# Geometric invariants for a class of submodules of analytic Hilbert modules

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# Declaration

I hereby declare that the work reported in this thesis is entirely original and has been carried out by me under the supervision of Prof. Gadadhar Misra at the Department of Mathematics, Indian Institute of Science, Bangalore. I further declare that this work has not been the basis for the award of any degree, diploma, fellowship, associateship or similar title of any University or Institution.

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**DEDICATED TO ALL THE WELL-WISHERS**

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# Abstract

Let $\Omega \subseteq \mathbb{C}^m$ be a bounded connected open set and $\mathscr{H} \subseteq \mathcal{O}(\Omega)$ be an analytic Hilbert module, i.e., the Hilbert space $\mathscr{H}$ possesses a reproducing kernel $K$, the polynomial ring $\mathbb{C}[\underline{z}] \subseteq \mathscr{H}$ is dense and the point-wise multiplication induced by $p \in \mathbb{C}[\underline{z}]$ is bounded on $\mathscr{H}$. We fix an ideal $\mathscr{I} \subseteq \mathbb{C}[\underline{z}]$ generated by $p_1, \ldots, p_t$ and let $[\mathscr{I}]$ denote the completion of $\mathscr{I}$ in $\mathscr{H}$. Let $X : [\mathscr{I}] \to \mathscr{H}$ be the inclusion map. Thus we have a short exact sequence of Hilbert modules $0 \longrightarrow [\mathscr{I}] \xrightarrow{X} \mathscr{H} \xrightarrow{\pi} \mathscr{Q} \longrightarrow 0,$ where the module multiplication in the quotient $\mathscr{Q} := [\mathscr{I}]^\perp$ is given by the formula $m_p f = P_{[\mathscr{I}]^\perp}(pf)$, $p \in \mathbb{C}[\underline{z}]$, $f \in \mathscr{Q}$. The analytic Hilbert module $\mathscr{H}$ defines a subsheaf $\mathscr{S}^{\mathscr{H}}$ of the sheaf $\mathcal{O}(\Omega)$ of holomorphic functions defined on $\Omega$. For any open $U \subset \Omega$, it is obtained by setting

$$\mathscr{S}^{\mathscr{H}}(U) := \Big\{ \sum_{i=1}^{n} (f_i|_U) h_i : f_i \in \mathscr{H}, h_i \in \mathcal{O}(U), n \in \mathbb{N} \Big\}.$$

This is locally free and naturally gives rise to a holomorphic line bundle on $\Omega$. However, in general, the sheaf corresponding to the sub-module $[\mathscr{I}]$ is not locally free but only coherent.

Building on the earlier work of S. Biswas, a decomposition theorem is obtained for the kernel $K_{[\mathscr{I}]}$ along the zero set $V_{[\mathscr{I}]} := \{z \in \mathbb{C}^m : f(z) = 0, f \in [\mathscr{I}]\}$ which is assumed to be a submanifold of codimension $t$: There exists anti-holomorphic maps $F_1, \ldots, F_t : V_{[\mathscr{I}]} \to [\mathscr{I}]$ such that

$$K_{[\mathscr{I}]}(\cdot, u) = \overline{p_1(u)} F_w^1(u) + \cdots \overline{p_t(u)} F_w^t(u),\ u \in \Omega_w,$$

in some neighbourhood $\Omega_w$ of each fixed but arbitrary $w \in V_{[\mathscr{I}]}$ for some anti-holomorphic maps $F_w^1, \ldots, F_w^t : \Omega_w \to [\mathscr{I}]$ extending $F_1, \ldots, F_t$. The anti-holomorphic maps $F_1, \ldots, F_t$ are linearly independent on $V_{[\mathscr{I}]}$, defining a rank $t$ anti-holomorphic Hermitian vector bundle on it. This gives rise to complex geometric invariants for the pair $([\mathscr{I}], \mathscr{H})$.

Next, using a decomposition formula obtained from an earlier work of Douglas, Misra and Varughese, the maps $F_1, \ldots, F_t : V_{[\mathscr{I}]} \to [\mathscr{I}]$ are explicitly determined with the additional

assumption that $p_i, p_j$ are relatively prime for $i \neq j$. Using this, a line bundle on $V_{[\mathscr{I}]} \times \mathbb{P}^{t-1}$ is constructed via the monoidal transformation around $V_{[\mathscr{I}]}$ which provides useful invariants for $([\mathscr{I}], \mathscr{H})$.

Localising the modules $[\mathscr{I}]$ and $\mathscr{H}$ at $w \in \Omega$, we obtain the localization $X(w)$ of the module map $X$. The localizations are nothing but the quotient modules $[\mathscr{I}]/[\mathscr{I}]_w$ and $\mathscr{H}/\mathscr{H}_w$, where $[\mathscr{I}]_w$ and $\mathscr{H}_w$ are the maximal sub-modules of functions vanishing at $w$. These clearly define anti-holomorphic line bundles $E_{[\mathscr{I}]}$ and $E_{\mathscr{H}}$, respectively, on $\Omega \setminus V_{[\mathscr{I}]}$. However, there is a third line bundle, namely, $\mathrm{Hom}(E_{\mathscr{H}}, E_{[\mathscr{I}]})$ defined by the anti-holomorphic map $X(w)^*$. The curvature of a holomorphic line bundle $\mathscr{L}$ on $\Omega$, computed with respect to a holomorphic frame $\gamma$ is given by the formula

$$\mathscr{K}_{\mathscr{L}}(z) = \sum_{i,j=1}^{m} \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log \|\gamma(z)\|^2 dz_i \wedge d\bar{z}_j.$$

It is a complete invariant for the line bundle $\mathscr{L}$. The alternating sum

$$\mathscr{A}_{[\mathscr{I}],\mathscr{H}}(w) := \mathscr{K}_X(w) - \mathscr{K}_{[\mathscr{I}]}(w) + \mathscr{K}_{\mathscr{H}}(w) = 0, \ w \in \Omega \setminus V_{[\mathscr{I}]},$$

where $\mathscr{K}_X$, $\mathscr{K}_{[\mathscr{I}]}$ and $\mathscr{K}_{\mathscr{H}}$ denote the curvature $(1,1)$ form of the line bundles $E_X$, $E_{[\mathscr{I}]}$ and $E_{\mathscr{H}}$, respectively. Thus it is an invariant for the pair $([\mathscr{I}], \mathscr{H})$. However, when $\mathscr{I}$ is principal, by taking distributional derivatives, $\mathscr{A}_{[\mathscr{I}],\mathscr{H}}(w)$ extends to all of $\Omega$ as a $(1,1)$ current. Consider the following diagram of short exact sequences of Hilbert modules:

$$(1) \quad \begin{array}{ccccccccc} 0 & \longrightarrow & [\mathscr{I}] & \xrightarrow{X} & \mathscr{H} & \xrightarrow{\pi} & \mathscr{Q} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow L & & \downarrow & & \\ 0 & \longrightarrow & [\widetilde{\mathscr{I}}] & \xrightarrow{\tilde{X}} & \widetilde{\mathscr{H}} & \xrightarrow{\tilde{\pi}} & \widetilde{\mathscr{Q}} & \longrightarrow & 0, \end{array} \qquad (2) \quad \begin{array}{ccc} [\mathscr{I}] & \xrightarrow{X} & \mathscr{H} \\ \downarrow & & \downarrow L \\ [\widetilde{\mathscr{I}}] & \xrightarrow{\tilde{X}} & \widetilde{\mathscr{H}} \end{array}$$

It is shown that if $\mathscr{A}_{[\mathscr{I}],\mathscr{H}}(w) = \mathscr{A}_{[\widetilde{\mathscr{I}}],\widetilde{\mathscr{H}}}(w)$, then $L|_{[\mathscr{I}]}$ makes the second diagram commute. Hence, if $L$ is bijective, then $[\mathscr{I}]$ and $[\widetilde{\mathscr{I}}]$ are equivalent as Hilbert modules. It follows that the alternating sum is an invariant for the "rigidity" phenomenon.

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# Chapter 1

# Introduction

## 1.1 Preliminaries

The notion of a Hilbert module over a function algebra was introduced by R. G. Douglas in the late eighties. Over the past couple of decades, problems of multi-variate operator theory have been discussed using the language of these Hilbert modules. In this thesis, we continue this tradition. Let us begin by setting up some conventions that will be in force throughout.

1. $\mathbb{C}[\underline{z}] := \mathbb{C}[z_1, \ldots, z_m]$ is the polynomial ring in $m$ variables.

2. $\Omega \subseteq \mathbb{C}^m$ is an open connected and bounded set.

3. $\mathscr{O}(\Omega)$ is the ring of holomorphic functions on the bounded domain $\Omega$.

4. $\mathscr{H}$ is a complex separable Hilbert space and $\mathscr{L}(\mathscr{H})$ is the algebra of bounded linear operators on $\mathscr{H}$.

**Definition 1.1.1.** For $1 \leq i \leq m$, let $T_i : \mathscr{H} \to \mathscr{H}$ be a commuting set of bounded linear operators on the Hilbert space $\mathscr{H}$. Set $T = (T_1, \ldots, T_m)$. For any polynomial $p$, the map $(p, h) \to p(T)h$, $h \in \mathscr{H}$, is clearly a module multiplication, that is, $p \to m_p := p(T)$ is an algebra homomorphism from $\mathbb{C}[z_1, \ldots, z_m]$ to $\mathscr{L}(\mathscr{H})$. The Hilbert space $\mathscr{H}$ is said to be a *Hilbert module* over the polynomial ring $\mathbb{C}[z_1, \ldots, z_m]$. A closed subspace $\mathscr{H}_0 \subseteq \mathscr{H}$ is said to be a *sub-module* of $\mathscr{H}$ if it is invariant under the module multiplication, i.e., $m_p f \in \mathscr{H}_0$ for all $f \in \mathscr{H}_0$. The *quotient module* $\mathscr{Q}$ is the quotient space $\mathscr{H} / \mathscr{H}_0$, which is the ortho-compliment of $\mathscr{H}_0$ in $\mathscr{H}$. The module multiplication on this space is defined by compression of the multiplication on $\mathscr{H}$ to $\mathscr{Q}$, i.e., $\mathfrak{m}_p f = P_{\mathscr{H}_0^\perp}\big(m_p f\big)$, $f \in \mathscr{Q}$.

(The original definition of the Hilbert module required the module map to be continuous in both the variables, however, we won't require this. )

Two Hilbert modules $\mathscr{H}$ and $\tilde{\mathscr{H}}$ are said to be "unitarily" equivalent if there exists a unitary module map $\theta : \mathscr{H} \to \tilde{\mathscr{H}}$ intertwining the module maps, $m$ and $\tilde{m}$, that is, $\tilde{m}_p\theta = \theta m_p$.

In this thesis, we will be studying the Hilbert modules closely related to the Hilbert modules in the Cowen-Douglas class $B_1(\Omega)$, namely, *analytic Hilbert modules*. To describe these, we first recall the notion of a kernel function which is an essential tool for this work.

**Definition 1.1.2.** Let $K : \Omega\times\Omega \to \mathbb{C}$ be a function holomorphic in the first and anti-holomorphic in the second variable. Assume that $K(z,w) = \overline{K(w,z)}$ and that it is *non-negative definite*:

$$\big\langle \big(\big(K(z_i,z_j)\big)\big)\underline{x}, \underline{x}\big\rangle \geq 0,\ \{z_1,\ldots,z_n\} \subseteq \Omega,\ \underline{x} \in \mathbb{C}^n,\ n \in \mathbb{N}.$$

Let $k_w$ be the holomorphic function defined by $k_w(z) := K(z,w)$.

Let $\mathscr{H}^0$ be the linear span of the vectors $\{k_w : w \in \Omega\}$. For any finite subset $\{z_1,\ldots,z_n\}$ of $\Omega$ and complex numbers $x_1,\ldots,x_n$, set

$$\Big\| \sum_{i=1}^{n} x_j k_{z_j} \Big\|^2 := \big\langle \big(\big(K(z_i,z_j)\big)\big)\underline{x}, \underline{x}\big\rangle,$$

where $\underline{x}$ is the vector whose $i$-th coordinate is $x_i$. Since $K$ is assumed to be non-negative definite, this defines a semi-norm on the linear space $\mathscr{H}^0$.

Now, the sesquilinear form $K(z,w) = \langle K(\cdot,w), K(\cdot,z)\rangle$ is non-negative definite by assumption. However, for $f \in \mathscr{H}^0$, Cauchy-Schwarz gives

$$|f(w)|^2 = |\langle f, K(\cdot,w)\rangle|^2 \leq \|f\|^2 K(w,w),\ w \in \Omega.$$

It follows that if $\|f\| = 0$, then $f$ is the zero vector in $\mathscr{H}^0$. Thus the semi-norm defined by $K$, as above, is indeed a norm on $\mathscr{H}^0$. The completion of $\mathscr{H}^0$ equipped with this norm is a Hilbert space, which consists of holomorphic functions on $\Omega$ (cf. [2]). The function $k_w := K(\cdot,w)$, then has the reproducing property, namely,

$$\langle f, k_w\rangle = f(w),\ f \in \mathscr{H},\ w \in \Omega.$$

Conversely, assume that the point evaluation $e_w : \mathscr{H} \to \mathbb{C}$, $w \in \Omega$, on a Hilbert space $\mathscr{H} \subseteq \mathscr{O}(\Omega)$ is bounded, that is, $|f(w)| \leq C\|f\|$, $f \in \mathscr{H}$. Then $f(w) = \langle f, k_w\rangle$ for some vector $k_w \in \mathscr{H}$. It follows that $e_w^* = k_w$. Let $K(z,w) = e_z k_w = e_z e_w^*$. The function $K$ is holomorphic in the first variable and anti-holomoprhic in the second. Also, $\overline{K(z,w)} = K(w,z)$. Finally, for any finite subset $\{z_1,\ldots,z_n\}$ of $\Omega$, we have

$$0 \leq \Big\| \sum_{i=1}^{n} x_j k_{z_j} \Big\|^2 = \sum_{i,j=1}^{n} \bar{x}_i x_j K(z_i,z_j) = \big\langle \big(\big(K(z_i,z_j)\big)\big)\underline{x}, \underline{x}\big\rangle \tag{1.1}$$

The non-negative definite function $K$ is said to be the *reproducing kernel* of the Hilbert space $\mathscr{H}$.

We will be studying a class $\mathfrak{B}_1(\Omega)$ of Hilbert modules closely related to analytic Hilbert modules. We complete the study of these Hilbert modules, which was initiated in [3], in some respects. First we recall the notion of an *analytic Hilbert module*.

**Definition 1.1.3.** A Hilbert module $\mathscr{H} \subseteq \mathscr{O}(\Omega)$ over the polynomial ring $\mathbb{C}[\underline{z}]$ is said to be an *analytic Hilbert module* if it possesses a reproducing kernel $K$ and the polynomial ring $\mathbb{C}[\underline{z}]$ is included in $\mathscr{H}$ and it is dense. In particular, $K(w,w) \neq 0$, $w \in \Omega$.

There are several notions, namely, locally free modules [7], modules with sharp kernels [1], quasi-free modules [16], which are closely related to the notion of analytic Hilbert modules. In all of these variants, the definition ensures the existence of a holomorphic Hermitian vector bundle corresponding to these Hilbert modules. The fundamental theorem of Cowen and Douglas then applies and says that the equivalence class of the Hilbert modules and those of the vector bundles determine each other. Finding tractable invariants for these remains a challenge.

It is easy to verify that $M_p^* k_w = \overline{p(w)} k_w$, or equivalently, $k_w$ is in $\ker(M_p - p(w))^*$. If $\mathscr{H}$ is an analytic Hilbert module, then it follows that the $\dim \cap_{i=1}^m \ker(M_i - w_i)^* = 1$, $w \in \Omega$, where $M_i$ is the operator of multiplication by the coordinate function $z_i$ on $\mathscr{H}$. This is easily verified as follows. For any $f \in \cap_{i=1}^m \ker(M_i - w_i)^*$, $p \in \mathbb{C}[\underline{z}]$, we have

$$\langle f, p \rangle = \langle M_p^* f, 1 \rangle = \langle \overline{p(w)} f, 1 \rangle = \langle a k_w, p \rangle,$$

where $a = \langle f, 1 \rangle$. Therefore, if $\mathscr{H}$ is an analytic Hilbert module, then the dimension of the joint kernel $\cap_{i=1}^m \ker(M_i - w_i)^*$ is 1 and is spanned by the vector $k_w$. Hence the map $\gamma : \Omega^* \to \mathscr{H}$, $\gamma(w) = k_{\bar{w}}$ is holomorphic for $w \in \Omega^* := \{w \in \mathbb{C}^m : \bar{w} \in \Omega\}$. Thus it defines a holomorphic Hermitian line bundle $\mathscr{L}$ on $\Omega^*$. If $\alpha$ is a non-vanishing holomorphic function defined on $\Omega^*$, then $\alpha(w)\gamma(w)$ serves as a holomorphic frame for the line bundle $\mathscr{L}$ as well. The Hermitian structures induced by these two holomorphic frames are $\|\gamma(w)\|^2$ and $|\alpha(w)|^2\|\gamma(w)\|^2$, respectively. These differ by the absolute square of a non-vanishing holomorphic function. However, the curvature $\mathscr{K}_{\mathscr{L}}$ defined relative to either one of these two frames is the same and therefore serves as an invariant for the holomorphic Hermitian line bundle $\mathscr{L}$. Recall that the curvature of $\mathscr{L}$ is defined to be the $(1,1)$ form:

$$\mathscr{K}_{\mathscr{L}}(z) := \sum_{i,j=1}^n \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log \|\gamma(z)\|^2 dz_i \wedge d\bar{z}_j.$$

The fundamental theorem proved by Cowen and Douglas in [10], then says that two analytic Hilbert modules are equivalent if and only if their curvatures are equal. However, there is a large class of Hilbert modules, where the dimension of the joint kernel $\cap_{i=1}^m \ker(M_i - w_i)^*$ is not constant.

**Example 1.1.4.** The basic example of this phenomenon is the sub-module $H^2_{(0,0)}(\mathbb{D}^2)$ of functions vanishing at $(0,0)$ of the Hardy module $H^2(\mathbb{D}^2)$. In this case, it is easy to verify (cf. [18]) that

$$\bigcap_{i=1}^{2} \ker(M_i - w_i)^* = \begin{cases} \frac{1}{(1-\bar{w}_1 z_1)(1-\bar{w}_2 z_2)} & \text{if } (w_1, w_2) \neq (0,0) \\ \{a_1 z_1 + a_2 z_2 : a_1, a_2 \in \mathbb{C}\} & \text{if } (w_1, w_2) = (0,0). \end{cases}$$

We investigate a class of sub-modules of analytic Hilbert modules like the sub-module $H^2_{(0,0)}(\mathbb{D}^2)$ of the Hardy module $H^2(\mathbb{D}^2)$. First, we recall the following definition from [3].

**Definition 1.1.5.** The class $\mathfrak{B}_1(\Omega)$ consists of Hilbert modules $\mathcal{H} \subseteq \mathcal{O}(\Omega)$ possessing a reproducing kernel $K$ and such that $\dim \cap_{i=1}^m \ker(M_i - w_i)^* < \infty$, $w \in \Omega$.

All the analytic Hilbert modules $\mathcal{H} \subseteq \mathcal{O}(\Omega)$ are in $\mathfrak{B}_1(\Omega)$. However, the reproducing kernel $K$ of a Hilbert module in $\mathfrak{B}_1(\Omega)$ may vanish – $K(w,w) = 0$ for $w$ in some closed subset of $\Omega$ – unlike the case of the analytic Hilbert modules. Indeed the modules in this class are the ones where the dimension of the joint kernel $\cap_{i=1}^m \ker(M_i - w_i)^*$ of the module multiplication is not necessarily constant. Therefore the techniques from complex geometry developed in [10, 12] do not apply directly.

Cowen and Douglas had observed in [11] that all sub-modules of the Hardy module $H^2(\mathbb{D})$ are equivalent. However, in more than one more variable, this is no longer true. Indeed, $H^2_{(0,0)}(\mathbb{D}^2)$ is not equivalent to the Hardy module $H^2(\mathbb{D}^2)$. Thus it is natural to ask when two sub-modules of a Hilbert module are equivalent. This was studied vigorously giving rise to the rigidity phenomenon, see [20]. One of the useful techniques here is the method of "localization", which is described below.

Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two Hilbert modules over the polynomial ring $\mathbb{C}[\underline{z}]$. The Hilbert space tensor product $\mathcal{M}_1 \otimes \mathcal{M}_2$ of these two Hilbert modules has two natural module multiplications, namely, $m_p \otimes \mathrm{Id}(f_1 \otimes f_2) = m_p(f_1) \otimes f_2$ and $\mathrm{Id} \otimes m_p(f_1 \otimes f_2) = f_1 \otimes m_p(f_2)$. The module tensor product $\mathcal{M}_1 \otimes_{\mathbb{C}[\underline{z}]} \mathcal{M}_2$ is obtained by identifying the space on which these two multiplications coincide. Set

$$\mathcal{N} := \{m_p f_1 \otimes f_2 - f_1 \otimes m_p f_2 : f_1 \in \mathcal{M}_1, f_2 \in \mathcal{M}_2, p \in \mathbb{C}[\underline{z}]\}.$$

The subspace $\mathcal{N}$ is a sub-module for both the left and the right multiplications: $m_p \otimes \mathrm{Id}$ and $\mathrm{Id} \otimes m_p$. On the quotient $\mathcal{N}^\perp = (\mathcal{M}_1 \otimes \mathcal{M}_2) \ominus \mathcal{N}$, these two module multiplications coincide (cf. [19]). The quotient Hilbert space $\mathcal{N}^\perp$ equipped with this multiplication is the module tensor product.

We consider the special case $\mathcal{H} \otimes_{\mathbb{C}[\underline{z}]} \mathbb{C}_w$, where $\mathbb{C}_w$ is the evaluation module, the one dimensional Hilbert module, where the module multiplication is defined by evaluation at $w$: $m_p(\lambda) = p(w)\lambda$, $w \in \Omega$, $p \in \mathbb{C}[\underline{z}]$. In $\mathcal{H}$, let $\mathcal{J}(w)$ denote the joint kernel $\cap_{i=1}^m \ker(M_i - w_i)^* = \cap_{p \in \mathbb{C}[\underline{z}]} \ker(M_p - p(w))^*$. We have the following useful lemma.

**Lemma 1.1.6.** *Let $\mathcal{H} \subseteq \mathcal{O}(\Omega)$ be a Hilbert module. For any $w \in \Omega$, we have the equality*

$$\mathcal{H} \otimes_{\mathbb{C}[\underline{z}]} \mathbb{C}_w = \Big(\mathcal{J}(w)\Big) \otimes \mathbb{C}.$$

*Proof.* The proof consists of the following string of equalities:

$$\begin{aligned}
\mathcal{H} \otimes_{\mathbb{C}[\underline{z}]} \mathbb{C}_w &= \big(\mathcal{H} \otimes \mathbb{C}\big)/\big\{pf \otimes \lambda - f \otimes p(w)\lambda : f \in \mathcal{H}, p \in \mathbb{C}[\underline{z}], \lambda \in \mathbb{C}\big\} \\
&= \big\{(p - p(w))f \otimes \lambda : f \in \mathcal{H}, p \in \mathbb{C}[\underline{z}], \lambda \in \mathbb{C}\big\}^{\perp} \\
&= \big\{g \otimes \mu \in \mathcal{H} \otimes \mathbb{C} : \langle g, (p - p(w))f\rangle \mu\bar{\lambda} = 0,\ f \in \mathcal{H}, p \in \mathbb{C}[\underline{z}], \lambda \in \mathbb{C}\big\} \\
&= \big\{g \otimes \mu : \langle M^*_{p-p(w)} g, f\rangle = 0,\ f \in \mathcal{H}, p \in \mathbb{C}[\underline{z}], \lambda \in \mathbb{C}\big\} \\
&= \Big(\mathcal{J}(w)\Big) \otimes \mathbb{C}.
\end{aligned}$$

These equalities are easily verified. □

Any module map $L : \mathcal{H} \to \widetilde{\mathcal{H}}$ must map the joint kernel $\mathcal{J}(w) \subseteq \mathcal{H}$ into the joint kernel $\widetilde{\mathcal{J}}(w) \subseteq \widetilde{\mathcal{H}}$. If the map $L$ is assumed to be invertible then its restriction to the kernel $\mathcal{J}(w) \subseteq \mathcal{H}$ is evidently an isomorphism. Thus we have proved the following Proposition.

