ and

$$P_r E_{t^n} \sim E_r.$$

Then it follows that $\Gamma$ is surjective.

Finally, we prove that $\Gamma$ is also injective. Let $P, Q \in \mathcal{A}'(E_{t^n})$. Suppose that there exists an invertible operator $X \in \mathcal{A}'(E_{t^n})$ such that

$$XPE_{t^n} = QE_{t^n}.$$

Let $\{p_1, p_2, \cdots, p_m\}$ be a decomposition of $P$. Then $\{Xp_1X^{-1}, Xp_2X^{-1}, \cdots, Xp_mX^{-1}\}$ be a decomposition of $Q$. In fact, we have

$$\begin{aligned} Xp_1X^{-1}QE_{t^n} + Xp_2X^{-1}QE_{t^n} + \cdots + Xp_mX^{-1}QE_{t^n} &= Xp_1E_{t^n} + Xp_2E_{t^n} + \cdots + Xp_mE_{t^n} \\ &= XPE_{t^n} \\ &= QE_{t^n}. \end{aligned}$$

Suppose that $\{p_{m+1}, p_{m+2}, \cdots, p_N\}$ and $\{q_{m+1}, q_{m+2}, \cdots, q_N\}$ be the decompositions of $(I-P)E_{t^n}$ and $(I-Q)E_{t^n}$ respectively. Then we have

$$\{p_1, p_2, \cdots, p_N\} \text{ and, } \{Xp_1X^{-1}, Xp_2X^{-1}, \cdots, Xp_mX^{-1}, q_{m+1}, q_{m+2}, \cdots, q_N\}$$

are two different decompositions of $E_{t^n}$. By the uniqueness of decomposition of $E_{t^n}$, there exists an invertible bounded linear operator $Y \in \mathcal{A}'(E_{t^n})$ such that $\{Y^{-1}P_iY\}$ is a rearrangement of

$$\{Xp_1X^{-1}, Xp_2X^{-1}, \cdots, Xp_mX^{-1}, q_{m+1}, q_{m+2}, \cdots, q_N\}.$$

By [12, Lemma 2.6]), for any $v \in \{m+1, m+2, \cdots, N\}$, we can find $Z_v$ in $GL(L(q_v\mathcal{H}, p_v\mathcal{H}))$ and $p_{v'}, v' \in \{m+1, \cdots, N\}$ such that

$$Z_vq_vE_{t^n} = p_{v'}E_{t^n}, v_1' = v_2', \text{ when } v_1 = v_2.$$

Set $Z_k = X^{-1}|_{Xp_kX^{-1}\mathcal{H}}, k = 1, 2, \cdots m$, then we have that

$$Z = \sum_{k=1}^{m} Z_k + \sum_{v=m+1}^{N} Z_v \in GL\mathcal{A}'(E_{t^n}),$$

and

$$ZPZ^{-1} = Q.$$

It follows that $\Gamma$ is injective. Since $\Gamma$ is also a homomorphism, then we have

$$\mathrm{Vect}^0(E_t) \cong \mathrm{Vect}(\mathcal{A}'(E_{t^n}), K^0(E_t) \cong K_0(\mathcal{A}'(E_t)).$$

$\square$

5.3. The Halmos' question. The well-known question of Halmos asks if $\varrho : \mathbb{C}[z] \to \mathcal{L}(\mathcal{H})$ is a continuous (for $p \in \mathbb{C}[z]$, the norm $\|p\| = \sup_{z\in\mathbb{D}} |p(z)|$) algebra homomorphism induced by an operator $S$, that is, $\varrho(p) = p(S)$, then does there exist an invertible linear operator $L$ and a contraction $T$ on the Hilbert space $\mathcal{H}$ so that $S = LTL^{-1}$. After the question was raised in [6, Problem 6], an affirmative answer for several classes of operators were given. A counter example was found by Pisier in 1996 (cf. [22]). It was pointed out in a recent paper of the third author with Korányi [17] that the Halmos' question has an affirmative answer for homogeneous operators in the Cowen-Douglas class $B_n(\mathbb{D})$. Thus it is natural to ask if the Halmos' question has an affirmative answer for quasi-homogeneous operators. If $\Lambda(t) \geq 2$, the answer is evidently "yes":