**Proposition 1.1.7.** *Suppose $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ are two Hilbert module in $\mathcal{O}(\Omega)$, which are isomorphic via an invertible module map. Then $\mathcal{H} \otimes_{\mathbb{C}[\underline{z}]} \mathbb{C}_w$ and $\widetilde{\mathcal{H}} \otimes_{\mathbb{C}[\underline{z}]} \mathbb{C}_w$ are isomorphic for each $w \in \Omega$.*

Therefore, $\dim \mathcal{J}(w)$ is clearly an invariant for the class of Hilbert modules in $\mathcal{O}(\Omega)$. For an analytic Hilbert module, this is a constant function.

Now we observe (as in [18]), for the Hardy module $H^2(\mathbb{D}^2)$, $\dim \mathcal{J}(w)$ is identically 1 for all $w \in \mathbb{D}^2$ while for the sub-module $H^2_{(0,0)}(\mathbb{D}^2)$, it equals 1 for $w \neq (0,0)$ but is equal to 2 at $(0,0)$. Thus $H^2(\mathbb{D}^2)$ and $H^2_{(0,0)}(\mathbb{D}^2)$ are not equivalent via any invertible module map.

The module tensor product $\mathcal{H} \otimes_{\mathbb{C}[\underline{z}]} \mathbb{C}_w$ is said to be the *localization* of $\mathcal{H}$ at $w$ and the set $\mathrm{Sp}(\mathcal{H}) := \big\{\mathcal{H} \otimes_{\mathbb{C}[\underline{z}]} \mathbb{C}_w : w \in \Omega\big\}$ is said to be the spectral sheaf. When $\mathcal{H} \subseteq \mathcal{O}(\Omega)$ is an analytic Hilbert module, the spectral sheaf determines an anti-holomorphic line bundle via the frame $1 \otimes_{\mathbb{C}[\underline{z}]} 1_w$. The Hermitian structure is induced from $\mathcal{H} \otimes_{\mathbb{C}[\underline{z}]} \mathbb{C}_w$. In general, however, the spectral sheaf is a direct sum of $k$ copies of $\mathbb{C}_w$, where $k$ is between 1 and $t$, which is the rank of $\mathcal{H}$, see below. In what follows, it will be convenient to use the notion of *locally free* module of rank $n$ over $\Omega^* := \{w \in \mathbb{C}^m : \bar{w} \in \Omega\}$, where $\Omega$ is some open bounded subset of $\mathbb{C}^m$.

**Definition 1.1.8** (Definition 1.4, [10])**.** Let $\mathcal{H}$ be a Hilbert module over $\mathbb{C}[\underline{z}]$. Let $\Omega$ be a bounded open connected subset of $\mathbb{C}^m$. We say $\mathcal{H}$ is locally free of rank $n$ at $w_0$ in $\Omega^*$ if there exists a neighbourhood $\Omega_0^*$ of $w_0$ and holomorphic functions $\gamma_1, \gamma_2, \ldots, \gamma_n : \Omega_0^* \to \mathcal{H}$ such that the linear span of the set of $n$ vectors $\{\gamma_1(w), \ldots, \gamma_n(w)\}$ is the module tensor product $\mathcal{H} \otimes_{\mathbb{C}[\underline{z}]} \mathbb{C}_{\bar{w}}$. Following the terminology of [7], we say that a module $\mathcal{H}$ is *l*ocally free on $\Omega^*$ of rank $n$ if it is locally free of rank $n$ at every $w$ in $\Omega^*$.

Thus an analytic Hilbert module $\mathscr{H} \subseteq \mathscr{O}(\Omega)$ is locally free on $\Omega^*$ of rank $1$. On the other hand, if $\mathscr{H}$ is locally free on $\Omega$, we assume for simplicity, of rank $1$, then, for each $w_0 \in \Omega$, there exists a neighbourhood $\Omega_0$ such that $\mathscr{H}$ is in $\mathfrak{B}_1(\Omega_0^*)$. Moreover, in this case, $\dim \mathscr{J}(w) = 1$, $w \in \Omega$.

The typical example that we will be considering is the one where $\mathscr{H}_0$ is a sub-module of an analytic Hilbert module $\mathscr{H}$ with $\mathscr{H}_0$ of the form $[\mathscr{I}]$, the completion of the polynomial ideal $\mathscr{I}$ in the norm topology of $\mathscr{H}$.

**Proposition 1.1.9.** *The sub-module $[\mathscr{I}]$ of an analytic Hilbert module $\mathscr{H} \subseteq \mathscr{O}(\Omega)$ is in $\mathfrak{B}_1(\Omega)$.*

*Proof.* We observe that

$$\begin{aligned}\dim \bigcap_{i=1}^{m} \ker(M_i - w_i)^* &= \dim \mathscr{J}(w)\\ &= \dim\big([\mathscr{I}] \otimes_{\mathbb{C}[\underline{z}]} \mathbb{C}_w\big).\end{aligned}$$

Since the polynomial ring is Noetherian, it follows that $[\mathscr{I}]$ is finitely generated. Now, from [19, Lemma 5.11], it follows that the $\dim\big([\mathscr{I}] \otimes_{\mathbb{C}[\underline{z}]} \mathbb{C}_w\big)$ is finite, completing the proof. □

In these examples, we have the following Lemma from [5, Lemma 1.3]. Set $V_{[\mathscr{I}]} := \big\{z \in \Omega : f(z) = 0, f \in [\mathscr{I}]\big\}$.

**Lemma 1.1.10.** *The sub-module $[\mathscr{I}]$ of an analytic Hilbert module $\mathscr{H} \subseteq \mathscr{O}(\Omega)$ is locally free on $\Omega^*$ of rank $1$ if the ideal $\mathscr{I}$ is principal while if $p_1, \ldots p_t$, $t > 1$, is a minimal set of generators for $\mathscr{I}$, then $[\mathscr{I}]$ is locally free on $(\Omega \setminus V_{[\mathscr{I}]})^*$ of rank $1$.*

Now, we have the following description of the spectral sheaf for a Hilbert module of the form $[\mathscr{I}]$ possessing a minimal set of generators, say, $\{p_1, \ldots, p_t\}$. For $w \in \Omega \setminus V_{[\mathscr{I}]}$, we have $[\mathscr{I}] \otimes_{\mathbb{C}[\underline{z}]} \mathbb{C}_w = p_i \otimes_{\mathbb{C}[\underline{z}]} 1_w$, $1 \le i \le t$. However, note that

$$\begin{aligned}p_i \otimes_{\mathbb{C}[\underline{z}]} 1_w &= P_{\mathscr{J}(w) \otimes \mathbb{C}}(p_i \otimes 1)\\ &= \big(P_{\mathbb{C}[K_{[\mathscr{I}]}(\cdot, w)]} \otimes 1\big)(p_i \otimes 1)\\ &= \tfrac{p_i(w)}{K_{[\mathscr{I}]}(w,w)} K_{[\mathscr{I}]}(\cdot, w) \otimes 1.\end{aligned}$$

Here $\mathbb{C}[K_{[\mathscr{I}]}(\cdot, w)]$ denotes the one dimensional space spanned by the vector $K(\cdot, w)$. Thus the set of vectors $p_i \otimes_{\mathbb{C}[\underline{z}]} 1_w$ are linearly dependent and therefore $\dim [\mathscr{I}] \otimes_{\mathbb{C}[\underline{z}]} \mathbb{C}_w = 1$ for $w \in \Omega \setminus V_{[\mathscr{I}]}$. Based on this observation and explicit computations in simple examples, it was conjectured in [18] that

$$\dim [\mathscr{I}] \otimes_{\mathbb{C}[\underline{z}]} \mathbb{C}_w = \begin{cases} 1 & \text{for } w \in \Omega \setminus V_{[\mathscr{I}]} \\ \text{codim of } V_{[\mathscr{I}]} & \text{for } w \in V_{[\mathscr{I}]} \end{cases}$$

This formula is shown to be false in general by means of several examples by Duan and Guo, in the paper [21]. They show that the formula given above is valid if the ideal $\mathscr{I}$ has any one of the following properties:

1. $\mathscr{I}$ is singly generated,

2. $\mathscr{I}$ is a prime ideal in $\mathbb{C}[z_1, z_2]$

3. $\mathscr{I}$ is a prime ideal in $\mathbb{C}[z_1, \ldots, z_m]$, $m > 2$ and $w$ is a smooth point of $V_{[\mathscr{I}]}$.

For instance, if $[\mathscr{I}] \subseteq H^2(\mathbb{D}^2)$ and $\mathscr{I}$ is generated by $z_1, z_2$, then it is a prime ideal and the dimension formula is valid from the Duan-Guo criterion. However, observe that $[\mathscr{I}] = H^2_{(0,0)}(\mathbb{D}^2)$ for which we have shown the result to be true by direct computation earlier.

One of the main problems now is to distinguish two sub-modules, say $[\mathscr{I}_1]$ and $[\mathscr{I}_2]$ in an analytic Hilbert module $\mathscr{H}$. This was studied vigorously decades ago and several rigidity theorems were proved, see [20]. It is also possible to investigate this using the sheaf model developed in [3] which produces a slightly different proof of the rigidity theorem [4]. Here some of the results from [3] are refined and generalized to obtain a set of new invariants. We begin by recalling the sheaf model from [3].

Let $\mathscr{H}$ be a Hilbert module in $\mathfrak{B}_1(\Omega)$. Define the sheaf $\mathscr{S}^{\mathscr{H}}(\Omega)$ to be the sub-sheaf of the sheaf of holomorphic functions by setting

$$\mathscr{S}^{\mathscr{H}}(U) = \Big\{ \sum_{i=1}^{n} (f_i|_U) h_i : f_i \in \mathscr{H},\ h_i \in \mathscr{O}(U), n \in \mathbb{N} \Big\},$$

where $U$ is a fixed but arbitrary open subset of $\Omega$. If $\mathscr{H} \subseteq \mathscr{O}(\Omega)$ is an analytic Hilbert module, then the sheaf $\mathscr{S}^{\mathscr{H}}(\Omega)$ coincides with $\mathscr{O}(\Omega)$, therefore it is locally free on itself. Thus it defines a holomorphic vector bundle on $\Omega$, see [29, Theorem 1.13]. However, if $\mathscr{H}$ is in $\mathfrak{B}_1(\Omega)$, then the sheaf $\mathscr{S}^{\mathscr{H}}(\Omega)$ is not necessarily locally free, however, it is shown in [3] that it is a coherent sheaf. This implies that the stalk $\mathscr{S}^{\mathscr{H}}_w$ is finitely generated at any fixed but arbitrary $w_0 \in \Omega$. One of the main theorems of [3] says: There exists a neighbourhood $\Omega_0$ of $w_0$ such that

$$K(\cdot, w) = \overline{g_0^1(w)} K_0^{(1)}(\cdot, w) + \cdots + \overline{g_0^r(w)} K_0^{(r)}(\cdot, w),\ w \in \Omega_0,$$

where $g_0^i \in \mathscr{O}(\Omega_0)$, $1 \le i \le r$, their germs at $w_0$ is a minimal set of generators for $\mathscr{S}^{\mathscr{H}}_{w_0}$ and $K$ is the reproducing kernel of $\mathscr{H}$. Furthermore,

1. The vectors $K_0^{(1)}(\cdot, w_0), \ldots, K_0^{(r)}(\cdot, w_0)$ are uniquely determined,

2. The linear span of the vectors $K_0^{(1)}(\cdot, w_0), \ldots, K_0^{(r)}(\cdot, w_0)$ is a subspace of the joint kernel $\mathscr{J}(w_0)$ of the Hilbert module $\mathscr{H}$;

3. The vectors $K_0^{(1)}(\cdot, w), \ldots, K_0^{(r)}(\cdot, w)$ are linearly independent for each $w \in \Omega_0$.

We point out that if $w_0 \in \Omega_0 \setminus \mathcal{Z}(g_0^1, \dots g_0^r)$, where $\mathcal{Z}(g_0^1, \dots g_0^r)$ denotes the common zero set of $g_0^1, \dots g_0^r$, then, on the neighbourhood $\Omega_0 \setminus \mathcal{Z}(g_0^1, \dots g_0^r)$ of $w_0$, $\mathcal{S}^{\mathcal{H}}$ is singly generated by 1. Hence, in this case, there is no non-trivial decomposition of $K(\cdot, w)$, $w \in \Omega_0 \setminus \mathcal{Z}(g_0^1, \dots g_0^r)$.

## 1.2 Results

In the second chapter of this thesis we generalize these ideas and obtain, what may be viewed as a global version of these statements. This is Theorem 2.1.4 which is stated below.

**Theorem** (Theorem 2.1.4)**.** Let $\mathcal{H} \subseteq \mathcal{O}(\Omega)$ be an analytic Hilbert module for some bounded domain $\Omega \subseteq \mathbb{C}^m$. Let $[\mathcal{I}]$ be the completion of some polynomial ideal $\mathcal{I} \subseteq \mathcal{H}$ with generators $p_1, \dots, p_t$. Furthermore, assume that $V_{[\mathcal{I}]}$ is a submanifold of codimension $t$. Then there exist anti-holomorphic maps $F_1, \dots, F_t : V_{[\mathcal{I}]} \to [\mathcal{I}]$ such that we have the following.

1. For each $w \in V_{[\mathcal{I}]}$, there exists a neighbourhood $\Omega_w$ of $w$ in $\Omega$ and anti-holomorphic maps $F^1_{\Omega_w}, \dots, F^t_{\Omega_w} : \Omega_w \to [\mathcal{I}]$ with the properties listed below.

   a) $F^j_{\Omega_w}(v) = F_j(v)$, for all $v \in V_{[\mathcal{I}]} \cap \Omega_w$, $j \in \{1, \dots, t\}$.

   b) $k_u = \sum_{j=1}^t \overline{p_j(u)} F^j_{\Omega_w}(u)$, for all $u \in \Omega_w$, where $k_w := K_{[\mathcal{I}]}(\cdot, w)$, $w \in \Omega$, with $K_{[\mathcal{I}]}$ being the reproducing kernel of the submodule $[\mathcal{I}]$.

   c) $\{F^1_{\Omega_w}(u), \dots, F^t_{\Omega_w}(u)\}$ is a linearly independent set for each $u \in \Omega_w$.

2. The set $\{F_1, \dots, F_t\}$ is uniquely determined by $\{p_1, \dots, p_t\}$, that is, if $G_1, \dots, G_t$ is another collection of anti-holomorphic maps from $V_{[\mathcal{I}]}$ to $[\mathcal{I}]$ satisfying 1. a) and 1. b), then $G_j = F_j$, $1 \le j \le t$.

3. $M_p^* F_j(v) = \overline{p(v)} F_j(v)$, for all $j = 1, \dots, t$, $v \in V_{\mathcal{I}}$, where $M_p$ is the multiplication by the polynomial $p$.

4. For each $v \in V_{[\mathcal{I}]}$, the linear span of the set of vectors $\{F_1(v), \dots, F_t(v)\}$ in $[\mathcal{I}]$ is independent of the choice of the generators $p_1, \dots, p_t$, that is, if $\{q_1, \dots, q_t\}$ is another set of generators in $[\mathcal{I}]$ and $G_1, \dots, G_t : V_{[\mathcal{I}]} \mapsto [\mathcal{I}]$ are the anti-holomorphic maps determined by $\{q_1, \dots, q_t\}$ satisfying condition 1 to 3 already listed, then $\mathrm{Span}\{F_1(v), \dots, F_t(v)\} = \mathrm{Span}\{G_1(v), \dots, G_t(v)\}$.

One consequence of this result is that the modules in $\mathfrak{B}_1(\Omega)$ are made up of locally free ones except that the rank on the set $V^*_{[\mathcal{I}]}$ is not the same as the rank on $(\Omega \setminus V_{[\mathcal{I}]})^*$.

Since Hilbert modules of the form $[\mathcal{I}]$ are in $\mathfrak{B}_1(\Omega)$ and are locally free, of rank 1, on $(\Omega \setminus V_{[\mathcal{I}]})^*$, the curvature of the anti-holomorphic Hermitian vector bundle is a complete invariant for such modules. However, computing the curvature might be cumbersome, in general. Therefore, finding invariants which may be more tractable is worthwhile. The first such attempt goes back to [18] and has been the main topic of [3–5]. This has been the topic of the more recent paper [27]. The essential tool in these papers is the “blow-up" technique, which we describe below. Here, we take this further in a somewhat different direction.

Suppose $f_1,\ldots,f_r$ are holomorphic functions defined on $\Omega\subseteq\mathbb{C}^m$ and $\mathcal{Z}$ is their common zero set. With a slight abuse of language, we set $\mathcal{Z}\cap\Omega$ to be $\mathcal{Z}\subseteq\Omega$. Let $\mathbf{f}:\Omega\to\mathbb{C}^r$ be the function $\mathbf{f}=(f_1,\ldots,f_r)$. Following [22, p. 241], recall that

$$\widehat{\Omega}=\left\{(\underline{z},x)\in\Omega\times\mathbb{P}^{r-1}:\mathbf{f}(\underline{z})\in\ell(x)\right\}.$$

Here $x\in\mathbb{P}^{r-1}$ determines a line in $\mathbb{C}^r$, i.e., $\ell(x)=\pi^{-1}(x)\cup\{0\}$, where $\pi:\mathbb{C}^r\setminus\{0\}\to\mathbb{P}^{r-1}$ is the canonical projection. The set $\widehat{\Omega}$ is called the *monoidal transform* with center $\mathcal{Z}$.

The set of vectors $K_0^{(i)}(\cdot,w)$, $1\le i\le r$, does not immediately yield invariants for the Hilbert module $\mathcal{H}$. However, there exists a canonical choice, as noted in [3] prompted by the work in [12], of an anti-holomorphic frame $\left\{K_0^{(1)}(\cdot,w),\ldots,K_0^{(r)}(\cdot,w)\right\}$ on $\Omega_0$ which defines an anti-holomorphic Hermitian vector bundle of rank $r$ on $\Omega_0$. On the other hand, due to the uniqueness of the set of vectors $K_0^{(i)}(\cdot,w_0)$, $1\le i\le r$, we also obtain an anti-holomorphic Hermitian line bundle on $\{w_0\}\times\mathbb{P}^{r-1}$, $w_0\in\mathcal{Z}$. If $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ are two Hilbert modules which are completions of a polynomial ideal $\mathcal{I}$ in two different inner products, and they are equivalent, then the anti-holomorphic Hermitian vector bundles of rank $r$ on $\Omega_0$ as well as the anti-holomorphic Hermitian line bundles on $\{w_0\}\times\mathbb{P}^{r-1}$ they determine must be equivalent. These are Theorem 1.10 of [5] and Theorem 3.4 of [4].

In the third chapter of this thesis, using Theorem 2.1.4, starting with a Hilbert module $\mathcal{H}$ in $\mathfrak{B}_1(\Omega)$, an anti-holomorphic line bundle on $\mathcal{Z}\times\mathbb{P}^{r-1}$ is produced with the property that if two such Hilbert modules are equivalent, then the corresponding line bundles are equivalent. Therefore, complex geometric invariants of the restriction of this line bundle to $\mathcal{Z}\times\{p\}$ provide invariants for the Hilbert modules in the class $\mathfrak{B}_1(\Omega)$. These invariants are often more effective in determining when two such Hilbert modules are inequivalent as demonstrated in Proposition 3.4.5.

Let $\mathcal{H}_i$, $i=1,2$, be two Hilbert spaces possessing reproducing kernels $K_1$ and $K_2$ respectively. Let $\mathcal{H}_{12}$ be their intersection and $K_{12}$ be its kernel function. Without giving precise conditions on these spaces, it was stated in [18] that the kernel function $K$ of

$$\mathcal{H}:=\bigvee\{f_1+f_2:f_1\in\mathcal{H}_1,f_2\in\mathcal{H}_2\}=\overline{\mathcal{H}_1+\mathcal{H}_2}$$

is of the form: $K=K_1+K_2-K_{12}$. Here we provide several necessary and sufficient conditions on $\mathcal{H}_1,\mathcal{H}_2$ and $\mathcal{H}_{12}$ to ensure this formula. Furthermore, when $\mathcal{H}:=\overline{\mathcal{H}_1+\cdots+\mathcal{H}_n}$, $n\ge 3$, this formula has been generalized and a sufficient condition has been provided. Having described several such situations explicitly, we extract some consequences when $K$ has this form. In particular, the following Proposition is proved.

**Proposition** (Proposition 3.4.1)**.** Let $\mathscr{H} \subseteq \mathscr{O}(\Omega)$ be an analytic Hilbert module for some bounded domain $\Omega \subseteq \mathbb{C}^m$. Also, let $[\mathscr{I}]$ is the closure of the polynomial ideal generated by $\{p_1, \dots, p_t\}$ in $\mathscr{H}$, where

i) $V_{[\mathscr{I}]}$ is a submanifold of codimension $t (\geq 2)$

ii) $p_i, p_j$ are relatively prime for $i \neq j$, $1 \leq i, j \leq t$.

Then we can find anti-holomorphic maps $F_1, \dots, F_t : V_{[\mathscr{I}]} \to [\mathscr{I}]$ which satisfy conditions 1) to 4) of Theorem 2.1.4. For $i = 1, \dots, t$, let $\mathscr{I}_i$ be the principal ideal generated by $p_i$ and set $[\mathscr{I}_i]$ to be the closure of $\mathscr{I}_i$ in $\mathscr{H}$ with reproducing kernel $K_i$. Suppose $K_{[\mathscr{I}]}$ admits a decomposition as in Equation (3.3). Then $F_i(v) = M_{p_i}\chi_i(\cdot, v)$ for all $v \in Z(p_1, \dots, p_t) \cap \Omega$, where $\chi_i$ is taken from the equation $K_i(z, w) = p_i(z)\overline{p_i(w)}\chi_i(z, w)$, $z, w \in \Omega$.

To describe the results of the fourth chapter, we first recall the notion of the tensor product of two Hilbert modules over the polynomial ring and tensor product of module maps between two of these. If $X : \mathscr{H} \to \widetilde{\mathscr{H}}$ is any module map, then

$$X \otimes_{\mathbb{C}[\underline{z}]} 1_w : \mathscr{H} \otimes_{\mathbb{C}[\underline{z}]} \mathbb{C}_w \to \widetilde{\mathscr{H}} \otimes_{\mathbb{C}[\underline{z}]} \mathbb{C}_w,\ w \in \Omega,$$

defined by the rule $X \otimes_{\mathbb{C}[\underline{z}]} 1_w := P_{\widetilde{\mathscr{N}}^\perp}\big(X \otimes 1_w\big|_{\mathscr{N}^\perp}\big)$, where $1_w : \mathbb{C}_w \to \mathbb{C}_w$ is the identity operator, is again a module map, called the localization of $X$ at $w$.

By definition, we have $X \otimes_{\mathbb{C}[\underline{z}]} 1_w = P_{\widetilde{\mathscr{N}}^\perp}\big(X \otimes 1\big)\big|_{\mathscr{N}^\perp}$. But

$$\widetilde{\mathscr{N}}^\perp = \big(\widetilde{\mathscr{J}}(w)\big) \otimes \mathbb{C} \subseteq \widetilde{\mathscr{H}} \otimes \mathbb{C}$$

and similarly,

$$\mathscr{N}^\perp = \big(\mathscr{J}(w)\big) \otimes \mathbb{C} \subseteq \mathscr{H} \otimes \mathbb{C}.$$

It follows that

$$P_{\widetilde{\mathscr{N}}^\perp} = P_{\widetilde{\mathscr{J}}(w)} \otimes 1.$$

Hence, for $h \in \mathscr{J}(w)$, we have

$$\begin{aligned} X \otimes_{\mathbb{C}[\underline{z}]} 1_w(h \otimes \lambda) &= P_{\widetilde{\mathscr{N}}^\perp}(Xh \otimes \lambda) \\ &= \big(P_{\widetilde{\mathscr{J}}(w)} \otimes 1\big)(Xh \otimes \lambda) \\ &= P_{\widetilde{\mathscr{J}}(w)}(Xh) \otimes \lambda. \end{aligned}$$

Thus

$$X \otimes_{\mathbb{C}[\underline{z}]} 1_w = \big(P_{\widetilde{\mathscr{J}}(w)} X|_{\mathscr{J}(w)}\big) \otimes 1.$$

On the other hand, since $X^*\big(\widetilde{\mathscr{J}}(w)\big) \subseteq \mathscr{J}(w)$, we have

$$\big(X \otimes_{\mathbb{C}[\underline{z}]} 1_w\big)^* = \big(X^*|_{\widetilde{\mathscr{J}}(w)}\big) \otimes 1 = X^* \otimes_{\mathbb{C}[\underline{z}]} 1_w.$$

For modules in $\mathfrak{B}_1(\Omega)$, we can obtain more precise information. Let $\mathscr{H} \subseteq \mathscr{O}(\Omega)$ be an analytic Hilbert module and $K$ be its reproducing kernel. Recall that $\mathscr{H}_0 \xrightarrow{X} \mathscr{H} \xrightarrow{\pi} \mathscr{Q}$ is said to be *topologically exact* at $\mathscr{H}$ if $\overline{\operatorname{ran X}} = \ker \pi$ and a complex of Hilbert modules $0 \longrightarrow \mathscr{H}_0 \xrightarrow{X} \mathscr{H} \xrightarrow{\pi} \mathscr{Q} \longrightarrow 0$ is said to be *topologically (short) exact* if $X$ is injective, $\overline{\operatorname{ran X}} = \ker \pi$ and $\pi$ is surjective. Now, fix a submodule of an analytic Hilbert module $\mathscr{H}$ of the form $[\mathscr{I}]$, where $[\mathscr{I}]$ is the closure of the ideal $\mathscr{I} \subseteq \mathbb{C}[\underline{z}]$ in $\mathscr{H}$ generated by $p_1, \ldots p_t$. Let $K_{[\mathscr{I}]}$ denote the reproducing kernel of the sub-module $[\mathscr{I}]$. In these examples, setting $X : [\mathscr{I}] \to \mathscr{H}$ to be the inclusion map, we see that $X^* K(\cdot, w) = K_{[\mathscr{I}]}(\cdot, w)$, $w \notin V_{[\mathscr{I}]}$. Let $X(w)$ denote the map $X \otimes_{\mathbb{C}[\underline{z}]} 1_w$. In this notation, $X(w)^* K(\cdot, w) = K_{[\mathscr{I}]}(\cdot, w)$. Thus the module map defines an anti-holomorphic frame on $\Omega \setminus V_{[\mathscr{I}]}$, namely, $X(w)^*$. Also,

$$X(w) = \big(\tfrac{K_{[\mathscr{I}]}(w,w)}{K(w,w)}\big)^{1/2} = X(w)^*,\ w \notin V_{[\mathscr{I}]},$$

relative to the normalized frames $\frac{K(\cdot,w)}{\sqrt{K(w,w)}}$ and $\frac{K_{[\mathscr{I}]}(\cdot,w)}{\sqrt{K_{[\mathscr{I}]}(w,w)}}$. Both the operators $X(w)$ and $X(w)^*$ are zero for $w \in V_{[\mathscr{I}]}$. On the other hand, if $\mathscr{H}$ and $\widetilde{\mathscr{H}}$ are analytic Hilbert modules over $\Omega$ and $X : \mathscr{H} \to \widetilde{\mathscr{H}}$ is the operator of multiplication by a polynomial $p$, then $\overline{M_p(\mathscr{H})} \subseteq \widetilde{\mathscr{H}}$. Since $X$ is a module map, we have $X^* \widetilde{K}(\cdot, w) = \overline{p(w)} K(\cdot, w)$, like before. It follows that

$$X(w)X(w)^* = |p(w)|^2 \tfrac{K(w,w)}{\widetilde{K}(w,w)},\ w \in \Omega,$$

again, relative to the normalized bases $\frac{K(\cdot,w)}{\sqrt{K(w,w)}}$ and $\frac{\widetilde{K}(\cdot,w)}{\sqrt{\widetilde{K}(w,w)}}$. The Hilbert modules $[\mathscr{I}]$ and $\mathscr{H}$ define anti-holomorphic Hermitian line bundles on $\Omega \setminus V_{[\mathscr{I}]}$, say $E_{[\mathscr{I}]}$ and $E_{\mathscr{H}}$, determined by the localization $\mathscr{H} \otimes_{\mathbb{C}[\underline{z}]} \mathbb{C}_w$ and $[\mathscr{I}] \otimes_{\mathbb{C}[\underline{z}]} \mathbb{C}_w$, $w \in \Omega \setminus V_{[\mathscr{I}]}$, respectively. However, there is a third bundle $E_X$ defined via the localisation of the inclusion map $X$, namely, $X(w)^*$. This is the anti-holomorphic line bundle $\operatorname{Hom}(E_{\mathscr{H}}, E_{[\mathscr{I}]})$. The Hermitian structure is induced by noting that the fibre at $w$, $w \in \Omega \setminus V_{[\mathscr{I}]}$, is spanned by $X(w)^*$ and $\|X(w)^*\|^2 = X(w)X(w)^* = \frac{K_{[\mathscr{I}]}(w,w)}{K(w,w)}$. Therefore the alternating sum

$$\mathscr{A}_{[\mathscr{I}],\mathscr{H}}(w) := \mathscr{K}_X(w) - \mathscr{K}_{[\mathscr{I}]}(w) + \mathscr{K}_{\mathscr{H}}(w) = 0,\ w \in \Omega \setminus V_{[\mathscr{I}]},$$

where $\mathscr{K}_X$, $\mathscr{K}_{[\mathscr{I}]}$ and $\mathscr{K}_{\mathscr{H}}$ denote the curvature $(1,1)$ forms of the line bundles $E_X$, $E_{[\mathscr{I}]}$ and $E_{\mathscr{H}}$, respectively. When $\mathscr{I}$ is principal, one may, however, evaluate this alternating sum $\mathscr{A}_{[\mathscr{I}],\mathscr{H}}$ on all of $\Omega$ in the sense of distributions obtaining a current. In what follows, we will assume that $\mathscr{A}_{[\mathscr{I}],\mathscr{H}}$ is a current defined on all of $\Omega$. Restricting only to the case of pairs $([\mathscr{I}], \mathscr{H})$

of Hilbert modules, the notion of "topological" exactness coincides with ordinary exactness. Therefore, [15, Theorem 1], in particular, shows that if the diagram

$$
\begin{tikzcd}
0 \arrow[r] & {[\mathscr{I}]} \arrow[r, "X"] \arrow[d] & \mathscr{H} \arrow[r, "\pi"] \arrow[d, "L"] & \mathscr{Q} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & {[\widetilde{\mathscr{I}}]} \arrow[r, "\widetilde{X}"] & \widetilde{\mathscr{H}} \arrow[r, "\tilde{\pi}"] & \widetilde{\mathscr{Q}} \arrow[r] & 0,
\end{tikzcd}
$$

is exact, then $\mathscr{A}_{[\mathscr{I}],\mathscr{H}}(w) = \mathscr{A}_{[\widetilde{\mathscr{I}}],\widetilde{\mathscr{H}}}(w)$. The alternating sum of $(1,1)$ forms therefore is an invariant for the pair $([\mathscr{I}],\mathscr{H})$. It was observed in [14, Theorem 1.4] that $\mathscr{A}_{[\mathscr{I}],\mathscr{H}}$ represents the fundamental class of $V_{[\mathscr{I}]}$ when the codimension of $V_{[\mathscr{I}]}$ in $\Omega$ is 1. Generalization of this result was given in [17, 18].

In chapter 4, after making suitable assumptions, among other things, we show that $\mathscr{A}_{[\mathscr{I}],\mathscr{H}}$ is a complete invariant for $[\mathscr{I}]$. Firstly, we will consider the case when $\operatorname{codim} V_{[\mathscr{I}]} = 1$. This is included in Theorem 4.3.4 where $\mathscr{I}$ is a principal ideal. Finally, Theorem 4.3.7 describes what happens if the codimension of the zero set is $> 1$.

**Theorem** (Theorem 4.3.4)**.** Let $\Omega$ be a bounded domain in $\mathbb{C}^m$ and let $\mathscr{H}, \widetilde{\mathscr{H}}$ be analytic Hilbert modules in $\mathscr{O}(\Omega)$. Also, let $\mathscr{I}, \widetilde{\mathscr{I}}$ be two principal ideals in $\mathbb{C}[z_1, \ldots, z_m]$ generated by $p, \widetilde{p}$ respectively and assume that the zero set of each irreducible component of $p, \widetilde{p}$ intersects $\Omega$. Define $[\mathscr{I}], [\widetilde{\mathscr{I}}]$ as the closure of the polynomial ideals $\mathscr{I}, \widetilde{\mathscr{I}}$ in $\mathscr{H}, \widetilde{\mathscr{H}}$, respectively. If $L : \mathscr{H} \mapsto \widetilde{\mathscr{H}}$ is a bijective module map, then the following are equivalent:

a) $L([\mathscr{I}]) = [\widetilde{\mathscr{I}}]$;

b) $\mathscr{K}_X - \mathscr{K}_{[\mathscr{I}]} + \mathscr{K}_{\mathscr{H}} = \mathscr{K}_{\widetilde{X}} - \mathscr{K}_{[\widetilde{\mathscr{I}}]} + \mathscr{K}_{\widetilde{\mathscr{H}}}$ as (1,1) currents on $\Omega$;

c) $\mathscr{I} = \widetilde{\mathscr{I}}$.

**Theorem** (Theorem 4.3.7)**.** Let $\Omega$ be a bounded domain in $\mathbb{C}^m$ and let $\mathscr{H}, \widetilde{\mathscr{H}}$ be analytic Hilbert modules in $\mathscr{O}(\Omega)$. Also, let $\varphi := (\varphi_1, \ldots, \varphi_r), \psi := (\psi_1, \ldots, \psi_r)$ be holomorphic maps from $\Omega$ to $\mathbb{C}^r$ that satisfy the following:

i) for each $i$, $1 \le i \le r$, $\varphi_i \in \mathscr{H}, \psi_i \in \widetilde{\mathscr{H}}$ and they define $Z(\varphi_i), Z(\psi_i)$ respectively;

ii) $Z(\varphi), Z(\psi)$ are complete intersections, where $Z(\varphi) := Z(\varphi_1) \cap \cdots \cap Z(\varphi_r)$ and $Z(\psi) := Z(\psi_1) \cap \cdots \cap Z(\psi_r)$;

iii) $Z(\varphi_i), Z(\psi_i), Z(\varphi), Z(\psi)$ are connected subsets of $\Omega$ for all $i = 1, \ldots, r$.

Define $\mathscr{M}_i = \{f \in \mathscr{H} : f = 0 \text{ on } Z(\varphi_i)\}$, $\mathscr{M} = \{f \in \mathscr{H} : f = 0 \text{ on } Z(\varphi)\}$, $\widetilde{\mathscr{M}_i} = \{g \in \widetilde{\mathscr{H}} : g = 0 \text{ on } Z(\psi_i)\}$, $\widetilde{\mathscr{M}} = \{g \in \widetilde{\mathscr{H}} : g = 0 \text{ on } Z(\psi)\}$, for $i = 1, \ldots, r$ and assume that $\operatorname{rank}(\mathscr{M}_i) = \operatorname{rank}(\widetilde{\mathscr{M}_i}) = 1$. If $L : \mathscr{H} \mapsto \widetilde{\mathscr{H}}$ is a bijective module map, then the following are equivalent:

a) $L(\mathscr{M}) = \widetilde{\mathscr{M}}$;

b) $Z(\varphi) = Z(\psi)$;

c) $\bigwedge_{i=1}^{r} \left(\mathscr{K}_{X_i} - \mathscr{K}_{\mathscr{M}_i} + \mathscr{K}_{\mathscr{H}}\right) = \bigwedge_{i=1}^{r} \left(\mathscr{K}_{\widetilde{X}_i} - \mathscr{K}_{\widetilde{\mathscr{M}_i}} + \mathscr{K}_{\widetilde{\mathscr{H}}}\right)$ as $(r, r)$ currents on $\Omega$, where $X_i : \mathscr{M}_i \mapsto \mathscr{H}, \widetilde{X}_i : \widetilde{\mathscr{M}_i} \mapsto \widetilde{\mathscr{H}}$ are the canonical inclusion maps for all $i = 1, \ldots, r$.

# Chapter 2

# The joint kernel for a class of submodules along their common zero sets

## 2.1 Douglas, Misra and Varughese conjecture and a decomposition theorem

Let $\Omega$ be a bounded domain in $\mathbb{C}^m$ and $\mathcal{M}$ be a Hilbert module over the polynomial ring $\mathbb{C}[z_1,\ldots,z_m]$, in the class $\mathfrak{B}_1(\Omega)$. We construct a sheaf $\mathcal{S}^{\mathcal{M}}$ for the Hilbert module $\mathcal{M}$ as follows:

$$\mathcal{S}^{\mathcal{M}}(U) = \Big\{\sum_{i=1}^{n}(f_i|_U)g_i : f_i \in \mathcal{M}, g_i \in \mathcal{O}(U), n \in \mathbb{N}\Big\},\ U \text{ open in } \Omega$$

or equivalently,

$$\mathcal{S}^{\mathcal{M}}_w = \big\{(f_1)_w\mathcal{O}_w + \cdots + (f_n)_w\mathcal{O}_w : f_1,\ldots,f_n \in \mathcal{M}, n \in \mathbb{N}\big\},\ w \in \Omega.$$

Clearly, $\mathcal{S}^{\mathcal{M}}$ is a subsheaf of the sheaf of holomorphic functions $\mathcal{O}_\Omega$. From [3, Proposition 2.1], it follows that $\mathcal{S}^{\mathcal{M}}$ is coherent. In particular, for each fixed $w \in \Omega$, $\mathcal{S}^{\mathcal{M}}_w$ is generated by finitely many elements from $\mathcal{O}_w$. We now state the decomposition theorem given in [3, Theorem 2.3].

**Theorem 2.1.1.** *Suppose $g_i^0$, $1 \le i \le d$, is a minimal set of generators for the stalk $\mathcal{S}^{\mathcal{M}}_{w_0}$, and $K$ is the reproducing kernel of $\mathcal{M}$. Then*

(i) *there exists an open neighbourhood $\Omega_0$ of $w_0$ such that*

$$K(\cdot,w) = \overline{g_1^0(w)}K^{(1)}(w) + \cdots + \overline{g_d^0(w)}K^{(d)}(w),\ w \in \Omega_0$$

*for some choice of anti-holomorphic maps $K^{(1)},\ldots,K^{(d)} : \Omega_0 \to \mathcal{M}$,*

(ii) *the vectors* $K^{(i)}(w)$, $1 \le i \le d$, *are linearly independent in* $\mathscr{M}$ *for* $w$ *in some small neighbourhood of* $w_0$,

(iii) *the vectors* $K^{(i)}(w_0)$, $1 \le i \le d$, *are uniquely determined by the generators* $g_1^0, \dots, g_d^0$,

(iv) *the linear span of the set of vectors* $\{K^{(i)}(w_0) : 1 \le i \le d\}$ *in* $\mathscr{M}$ *is independent of the choice of generators* $g_1^0, \dots, g_d^0$, *and*

(v) $M_p^* K^{(i)}(w_0) = \overline{p(w_0)} K^{(i)}(w_0)$, *for* $i = 1, \dots, d$, *where* $M_p$ *denotes the module multiplication by the polynomial* $p$.

Now, assume furthermore that $\mathscr{H}$ is an analytic Hilbert module in $\mathscr{O}(\Omega)$ and $\mathscr{M}$ is the closure of a polynomial ideal $\mathscr{I}$ in $\mathscr{H}$ generated by $\{p_1, \dots, p_t\}$. From [8, Lemma 2.3.2] we obtain that for each $w \in \Omega$, $\{(p_1)_w, \dots, (p_t)_w\}$ generates the stalk $\mathscr{S}_w^{\mathscr{M}}$. In the following lemma we provide a sufficient condition for the minimality of such a generator. We let $Z(p_1, \dots, p_t)$ denote the common zero set of the polynomials $p_1, \dots, p_t$.

**Lemma 2.1.2.** *If* $V(\mathscr{M}) := Z(p_1, \dots, p_t) \cap \Omega$ *is a submanifold of codimension* $t$, *then*

a) $\{p_1, \dots, p_t\}$ *is a minimal generator of* $\mathscr{I}$ *and*

b) *for each* $w \in V(\mathscr{M})$, $(p_1)_w, \dots, (p_t)_w$ *is a minimal generator of* $\mathscr{S}_w^{\mathscr{M}}$.

*Proof.* Assume that $p_t = q_1 p_1 + \dots + q_{t-1} p_{t-1}$, for $q_1, \dots, q_{t-1} \in \mathbb{C}[z_1, \dots, z_m]$. Then we have $V(\mathscr{M}) = Z(p_1, \dots, p_{t-1}) \cap \Omega$. Since $p_1|_\Omega, \dots, p_{t-1}|_\Omega$ are $t-1$ holomorphic functions from $\Omega$ to $\mathbb{C}$, applying [9, Section 3.5] we obtain that $\operatorname{codim}(V(\mathscr{M}))$ is at most $t-1$ which contradicts the hypothesis of the lemma. This proves part a).

To prove part b), assume that there exists a point $w_0 \in V(\mathscr{M})$ such that

$$(p_t)_{w_0} = (a_1)_{w_0} (p_1)_{w_0} + \dots + (a_{t-1})_{w_0} (p_{t-1})_{w_0},$$

where $a_1, \dots, a_{t-1}$ are holomorphic functions defined on some neighbourhood $N_{w_0}$ of $w_0$ in $\Omega$. Going to a smaller neighbourhood if necessary, we have $p_t = a_1 p_1 + \dots + a_{t-1} p_{t-1}$ on $N_{w_0}$. As a result, $V(\mathscr{M}) \cap N_{w_0} = Z(p_1, \dots, p_{t-1}) \cap N_{w_0}$. Now, since $p_1|_{N_{w_0}}, \dots, p_{t-1}|_{N_{w_0}}$ are $t-1$ holomorphic functions on $N_{w_0}$ and $Z(p_1|_{N_{w_0}}, \dots, p_{t-1}|_{N_{w_0}}) = V(\mathscr{M}) \cap N_{w_0}$ is a submanifold, from [9, Section 3.5] it follows that $\operatorname{codim}(V(\mathscr{M}) \cap N_{w_0})$ is at most $t-1$. Again, this is a contradiction to the hypothesis saying $\operatorname{codim}(V(\mathscr{M}) \cap N_{w_0}) = \operatorname{codim}(V(\mathscr{M})) = t$. □

**Remark 2.1.3.** Suppose $Z(p_1, \dots, p_t)$ is a complete intersection, that is, the tuple $(p_1, \dots, p_t) : \Omega \to \mathbb{C}^t$ is a submersion at every point $w \in V(\mathscr{M})$. Then $V(\mathscr{M})$ is a submanifold of codimension $t$. In this case, we can give a more direct proof of Lemma 2.1.2 as follows.

If there exists a point $w_0 \in V(\mathscr{M})$ such that $p_t = a_1 p_1 + \cdots + a_{t-1} p_{t-1}$ on $N_{w_0}$, then, for $j = 1, \ldots, m$,

$$\begin{aligned}\frac{\partial p_t}{\partial w_j}(w_0) &= \frac{\partial (a_1 p_1)}{\partial w_j}(w_0) + \cdots + \frac{\partial (a_{t-1} p_{t-1})}{\partial w_j}(w_0)\\ &= a_1(w_0)\frac{\partial p_1}{\partial w_j}(w_0) + \cdots + a_{t-1}(w_0)\frac{\partial p_{t-1}}{\partial w_j}(w_0).\end{aligned}$$

As a result, the matrix

$$\begin{pmatrix} \frac{\partial p_1}{\partial w_1}(w_0) & \ldots & \frac{\partial p_1}{\partial w_t}(w_0) \\ \vdots & & \vdots \\ \frac{\partial p_t}{\partial w_1}(w_0) & \ldots & \frac{\partial p_t}{\partial w_t}(w_0) \end{pmatrix} = \begin{pmatrix} \frac{\partial p_1}{\partial w_1}(w_0) & \ldots & \frac{\partial p_1}{\partial w_t}(w_0) \\ \vdots & & \vdots \\ \frac{\partial p_{t-1}}{\partial w_1}(w_0) & \ldots & \frac{\partial p_{t-1}}{\partial w_t}(w_0) \\ \sum_{i=1}^{t-1} a_i(w_0)\frac{\partial p_i}{\partial w_1}(w_0) & \ldots & \sum_{i=1}^{t-1} a_i(w_0)\frac{\partial p_i}{\partial w_t}(w_0) \end{pmatrix}$$

has rank at most $t-1$. This contradicts the fact that $Z(p_1, \ldots, p_t)$ is a complete intersection at $w_0$.

The following Theorem is a generalization of Theorem 2.1.1.

**Theorem 2.1.4.** *Let $\mathscr{H} \subseteq \mathscr{O}(\Omega)$ be an analytic Hilbert module for some bounded domain $\Omega$ in $\mathbb{C}^m$. Also, let $\mathscr{M}$ be a submodule of $\mathscr{H}$ of the form $[\mathscr{I}]$, that is, $\mathscr{M}$ is the completion of some polynomial ideal $\mathscr{I} \subseteq \mathscr{H}$ with generators $p_1, \ldots, p_t$. Furthermore, assume that $V(\mathscr{M})$ is a submanifold of codimension $t$. Then there exist anti-holomorphic maps $F_1, \ldots, F_t : V(\mathscr{M}) \to \mathscr{M}$ such that we have the following.*

1. *For each $w \in V(\mathscr{M})$, there exists a neighbourhood $\Omega_w$ of $w$ in $\Omega$, anti-holomorphic maps $F^1_{\Omega_w}, \ldots, F^t_{\Omega_w} : \Omega_w \to \mathscr{M}$ with the properties listed below.*

   a) *$F^j_{\Omega_w}(v) = F_j(v)$, for all $v \in V(\mathscr{M}) \cap \Omega_w$, $j \in \{1, \ldots, t\}$.*

   b) *$k_u = \sum_{j=1}^t \overline{p_j(u)} F^j_{\Omega_w}(u)$, for all $u \in \Omega_w$, where $k_w := K(\cdot, w)$, $w \in \Omega$, with $K$ being the reproducing kernel of the submodule $\mathscr{M}$.*

   c) *$\{F^1_{\Omega_w}(u), \ldots, F^t_{\Omega_w}(u)\}$ is a linearly independent set for each $u \in \Omega_w$.*

2. *The set $\{F_1, \ldots, F_t\}$ is uniquely determined by $\{p_1, \ldots, p_t\}$, that is, if $G_1, \ldots, G_t$ is another collection of anti-holomorphic maps from $V(\mathscr{M})$ to $\mathscr{M}$ satisfying* 1. a) *and* 1. b)*, then $G_j = F_j$, $1 \le j \le t$.*

3. *$M_p^* F_j(v) = \overline{p(v)} F_j(v)$, for all $j = 1, \ldots, t$, $v \in V(\mathscr{M})$, where $M_p$ is the multiplication by the polynomial $p$.*

4. *For each $v \in V(\mathscr{M})$, the linear span of the set of vectors $\{F_1(v), \dots, F_t(v)\}$ in $\mathscr{M}$ is the joint kernel of $\mathscr{M}$ at $v$ and hence is independent of the choice of generators $p_1, \dots, p_t$.*

*Proof.* Pick an arbitrary point $w \in V(\mathscr{M})$. From Lemma 2.1.2, $(p_1)_w, \dots, (p_t)_w$ is a minimal set of generator of $\mathscr{S}_w^{\mathscr{M}}$. Consequently, there exists a neighbourhood $\Omega_w$ of $w$ in $\Omega$ such that for all $u \in \Omega_w$, $K(\cdot, u) = \sum_{j=1}^t \overline{p_j(u)} F_{\Omega_w}^j(u)$, where $F_{\Omega_w}^1, \dots, F_{\Omega_w}^t$ are anti-holomorphic maps from $\Omega_w$ to $\mathscr{M}$ satisfying conditions (ii) to (v) of Theorem 2.1.1.