In this case, the quasi-homogeneous operator $T$ is similar to the $n$- fold direct sum of the homogeneous operators $T_i$ (adjoint of the multiplication operator) acting on the weighted Bergman spaces $\mathbb{A}^{(\lambda_i)}(\mathbb{D})$, $i = 0, 1, \ldots, n-1$. Now, if $\lambda_0 \geq 1$, this direct sum is contractive and we are done. If $\lambda_0 < 1$, then $T_0$ is not even power bounded and therefore neither is the operator $T$. So, there is nothing to prove when $\lambda_0 < 1$.

If $\Lambda(t) < 2$, then the operator $T$ is strongly irreducible. Therefore, we can't answer the Halmos' question purely in terms of the atoms of the operator $T$. Never the less, the answer is affirmative even in this case. To show this, we first prove the following useful lemma.

For $i = 1, 2$, let $\mathcal{H}_i$ be a Hilbert space of holomorphic function on $\mathbb{D}$ possessing a reproducing kernel, say $K_i$, and $T_i$ be the adjoint of the multiplication operator on $\mathcal{H}_i$. Assume that $\mathcal{H}_0 \subseteq \mathcal{H}_1$ and let $\iota : \mathcal{H}_0 \to \mathcal{H}_1$ be the inclusion map. Then the adjoint $\iota^*$ of the inclusion map has the property $\iota^*(K_1(\cdot, w)) = K_0(\cdot, w),\ w \in \mathbb{D}$.

**Lemma 5.8.** *Assume that* $K_i(z,w) = \frac{1}{(1-z\overline{w})^{\lambda_i}}, i = 0, 1$. *Suppose that* $S : \mathcal{H}_0 \to \mathcal{H}_1$ *is a bounded linear operator with the intertwining property* $T_0S = ST_1$. *Then there exists a holomorphic function* $\phi$ *such that* $S = \phi(T_0)\iota^*$.

*Proof.* The operators $T_i,\ i = 0, 1$ are in $B_1(\mathbb{D})$. If $S : \mathcal{H}_0 \to \mathcal{H}_1$ is a bounded linear operator and $T_0S = ST_1$, then there exists a holomorphic function $\psi$ such that $S^* = M_\psi$. This is easily proved as in [19, Section 5]. Let $\phi$ be the holomorphic function defined on the unit disc by the formula $\overline{\phi(\overline{w})} = \psi(w),\ w \in \mathbb{D}$. For any $f \in \mathcal{H}_0$, we have that

$$\begin{aligned}\langle f(z), \phi(T_0)\iota^*(K_1(z,w))\rangle &= \langle f(z), \phi(\overline{w})K_0(z,w)\rangle\\ &= \overline{\phi(\overline{w})}\langle f(z), K_0(z,w)\rangle\\ &= \langle f(z), M_\psi^*(K_1(z,w))\rangle\\ &= \langle f(z), S(K_1(z,w))\rangle.\end{aligned}$$

Consequently, $S = \phi(T_0)\iota^*$. □

**Lemma 5.9.** *Suppose that $t$ is a quasi-homogeneous holomorphic curve. Assume that $\Lambda(t) < 2$ and $\lambda_0 \geq 1$. Then the operator $T$ is not power bounded.*

*Proof.* The top $2 \times 2$ block in the atomic decomposition of the quasi-homogeneous operator $T$ is of the form $\begin{pmatrix} T_0 & S_{0,1} \\ 0 & T_1 \end{pmatrix}$. As always, we assume that the operators $T_0$ and $T_1$ are the adjoints of the multiplication operator on the weighted Bergman spaces $\mathbb{A}^{(\lambda_0)}(\mathbb{D})$ and $\mathbb{A}^{(\lambda_1)}(\mathbb{D})$, respectively. The operator $S_{0,1}$ has the intertwining property $T_0 S_{0,1} = S_{0,1} T_1$.