Take $w_1, w_2 \in V(\mathscr{M})$ such that $\Omega_{w_1} \cap \Omega_{w_2} \cap V(\mathscr{M})$ is non-empty. Now, for each $u \in \Omega_{w_1} \cap \Omega_{w_2}$ we have

$$K(\cdot, u) = \sum_{j=1}^t \overline{p_j(u)} F_{\Omega_{w_1}}^j(u) \quad \text{and} \quad K(\cdot, u) = \sum_{j=1}^t \overline{p_j(u)} \tilde{F}_{\Omega_{w_2}}^j(u).$$

This implies

$$\sum_{j=1}^t \overline{p_j(u)} \big( F_{\Omega_{w_1}}^j(u) - \tilde{F}_{\Omega_{w_2}}^j(u) \big) = 0.$$

For each $u \in \Omega_{w_1} \cap \Omega_{w_2}$, $1 \le j \le t$, define $\alpha_j(u) = \overline{\big( F_{\Omega_{w_1}}^j(u) - \tilde{F}_{\Omega_{w_2}}^j(u) \big)}$. As a result, we have $\sum_{j=1}^t p_j(u) \alpha_j(u) = 0$. Now, fix an arbitrary $v \in \Omega_{w_1} \cap \Omega_{w_2} \cap V(\mathscr{M})$ and assume that $\alpha_1(v) \neq 0$. This gives $\sum_{j=1}^t (p_j)_v (\alpha_j)_v = 0$ in $\mathscr{O}_{\mathbb{C}^m, v}$ and $(\alpha_1)_v$ is an unit in $\mathscr{O}_{\mathbb{C}^m, v}$. Consequently, $(p_1)_v = -\sum_{j=2}^t \big( (\alpha_1)_v^{-1} (\alpha_j)_v \big) (p_j)_v$ which says that $\{(p_1)_v, \dots, (p_t)_v\}$ is not a minimal set of generators of $\mathscr{S}_v^{\mathscr{M}}$ contradicting Lemma 2.1.2. Thus,

$$\alpha_j(v) = 0 \Leftrightarrow F_{\Omega_{w_1}}^j(v) = \tilde{F}_{\Omega_{w_2}}^j(v), \forall v \in \Omega_{w_1} \cap \Omega_{w_2} \cap V(\mathscr{M}), 1 \le j \le t. \tag{2.1}$$

Since $\{\Omega_w \cap V(\mathscr{M})\}_{w \in V(\mathscr{M})}$ is an open cover of $V(\mathscr{M})$, for each $j = 1, \dots, t$, we define $F_j : V(\mathscr{M}) \to \mathscr{M}$ as follows:

$$F_j|_{\Omega_w \cap V(\mathscr{M})}(v) := F_{\Omega_w}^j(v), \forall v \in \Omega_w \cap V(\mathscr{M}).$$

From 2.1 it follows that for each $j = 1, \dots, t$, $F_j$ is a well-defined, anti-holomorphic map satisfying 1.a),1.b) and 1.c).

To prove 2., assume that for each $w \in V(\mathscr{M})$, there exist a neighbourhood $N_w$ of $w$ in $\Omega$, anti-holomorphic maps $G_{N_w}^1, \dots, G_{N_w}^t : N_w \to \mathscr{M}$ such that a) $G_{N_w}^j(v) = G_j(v)$, $\forall v \in N_w \cap V(\mathscr{M})$ and b) $K(\cdot, u) = \sum_{j=1}^t \overline{p_j(u)} G_{N_w}^j(u)$, $\forall u \in N_w$. This gives

$$\sum_{j=1}^t \overline{p_j(u)} \big( F_{\Omega_w}^j(u) - G_{N_w}^j(u) \big) = 0, \forall u \in \Omega_w \cap N_w.$$

Following similar arguments as given above, for each $v \in \Omega_w \cap N_w \cap V(\mathscr{M})$, we have $G_j(v) = G_{N_w}^j(v) = F_{\Omega_w}^j(v) = F_j(v)$. In particular, $F_j(w) = G_j(w)$, for all $w \in V(\mathscr{M})$.

The proof of Part 3 is straightforward from condition (v) of Theorem 2.1.1 and from the observation that for each $w \in V(\mathscr{M})$, $F_j(w) = F^j_{\Omega_w}(w)$. From the same observation we obtain that $\mathrm{Span}\{F_1(w),\dots,F_t(w)\}$ is a subspace of $\cap_{p\in\mathbb{C}[\underline{z}]}\mathrm{Ker}(M_p^* - \overline{p(w)})$, for each $w \in V(\mathscr{M})$. Now, from 1.a) and 1.c) it follows that the dimension of this subspace is at least $t$. On the other hand, from [19, Lemma 5.11] it follows that $\dim(\mathscr{M}\otimes_{\mathbb{C}[\underline{z}]}\mathbb{C}_w) = \dim\big(\cap_{p\in\mathbb{C}[\underline{z}]}\ker(M_p - p(w))^*\otimes\mathbb{C}\big) \le t$. So, for each $w \in V(\mathscr{M})$, $\mathrm{Span}\{F_1(w),\dots,F_t(w)\} = \cap_{p\in\mathbb{C}[\underline{z}]}\ker(M_p^* - \overline{p(w)})$ proving Part 4. □

**Corollary 2.1.5.** *With the hypotheses of the previous theorem, we have*

$$\dim\Big(\bigcap_{i=1}^{m}\ker(M_{z_j} - w_j)^*\Big) = \begin{cases} 1 & \text{for } w \notin V(\mathscr{I})\cap\Omega \\ \text{codimension of } V(\mathscr{I}) & \text{for } w \in V(\mathscr{I})\cap\Omega. \end{cases}$$

*Proof.* From the proof of Part 4., Theorem 2.1.4, we obtain that for each $w \in V(\mathscr{M}) = V(\mathscr{I})\cap\Omega$,

$$\dim\Big(\bigcap_{i=1}^{m}\ker(M_{z_j} - w_j)^*\Big) = \dim\Big(\bigcap_{p\in\mathbb{C}[\underline{z}]}\ker(M_p^* - \overline{p(w)})\Big) = t.$$

Furthermore, clearly $\mathscr{M} \in \mathfrak{B}_1(\Omega)$. So, from [3, Lemma 1.11] it follows that $\mathscr{M}$ is locally free on $(\Omega \setminus V(\mathscr{M}))^*$ with the result that $\dim\big(\cap_{i=1}^m \ker(M_{z_j} - w_j)^*\big) = 1$, for $w \notin V(\mathscr{M})$. □

Observe that we do not need $\mathscr{I}$ to be a prime ideal to prove Corollary 2.1.5. This enables us to consider examples that satisfy the conjecture of Douglas, Misra and Varughese [18] but don't follow from [21, Theorem 2.3]. We will discuss an explicit example below to demonstrate this.

## 2.2 An important example

**Example 2.2.1.** In what follows, we let $<\{p_1,\dots,p_t\}>$ denote the ideal generated by the polynomials $p_1,\dots,p_t$. Consider the ideal $\mathscr{I} =<\{z_1z_2, z_1 - z_2\}>$ in $\mathbb{C}[z_1,\dots,z_m]$ and define $\mathscr{M} = [\mathscr{I}]$ in $\mathscr{H} = H^2(\mathbb{D}^m)$, where $m \ge 2$. If $z \in V(\mathscr{I})$, then $z_1z_2 = 0$ and $z_1 - z_2 = 0$. Thus, $V(\mathscr{I}) \subseteq \{z_1 = z_2 = 0\}$. Furthermore, observe that

$$\mathscr{I} \subseteq<\{z_1, z_1 - z_2\}>=<\{z_1, z_2\}>.$$

Note that for two ideals $\mathscr{I}_1, \mathscr{I}_2$ in $\mathbb{C}[z_1,\dots,z_m]$, if $\mathscr{I}_1 \subseteq \mathscr{I}_2$, then $V(\mathscr{I}_2) \subseteq V(\mathscr{I}_{1.})$. To see this, choose an arbitrary point $z \in V(\mathscr{I}_2)$. This means, for any $q \in \mathscr{I}_2$, $q(z) = 0$. In particular, for any $q \in \mathscr{I}_1$, $q(z) = 0$ showing that $z \in V(\mathscr{I}_1)$.

As a result, we obtain $V(\mathscr{I}) = \{(z_1,\dots,z_m) : z_1 = z_2 = 0\}$ which implies that $V(\mathscr{M}) = \{(z_1,\dots,z_m) \in \mathbb{D}^m : z_1 = z_2 = 0\}$. So, by Corollary 2.1.5 we have

$$\dim(\mathscr{M}\otimes_{\mathbb{C}[\underline{z}]}\mathbb{C}_w) = \begin{cases} 1 & \text{for } w \notin \{w_1 = w_2 = 0\}, \\ 2 & \text{for } w \in \{w_1 = w_2 = 0\}\cap\mathbb{D}^m. \end{cases}$$

Now, we claim that $\mathscr{I}$ is not prime. To see this, it is enough to show that neither $z_1$ nor $z_2$ belongs to $\mathscr{I}$. Assume $z_1 \in \mathscr{I}$. Then, there exist $p_1, p_2 \in \mathbb{C}[z_1, \ldots, z_m]$ such that

$$z_1 = p_1 z_1 z_2 + p_2(z_1 - z_2)$$

which implies

$$z_1(1 - p_2 - p_1 z_2) = -p_2 z_2.$$

This means $z_1$ divides $p_2 z_2$. But $z_1$ is a prime element of $\mathbb{C}[z_1, \ldots, z_m]$ and $z_1$ does not divide $z_2$. So, $z_1$ divides $p_2$ and we will write $p_2 = q z_1$, for some $q \in \mathbb{C}[z_1, \ldots, z_m]$. Thus, we have

$$z_1 = p_1 z_1 z_2 + q z_1(z_1 - z_2).$$

Finally, dividing both sides by $z_1$ we have

$$1 = p_1 z_2 + q(z_1 - z_2).$$

This is a contradiction because the right hand side vanishes at the origin whereas the left hand side does not. This proves $z_1 \notin \mathscr{I}$. Following similar arguments one can show that $z_2 \notin \mathscr{I}$.

Let $T = (T_1, \ldots, T_m)$ be any commuting $m$ tuple of bounded linear operators on $\mathscr{H}$ and let $D_T : \mathscr{H} \to \mathscr{H} \oplus \cdots \oplus \mathscr{H}$ be the operator $D_T : f \mapsto (T_1 f, \ldots T_m f)$, $f \in \mathscr{H}$. For any polynomial $q$, let $q^*$ denote the polynomial defined by the formula $q^*(z) = \overline{q(\bar{z})}$. Set $q^*(\bar{D}) = q^*(\frac{\partial}{\partial \bar{w}_1}, \ldots, \frac{\partial}{\partial \bar{w}_m})$.

For each $w \in V(\mathscr{M})$, we now find a basis for $\ker D_{(M-w)^*} := \cap_{j=1}^m \ker(M_{z_j} - w_j)^*$. First consider the case of $m = 2$. In this case, $V(\mathscr{M}) = \{(0,0)\}$. Since $\mathscr{I}$ is generated by two homogeneous polynomials of different degree, from [3, Proposition 4.9], we obtain two polynomials $q_1, q_2$ which generate the ideal $\mathscr{I}$ and such that

$$\left\{q_1^*(\bar{D})K(.,w)|_{w=0}, q_2^*(\bar{D})K(.,w)|_{w=0}\right\}$$

is a basis for $\ker D_{M^*} = \ker M_{z_1}^* \cap \ker M_{z_2}^*$. Following the proof of the proposition, we get $q_1 = p_1$ and $q_2 = p_2 + (\gamma_{10} z_1 + \gamma_{01} z_2) p_1$, where $p_1 = z_1 - z_2$, $p_2 = z_1 z_2$ and $\gamma_{10}, \gamma_{01}$ are two complex numbers satisfying the matrix equation

$$\begin{pmatrix} \|\partial_1 p_1\|_0^2 + \|p_1\|_0^2 & \langle \partial_2 p_1, \partial_1 p_1 \rangle_0 \\ \langle \partial_1 p_1, \partial_2 p_1 \rangle_0 & \|\partial_2 p_1\|_0^2 + \|p_1\|_0^2 \end{pmatrix} \begin{pmatrix} \gamma_{10} \\ \gamma_{01} \end{pmatrix} = - \begin{pmatrix} \langle p_1, \partial_1 p_2 \rangle_0 \\ \langle p_1, \partial_2 p_2 \rangle_0 \end{pmatrix}.$$

Applying the Cauchy-Schwartz inequality, we see that the determinant

$$\|p_1\|_0^4 + \|\partial_1 p_1\|_0^2 \|p_1\|_0^2 + \|\partial_2 p_1\|_0^2 \|p_1\|_0^2 + \left(\|\partial_1 p_1\|_0^2 \|\partial_2 p_1\|_0^2 - |\langle \partial_1 p_1, \partial_2 p_1 \rangle_0|^2\right)$$

of the coefficient matrix is positive and hence it is invertible. Solving the system of linear equations, we get $\gamma_{10} = \frac{1}{4}, \gamma_{01} = -\frac{1}{4}$ and hence $q_1 = z_1 - z_2, q_2 = \frac{(z_1+z_2)^2}{4}$. Ignoring the constant, we will take $q_2 = (z_1+z_2)^2$. The fact that $\mathscr{I} = <\{z_1 - z_2, (z_1+z_2)^2\}>$ follows from the equality: $z_1 z_2 = \frac{1}{4}\big((z_1+z_2)^2 - (z_1-z_2)^2\big)$.

Since $z_1^2 = z_1 z_2 + z_1(z_1 - z_2)$, $z_1^2 \in \mathscr{I}$. Similarly, $z_2^2 \in \mathscr{I}$. Thus,

$$\{z_1^{\alpha_1} z_2^{\alpha_2} : \alpha_1 + \alpha_2 \geq 2, \alpha_1, \alpha_2 \in \mathbb{N} \cup \{0\}\} \subseteq \mathscr{I},$$

which implies

$$\overline{\bigvee}\{z_1^{\alpha_1} z_2^{\alpha_2} : \alpha_1 + \alpha_2 \geq 2, \alpha_1, \alpha_2 \in \mathbb{N} \cup \{0\}\} \subseteq \mathscr{M}.$$

So, for $f \in H^2(\mathbb{D}^2) \ominus \mathscr{M}$ arbitrary, we have $f = a + bz_1 + cz_2$, $a, b, c \in \mathbb{C}$. Furthermore,

$$0 = \langle f, z_1 - z_2 \rangle = \langle a + bz_1 + cz_2, z_1 - z_2 \rangle = b - c,$$

which gives $f = a + b(z_1 + z_2)$. On the other hand, it can be easily checked that the linear subspace of $H^2(\mathbb{D}^2)$ which is generated by 1 and $z_1 + z_2$ is orthogonal to $\mathscr{M}$. It follows that $\{1, \frac{z_1+z_2}{\sqrt{2}}\}$ is an orthonormal basis of $\mathscr{M}^\perp$ in $H^2(\mathbb{D}^2)$. The reproducing kernel $K$ of $\mathscr{M}$ is then given by

$$\begin{aligned}
K(z,w) &= \frac{1}{(1 - z_1\bar{w}_1)(1 - z_2\bar{w}_2)} - \frac{(z_1+z_2)(\bar{w}_1 + \bar{w}_2)}{2} - 1 \\
&= \sum_{i,j\geq 0} z_1^i z_2^j \bar{w}_1^i \bar{w}_2^j - \Big(\frac{z_1\bar{w}_1 + z_1\bar{w}_2 + z_2\bar{w}_1 + z_2\bar{w}_2}{2}\Big) - 1 \\
&= \sum_{i,j\geq 0, i+j\geq 1} z_1^i z_2^j \bar{w}_1^i \bar{w}_2^j - \Big(\frac{z_1\bar{w}_1 + z_1\bar{w}_2 + z_2\bar{w}_1 + z_2\bar{w}_2}{2}\Big) \\
&= (z_1\bar{w}_1 + z_2\bar{w}_2) - \Big(\frac{z_1\bar{w}_1 + z_1\bar{w}_2 + z_2\bar{w}_1 + z_2\bar{w}_2}{2}\Big) + \sum_{i,j\geq 0, i+j\geq 2} z_1^i z_2^j \bar{w}_1^i \bar{w}_2^j \\
&= \frac{(z_1 - z_2)(\bar{w}_1 - \bar{w}_2)}{2} + \sum_{i,j\geq 0, i+j\geq 2} z_1^i z_2^j \bar{w}_1^i \bar{w}_2^j.
\end{aligned}$$

As a result,

$$q_1^*(\bar{D})K(\cdot, w)|_{w=0} = (\bar{\partial}_1 - \bar{\partial}_2)K(\cdot, w)|_{w=0} = z_1 - z_2$$

and

$$q_2^*(\bar{D})K(\cdot, w)|_{w=0} = (\bar{\partial}_1 + \bar{\partial}_2)^2 K(\cdot, w)|_{w=0} = 2(z_1^2 + z_1 z_2 + z_2^2).$$

Now, consider the case of $m \geq 3$. In this case, we show that for any $w_0 \in V(\mathscr{M}) = \{w \in \mathbb{C}^m : w_1 = w_2 = 0\}$,

$$\big\{(z_1 - z_2)K_{\mathscr{H}}(\cdot, w_0), (z_1^2 + z_1 z_2 + z_2^2)K_{\mathscr{H}}(\cdot, w_0)\big\}$$

is a basis for $\ker D_{(M-w_0)^*} = \ker M_{z_1}^* \cap \ker M_{z_2}^* \cap \big(\cap_{j=3}^m \ker(M_{z_j} - w_{0j})^*\big)$. First, note that

$$M_{z_1}^*\big((z_1 - z_2)K_{\mathscr{H}}(\cdot, w_0)\big) = M_{z_1}^*\Big(\frac{z_1 - z_2}{\prod_{i=3}^m (1 - z_i\bar{w}_{0i})}\Big)$$

$$\begin{aligned}
&= P_{\mathcal{M}}(M_{z_1}^{\mathcal{H}})^*\Big(\frac{z_1 - z_2}{\prod_{i=3}^m (1 - z_i \bar{w}_{0i})}\Big)\\
&= P_{\mathcal{M}}(M_{z_1}^{\mathcal{H}})^*\Big(\frac{z_1}{\prod_{i=3}^m (1 - z_i \bar{w}_{0i})}\Big) - P_{\mathcal{M}}(M_{z_1}^{\mathcal{H}})^*\Big(\frac{z_2}{\prod_{i=3}^m (1 - z_i \bar{w}_{0i})}\Big)\\
&= P_{\mathcal{M}}\Big(\frac{1}{\prod_{i=3}^m (1 - z_i \bar{w}_{0i})}\Big)\\
&= 0,
\end{aligned}$$

where $M_{z_1}^{\mathcal{H}}$ is the module multiplication by $z_1$ on $\mathcal{H}$. The fourth equality follows since

$$\Big\langle (M_{z_1}^{\mathcal{H}})^*\Big(\frac{z_2}{\prod_{i=3}^m (1 - z_i \bar{w}_{0i})}\Big), h\Big\rangle_{\mathcal{H}} = \Big\langle \Big(\frac{z_2}{\prod_{i=3}^m (1 - z_i \bar{w}_{0i})}\Big), z_1 h\Big\rangle_{\mathcal{H}} = 0, \, h \in \mathcal{H}.$$

Also, for each $w_0 \in V(\mathcal{M})$, $K_{\mathcal{H}}(\cdot, w_0) = \frac{1}{\prod_{i=3}^m (1 - z_i \bar{w}_{0i})}$ is orthogonal to the space $\{f \in H^2(\mathbb{D}^m) : f = 0 \text{ on } z_1 = z_2 = 0\}$ which contains $\mathcal{M}$. Consequently, $P_{\mathcal{M}}\Big(\frac{1}{\prod_{i=3}^m (1 - z_i \bar{w}_{0i})}\Big) = 0$. Similarly,

$$M_{z_2}^*\big((z_1 - z_2)K_{\mathcal{H}}(\cdot, w_0)\big) = -P_{\mathcal{M}}(M_{z_2}^{\mathcal{H}})^*\Big(\frac{z_2}{\prod_{i=3}^m (1 - z_i \bar{w}_{0i})}\Big) = -P_{\mathcal{M}}\Big(\frac{1}{\prod_{i=3}^m (1 - z_i \bar{w}_{0i})}\Big) = 0.$$

Next, for any $m-2$ tuple $\alpha = (\alpha_3, \ldots, \alpha_m) \in (\mathbb{N} \cup \{0\})^{m-2}$, define $z^\alpha = z_3^{\alpha_3} \cdots z_m^{\alpha_m}$. Then, for each fixed $j \in \{3, \ldots m\}$,

$$\begin{aligned}
M_{z_j}^*\big((z_1 - z_2)K_{\mathcal{H}}(\cdot, w_0)\big) &= P_{\mathcal{M}}(M_{z_j}^{\mathcal{H}})^*\Big(\frac{z_1 - z_2}{\prod_{i=3}^m (1 - z_i \bar{w}_{0i})}\Big)\\
&= P_{\mathcal{M}}(M_{z_j}^{\mathcal{H}})^*\Big((z_1 - z_2)\sum_{\alpha} z^\alpha \bar{w}_0^\alpha\Big)\\
&= P_{\mathcal{M}}(M_{z_j}^{\mathcal{H}})^*\Big((z_1 - z_2)\sum_{\alpha : \alpha_j \geq 1} z_3^{\alpha_3} \cdots z_j^{\alpha_j} \cdots z_m^{\alpha_m} \bar{w}_{03}^{\alpha_3} \cdots \bar{w}_{0j}^{\alpha_j} \cdots \bar{w}_{0m}^{\alpha_m}\Big)\\
&= P_{\mathcal{M}}\Big((z_1 - z_2)\sum_{\alpha : \alpha_j \geq 1} z_3^{\alpha_3} \cdots z_j^{\alpha_j - 1} \cdots z_m^{\alpha_m} \bar{w}_{03}^{\alpha_3} \cdots \bar{w}_{0j}^{\alpha_j} \cdots \bar{w}_{0m}^{\alpha_m}\Big)\\
&= \bar{w}_{0j} P_{\mathcal{M}}\Big((z_1 - z_2)\sum_{\alpha : \alpha_j \geq 1} z_3^{\alpha_3} \cdots z_j^{\alpha_j - 1} \cdots z_m^{\alpha_m} \bar{w}_{03}^{\alpha_3} \cdots \bar{w}_{0j}^{\alpha_j - 1} \cdots \bar{w}_{0m}^{\alpha_m}\Big).
\end{aligned}$$

Now, define $\beta = (\beta_3, \ldots, \beta_m) \in (\mathbb{N} \cup \{0\})^{m-2}$ such that $\beta_j = \alpha_j - 1$ and $\beta_i = \alpha_i$, for $i \neq j$. Then we have

$$\begin{aligned}
M_{z_j}^*\big((z_1 - z_2)K_{\mathcal{H}}(\cdot, w_0)\big) &= \bar{w}_{0j} P_{\mathcal{M}}\Big((z_1 - z_2)\sum_{\beta} z^\beta \bar{w}_0^\beta\Big)\\
&= \bar{w}_{0j} P_{\mathcal{M}}\big((z_1 - z_2)K_{\mathcal{H}}(\cdot, w_0)\big)\\
&= \bar{w}_{0j}\big((z_1 - z_2)K_{\mathcal{H}}(\cdot, w_0)\big).
\end{aligned}$$

Thus, $(z_1 - z_2)K_{\mathcal{H}}(\cdot, w_0) \in \ker D_{(M - w_0)^*}$. For each $j \in \{3, \ldots, m\}$, following similar arguments as above, we can show that

$$M_{z_j}^*\big((z_1^2 + z_1 z_2 + z_2^2)K_{\mathcal{H}}(\cdot, w_0)\big) = \bar{w}_{0j}\big((z_1^2 + z_1 z_2 + z_2^2)K_{\mathcal{H}}(\cdot, w_0)\big).$$

Finally, observe that

$$\begin{aligned}
M_{z_1}^*\big((z_1^2+z_1z_2+z_2^2)K_{\mathcal{H}}(\cdot,w_0)\big) &= P_{\mathcal{M}}(M_{z_1}^{\mathcal{H}})^*\big((z_1^2+z_1z_2+z_2^2)K_{\mathcal{H}}(\cdot,w_0)\big)\\
&= P_{\mathcal{M}}\big((z_1+z_2)K_{\mathcal{H}}(\cdot,w_0)\big)\\
&= 0.
\end{aligned}$$

The validity of the last equality follows from the observation: For any two polynomials $p$ and $q$,

$$\begin{aligned}
\big\langle (z_1+z_2)K_{\mathcal{H}}(\cdot,w_0), z_1z_2p\big\rangle_{\mathcal{H}} &= \big\langle z_1K_{\mathcal{H}}(\cdot,w_0), z_1z_2p\big\rangle_{\mathcal{H}} + \big\langle z_2K_{\mathcal{H}}(\cdot,w_0), z_1z_2p\big\rangle_{\mathcal{H}}\\
&= \big\langle K_{\mathcal{H}}(\cdot,w_0), z_2p\big\rangle_{\mathcal{H}} + \big\langle K_{\mathcal{H}}(\cdot,w_0), z_1p\big\rangle_{\mathcal{H}}\\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
\big\langle (z_1+z_2)K_{\mathcal{H}}(\cdot,w_0), (z_1-z_2)q\big\rangle_{\mathcal{H}} &= \big\langle z_1K_{\mathcal{H}}(\cdot,w_0), z_1q\big\rangle_{\mathcal{H}} - \big\langle z_2K_{\mathcal{H}}(\cdot,w_0), z_2q\big\rangle_{\mathcal{H}}\\
&= \big\langle K_{\mathcal{H}}(\cdot,w_0), q\big\rangle_{\mathcal{H}} - \big\langle K_{\mathcal{H}}(\cdot,w_0), q\big\rangle_{\mathcal{H}}\\
&= 0,
\end{aligned}$$

which implies $(z_1+z_2)K_{\mathcal{H}}(\cdot,w_0)\perp\mathcal{M}$. Similarly, $M_{z_2}^*\big((z_1^2+z_1z_2+z_2^2)K_{\mathcal{H}}(\cdot,w_0)\big)=0$, proving $(z_1^2+z_1z_2+z_2^2)K_{\mathcal{H}}(\cdot,w_0)\in\ker D_{(M-w_0)^*}$. Then, the fact that

$$\big\{(z_1-z_2)K_{\mathcal{H}}(\cdot,w_0), (z_1^2+z_1z_2+z_2^2)K_{\mathcal{H}}(\cdot,w_0)\big\}$$

is a basis of $\ker D_{(M-w_0)^*}$ follows from

a) $(z_1-z_2)K_{\mathcal{H}}(\cdot,w_0)$ and $(z_1^2+z_1z_2+z_2^2)K_{\mathcal{H}}(\cdot,w_0)$ are linearly independent vectors in $H^2(\mathbb{D}^m)$ (can be checked easily) and

b) $\dim\ker D_{(M-w_0)^*}=\dim(\mathcal{M}\otimes_{\mathbb{C}[\underline{z}]}\mathbb{C}_{w_0})=2$.