Let $\iota$ denote the inclusion map from $\mathbb{A}^{(\lambda_0)}(\mathbb{D})$ to $\mathbb{A}^{(\lambda_1)}(\mathbb{D})$. Then $\iota^*(t_1)(w) = t_0(w),\ w \in \mathbb{D}$, and the operator $S_{0,1}$ must be of the form $\phi(T_0)\iota^*$ for some holomorphic function $\phi$ on the unit disc $\mathbb{D}$, as we have shown in Lemma 5.8. Indeed, $S_{0,1}(t_1(w)) = \phi(w)t_1(w) = \phi(T_0)\iota^*(t_1(w))$.

Without loss of generality, we assume that $\phi(w) = \sum\limits_{i=0}^{\infty} \phi_i w^i$ and $\phi_0 \neq 0$. For $j = 0, 1$, the set of vectors $e_\ell^{(\lambda_j)} := \sqrt{a_\ell(\lambda_j)}\, z^\ell,\ \ell \geq 0$, is an orthonormal basis in $\mathbb{A}^{(\lambda_j)}(\mathbb{D})$. Then we have that

$$T_0^{n-1}(e_\ell(\lambda_0)) = \prod_{i=\ell-n+1}^{\ell-1} w_i(\lambda_0) e_{\ell-n+1}(\lambda_0), S_{0,1}(e_\ell(\lambda_1)) = \phi_0 \frac{\prod\limits_{i=0}^{\ell-1} w_i(\lambda_1)}{\prod\limits_{i=0}^{\ell-1} w_i(\lambda_0)} e_\ell(\lambda_0).$$

Consequently,

$$nT_0^{n-1}S_{0,1}(e_\ell(\lambda_1)) = n\phi_0 \frac{\prod\limits_{i=0}^{\ell-1} w_i(\lambda_1)}{\prod\limits_{i=0}^{\ell-n} w_i(\lambda_0)} e_{\ell-n+1}(\lambda_0).$$

It is then easily deduced that $||nT_0^{n-1}S_{0,1}|| \to \infty$ as $n \to \infty$.

Let $T_{|2\times 2}$ denote the top $2 \times 2$ block $\begin{pmatrix} T_0 & S_{0,1} \\ 0 & T_1 \end{pmatrix}$ in the operator $T$. Since $T_{|2\times 2}^n = \begin{pmatrix} T_0^n & nT_0^{n-1}S_{0,1} \\ 0 & T_1^n \end{pmatrix}$, and $||T_{|2\times 2}^n|| \geq ||nT_0^{n-1}S_{0,1}||$, it follows that $||T_{|2\times 2}^n|| \to \infty$ as $n \to \infty$. Clearly, $\|T^n\| \geq \|T_{|2\times 2}^n\|$ completing the proof. □

Since a quasi-homogeneous operator for which $\lambda_0 < 1$ can't be power bounded, the lemma we have just proved shows that if $T$ is quasi-homogeneous and $\Lambda(t) < 2$, then the operator $T$ is not power bounded. Therefore we have proved the following theorem answering the Halmos' question in the affirmative.

**Theorem 5.10.** *If a quasi-homogeneous operator $T$ has the property $\|p(T)\|_{\mathrm{op}} \leq K\|p\|_{\infty,\mathbb{D}}$, $p \in \mathbb{C}[z]$, then it must be similar to a contraction.*

**Acknowledgement.** We thank the Indian Institute of Science and the Hebei Normal University for providing excellent environment for our collaborative research.