Clearly, $\big\{(z_1-z_2)K_{\mathcal{H}}(\cdot,w_0), 2(z_1^2+z_1z_2+z_2^2)K_{\mathcal{H}}(\cdot,w_0)\big\}$ is also a basis of $\ker D_{(M-w_0)^*}$. Now, if $m\geq 3$, we claim that

$$\Big\{z^\alpha, \frac{(z_1+z_2)}{\sqrt{2}}z^\beta : \alpha,\beta\in(\mathbb{N}\cup\{0\})^{m-2}\Big\}$$

is an orthonormal basis of $\mathcal{M}^\perp$ in $\mathcal{H}^2(\mathbb{D}^m)$ which follows from the sequence of observations given below.

i) Following similar arguments as given in the case $m=2$, we have

$$\bigvee\{z_1^{\alpha_1}\cdots z_m^{\alpha_m} : \alpha_1+\alpha_2\geq 2, \alpha_i\in\mathbb{N}\cup\{0\}, i=1,\ldots,m\}\subseteq\mathcal{M}.$$

ii) Let $f = a(z_3,\dots,z_m)+b(z_3,\dots,z_m)z_1+c(z_3,\dots,z_m)z_2$ be an arbitrary element of $H^2(\mathbb{D}^m)\ominus\mathscr{M}$, where $a,b,c\in H^2(\mathbb{D}^m)$. Then, for any $r=r(z_1,\dots,z_m)\in H^2(\mathbb{D}^m)$,

$$\begin{aligned}0 =& \langle f, r(z_1-z_2)\rangle\\ &= \langle a, r(z_1-z_2)\rangle + \langle bz_1, r(z_1-z_2)\rangle + \langle cz_2, r(z_1-z_2)\rangle\\ &= \langle bz_1, rz_1\rangle - \langle cz_2, rz_2\rangle\\ &= \langle b-c, r\rangle.\end{aligned}$$

Consequently, $b=c$ and hence $f=a+b(z_1+z_2)$.

iii) For any $\alpha=(\alpha_3,\dots,\alpha_m), \beta=(\beta_3,\dots,\beta_m)$, the closed linear span generated by $z^\alpha$ and $(z_1+z_2)z^\beta$ is orthogonal to $\mathscr{M}$. This can be checked through direct computation.

Thus, for $z,w\in\mathbb{D}^m$,

$$\begin{aligned}K(z,w) &= \left(\frac{1}{(1-z_1\bar w_1)(1-z_2\bar w_2)} - \frac{(z_1+z_2)(\bar w_1+\bar w_2)}{2} - 1\right)\frac{1}{\prod_{i=3}^m(1-z_i\bar w_i)}\\ &= \left(\frac{(z_1-z_2)(\bar w_1-\bar w_2)}{2} + \sum_{i,j\ge 0, i+j\ge 2} z_1^i z_2^j \bar w_1^i \bar w_2^j\right)\frac{1}{\prod_{i=3}^m(1-z_i\bar w_i)}.\end{aligned}$$

If we choose $q_1=z_1-z_2$ and $q_2=(z_1+z_2)^2$, then

$$\begin{aligned}q_1^*(\bar D)K(\cdot,w)|_{w=w_0} &= (\bar\partial_1-\bar\partial_2)K(\cdot,w)|_{w=w_0}\\ &= \frac{1}{\prod_{i=3}^m(1-z_i\bar w_{0i})}(\bar\partial_1-\bar\partial_2)|_{w=w_0}\left(\frac{(z_1-z_2)(\bar w_1-\bar w_2)}{2} + \sum_{i,j\ge 0, i+j\ge 2} z_1^i z_2^j \bar w_1^i \bar w_2^j\right)\\ &= \frac{z_1-z_2}{\prod_{i=3}^m(1-z_i\bar w_{0i})}\\ &= (z_1-z_2)K_{\mathscr{H}}(\cdot,w_0)\end{aligned}$$

and similarly,

$$\begin{aligned}q_2^*(\bar D)K(\cdot,w)|_{w=w_0} &= (\bar\partial_1+\bar\partial_2)^2K(\cdot,w)|_{w=w_0}\\ &= \frac{1}{\prod_{i=3}^m(1-z_i\bar w_{0i})}(\bar\partial_1+\bar\partial_2)^2|_{w=w_0}\left(\frac{(z_1-z_2)(\bar w_1-\bar w_2)}{2} + \sum_{i,j\ge 0, i+j\ge 2} z_1^i z_2^j \bar w_1^i \bar w_2^j\right)\\ &= 2(z_1^2+z_1z_2+z_2^2)K_{\mathscr{H}}(\cdot,w_0)\end{aligned}$$

Thus, in this case, we have obtained two polynomials $q_1,q_2$ independent of $w_0$ such that they generates $\mathscr{I}$ and

$$\{q_1^*(\bar D)K(\cdot,w)|_{w=w_0}, q_2^*(\bar D)K(\cdot,w)|_{w=w_0}\}$$

is a basis of $\ker D_{(M-w_0)^*}$ for all $w_0\in V(\mathscr{M})$.

## 2.3 The vector bundle associated to the joint kernel and its curvature

From Theorem 2.1.4 we obtain a rank $t$, trivial, anti-holomorphic bundle $E_{\mathcal{M}}$ on $V(\mathcal{M}) = V(\mathcal{I}) \cap \Omega$ corresponding to the set $\{p_1, \ldots, p_t\}$ given by

$$E_{\mathcal{M}} = \bigsqcup_{w \in V(\mathcal{M})} < \{F_1(w), \ldots, F_t(w)\} > .$$

Since, for each $w \in V(\mathcal{M})$, $(E_{\mathcal{M}})_w$ is a subspace of $\mathcal{M}$, we can give a Hermitian structure on $E_{\mathcal{M}}$ which is canonically induced by the inner product of $\mathcal{M}$. This observation leads to the following theorem.

**Theorem 2.3.1.** *Let $\Omega$ be a bounded domain in $\mathbb{C}^m$ and $\mathcal{I}$ be a polynomial ideal with generators $p_1, \ldots, p_t$. Also, let $\mathcal{H}, \mathcal{H}'$ be analytic Hilbert modules in $\mathcal{O}(\Omega)$ and $\mathcal{M}, \mathcal{M}'$ be the closure of $\mathcal{I}$ in $\mathcal{H}, \mathcal{H}'$, respectively, with the property that* $\operatorname{codim} V(\mathcal{M}) = \operatorname{codim} V(\mathcal{M}') = t$. *If the modules $\mathcal{M}, \mathcal{M}'$ are "unitarily" equivalent, then we have the following:*

(a) *$E_{\mathcal{M}}$ is equivalent to $E_{\mathcal{M}'}$, where $E_{\mathcal{M}}, E_{\mathcal{M}'}$ are two bundles on $V(\mathcal{M}) = V(\mathcal{M}') = V(\mathcal{I}) \cap \Omega$ obtained from Theorem* 2.1.4 *and the discussion above.*

(b) *If $F_{\mathcal{M}} := \{F_1, \ldots, F_t\}, F_{\mathcal{M}'} := \{F_1^{'}, \ldots, F_t^{'}\}$ are the global frames of $E_{\mathcal{M}}, E_{\mathcal{M}'}$ respectively that are obtained from applying Theorem* 2.1.4 *on $\mathcal{M}, \mathcal{M}'$ with respect to the set $\{p_1, \ldots, p_t\}$, then*

$$\mathcal{K}_{E_{\mathcal{M}}}(F_{\mathcal{M}}) = \mathcal{K}_{E_{\mathcal{M}'}}(F_{\mathcal{M}'}).$$

*Here $\mathcal{K}_{E_{\mathcal{M}}}(F_{\mathcal{M}}), \mathcal{K}_{E_{\mathcal{M}'}}(F_{\mathcal{M}'})$ are the curvature matrices of $E_{\mathcal{M}}, E_{\mathcal{M}'}$ with respect to the frames $F_{\mathcal{M}}, F_{\mathcal{M}'}$, respectively.*

*Proof.* From Theorem 2.1.4, it follows that for each $w \in V(\mathcal{I}) \cap \Omega$, $(E_{\mathcal{M}})_w = \cap_{i=1}^m \ker(M_{z_i} - w_i)^*$ and $(E_{\mathcal{M}'})_w = \cap_{i=1}^m \ker(M_{z_i}^{'} - w_i)^*$, where $M_{z_i}, M_{z_i}^{'}$ are pointwise multiplication by $z_i$ on $\mathcal{M}, \mathcal{M}'$ respectively. If $L : \mathcal{M} \to \mathcal{M}'$ is an unitary module map, then, for any $g \in \cap_{i=1}^m \ker(M_{z_i} - w_i)^*$, $(M_{z_i}^{'})^*(Lg) = L(M_{z_i}^* g) = L(\bar{w}_i g) = \bar{w}_i(Lg)$. Thus,

$$L\Big(\cap_{i=1}^m \ker(M_{z_i} - w_i)^*\Big) \subseteq \cap_{i=1}^m \ker(M_{z_i}^{'} - w_i)^*.$$

Similarly, it can be shown that

$$L^{-1}\Big(\cap_{i=1}^m \ker(M_{z_i}^{'} - w_i)^*\Big) \subseteq \cap_{i=1}^m \ker(M_{z_i} - w_i)^*,$$

which is equivalent to

$$\cap_{i=1}^m \ker(M_{z_i}^{'} - w_i)^* \subseteq L\Big(\cap_{i=1}^m \ker(M_{z_i} - w_i)^*\Big).$$

Consequently, $L\big(\cap_{i=1}^m \ker(M_{z_i}-w_i)^*\big)=\cap_{i=1}^m \ker(M^{'}_{z_i}-w_i)^*$ and hence $L$ is an isometric isomorphism between $(E_{\mathcal{M}})_w$ and $(E_{\mathcal{M}'})_w$ for all $w\in V(\mathscr{I})\cap\Omega$. Moreover, there exists an anti-holomorphic map $A: V(\mathscr{I})\cap\Omega\mapsto GL(t,\mathbb{C})$ such that for each $w$, the following matrix equality is true:

$$\begin{bmatrix} LF_1(w) & \cdots & LF_t(w)\end{bmatrix}=\begin{bmatrix} F^{'}_1(w) & \cdots & F^{'}_t(w)\end{bmatrix}A(w). \tag{2.2}$$

Thus, $L$ induces a bundle isomorphism between $E_{\mathcal{M}}$ and $E_{\mathcal{M}'}$ which proves part (a).

Next, observe that part (b) follows trivially from part (a) when $t=1$. This is because from part (a) we obtain the equality of the first Chern forms on $E_{\mathcal{M}}, E_{\mathcal{M}'}$ given by $c_1(E_{\mathcal{M}}, N_{\mathcal{M}})=c_1(E_{\mathcal{M}'}, N_{\mathcal{M}'})$ which implies $\frac{i}{2\pi}\mathscr{K}_{E_{\mathcal{M}}}(F_{\mathcal{M}})=\frac{i}{2\pi}\mathscr{K}_{E_{\mathcal{M}'}}(F_{\mathcal{M}'})\Leftrightarrow \mathscr{K}_{E_{\mathcal{M}}}(F_{\mathcal{M}})=\mathscr{K}_{E_{\mathcal{M}'}}(F_{\mathcal{M}'})$. Here $N_{\mathcal{M}}, N_{\mathcal{M}'}$ are the Hermitian metrics on $E_{\mathcal{M}}, E_{\mathcal{M}'}$ induced by the inner products of $\mathcal{M}, \mathcal{M}'$, respectively.

For the case where $t\geq 2$, note that $\mathcal{M}, \mathcal{M}'\in\mathfrak{B}_1(\Omega)$. As a result, following [3, Lemma 1.11] we obtain that the reproducing kernels of $\mathcal{M}, \mathcal{M}'$ are sharp on $\Omega\setminus V(\mathscr{I})$. Since $L$ is unitary, there exists a non-vanishing, holomorphic function $\phi$ on $\Omega\setminus V(\mathscr{I})$ such that

$$LK(\cdot,w)=\overline{\phi(w)}K'(\cdot,w), \tag{2.3}$$

for all $w\in\Omega\setminus V(\mathscr{I})$ [12, Theorem 3.7], where $K, K'$ are the reproducing kernels of $\mathcal{M}, \mathcal{M}'$, respectively. But $\operatorname{codim} V(\mathcal{M})\geq 2$. So, by the Hartog's Extension Theorem [26, Page 198] $\phi$ can be uniquely extended to $\Omega$ as a holomorphic function. Since an analytic function is unambiguously determined from its definition on any open set, it follows that Equation (2.3) is true for all $w\in\Omega$.

Now, fix an arbitrary point $w_0\in V(\mathscr{I})\cap\Omega$. Then, from condition b) of Theorem 2.1.4 we have

$$K(\cdot,u)=\sum_{j=1}^t \overline{p_j(u)}F^j_{\Omega_{w_0}}(u) \text{ and } K'(\cdot,u)=\sum_{j=1}^t \overline{p_j(u)}F'^j_{\Omega_{w_0}}(u), \tag{2.4}$$

for all $u\in\Omega_{w_0}$. Applying $L$ to the first equality of the Equation (2.4) we obtain that

$$\overline{\phi(u)}K'(\cdot,u)=\sum_{j=1}^t \overline{p_j(u)}LF^j_{\Omega_{w_0}}(u)$$

or, equivalently,

$$K'(\cdot,u)=\sum_{j=1}^t \overline{p_j(u)}\frac{LF^j_{\Omega_{w_0}}(u)}{\overline{\phi(u)}}.$$

Since, $v\mapsto \frac{LF_j(v)}{\overline{\phi(v)}}, v\mapsto F^{'}_j(v)$ are anti-holomorphic maps from $V(\mathcal{M}')$ to $\mathcal{M}'$ for each $j=1,\ldots,t$, satisfying 1.a) and 1.b) of Theorem 2.1.4, by the condition 2 of the same theorem, we have

$LF_j(v) = \overline{\phi(v)}F_j'(v)$, for all $v \in V(\mathscr{I}) \cap \Omega$. As a result, from Equation (2.2) we have $A(v) = \overline{\phi(v)} I_{t\times t}$, for all $v \in V(\mathscr{I}) \cap \Omega$. Finally,

$$\begin{aligned}
\big(\mathscr{K}_{E_{\mathscr{M}}}(F_{\mathscr{M}})\big)(v) &= \partial\Big(\big(N_{\mathscr{M}}(F_{\mathscr{M}})\big)^{-1}\bar{\partial}\big(N_{\mathscr{M}}(F_{\mathscr{M}})\big)\Big)(v)\\
&= \partial\Big(\big(N_{\mathscr{M}'}(LF_{\mathscr{M}})\big)^{-1}\bar{\partial}\big(N_{\mathscr{M}'}(LF_{\mathscr{M}})\big)\Big)(v)\\
&= \big(\mathscr{K}_{E_{\mathscr{M}'}}(LF_{\mathscr{M}})\big)(v)\\
&= \big(\mathscr{K}_{E_{\mathscr{M}'}}(F_{\mathscr{M}'}A)\big)(v)\\
&= A(v)^{-1}\cdot\big(\mathscr{K}_{E_{\mathscr{M}'}}(F_{\mathscr{M}'})\big)(v)\cdot A(v)\\
&= \big(\mathscr{K}_{E_{\mathscr{M}'}}(F_{\mathscr{M}'})\big)(v),
\end{aligned}$$

which proves part (b) of the Theorem. □

**Remark 2.3.2.** Let $\mathscr{H}$ be an analytic Hilbert module in $\mathscr{O}(\Omega)$ with reproducing kernel $K_{\mathscr{H}}$. Consider a non-vanishing polynomial $p:\Omega \to \mathbb{C}$ with $1/p \in \mathscr{H}$ and for all $z, w \in \Omega$, define the sesquianalytic function $K_1 : \Omega\times\Omega \to \mathbb{C}$ as $K_1(z,w) := p(z)K_{\mathscr{H}}(z,w)\overline{p(w)}$. Clearly, $K_1$ is a non-negative definite function on $\Omega$. Consequently, there exists a reproducing kernel Hilbert space $\mathscr{H}'$ in $\mathscr{O}(\Omega)$ whose reproducing kernel is $K_1$. Let us denote $K_1$ by $K_{\mathscr{H}'}$. Then one can check that the following are true:

i) The pointwise multiplication operator by the polynomial $p$ which we will denote by $M_p$, is an unitary operator from $\mathscr{H}$ to $\mathscr{H}'$ with $M_p^* = M_{1/p}$.

ii) $\mathscr{H}'$ is a Hilbert module over $\mathbb{C}[z_1,\dots,z_m]$.

iii) $M_p$(similarly $M_{1/p}$) is a module map.

Next, observe that $\mathbb{C}[z_1,\dots,z_m] \subseteq \mathscr{H}'$. To see this, take an arbitrary polynomial $q$. Since $1/p \in \mathscr{H}$ and $\mathscr{H}$ is a module over $\mathbb{C}[z_1,\dots,z_m]$, $q/p = M_q(1/p) \in \mathscr{H}$. As a result, $q = pq/p = M_p(q/p) \in \mathscr{H}'$.

Let $\mathscr{I}$ be an arbitrary polynomial ideal in $\mathbb{C}[z_1,\dots,z_m]$ and define $\mathscr{M}, \mathscr{M}'$ as the closure of the polynomial ideal $\mathscr{I}$ in $\mathscr{H}, \mathscr{H}'$, respectively. Then we claim that $M_p(\mathscr{M}) = \mathscr{M}'$. Firstly, we will show that $M_p(\mathscr{M}) \subseteq \mathscr{M}'$. Take an arbitrary element $m \in \mathscr{M}$. Then there exists a sequence $q_n \in \mathscr{I}$ such that $q_n$ converges to $m$ in $\mathscr{H}$. This implies $pq_n$ converges to $pm$ in $\mathscr{H}'$. Thus $pm \in \mathscr{M}'$ which shows $M_p(\mathscr{M}) \subseteq \mathscr{M}'$. To prove the converse part, take an arbitrary element $m' \in \mathscr{M}'$. Then there exists a sequence $r_n \in \mathscr{I}$ such that $r_n$ converges to $m'$ in $\mathscr{H}'$. This implies $r_n/p = M_p^*(r_n)$ converges to $m'/p = M_p^*(m')$ in $\mathscr{H}$. But $r_n/p \in \mathscr{M}$, for each n. To see this, note that there exists a sequence of polynomials $a_m$ that converges to $1/p$ in $\mathscr{H}$(this is because $\mathscr{H}$ is an analytic Hilbert module). Using the fact that $\mathscr{H}$ is a Hilbert module over $\mathbb{C}[z_1,\dots,z_m]$, it follows that $r_n a_m$ converges to $r_n/p$ in $\mathscr{H}$. Since $r_n a_m \in \mathscr{I}$, for all $m$, we have $r_n/p \in \mathscr{M}$. As a result, $m'/p \in \mathscr{M}$ proving $M_p^*(\mathscr{M}') \subseteq \mathscr{M}$ or equivalently, $M_p(\mathscr{M}) \supseteq \mathscr{M}'$.

In particular, if we consider $\mathscr{I}$ to be the principal ideal generated by 1, we can easily show that $\mathscr{H}'$ is an analytic Hilbert module. With these observations, we now consider the following example.

**Example 2.3.3.** Let $\mathscr{H}_0$ be the Hardy module $H^2(\mathbb{D}^3)$. Consider the set

$$\mathscr{H} = \Big\{ f \in \mathscr{O}(\mathbb{D}^3) : \text{ if } f(z_1, z_2, z_3) = \sum_{i,j,k\geq 0} \hat{f}(i,j,k) z_1^i z_2^j z_3^k, \text{ then } \sum_{i,j,k\geq 0} \frac{|\hat{f}(i,j,k)|^2}{w_{i,j,k}} < \infty \Big\},$$

where $w_{i,j,k} = k+1$ whenever $i = j = 0$ and is 1 otherwise. Define the inner product on $\mathscr{H}$ as follows:

$$\langle f, g \rangle_{\mathscr{H}} := \sum_{i,j,k\geq 0} \frac{\hat{f}(i,j,k)\overline{\hat{g}(i,j,k)}}{w_{i,j,k}},$$

where, $g(z_1, z_2, z_3) = \sum_{i,j,k\geq 0} \hat{g}(i,j,k) z_1^i z_2^j z_3^k$. If we define $e_{i,j,k}(z_1, z_2, z_3) = \sqrt{w_{i,j,k}} z_1^i z_2^j z_3^k$, then the set $\{e_{i,j,k} : i, j, k \in \mathbb{N} \cup \{0\}\}$ forms an orthonormal basis of $\mathscr{H}$. Consequently, it follows that $\mathscr{H}$ is a reproducing kernel Hilbert space with kernel

$$K_{\mathscr{H}}(z, w) = \sum_{i,j,k\geq 0} e_{i,j,k}(z)\overline{e_{i,j,k}(w)} = \frac{1}{(1 - z_3\bar{w}_3)^2} + \frac{1}{(1 - z_1\bar{w}_1)(1 - z_2\bar{w}_2)(1 - z_3\bar{w}_3)} + \frac{1}{1 - z_3\bar{w}_3},$$

for all $z = (z_1, z_2, z_3), w = (w_1, w_2, w_3) \in \mathbb{D}^3$. It also follows that $\mathbb{C}[z_1, z_2, z_3]$ is a dense subset of $\mathscr{H}$ and $\mathscr{H}$ is a Hilbert module on it, i.e., $\mathscr{H}$ is an analytic Hilbert module. However, note that $\mathscr{H}_0$ is not unitarily equivalent to $\mathscr{H}$. Otherwise, if $U$ is an unitary module map between $\mathscr{H}_0$ and $\mathscr{H}$, then $U$ intertwines $M_{z_3}^{\mathscr{H}_0}$ and $M_{z_3}^{\mathscr{H}}$, where $M_{z_3}^{\mathscr{H}_0}, M_{z_3}^{\mathscr{H}}$ denote the pointwise multiplication operators by $z_3$ on $\mathscr{H}_0, \mathscr{H}$, respectively. As a result, from [28, Theorem 1.(b)], it follows that $|w_{0,0,k}| = 1$, for all $k \in \mathbb{N} \cup \{0\}$ which is a contradiction. Next, consider the ideal $\mathscr{I}$ in $\mathbb{C}[z_1, z_2, z_3]$ generated by $z_1$ and $z_2$. If $\mathscr{M}_0, \mathscr{M}$ denote the closure of $\mathscr{I}$ in $\mathscr{H}_0, \mathscr{H}$ respectively, then the set $\{z_1^i z_2^j z_3^k : i + j \geq 1\}$ is an orthonormal basis of both $\mathscr{M}_0$ and $\mathscr{M}$. Consequently, their reproducing kernels $K_0$ and $K$ are the same and the identity map $Id$ becomes an unitary module map from $\mathscr{M}_0$ to $\mathscr{M}$.

Finally, consider the polynomial $p = (z_3 - 2)^2$. It can be easily checked that $1/p \in \mathscr{H}$. Let $\mathscr{H}'$ be the Hilbert module obtained from $(z_3 - 2)^2$ following the previous remark. If we call the closure of $\mathscr{I}$ in $\mathscr{H}'$ as $\mathscr{M}'$, then we obtain that $\mathscr{M}, \mathscr{M}'$ are "unitarily" equivalent via $M_p$. This implies that $\mathscr{M}_0$ and $\mathscr{M}'$ are unitarily equivalent via $M_p \circ Id = M_p$. However, note that $\mathscr{H}_0$ and $\mathscr{H}'$ are not unitarily equivalent. Now, applying theorem 2.1.4 to $\mathscr{M}_0$, we get anti-holomorphic maps $F_1, F_2 : \{(0,0,w_3) : w_3 \in \mathbb{D}\} \to \mathscr{M}_0$ given by $F_i(0,0,w_3) = M_{z_i} K_{\mathscr{H}_0}(0,0,w_3) = \frac{z_i}{1 - z_3\bar{w}_3}$, $i = 1,2$. If we denote the set $\{F_1, F_2\}$ by $F_{\mathscr{M}_0}$, then, from theorem 2.3.1, $F_{\mathscr{M}'} = \{F_1', F_2'\}$, where $F_i'(0,0,w_3) = \frac{z_i p}{p(0,0,w_3)(1 - z_3\bar{w}_3)} = \frac{z_i(z_3 - 2)}{(\bar{w}_3 - 2)(1 - z_3\bar{w}_3)}$, $i = 1,2$. Consequently, for each $w_3 \in \mathbb{D}$,

$$\mathscr{K}_{E_{\mathscr{M}_0}}(F_{\mathscr{M}_0})(0,0,w_3) = \mathscr{K}_{E_{\mathscr{M}'}}(F_{\mathscr{M}'})(0,0,w_3) = \begin{pmatrix} \partial_{w_3}\bar{\partial}_{w_3} \log \frac{1}{1-|w_3|^2} & 0 \\ 0 & \partial_{w_3}\bar{\partial}_{w_3} \log \frac{1}{1-|w_3|^2} \end{pmatrix}.$$

Now, assume that $\Omega$ is a bounded domain containing the zero vector in $\mathbb{C}^m$ and $\{p_1,\dots,p_t\}$, $\{q_1,\dots,q_t\}$ are two sets of generators of the polynomial ideal $\mathscr{I}$ consisting of homogeneous polynomials of the same degree. Let $\mathscr{H}$ be an analytic Hilbert module in $\mathscr{O}(\Omega)$ and $\mathscr{M}$ be the closure of $\mathscr{I}$ in $\mathscr{H}$ with the property that $\operatorname{codim} V(\mathscr{M}) = t$. Then, by [3, Lemma 4.7] we obtain that

$$\big\{p_1(\bar{D})K(\cdot,w)|_{w=0},\dots,p_t(\bar{D})K(\cdot,w)|_{w=0}\big\} \text{ and } \big\{q_1(\bar{D})K(\cdot,w)|_{w=0},\dots,q_t(\bar{D})K(\cdot,w)|_{w=0}\big\}$$

are two bases of $\ker D_{M^*}$. As a result, from [3, Lemma 4.2] and [3, Proposition 4.11] it follows that there exists a constant invertible matrix $A=(a_{ij})_{i,j=1}^t$ such that,

$$q_j=\sum_{i=1}^t a_{ij}p_i,\ 1\le j\le t. \tag{2.5}$$

Applying Theorem 2.1.4 with $\{p_1,\dots,p_t\},\{q_1,\dots,q_t\}$ we obtain the sets $\{F_1^p,\dots,F_t^p\},\{F_1^q,\dots,F_t^q\}$, respectively which consist of the anti-holomorphic maps from $V(\mathscr{M})$ to $\mathscr{M}$ satisfying conditions 1) to 4) of the theorem. Now, we claim the following.

**Lemma 2.3.4.** *For each $w\in V(\mathscr{M})$,* $\begin{bmatrix}F_1^p(w) & \dots & F_t^p(w)\end{bmatrix}=\begin{bmatrix}F_1^q(w) & \dots & F_t^q(w)\end{bmatrix}A^*$.

*Proof.* Fix an arbitrary point $w_0\in V(\mathscr{M})$. Then from condition 1 of Theorem 2.1.4 there exist a neighbourhood $\Omega_{w_0}$ of $w_0$ in $\Omega$, anti-holomorphic maps $F^k_{\Omega_{w_0},p},F^k_{\Omega_{w_0},q}:V(\mathscr{M})\to\mathscr{M}$, $k=1,\dots,t$ such that

a) $F^k_{\Omega_{w_0},p}(v)=F^p_k(v), F^k_{\Omega_{w_0},q}(v)=F^q_k(v)$, for all $v\in V(\mathscr{M})\cap\Omega_{w_0}$, $k=1,\dots,t$ and

b) $K(\cdot,u)=\sum_{i=1}^t\overline{p_i(u)}F^i_{\Omega_{w_0},p}(u)$, $K(\cdot,u)=\sum_{j=1}^t\overline{q_j(u)}F^j_{\Omega_{w_0},q}(u)$, for all $u\in\Omega_{w_0}$.

Now, applying Equation (2.5) to the second equality of b) we obtain that

$$K(\cdot,u)=\sum_{i=1}^t\overline{p_i(u)}\Big(\sum_{j=1}^t\bar{a}_{ij}F^j_{\Omega_{w_0},q}(u)\Big),$$

for all $u\in\Omega_{w_0}$. Finally, observe that, for each $i\in\{1,\dots,t\}$, $v\mapsto\sum_{j=1}^t\bar{a}_{ij}F_j^q(v)$ is an anti-holomorphic map from $V(\mathscr{M})\to\mathscr{M}$ satisfying conditions 1.a) and 1.b) of Theorem 2.1.4 with respect to the set $\{p_1,\dots,p_t\}$. So, from condition 2 of Theorem 2.1.4, it follows that, for each $i=1,\dots,t$, $v\in V(\mathscr{M})$,

$$F_i^p(v)=\sum_{j=1}^t\bar{a}_{ij}F_j^q(v),$$

proving the lemma. □

Note that each of the sets $\{F_1^p,\ldots,F_t^p\}$ and $\{F_1^q,\ldots,F_t^q\}$ canonically induces an anti-holomorphic frame of $E_{\mathcal{M}}$ on $V(\mathcal{M})$. If we denote the frames as $F^p_{\mathcal{M}}$ and $F^q_{\mathcal{M}}$ respectively, then we have

$$\mathcal{K}_{E_{\mathcal{M}}}(F^p_{\mathcal{M}}) = \mathcal{K}_{E_{\mathcal{M}}}(F^q_{\mathcal{M}}A^*) = (A^*)^{-1}\cdot\mathcal{K}_{E_{\mathcal{M}}}(F^q_{\mathcal{M}})\cdot A^*,$$

where $\mathcal{K}_{E_{\mathcal{M}}}(F^p_{\mathcal{M}}),\mathcal{K}_{E_{\mathcal{M}}}(F^q_{\mathcal{M}})$ are the curvature matrices of the bundle $E_{\mathcal{M}}$ with respect to the frames $F^p_{\mathcal{M}},F^q_{\mathcal{M}}$, respectively. Thus, we have proved the following.

**Proposition 2.3.5.** *Let $\Omega$ be a bounded domain in $\mathbb{C}^m$ and $\{p_1,\ldots,p_t\},\{q_1,\ldots,q_t\}$ be two generators of the polynomial ideal $\mathcal{I}$ consisting of homogeneous polynomials of same degree. Also, let $\mathcal{H}\subseteq\mathcal{O}(\Omega)$ be an analytic Hilbert module and $\mathcal{M}$ be the closure of $\mathcal{I}$ in $\mathcal{H}$ with the property that* $\operatorname{codim} V(\mathcal{M}) = t$. *Furthermore, assume that $F^p_{\mathcal{M}},F^q_{\mathcal{M}}$ are the global frames of $E_{\mathcal{M}}$ obtained by applying Theorem* 2.1.4 *on $\mathcal{M}$ with respect to the generators mentioned above. Then there exists a constant invertible matrix $A$ such that*

$$\mathcal{K}_{E_{\mathcal{M}}}(F^p_{\mathcal{M}}) = (A^*)^{-1}\cdot\mathcal{K}_{E_{\mathcal{M}}}(F^q_{\mathcal{M}})\cdot A^*.$$

**Corollary 2.3.6.** *Let $\Omega,\mathcal{I}$ be as above, $\mathcal{H},\mathcal{H}'\subseteq\mathcal{O}(\Omega)$ be two analytic Hilbert modules and $\mathcal{M},\mathcal{M}'$ be the closure of $\mathcal{I}$ in $\mathcal{H},\mathcal{H}'$ respectively with* $\operatorname{codim} V(\mathcal{M}) = \operatorname{codim} V(\mathcal{M}') = t$. *Suppose $F^p_{\mathcal{M}} := \{F_1^p,\ldots,F_t^p\}$, $F^q_{\mathcal{M}'} := \{F_1^{'q},\ldots,F_t^{'q}\}$ are the global frames of $E_{\mathcal{M}},E_{\mathcal{M}'}$ corresponding to the generators $\{p_1,\ldots,p_t\},\{q_1,\ldots,q_t\}$, respectively. If the modules $\mathcal{M}$ and $\mathcal{M}'$ are "unitarily" equivalent, then there exists a constant invertible matrix $A$ such that*

$$\mathcal{K}_{E_{\mathcal{M}}}(F^p_{\mathcal{M}}) = (A^*)^{-1}\cdot\mathcal{K}_{E_{\mathcal{M}'}}(F^q_{\mathcal{M}'})\cdot A^*,$$

*where $\mathcal{K}_{E_{\mathcal{M}}}(F^p_{\mathcal{M}}),\mathcal{K}_{E_{\mathcal{M}'}}(F^q_{\mathcal{M}'})$ are the curvature matrices of $E_{\mathcal{M}},E_{\mathcal{M}'}$ with respect to the frames $F^p_{\mathcal{M}},F^q_{\mathcal{M}'}$, respectively.*

*Proof.* If we apply Theorem 2.1.4 on $\mathcal{M}$ with respect to the generator $\{q_1,\ldots,q_t\}$, we will obtain a collection of anti-holomorphic maps $\{F_1^q,\ldots,F_t^q\}$ from $V(\mathcal{M})$ to $\mathcal{M}$. This set canonically induces a global frame of $E_{\mathcal{M}}$. Let us denote the frame by $F^q_{\mathcal{M}}$. Then, by Proposition 2.3.5 it follows that there exists a constant invertible matrix $A$ such that

$$\mathcal{K}_{E_{\mathcal{M}}}(F^p_{\mathcal{M}}) = (A^*)^{-1}\cdot\mathcal{K}_{E_{\mathcal{M}}}(F^q_{\mathcal{M}})\cdot A^*.$$

Finally, from Theorem 2.3.1 we obtain that

$$\mathcal{K}_{E_{\mathcal{M}}}(F^q_{\mathcal{M}}) = \mathcal{K}_{E_{\mathcal{M}'}}(F^q_{\mathcal{M}'})$$

which proves the corollary. $\square$

# Chapter 3

# The kernel decomposition formula and its applications

## 3.1 Vector bundles on the blow-up space and their invariants

Let $\Omega$ be a bounded domain in $\mathbb{C}^m$, $\mathscr{H}$ be an analytic Hilbert module in $\mathscr{O}(\Omega)$ and $\mathscr{M} \in \mathfrak{B}_1(\Omega)$ be the closure of some polynomial ideal $\mathscr{I}$ in $\mathscr{H}$. Fix an arbitrary point $w_0 \in V(\mathscr{I}) \cap \Omega$ and set $t = \dim \mathscr{S}^{\mathscr{M}}_{w_0} / m_{w_0} \mathscr{S}^{\mathscr{M}}_{w_0} = \dim \ker D_{(M-w_0)^*}$, where $m_{w_0}$ denotes the collection of all elements in $\mathscr{O}_{w_0}$ that vanish at $w_0$. If $K$ be the reproducing kernel of $\mathscr{M}$, then, from [3, Theorem 2.3], there exist a minimal set of generators $(g_1)_{w_0}, \ldots, (g_t)_{w_0}$ of $\mathscr{S}^{\mathscr{M}}_{w_0}$ and a positive real number $r$ such that

$$K(\cdot, w) = \sum_{j=1}^{t} \overline{g_j(w)} K^{(j)}(w),$$

for all $w \in \Delta(w_0, r) \subseteq \Omega$, where $\Delta(w_0, r) := \{z \in \mathbb{C}^m : |z_i - w_{0i}| < r, i = 1, \ldots, m\}$ and $K^{(1)}, \ldots, K^{(t)} : \Delta(w_0, r) \to \mathscr{M}$ are anti-holomorphic maps. Moreover, the above decomposition canonically gives rise to a rank $t$ anti-holomorphic bundle $E$ on $\Delta(w_0, r)$ given by,

$$E = \bigsqcup_{w \in \Delta(w_0, r)} < \{K^{(1)}(w), \ldots, K^{(t)}(w)\} > .$$

Here, for each $w \in \Delta(w_0, r)$, $< \{K^{(1)}(w), \ldots, K^{(t)}(w)\} >$ denotes the subspace of $\mathscr{M}$ generated by $K^{(1)}(w), \ldots, K^{(t)}(w)$. Assume that $Z := Z(g_1, \ldots, g_t) \cap \Delta(w_0, r)$ be a singularity free analytic subset of $\mathbb{C}^m$ of codimension $t \geq 2$. Define

$$\begin{aligned} \hat{\Delta}(w_0, r) &:= \big\{(w, \pi(u)) \in \Delta(w_0, r) \times \mathbb{P}^{t-1} : u_i g_j(w) - u_j g_i(w) = 0, i, j = 1, \ldots, t\big\}, \\ &= Z \times \mathbb{P}^{t-1} \bigcup \big\{(w, \pi(u)) \in (\Delta(w_0, r) \setminus Z) \times \mathbb{P}^{t-1} : u_i g_j(w) - u_j g_i(w) = 0, i, j = 1, \ldots, t\big\}, \end{aligned}$$

where $\pi : \mathbb{C}^t \setminus \{0\} \to \mathbb{P}^{t-1}$ is the canonical projection map. The space $\hat{\Delta}(w_0, r)$ is called the monoidal transformation with center $Z$ (or, the blow up of $\Delta(w_0, r)$ with center $Z$). For

$i = 1, \ldots, t$, consider the open sets $\hat{U}_i := \big(\Delta(w_0, r) \times \{u_i \neq 0\}\big) \cap \hat{\Delta}(w_0, r)$. Then, in particular,

$$\begin{aligned}\hat{U}_1 &= \big\{(w, \pi(1, \theta_2^1, \ldots \theta_t^1)) : g_j(w) = \theta_j^1 g_1(w), w \in \Delta(w_0, r), \theta_j^1 \in \mathbb{C}, j = 2, \ldots, t\big\}\\ &= Z \times \big\{\pi(1, \theta_2^1, \ldots \theta_t^1) : \theta_j^1 \in \mathbb{C}, j = 2, \ldots, t\big\}\\ &\bigcup \big\{(w, \pi(1, \theta_2^1, \ldots \theta_t^1)) : g_j(w) = \theta_j^1 g_1(w), w \in U_1, \theta_j^1 \in \mathbb{C}, j = 2, \ldots, t\big\},\end{aligned}$$

where $U_i = \{w \in \Delta(w_0, r) : g_i(w) \neq 0\}$, $i = 1, \ldots, t$. The anti-holomorphic map $K : \Delta(w_0, r) \to \mathscr{M}$ determines an anti-holomorphic line bundle $\hat{E}_1$ on $\hat{\Delta}(w_0, r)$. On $\hat{U}_1$, we can write

$$\hat{E}_1 = \bigsqcup_{(w, \pi(1, \theta_2^1, \ldots \theta_t^1)) \in \hat{U}_1} < \big\{s_1(w, \pi(1, \theta_2^1, \ldots \theta_t^1))\big\} >,$$

where $s_1(w, \pi(1, \theta_2^1, \ldots \theta_t^1)) := K^{(1)}(w) + \bar{\theta}_2^1 K^{(2)}(w) + \cdots + \bar{\theta}_t^1 K^{(t)}(w)$. On the other hand, $E$ determines the rank $t$ anti-holomorphic bundle $p^*E$ on $\hat{\Delta}(w_0, r)$, where $p : \hat{\Delta}(w_0, r) \to \Delta(w_0, r)$ be the canonical projection. If we denote $p^*E$ as $\hat{E}$, then, on $\hat{U}_1$,

$$\hat{E} = \bigsqcup_{(w, \pi(1, \theta_2^1, \ldots \theta_t^1)) \in \hat{U}_1} < \big\{s_1(w, \pi(1, \theta_2^1, \ldots \theta_t^1)), K^{(2)}(w), \ldots, K^{(t)}(w)\big\} > .$$

This is because on $\hat{U}_1$, we have the following equality:

$$\Big[s_1(w, \pi(\Theta)), K^{(2)}(w), \ldots, K^{(t)}(w)\Big] = \Big[K^{(1)}(w), K^{(2)}(w), \ldots, K^{(t)}(w)\Big] \cdot \begin{pmatrix} 1 & 0 & \cdots & 0\\ \bar{\theta}_2^1 & 1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ \bar{\theta}_t^1 & 0 & \cdots & 1\end{pmatrix},$$

where $\Theta = (1, \theta_2^1, \ldots \theta_t^1)$, $\theta_j^1 \in \mathbb{C}$, for $j = 2, \ldots, t$. Clearly, $\hat{E}_1$ is a sub-bundle of $\hat{E}$. If we denote the quotient bundle by $\hat{E}_2$, then

$$\hat{E}_2 = \bigsqcup_{(w, \pi(1, \theta_2^1, \ldots \theta_t^1)) \in \hat{U}_1} < \big\{[K^{(2)}(w)], \ldots, [K^{(t)}(w)]\big\} >$$

on $\hat{U}_1$, where, $[K^{(j)}(w)]$ denotes the section $(w, \pi(\Theta)) \mapsto K^{(j)}(w) + < \{s_1(w, \pi(\Theta))\} >$ on $\hat{U}_1$, $K^{(j)}(w) + < \{s_1(w, \pi(\Theta))\} >$ is the image of $K^{(j)}(w)$ under the canonical quotient map from $\hat{E}_{(w, \pi(\Theta))}$ to $\hat{E}_{(w, \pi(\Theta))} / (\hat{E}_1)_{(w, \pi(\Theta))}$, $(\hat{E}_1)_{(w, \pi(\Theta))}, \hat{E}_{(w, \pi(\Theta))}$ are the fibres of $\hat{E}_1, \hat{E}$ at $(w, \pi(\Theta))$. So, we have an exact sequence of anti-holomorphic vector bundles on $\hat{\Delta}(w_0, r)$ given by

$$0 \to \hat{E}_1 \to \hat{E} \to \hat{E}_2 \to 0.$$

$\hat{E}_1$ and $\hat{E}$ have canonical Hermitian structures induced by $\mathscr{M}$. This is because each of their fibre is a subspace of $\mathscr{M}$ of dimension 1 and $t$, respectively. Define the Hermitian structure of $\hat{E}_2$ by identifying it with $\hat{E}_1^\perp$ in $\hat{E}$, as smooth bundles. In particular, we will define

$$\|[K^{(j)}(w)]\|_{\hat{E}_2}^2 := \|P_{\hat{E}_1^\perp} K^{(j)}(w)\|_{\hat{E}}^2 = \|P_{\hat{E}_1^\perp} K^{(j)}(w)\|_{\mathscr{H}}^2,$$

for $j = 2, \ldots, t$, where $P_{\hat{E}_1^{\perp}} K^{(j)}(w)$ is the smooth section which is obtained by applying the canonical projection on $\hat{E}_1^{\perp}$ to the section $(w, \pi(u)) \mapsto K^{(j)}(w)$. On $\hat{U}_1$, it is given by the map

$$(w, \pi(\Theta)) \mapsto \left( K^{(j)}(w) - \frac{\left\langle K^{(j)}(w), s_1(w, \pi(\Theta)) \right\rangle_{\mathscr{M}}}{\| s_1(w, \pi(\Theta)) \|_{\mathscr{M}}^2} s_1(w, \pi(\Theta)) \right).$$

Thus, we obtain an exact sequence of Hermitian, anti-holomorphic bundles on $\hat{\Delta}(w_0, r)$. Next, we restrict $\hat{E}_1, \hat{E}$ and $\hat{E}_2$ on $Z \times \mathbb{P}^{t-1}$ and denote them as $\tilde{E}_1, \tilde{E}$ and $\tilde{E}_2$, respectively. This canonically gives an exact sequence of bundles on $Z \times \mathbb{P}^{t-1}$. Finally, for $w_0 \in Z$ define $\tilde{E}_1^{w_0} = \tilde{E}_1|_{w_0 \times \mathbb{P}^{t-1}}$, $\tilde{E}^{w_0} = \tilde{E}|_{w_0 \times \mathbb{P}^{t-1}}$ and $\tilde{E}_2^{w_0} = \tilde{E}_2|_{w_0 \times \mathbb{P}^{t-1}}$. Then

$$0 \to \tilde{E}_1^{w_0} \to \tilde{E}^{w_0} \to \tilde{E}_2^{w_0} \to 0$$

can be considered as an exact sequence of Hermitian, anti-holomorphic bundles on $\mathbb{P}^{t-1}$. This leads us to the following theorem.

**Theorem 3.1.1.** *Let $\mathscr{H}, \mathscr{H}' \subseteq \mathscr{O}(\Omega)$ be two analytic Hilbert modules for some bounded domain $\Omega$ in $\mathbb{C}^m$. Let $\mathscr{M}, \mathscr{M}'$ be the submodules of the form $[\mathscr{I}]$, which have been obtained by taking the closure of $\mathscr{I}$ in $\mathscr{H}, \mathscr{H}'$ respectively. For $w_0 \in Z$, set $t := \dim \mathscr{M} \otimes_{\mathbb{C}[\underline{z}]} \mathbb{C}_{w_0}$, $t' := \dim \mathscr{M}' \otimes_{\mathbb{C}[\underline{z}]} \mathbb{C}_{w_0}$. Assume that $t, t' \geq 2$. Then we have exact sequences of Hermitian, anti-holomorphic vector bundles $0 \to \tilde{E}_1^{w_0} \to \tilde{E}^{w_0} \to \tilde{E}_2^{w_0} \to 0$ and $0 \to \tilde{E}'^{w_0}_1 \to \tilde{E}'^{w_0} \to \tilde{E}'^{w_0}_2 \to 0$ on $\mathbb{P}^{t-1}$. If $L : \mathscr{M} \to \mathscr{M}'$ is an unitary module map, then*

(a) $t = t'$,

(b) $\tilde{E}_1^{w_0}, \tilde{E}'^{w_0}_1; \tilde{E}^{w_0}, \tilde{E}'^{w_0}$ *and* $\tilde{E}_2^{w_0}, \tilde{E}'^{w_0}_2$ *are equivalent as Hermitian anti-holomorphic bundles on* $\mathbb{P}^{t-1}$.