## References

[1] B. Bagchi and G. Misra, *The homogeneous shifts,* J. Funct. Anal., 204 (2003), 293 - 319.
[2] M. J. Cowen and R. G. Douglas, *Complex geometry and operator theory*, Acta Math. 141 (1978), 187 - 261.
[3] D. N. Clark and G. Misra, *On weighted shifts, curvature and similarity,* J. London Math. Soc., 31(1985), 357 - 368.
[4] R. Curto and N. Salinas, *Generalized Bergman kernels and the Cowen-Douglas theory,* Amer. J. Math. 106 (1984), 447 - 488.
[5] R. G. Douglas and G. Misra, *Equivalence of quotient Hilbert modules. II,* Trans. Amer. Math. Soc., 360 (2008), 2229 - 2264.
[6] P. R. Halmos, *Ten problems in Hilbert space*, Bull. Amer. Math. Soc., 76 (1970) 887 - 933.
[7] G. H. Hardy and J. E. Littlewood, *Tauberian Theorems Concerning Power Series and Dirichlet's Series whose Coefficients are Positive*, Proc. London Math. Soc., 13 (1914), 174 - 191.
[8] C. Jiang, *Similarity, reducibility and approximation of the Cowen-Douglas operators,* J. Operator Theory, 32 (1994), 77 - 89.
[9] C. Jiang, *Similarity classification of Cowen-Douglas operators,* Canad. J. Math., 56 (2004), 742 - 775.
[10] K. Ji, C. Jiang, D. K. Keshari and G. Misra, *Flag structure for operators in the Cowen-Douglas class*, Comptes rendus - Mathématique, 352 (2014), 511 - 514.
[11] K. Ji, C. Jiang, D. K. Keshari and G. Misra, *Rigidity of the flag structure for a class of Cowen-Douglas operators*, to appear, J. Func. Anal.
[12] C. Jiang and K. Ji, *Similarity classification of holomorphic curves,* Adv. Math., 215 (2007), 446 - 468.
[13] C. Jiang, X. Guo and K. Ji, *K-group and similarity classification of operators,* J. Funct. Anal., 225 (2005), 167 - 192.
[14] C. Jiang and Z. Wang, *Strongly irreducible operators on Hilbert space,* Pitman Research Notes in Mathematics Series, 389. Longman, Harlow, 1998. x+243 pp.
[15] C. Jiang and Z. Wang, *The spectral picture and the closure of the similarity orbit of strongly irreducible operators,* Integral Equations Operator Theory, 24 (1996), 81 - 105.
[16] A. Koranyi and G. Misra, *Homogeneous Hermitian holomorphic vector bundles and the Cowen-Douglas class over bounded symmetric domains*, Comptes rendus - Mathèmatique, 354 (2016), 291- 295.
[17] A. Korányi and G. Misra, *A classification of homogeneous operators in the Cowen-Douglas class,* Adv. Math., 226 (2011) 5338 - 5360.
[18] A. Korányi and G. Misra, *Multiplicity-free homogeneous operators in the Cowen-Douglas class,* Perspectives in mathematical sciences. II, 83-101, Stat. Sci. Interdiscip. Res., 8, World Sci. Publ., Hackensack, NJ, 2009.
[19] A. Korányi and G. Misra, *Homogeneous operators on Hilbert spaces of holomorphic functions*, J. Func. Anal., 254 (2008), 2419 - 2436.
[20] G. Misra and S. Shyam Roy, *The curvature invariant for a class of Homogeneous operators*, Proc. London Math.Soc., 99 (2009), 557 - 584.
[21] G. Misra and H. Upmeier, *Homogeenous vector bundles and intertwining operators for symmetric domains*, Adv. Math., 303 (2016) 1077 - 1121.
[22] G. Pisier, *A polynomially bounded operator on Hilbert space which is not similar to a contraction*, J. Amer. Math. Soc., 10 (1997), 351 - 369.
[23] Pit-Mann Wong and W. Stoll, *On Holomorphic Jet Bundles,* arXiv:math/0003226v1, 2000.
[24] A. L. Shields, *Weighted Shift Operators And Analytic Function Theory,* Mathematical Survey, 13 (1974), 49-128.

*Current address*, C. Jiang and K. Ji: Department of Mathematics, Hebei Normal University, Shijiazhuang, Hebei 050016, China

*Current address*, G. Misra: Department of Mathematics, Indian Institute of Science, Bangalore, 560 012, India

*E-mail address*, C. Jiang: `cljiang@hebtu.edu.cn`

*E-mail address*, K. Ji: `jikui@hebtu.edu.cn`

*E-mail address*, G. Misra: `gm@math.iisc.ernet.in`