*Proof.* Since $L : \mathscr{M} \to \mathscr{M}'$ is an unitary module map, for any $g \in \cap_{i=1}^m \ker(M_{z_i} - w_{0i})^*$, we have $(M'_{z_i})^*(Lg) = L(M_{z_i}^* g) = L(\bar{w}_{0i} g) = \bar{w}_{0i}(Lg)$. Thus,

$$L\Big( \cap_{i=1}^m \ker(M_{z_i} - w_{0i})^* \Big) \subseteq \cap_{i=1}^m \ker(M'_{z_i} - w_{0i})^*.$$

Similarly, one can show that

$$L^{-1}\Big( \cap_{i=1}^m \ker(M'_{z_i} - w_{0i})^* \Big) \subseteq \cap_{i=1}^m \ker(M_{z_i} - w_{0i})^*,$$

which is equivalent to

$$\cap_{i=1}^m \ker(M'_{z_i} - w_{0i})^* \subseteq L\Big( \cap_{i=1}^m \ker(M_{z_i} - w_{0i})^* \Big).$$

Consequently, $L\big( \cap_{i=1}^m \ker(M_{z_i} - w_{0i})^* \big) = \cap_{i=1}^m \ker(M'_{z_i} - w_{0i})^*$. But $\cap_{i=1}^m \ker(M_{z_i} - w_{0i})^* \subseteq \mathscr{M}$, $\cap_{i=1}^m \ker(M'_{z_i} - w_{0i})^* \subseteq \mathscr{M}'$ and $L : \mathscr{M} \to \mathscr{M}'$ is an unitary map. This implies that $L$ is an isomorphism between $\cap_{i=1}^m \ker(M_{z_i} - w_{0i})^*$ and $\cap_{i=1}^m \ker(M^{'}_{z_i} - w_{0i})^*$. As a result, $\dim \mathcal{M} \otimes_{\mathbb{C}[\underline{z}]} \mathbb{C}_{w_0} = \dim \mathcal{M}' \otimes_{\mathbb{C}[\underline{z}]} \mathbb{C}_{w_0}$ which is equivalent to $t = t'$, proving part (a) of the theorem.

Note that $\mathbb{P}^{t-1} = \cup_{i=1}^t \tilde{U}_i^{w_0}$, where $\tilde{U}_i^{w_0} = \{\pi(u_1,\dots,u_t) : u_i \neq 0\}$. In particular, $\tilde{U}_1^{w_0} = \{\pi(1,\theta_2^1,\dots,\theta_t^1) : \theta_j^1 \in \mathbb{C}, j = 2,\dots,t\}$. Firstly, we will show that on $\tilde{U}_1^{w_0}$, $\tilde{E}_1^{w_0}$ and $\tilde{E'}_1^{w_0}$ are equivalent. Since $\mathcal{M},\mathcal{M}'$ are locally free of rank 1 on $(\Omega \setminus V(\mathscr{I}))^*$ [3, Lemma 1.11] and $L$ is unitary, from [12, Theorem 3.7] it follows that there exists a non-vanishing, holomorphic function $\phi$ on $\Omega \setminus V(\mathscr{I})$ such that $LK(\cdot,w) = \overline{\phi(w)}K'(\cdot,w)$, for all $w \in \Omega \setminus V(\mathscr{I})$. But $\operatorname{codim} V(\mathscr{I}) \geq 2$. So, according to the Hartog's Extension Theorem [26, Page 198] $\phi$ has an unique extension to $\Omega$ as a non-vanishing, holomorphic function. As a result, the above equality holds for all $w \in \Omega$. This implies $L^* f = \phi f$, for all $f \in \mathcal{M}'$ resulting $\mathscr{S}_{w_0}^{\mathcal{M}} = \mathscr{S}_{w_0}^{\mathcal{M}'}$.

Let $(g_1^{'})_{w_0},\dots,(g_t^{'})_{w_0}$ be a minimal set of generators of $\mathscr{S}_{w_0}^{\mathcal{M}'}$. Then there exists a positive real number $r$ such that $K(\cdot,w) = \sum_{j=1}^t \overline{g_j^{'}(w)}K^{(j)}(w)$ and $K'(\cdot,w) = \sum_{j=1}^t \overline{g_j^{'}(w)}K'^{(j)}(w)$ for all $w \in \Delta(w_0,r)$. Applying $L$ to the first equality we get,

$$\overline{\phi(w)}K'(\cdot,w) = \sum_{j=1}^t \overline{g_j^{'}(w)}LK^{(j)}(w),$$

which implies

$$K'(\cdot,w) = \sum_{j=1}^t \overline{g_j^{'}(w)}\frac{LK^{(j)}(w)}{\overline{\phi(w)}}.$$

From [3, Theorem 2.3], $\{K'^{(1)}(w_0),\dots,K'^{(t)}(w_0)\}$ is uniquely determined by $(g_1^{'})_{w_0},\dots,(g_t^{'})_{w_0}$. So, comparing the above two expressions of $K'(\cdot,w)$ at $w_0$ we obtain, $K'^{(j)}(w_0) = \frac{LK^{(j)}(w_0)}{\overline{\phi(w_0)}}$, for $j = 1,\dots,t$. This implies $Ls_1(w_0,\pi(\Theta)) = \overline{\phi(w_0)}s_1'(w_0,\pi(\Theta))$, which proves that $L$ induces an isometry between the bundles $\tilde{E}_1^{w_0}$ and $\tilde{E'}_1^{w_0}$ on $\tilde{U}_1^{w_0}$. In other words, on $\tilde{U}_1^{w_0}$, the bundles are equivalent. Similarly, it can be shown that the same is true on $\tilde{U}_i^{w_0}$ for all $i = 2,\dots,t$. Hence $\tilde{E}_1^{w_0}$ and $\tilde{E'}_1^{w_0}$ are equivalent as Hermitian, anti-holomorphic bundles on $\mathbb{P}^{t-1}$.

$\tilde{E}^{w_0}$ and $\tilde{E'}^{w_0}$ have constant global frames on $\mathbb{P}^{t-1}$, namely, $\{K^{(1)}(w_0),\dots,K^{(t)}(w_0)\}$ and $\{K'^{(1)}(w_0),\dots,K'^{(t)}(w_0)\}$. So, from the equality $LK^{(j)}(w_0) = \overline{\phi(w_0)}K'^{(j)}(w_0)$, for all $j = 1,\dots,t$, it follows that $L$ induces an isometric isomorphism between $\tilde{E}^{w_0}$ and $\tilde{E'}^{w_0}$ on $\mathbb{P}^{t-1}$.

On $\tilde{U}_1^{w_0}$, $\tilde{E}_2^{w_0}$ is generated by $\{\pi(\Theta) \mapsto K^{(2)}(w_0) + (\tilde{E}_1^{w_0})_{\pi(\Theta)},\dots,\pi(\Theta) \mapsto K^{(t)}(w_0) + (\tilde{E}_1^{w_0})_{\pi(\Theta)}\}$, where $(\tilde{E}_1^{w_0})_{\pi(\Theta)}$ is the stalk of $\tilde{E}_1^{w_0}$ at $\pi(\Theta)$ and $K^{(j)}(w_0) + (\tilde{E}_1^{w_0})_{\pi(\Theta)}$ is the image of $K^{(j)}(w_0)$ under the quotient map $\tilde{E}_{\pi(\Theta)}^{w_0} \to \tilde{E}_{\pi(\Theta)}^{w_0}/(\tilde{E}_1^{w_0})_{\pi(\Theta)}$, for all $j = 1,\dots,t$. In short, we write

$$\tilde{E}_2^{w_0}|_{\tilde{U}_1^{w_0}} = \bigsqcup_{\theta_j^1 \in \mathbb{C}, j=2,\dots,t} < \{[K^{(2)}(w_0)],\dots,[K^{(t)}(w_0)]\} >,$$

where $\Theta = (1,\theta_2^1,\dots,\theta_t^1)$. Similarly,

$$\tilde{E'}_2^{w_0}|_{\tilde{U}_1^{w_0}} = \bigsqcup_{\theta_j^1 \in \mathbb{C}, j=2,\dots,t} < \{[K'^{(2)}(w_0)],\dots,[K'^{(t)}(w_0)]\} > .$$

Now, for each $j = 1, \ldots, t$,

$$L\Big(K^{(j)}(w_0) + (\tilde{E}_1^{w_0})_{\pi(\Theta)}\Big) = \overline{\phi(w_0)} K'^{(j)}(w_0) + (\tilde{E}'^{w_0}_1)_{\pi(\Theta)} = \overline{\phi(w_0)}\Big(K'^{(j)}(w_0) + (\tilde{E}'^{w_0}_1)_{\pi(\Theta)}\Big),$$

which implies $L\big([K^{(j)}(w_0)]\big) = \overline{\phi(w_0)}[K'^{(j)}(w_0)]$.

Next, we will show that $L$ induces an isometry between $\tilde{E}_2^{w_0}|_{\tilde{U}_1^{w_0}}$ and $\tilde{E}'^{w_0}_2|_{\tilde{U}_1^{w_0}}$. Note that unlike the previous cases this is not automatic from the above equality because $(\tilde{E}_2^{w_0})_{\pi(\Theta)}$ (respectively, $(\tilde{E}'^{w_0}_2)_{\pi(\Theta)}$) is not a subspace of $\mathcal{M}$ (respectively $\mathcal{M}'$). So, the Hermitian metrics on these bundles are not canonically induced from the inner products of $\mathcal{M}$ and $\mathcal{M}'$. However, the following set of equalities establishes the claim.

$$\begin{aligned}
&\Big\langle L\big([K^{(j)}(w_0)]\big), L\big([K^{(i)}(w_0)]\big)\Big\rangle_{\tilde{E}'^{w_0}_2}\\
&= |\phi(w_0)|^2 \Big\langle [K'^{(j)}(w_0)], [K'^{(i)}(w_0)]\Big\rangle_{\tilde{E}'^{w_0}_2}\\
&= |\phi(w_0)|^2 \Big\langle K'^{(j)}(w_0) - \frac{\big\langle K'^{(j)}(w_0), s_1'(w_0,\pi(\Theta))\big\rangle_{\mathcal{M}'}}{\|s_1'(w_0,\pi(\Theta))\|^2_{\mathcal{M}'}} s_1'(w_0,\pi(\Theta)),\\
&\qquad\qquad K'^{(i)}(w_0) - \frac{\big\langle K'^{(i)}(w_0), s_1'(w_0,\pi(\Theta))\big\rangle_{\mathcal{M}'}}{\|s_1'(w_0,\pi(\Theta))\|^2_{\mathcal{M}'}} s_1'(w_0,\pi(\Theta))\Big\rangle_{\mathcal{M}'}\\
&= |\phi(w_0)|^2 \left[\Big\langle K'^{(j)}(w_0), K'^{(i)}(w_0)\Big\rangle_{\mathcal{M}'} - \frac{\big\langle K'^{(j)}(w_0), s_1'(w_0,\pi(\Theta))\big\rangle_{\mathcal{M}'} \cdot \overline{\big\langle K'^{(i)}(w_0), s_1'(w_0,\pi(\Theta))\big\rangle_{\mathcal{M}'}}}{\|s_1'(w_0,\pi(\Theta))\|^2_{\mathcal{M}'}}\right]\\
&= |\phi(w_0)|^2 \Bigg[\frac{\big\langle LK^{(j)}(w_0), LK^{(i)}(w_0)\big\rangle_{\mathcal{M}'}}{|\phi(w_0)|^2} -\\
&\qquad\qquad \frac{\big\langle LK^{(j)}(w_0), Ls_1(w_0,\pi(\Theta))\big\rangle_{\mathcal{M}'} \cdot \overline{\big\langle LK^{(i)}(w_0), Ls_1(w_0,\pi(\Theta))\big\rangle_{\mathcal{M}'}}}{|\phi(w_0)|^2 \|Ls_1(w_0,\pi(\Theta))\|^2_{\mathcal{M}'}}\Bigg]\\
&= \Big\langle K^{(j)}(w_0), K^{(i)}(w_0)\Big\rangle_{\mathcal{M}} - \frac{\big\langle K^{(j)}(w_0), s_1(w_0,\pi(\Theta))\big\rangle_{\mathcal{M}} \cdot \overline{\big\langle K^{(i)}(w_0), s_1(w_0,\pi(\Theta))\big\rangle_{\mathcal{M}}}}{\|s_1(w_0,\pi(\Theta))\|^2_{\mathcal{M}}}\\
&= \Big\langle [K^{(j)}(w_0)], [K^{(i)}(w_0)]\Big\rangle_{\tilde{E}_2^{w_0}}.
\end{aligned}$$

Similarly, one can show that the above claim is true on $\tilde{U}_i^{w_0}$, for each $i = 2, \ldots, t$. Consequently, $\tilde{E}_2^{w_0}$ and $\tilde{E}'^{w_0}_2$ are equivalent as Hermitian, anti-holomorphic bundles on $\mathbb{P}^{t-1}$. $\square$

**Corollary 3.1.2.** *With the same hypothesis as in the previous theorem, we have the following equalities of the total Chern forms:*

i) $c(\tilde{E}_1^{w_0}, \tilde{N}_1^{w_0}) = c(\tilde{E}'^{w_0}_1, \tilde{N}'^{w_0}_1)$

ii) $c(\tilde{E}^{w_0}, \tilde{N}^{w_0}) = c(\tilde{E}'^{w_0}, \tilde{N}'^{w_0})$

iii) $c(\tilde{E}_2^{w_0}, \tilde{N}_2^{w_0}) = c(\tilde{E}'^{w_0}_2, \tilde{N}'^{w_0}_2)$, *where* $\tilde{N}_1^{w_0}$, $\tilde{N'}_1^{w_0}$, $\tilde{N}^{w_0}$, $\tilde{N'}^{w_0}$, $\tilde{N}_2^{w_0}$ *and* $\tilde{N'}_2^{w_0}$ *are the Hermitian norms on their respective bundles.*

*Proof.* Follows trivially from the equivalence of the bundles. □

**Lemma 3.1.3.** $c(\tilde{E}^{w_0}, \tilde{N}^{w_0}) = 0$ *on* $\mathbb{P}^{t-1}$.

*Proof.* $\mathbb{P}^{t-1} = \bigcup_{i=1}^{t} \tilde{U}_i^{w_0}$ and for each $i = 1 \dots t$, $\tilde{U}_i^{w_0}$ is a co-ordinate chart of $\mathbb{P}^{t-1}$. On $\tilde{U}_1^{w_0}$, if we denote the frame $\{\pi(\Theta) \mapsto K^{(j)}(w_0) : 1 \le j \le t\}$ by $s$, then, from [6],

$$c(\tilde{E}^{w_0}, \tilde{N}^{w_0})|_{\tilde{U}_1^{w_0}} = c(\tilde{E}^{w_0}, \tilde{N}^{w_0})(s) = \det\Big(I + \frac{i}{2\pi}\partial_\Theta\big((\tilde{N}^{w_0}(s))^{-1}\bar{\partial}_\Theta \tilde{N}^{w_0}(s)\big)\Big).$$

Now, $\tilde{N}^{w_0}(s) = \big(\langle K^{(j)}(w_0), K^{(i)}(w_0)\rangle\big)_{i,j=1}^{t}$, which is a constant function in $\theta_j^1$ variables, $j = 2 \dots t$. As a result, $c(\tilde{E}^{w_0}, \tilde{N}^{w_0})(s) = 0$. Similarly, one can show that $c(\tilde{E}^{w_0}, \tilde{N}^{w_0})|_{\tilde{U}_i^{w_0}} = 0$ for $i = 2, \dots, t$, proving the lemma. □

In particular, $c_1(\tilde{E}^{w_0}, \tilde{N}^{w_0}) = 0$, where $c_1(\tilde{E}^{w_0}, \tilde{N}^{w_0})$ is the first Chern form of $\tilde{E}^{w_0}$ relative to the Chern connection associated to the metric $\tilde{N}^{w_0}$. Now, following the work of Bott and Chern [6, Proposition 4.2], we have the equality $c_1(\tilde{E}^{w_0}, \tilde{N}^{w_0}) = c_1(\tilde{E}_1^{w_0}, \tilde{N}_1^{w_0}) + c_1(\tilde{E}_2^{w_0}, \tilde{N}_2^{w_0})$. Consequently, we obtain that $c_1(\tilde{E}_2^{w_0}, \tilde{N}_2^{w_0}) = -c_1(\tilde{E}_1^{w_0}, \tilde{N}_1^{w_0})$ as (1,1) forms on $\mathbb{P}^{t-1}$.

Next, assume that $\Omega$ is a bounded domain in $\mathbb{C}^m$, $\mathscr{H} \subseteq \mathscr{O}(\Omega)$ is an analytic Hilbert module, $\mathscr{M}$ is the closure of a polynomial ideal $\mathscr{I}$ in $\mathscr{H}$ generated by $\{p_1, \dots, p_t\}$ and $V(\mathscr{M}) := Z(p_1, \dots, p_t) \cap \Omega$ is a submanifold of codimension $t \ge 2$. Then, from Theorem 2.1.4, there exists anti-holomorphic maps $F_1, \dots, F_t : V(\mathscr{M}) \to \mathscr{M}$ which satisfies the following:

For each $w \in V(\mathscr{M})$, there exists a neighbourhood $\Omega_w$ of $w$ in $\Omega$, anti-holomorphic maps $F_{\Omega_w}^1, \dots, F_{\Omega_w}^t : \Omega_w \to \mathscr{M}$ such that

(a) $F_{\Omega_w}^j(v) = F_j(v)$, for $v \in V(\mathscr{M}) \cap \Omega_w, j = 1, \dots, t$;

(b) $K(\cdot, u) = \sum_{j=1}^{t} \overline{p_j(u)} F_{\Omega_w}^j(u)$, for $u \in \Omega_w$.

Fix an arbitrary point $w_0 \in V(\mathscr{M})$. Without loss of generality, going to a smaller neighbourhood if necessary, we may assume that $\Omega_{w_0} = \Delta(w_0, r)$. Define $Z = V(\mathscr{M}) \cap \Delta(w_0, r) = Z(p_1, \dots, p_t) \cap \Delta(w_0, r)$. Following the blow up construction prescribed above we have an exact sequence of Hermitian, anti-holomorphic vector bundles on $Z \times \mathbb{P}^{t-1}$ given by $0 \to \tilde{E}_1 \to \tilde{E} \to \tilde{E}_2 \to 0$. Note that $Z \times \mathbb{P}^{t-1} = \cup_{i=1}^{t} \tilde{U}_i$, where $\tilde{U}_i = Z \times \{\pi(u_1, \dots, u_t) : u_i \neq 0\} = Z \times \tilde{U}_i^{w_0}$. On $\tilde{U}_1$, $\tilde{E}_1, \tilde{E}$ and $\tilde{E}_2$ are described as follows:

$$\tilde{E}_1 = \bigsqcup_{(v, \pi(\Theta)) \in \tilde{U}_1} < \{s_1(v, \pi(\Theta))\} >,$$

where $s_1(v,\pi(\Theta)) = F^1_{\Omega_{w_0}}(v) + \bar\theta^1_2 F^2_{\Omega_{w_0}}(v) + \cdots + \bar\theta^1_t F^t_{\Omega_{w_0}}(v) = F_1(v) + \bar\theta^1_2 F_2(v) + \cdots + \bar\theta^1_t F_t(v)$ for any $(v,\pi(\Theta)) \in \tilde U_1$, $\Theta = (1,\theta^1_2,\ldots,\theta^1_t)$. Similarly,

$$\tilde E = \bigsqcup_{(v,\pi(\Theta))\in \tilde U_1} < \{s_1(v,\pi(\Theta)), F_2(v),\ldots,F_t(v)\} >$$

and

$$\tilde E_2 = \bigsqcup_{(v,\pi(\Theta))\in \tilde U_1} < \{[F_2(v)],\ldots,[F_t(v)]\} >,$$

where, $[F_j(v)]$ denotes the section $(v,\pi(\Theta)) \mapsto F_j(v) + < \{s_1(v,\pi(\Theta))\} >$ on $\tilde U_1$, for each $j = 2,\ldots,t$. Now, we will generalize Theorem 3.1.1.

**Theorem 3.1.4.** *Let $\Omega$ be a bounded domain in $\mathbb{C}^m$, $\mathscr{H},\mathscr{H}' \subseteq \mathscr{O}(\Omega)$ be analytic Hilbert modules, $\mathscr{I}$ be a polynomial ideal generated by $\{p_1,\ldots,p_t\}$ and $\mathscr{M},\mathscr{M}'$ be the closure of $\mathscr{I}$ in $\mathscr{H},\mathscr{H}'$, respectively. Then, following the above, we have two exact sequences of Hermitian, anti-holomorphic bundles on $Z\times\mathbb{P}^{t-1}$ given by $0\to\tilde E_1\to\tilde E\to\tilde E_2\to 0$ and $0\to\tilde E^{'}_1\to\tilde E^{'}\to\tilde E^{'}_2\to 0$. If $L:\mathscr{M}\to\mathscr{M}'$ is an unitary module map, then $\tilde E_1,\tilde E^{'}_1;\tilde E,\tilde E^{'}$ and $\tilde E_2,\tilde E^{'}_2$ are equivalent.*

*Proof.* Firstly, observe that $\mathscr{M},\mathscr{M}' \in \mathfrak{B}_1(\Omega)$. So, there exists a non-vanishing holomorphic function $\phi$ on $\Omega$ such that $LK(\cdot,w) = \overline{\phi(w)}K'(\cdot,w)$ for all $w\in\Omega$. Moreover, from condition (b) of Theorem 2.1.4 we have

$$K(\cdot,u) = \sum_{j=1}^t \overline{p_j(u)} F^j_{\Omega_{w_0}}(u) \text{ and } K'(\cdot,u) = \sum_{j=1}^t \overline{p_j(u)} {F'}^j_{\Omega_{w_0}}(u), \tag{3.1}$$

for all $u\in\Omega_{w_0} = \Delta(w_0,r)$. Now, applying $L$ to the first equality of 3.1 we obtain that

$$\overline{\phi(u)}K'(\cdot,u) = \sum_{j=1}^t \overline{p_j(u)} LF^j_{\Omega_{w_0}}(u)$$

which implies

$$K'(\cdot,u) = \sum_{j=1}^t \overline{p_j(u)} \frac{LF^j_{\Omega_{w_0}}(u)}{\overline{\phi(u)}}.$$

Note that $v\mapsto \frac{LF_j(v)}{\overline{\phi(v)}}$ and $v\mapsto F'_j(v)$, $j=1,\ldots,t$ are anti-holomorphic maps from $V(\mathscr{M})$ to $\mathscr{M}$ satisfying 1.a) and 1.b) of Theorem 2.1.4. So, by condition 2 of Theorem 2.1.4, we have $\frac{LF_j(v)}{\overline{\phi(v)}} = F'_j(v)$, or equivalently, $LF_j(v) = \overline{\phi(v)}F'_j(v)$, for all $j=1,\ldots,t$. As a result, $Ls_1(v,\pi(\Theta)) = \overline{\phi(v)}s'_1(v,\pi(\Theta))$ and $L\big([F^j(v)]\big) = \overline{\phi(v)}[F'_j(v)]$, for all $j=2,\ldots,t$. Finally, following similar arguments as given in Theorem 3.1.1, it can be shown that $L$ induces isometric isomorphisms between each pair of bundles. □

Consequently, we obtain the following Corollary generalizing Corollary 3.1.2.

**Corollary 3.1.5.** *Under the hypotheses of Theorem* 3.1.4*, we have*

i) $c(\tilde{E}_1, \tilde{N}_1) = c(\tilde{E}_1^{'}, \tilde{N}_1^{'})$

ii) $c(\tilde{E}, \tilde{N}) = c(\tilde{E}^{'}, \tilde{N}^{'})$

iii) $c(\tilde{E}_2, \tilde{N}_2) = c(\tilde{E}_2^{'}, \tilde{N}_2^{'})$,

*where* $\tilde{N}_1$*,* $\tilde{N}_1^{'}$*,* $\tilde{N}$*,* $\tilde{N}^{'}$*,* $\tilde{N}_2$ *and* $\tilde{N}_2^{'}$ *are the Hermitian norms on their respective bundles.*

*Proof.* Follows trivially from the equivalence of bundles proved in Theorem 3.1.4. □

**Remark 3.1.6.** If we fix an arbitrary point $p \in \mathbb{P}^{t-1}$ and restrict the bundles in Theorem 3.1.4 on $Z \times \{p\}$, we will obtain two exact sequences of Hermitian, anti-holomorphic bundles on $Z$. Consequently, the unitary module map $L : \mathcal{M} \to \mathcal{M}'$ makes the restrictions of $\tilde{E}_1, \tilde{E}_1^{'}$; $\tilde{E}, \tilde{E}^{'}$ and $\tilde{E}_2, \tilde{E}_2^{'}$ on $Z \times \{p\}$ equivalent.

## 3.2 The kernel decomposition formula

**Lemma 3.2.1.** *Let* $\mathcal{H}$ *be a Hilbert space. For any two closed subspaces* $\mathcal{M}_1, \mathcal{M}_2$ *of* $\mathcal{H}$*, the following are equivalent:*

(a) $\mathcal{M}_1 \ominus \mathcal{M}_{12} = \mathcal{M}_1 \ominus \mathcal{M}_2 := \{f \in \mathcal{M}_1 : \langle f, g \rangle_{\mathcal{H}} = 0 \; \forall g \in \mathcal{M}_2\}$, *where* $\mathcal{M}_{12} = \mathcal{M}_1 \cap \mathcal{M}_2$*;*

(b) $\mathcal{M}_1 \ominus \mathcal{M}_{12} \perp \mathcal{M}_2 \ominus \mathcal{M}_{12}$*;*

(c) $\overline{\mathcal{M}_1 + \mathcal{M}_2} \ominus \mathcal{M}_1 = \mathcal{M}_2 \ominus \mathcal{M}_{12}$.

*Proof.* (a)⇔(b) Let $f \in \mathcal{M}_1 \ominus \mathcal{M}_{12} = \mathcal{M}_1 \ominus \mathcal{M}_2$ be an arbitrary element. Then $\langle f, g \rangle_{\mathcal{H}} = 0$ for all $g \in \mathcal{M}_2$ and hence for all $g \in \mathcal{M}_2 \ominus \mathcal{M}_{12}$. Conversely, assume that $\mathcal{M}_1 \ominus \mathcal{M}_{12} \perp \mathcal{M}_2 \ominus \mathcal{M}_{12}$. Then for any $f \in \mathcal{M}_1 \ominus \mathcal{M}_{12}, g \in \mathcal{M}_2$,

$$\langle f, g \rangle_{\mathcal{H}} = \langle f, P_{\mathcal{M}_2 \ominus \mathcal{M}_{12}} g + P_{\mathcal{M}_{12}} g \rangle_{\mathcal{H}} = \langle f, P_{\mathcal{M}_2 \ominus \mathcal{M}_{12}} g \rangle_{\mathcal{H}} + \langle f, P_{\mathcal{M}_{12}} g \rangle_{\mathcal{H}} = 0.$$

So, $\mathcal{M}_1 \ominus \mathcal{M}_{12} \subseteq \mathcal{M}_1 \ominus \mathcal{M}_2$. On the other hand, from the description of $\mathcal{M}_1 \ominus \mathcal{M}_2$ it clearly follows that $\mathcal{M}_1 \ominus \mathcal{M}_2 \subseteq \mathcal{M}_1 \ominus \mathcal{M}_{12}$.

(b)⇔(c) From (c) we clearly have $\mathcal{M}_1 \perp \mathcal{M}_2 \ominus \mathcal{M}_{12}$. Since $\mathcal{M}_1 \ominus \mathcal{M}_{12} \subseteq \mathcal{M}_1$, part (b) follows. For the converse part, firstly, note that part (b) implies $\mathcal{M}_1 \perp \mathcal{M}_2 \ominus \mathcal{M}_{12}$. As a result, $\mathcal{M}_1 + (\mathcal{M}_2 \ominus \mathcal{M}_{12})$ is a closed subspace of $\mathcal{H}$. Also, observe that $\mathcal{M}_1 + \mathcal{M}_2 \subseteq \mathcal{M}_1 + (\mathcal{M}_2 \ominus \mathcal{M}_{12})$. This is because for each $f \in \mathcal{M}_1 + \mathcal{M}_2$, there exist $f_1 \in \mathcal{M}_1$, $f_2 \in \mathcal{M}_2$ such that $f = f_1 + f_2$ and

$$f_1 + f_2 = (f_1 + P_{\mathcal{M}_{12}} f_2) + P_{\mathcal{M}_2 \ominus \mathcal{M}_{12}} f_2 \in \mathcal{M}_1 + (\mathcal{M}_2 \ominus \mathcal{M}_{12}).$$

Now, for $f \in \overline{\mathcal{M}_1 + \mathcal{M}_2}$, there exists a sequence $f_n \in \mathcal{M}_1 + \mathcal{M}_2$ such that $f_n$ converges to $f$ in $\mathcal{H}$. But $f_n \in \mathcal{M}_1 + (\mathcal{M}_2 \ominus \mathcal{M}_{12})$ for each $n$ and $\mathcal{M}_1 + (\mathcal{M}_2 \ominus \mathcal{M}_{12})$ is closed. Consequently, $\overline{\mathcal{M}_1 + \mathcal{M}_2} \subseteq \mathcal{M}_1 + (\mathcal{M}_2 \ominus \mathcal{M}_{12})$. On the other hand, $\mathcal{M}_1 + (\mathcal{M}_2 \ominus \mathcal{M}_{12})$ is clearly a subset of $\mathcal{M}_1 + \mathcal{M}_2 \subseteq \overline{\mathcal{M}_1 + \mathcal{M}_2}$. As a result, $\overline{\mathcal{M}_1 + \mathcal{M}_2} = \mathcal{M}_1 + (\mathcal{M}_2 \ominus \mathcal{M}_{12})$. Since $\mathcal{M}_1 \perp \mathcal{M}_2 \ominus \mathcal{M}_{12}$, this is equivalent to $\overline{\mathcal{M}_1 + \mathcal{M}_2} \ominus \mathcal{M}_1 = \mathcal{M}_2 \ominus \mathcal{M}_{12}$. □

As a corollary, we have the following proposition.

**Proposition 3.2.2.** *Let $\Omega$ be a bounded domain in $\mathbb{C}^m$ and $\mathcal{H}$ be a "Reproducing Kernel Hilbert Space (RKHS)" consisting of complex valued holomorphic functions defined on $\Omega$. For $i = 1,2$, define $\mathcal{M}_i = \{f \in \mathcal{H} : f = 0 \text{ on } \mathcal{Z}_i\}$, where $\mathcal{Z}_i$ is a subset of $\Omega$, not necessarily open. Then the following are equivalent:*

(a) $\mathcal{M}_1 \ominus \mathcal{M}_{12} = \mathcal{M}_1 \ominus \mathcal{M}_2$;

(b) $\mathcal{M}_1 \ominus \mathcal{M}_{12} \perp \mathcal{M}_2 \ominus \mathcal{M}_{12}$;

(c) $\overline{\mathcal{M}_1 + \mathcal{M}_2} \ominus \mathcal{M}_1 = \mathcal{M}_2 \ominus \mathcal{M}_{12}$;

(d) $K_{\overline{\mathcal{M}_1+\mathcal{M}_2}}(w_1, w_2) = K_{\overline{\mathcal{M}_1+\mathcal{M}_2}}(w_2, w_1) = 0$, *where* $w_1 \in \mathcal{Z}_1$, $w_2 \in \mathcal{Z}_2$ *and* $K_{\overline{\mathcal{M}_1+\mathcal{M}_2}}$ *is the reproducing kernel of* $\overline{\mathcal{M}_1 + \mathcal{M}_2}$.

*Proof.* Proof of (a)⇔(b), (b)⇔(c) has already been done in the lemma above. Here we will show (b)⇔(d). To prove the forward part, observe that (b) implies $\overline{\mathcal{M}_1 + \mathcal{M}_2} \ominus \mathcal{M}_1 = \mathcal{M}_2 \ominus \mathcal{M}_{12}$ (similarly, $\overline{\mathcal{M}_1 + \mathcal{M}_2} \ominus \mathcal{M}_2 = \mathcal{M}_1 \ominus \mathcal{M}_{12}$). So, $\overline{\mathcal{M}_1 + \mathcal{M}_2} \ominus \mathcal{M}_1 \perp \overline{\mathcal{M}_1 + \mathcal{M}_2} \ominus \mathcal{M}_2$. Next, for $i = 1,2$, note that $\overline{\mathcal{M}_1 + \mathcal{M}_2} \ominus \mathcal{M}_i = \bigvee\{K_{\overline{\mathcal{M}_1+\mathcal{M}_2}}(\cdot, w_i) : w_i \in \mathcal{Z}_i\}$. From the line above, this means $K_{\overline{\mathcal{M}_1+\mathcal{M}_2}}(w_1, w_2) = K_{\overline{\mathcal{M}_1+\mathcal{M}_2}}(w_2, w_1) = 0$, $w_1 \in \mathcal{Z}_1$, $w_2 \in \mathcal{Z}_2$.

From the description of $\overline{\mathcal{M}_1 + \mathcal{M}_2} \ominus \mathcal{M}_i$, $i = 1,2$, it is clear that (d) implies $\overline{\mathcal{M}_1 + \mathcal{M}_2} \ominus \mathcal{M}_1$ and $\overline{\mathcal{M}_1 + \mathcal{M}_2} \ominus \mathcal{M}_2$ are pairwise orthogonal. Now, $\overline{\mathcal{M}_1 + \mathcal{M}_2} = \mathcal{M}_1 + (\overline{\mathcal{M}_1 + \mathcal{M}_2} \ominus \mathcal{M}_1)$. This means, $\overline{\mathcal{M}_1 + \mathcal{M}_2} \ominus \mathcal{M}_2 \subseteq \mathcal{M}_1$ (similarly, $\overline{\mathcal{M}_1 + \mathcal{M}_2} \ominus \mathcal{M}_1 \subseteq \mathcal{M}_2$). So, for $w_1 \in \mathcal{Z}_1$, $w_2 \in \mathcal{Z}_2$,

$$K_{\overline{\mathcal{M}_1+\mathcal{M}_2}}(\cdot, w_1) = P_{\mathcal{M}_2} K_{\overline{\mathcal{M}_1+\mathcal{M}_2}}(\cdot, w_1) = K_{\mathcal{M}_2}(\cdot, w_1)$$

and

$$K_{\overline{\mathcal{M}_1+\mathcal{M}_2}}(\cdot, w_2) = P_{\mathcal{M}_1} K_{\overline{\mathcal{M}_1+\mathcal{M}_2}}(\cdot, w_2) = K_{\mathcal{M}_1}(\cdot, w_2).$$

But $\mathcal{M}_2 \ominus \mathcal{M}_{12} = \bigvee\{K_{\mathcal{M}_2}(\cdot, w_1) : w_1 \in (\mathcal{Z}_1 \cup \mathcal{Z}_2) - \mathcal{Z}_2\} = \bigvee\{K_{\mathcal{M}_2}(\cdot, w_1) : w_1 \in \mathcal{Z}_1\}$. The second equality is true because for each $w_2 \in \mathcal{Z}_2$, $K_{\mathcal{M}_2}(\cdot, w_2) \in \mathcal{M}_2 \subseteq \mathcal{H}$. Consequently, $K_{\mathcal{M}_2}(w_2, w_2) = \|K_{\mathcal{M}_2}(\cdot, w_2)\|^2_{\mathcal{H}} = 0$ which implies $K_{\mathcal{M}_2}(\cdot, w_2) = 0$. Similarly, $\mathcal{M}_1 \ominus \mathcal{M}_{12} = \bigvee\{K_{\mathcal{M}_1}(\cdot, w_2) : w_2 \in \mathcal{Z}_2\}$. As a result, for $w_i \in \mathcal{Z}_i$, $i = 1,2$,

$$\left\langle K_{\mathcal{M}_1}(\cdot, w_2), K_{\mathcal{M}_2}(\cdot, w_1)\right\rangle_{\mathcal{H}} = \left\langle K_{\overline{\mathcal{M}_1+\mathcal{M}_2}}(\cdot, w_2), K_{\overline{\mathcal{M}_1+\mathcal{M}_2}}(\cdot, w_1)\right\rangle_{\mathcal{H}} = K_{\overline{\mathcal{M}_1+\mathcal{M}_2}}(w_1, w_2) = 0$$

which proves part (b). □

If any one of the conditions of the previous proposition is true, we obtain a decomposition formula for the reproducing kernel of $\overline{\mathcal{M}_1+\mathcal{M}_2}$ given by

$$K_{\overline{\mathcal{M}_1+\mathcal{M}_2}}(z,w)=K_{\mathcal{M}_1}(z,w)+K_{\mathcal{M}_2}(z,w)-K_{\mathcal{M}_{12}}(z,w). \tag{3.2}$$

More generally, if we assume $\Omega,\mathcal{H}$ as above, $\mathcal{M}_i=\{f\in\mathcal{H}: f=0 \text{ on } \mathcal{Z}_i\}$ for $i=1,\ldots,n$, then under suitable conditions we obtain the following

$$\begin{aligned}K(z,w)=\sum_{i=1}^{n}K_i(z,w)-\sum_{i<j,i,j=1}^{n}K_{ij}(z,w)+\cdots\\ +(-1)^{k-1}\sum_{i_1<\ldots<i_k,i_1,\ldots,i_k=1}^{n}K_{i_1\ldots i_k}(z,w)+\cdots+(-1)^{n-1}K_{1\ldots n}(z,w),\end{aligned} \tag{3.3}$$

where $K$ is the reproducing kernel of $\mathcal{M}:=\overline{\mathcal{M}_1+\cdots+\mathcal{M}_n}$ and for $i_1,\ldots,i_k\in\{1,\ldots,n\}$, $K_{i_1\ldots i_k}$ is the reproducing kernel of $\mathcal{M}_{i_1}\cap\cdots\cap\mathcal{M}_{i_k}$. In what follows, we will describe a set of sufficient conditions for the above equality to be true when $n=3$. Conditions for $n\geq 4$ can be obtained similarly.

When $n=3$, $\mathcal{M}=\overline{\mathcal{M}_1+\mathcal{M}_2+\mathcal{M}_3}=\overline{\mathcal{M}_1+(\mathcal{M}_2+\mathcal{M}_3)}$. Now, if we assume that $\overline{\mathcal{M}_2+\mathcal{M}_3}=\{f\in\mathcal{H}: f=0 \text{ on } \mathcal{Z}_2\cap\mathcal{Z}_3\}$ and $K(z,w)=0$, for all $z\in\mathcal{Z}_1, w\in\mathcal{Z}_2\cap\mathcal{Z}_3$, then from Equation (3.2) it follows that

$$K(z,w)=K_{\mathcal{M}_1}(z,w)+K_{\overline{\mathcal{M}_2+\mathcal{M}_3}}(z,w)-K_{\mathcal{M}_1\cap\overline{\mathcal{M}_2+\mathcal{M}_3}}(z,w). \tag{3.4}$$

Observe that $\overline{\mathcal{M}_{12}+\mathcal{M}_{13}}\subseteq\mathcal{M}_1\cap\overline{\mathcal{M}_2+\mathcal{M}_3}$, where $\mathcal{M}_{12},\mathcal{M}_{13}$ represents $\mathcal{M}_1\cap\mathcal{M}_2$, $\mathcal{M}_1\cap\mathcal{M}_3$ respectively. If we assume that these spaces are equal, then Equation (3.4) becomes

$$K(z,w)=K_{\mathcal{M}_1}(z,w)+K_{\overline{\mathcal{M}_2+\mathcal{M}_3}}(z,w)-K_{\overline{\mathcal{M}_{12}+\mathcal{M}_{13}}}(z,w).$$

Finally, if $K_{\overline{\mathcal{M}_2+\mathcal{M}_3}}(z,w)=0$, for all $z\in\mathcal{Z}_2, w\in\mathcal{Z}_3$ and $K_{\overline{\mathcal{M}_{12}+\mathcal{M}_{13}}}(z,w)=0$, for all $z\in\mathcal{Z}_1\cup\mathcal{Z}_2, w\in\mathcal{Z}_1\cup\mathcal{Z}_3$, then repeated application of Equation (3.2) gives

$$\begin{aligned}K(z,w)&=K_{\mathcal{M}_1}(z,w)+K_{\overline{\mathcal{M}_2+\mathcal{M}_3}}(z,w)-K_{\overline{\mathcal{M}_{12}+\mathcal{M}_{13}}}(z,w)\\&=K_1(z,w)+\big(K_2(z,w)+K_3(z,w)-K_{23}(z,w)\big)-\big(K_{12}(z,w)+K_{13}(z,w)-K_{123}(z,w)\big)\\&=K_1(z,w)+K_2(z,w)+K_3(z,w)-K_{12}(z,w)-K_{13}(z,w)-K_{23}(z,w)+K_{123}(z,w).\end{aligned}$$

## 3.3 Some examples

Now, we will consider the following examples.

**Example 3.3.1.** For $\lambda=(\lambda_1,\ldots,\lambda_m)\in\mathbb{R}_+^m$, $m\geq 3$, let $\mathcal{H}=H^\lambda(\mathbb{D}^m)$, where

$$H^\lambda(\mathbb{D}^m):=\Big\{f\in\mathcal{O}(\mathbb{D}^m): \text{if } f=\sum_\alpha f(\alpha)z^\alpha, \text{ then } \sum_\alpha\frac{|f(\alpha)|^2}{\binom{\lambda}{\alpha}}<\infty\Big\}$$

with the inner product $\langle f,g\rangle_{\mathscr{H}} = \sum_\alpha \frac{f(\alpha)\overline{g(\alpha)}}{\binom{\lambda}{\alpha}}$, $\alpha = (\alpha_1,\dots,\alpha_m) \in \big(\mathbb{N}\cup\{0\}\big)^m$, $\binom{\lambda}{\alpha} = \binom{\lambda_1}{\alpha_1}\cdots\binom{\lambda_m}{\alpha_m}$ and for each $k$, $1 \le k \le m$,

$$\begin{pmatrix}\lambda_k\\ \alpha_k\end{pmatrix} = \begin{cases} \frac{\lambda_k(\lambda_k+1)\cdots(\lambda_k+\alpha_k-1)}{(\alpha_k)!} & \text{when } \alpha_k \ge 1\\ 1 & \text{when } \alpha_k = 0. \end{cases}$$

Also, for each $i = 1,2,3$, let $\mathscr{M}_i := \{f \in \mathscr{H} : f = 0 \text{ on } z_i = 0\}$ and $\mathscr{M} := \{f \in \mathscr{H} : f = 0 \text{ on } z_1 = z_2 = z_3 = 0\}$. Then it can be checked that $\mathscr{M}_i, \mathscr{M}$ are the closure of the polynomial ideals in $\mathscr{H}$ generated by $\{z_i\}, \{z_1,z_2,z_3\}$ respectively. As a result, $\mathscr{M} = \overline{\mathscr{M}_1+\mathscr{M}_2+\mathscr{M}_3}$. Moreover,

$$\overline{\mathscr{M}_2+\mathscr{M}_3} = \overline{<\{z_2,z_3\}>} = \{f \in \mathscr{H} : f = 0 \text{ on } z_2 = z_3 = 0\}.$$

Consider an orthonormal basis $\big\{e_\alpha : \alpha \in \big(\mathbb{N}\cup\{0\}\big)^m\big\}$ of $\mathscr{H}$, where $e_\alpha(z) = \sqrt{\binom{\lambda}{\alpha}}z^\alpha$. Then an orthonormal basis of $\mathscr{M}$ is $\big\{e_\alpha : \alpha_1+\alpha_2+\alpha_3 \ge 1, \alpha \in \big(\mathbb{N}\cup\{0\}\big)^m\big\}$. So, we have

$$K(z,w) = \sum_{\alpha:\alpha_1+\alpha_2+\alpha_3\ge 1} e_\alpha(z)\overline{e_\alpha(w)} = \sum_{\alpha:\alpha_1+\alpha_2+\alpha_3\ge 1} \begin{pmatrix}\lambda\\ \alpha\end{pmatrix} z^\alpha \bar{w}^\alpha,$$

for all $z,w \in \mathbb{D}^m$. If $\alpha_1 \ge 1$, then $K(z,w) = 0$ for any $(z,w)$ with $z_1 = 0$. On the other hand, if $\alpha_1 = 0$ and $\alpha_2+\alpha_3 \ge 1$, then $K(z,w) = 0$ for $(z,w) \in \{w_2 = w_3 = 0\}$. Thus, $K(z,w) = 0$ for any $(z,w)$ where $z_1 = 0$ and $w_2 = w_3 = 0$. Consequently, $K$ satisfies Equation (3.4). Now, observe that for any $i, j \in \{1,2,3\}$,

$$\mathscr{M}_{ij} = \mathscr{M}_i \cap \mathscr{M}_j = \{f \in \mathscr{H} : f = 0 \text{ on } z_i z_j = 0\} = \overline{<\{z_i z_j\}>}.$$

As a result, it can be checked that

$$\overline{\mathscr{M}_{12}+\mathscr{M}_{13}} = \overline{<\{z_1z_2, z_1z_3\}>} = \{f \in \mathscr{H} : f = 0 \text{ on } z_1z_2 = z_1z_3 = 0\}.$$

On the other hand, it is clear that $\mathscr{M}_1 \cap \overline{\mathscr{M}_2+\mathscr{M}_3} = \{f \in \mathscr{H} : f = 0 \text{ on } z_1z_2 = z_1z_3 = 0\}$ which gives $\mathscr{M}_1 \cap \overline{\mathscr{M}_2+\mathscr{M}_3} = \overline{\mathscr{M}_{12}+\mathscr{M}_{13}}$. Finally, observe that $\big\{e_\alpha : \alpha_2+\alpha_3 \ge 1, \alpha \in \big(\mathbb{N}\cup\{0\}\big)^m\big\}$ and $\big\{e_\alpha : \alpha_1 \ge 1, \alpha_2+\alpha_3 \ge 1, \alpha \in \big(\mathbb{N}\cup\{0\}\big)^m\big\}$ are orthonormal bases of $\overline{\mathscr{M}_2+\mathscr{M}_3}$ and $\overline{\mathscr{M}_{12}+\mathscr{M}_{13}}$ respectively. So,

$$K_{\overline{\mathscr{M}_2+\mathscr{M}_3}}(z,w) = \sum_{\alpha:\alpha_2+\alpha_3\ge 1} \begin{pmatrix}\lambda\\ \alpha\end{pmatrix} z^\alpha \bar{w}^\alpha \quad \text{and} \quad K_{\overline{\mathscr{M}_{12}+\mathscr{M}_{13}}}(z,w) = \sum_{\alpha:\alpha\ge 1,\alpha_2+\alpha_3\ge 1} \begin{pmatrix}\lambda\\ \alpha\end{pmatrix} z^\alpha \bar{w}^\alpha,$$

for all $z,w \in \mathbb{D}^m$. As a result, $K_{\overline{\mathscr{M}_2+\mathscr{M}_3}}(z,w) = 0$, for all $(z,w)$ such that $z \in \{z_2 = 0\}, w \in \{w_3 = 0\}$ and $K_{\overline{\mathscr{M}_{12}+\mathscr{M}_{13}}}(z,w) = 0$, for all $(z,w)$ such that $z \in \{z_1z_2 = 0\}, w \in \{w_1w_3 = 0\}$. From the previous discussion, this gives

$$K(z,w) = K_1(z,w)+K_2(z,w)+K_3(z,w)-K_{12}(z,w)-K_{13}(z,w)-K_{23}(z,w)+K_{123}(z,w).$$

In general, following similar arguments as above, it can also be proved that when $n \le m$, $i_1,\dots,i_n \in \{1,\dots,m\}$ and $\mathscr{M} = \{f \in H^\lambda(\mathbb{D}^m) : f = 0 \text{ on } z_{i_1} = \cdots = z_{i_n} = 0\}$, the reproducing kernel $K$ of $\mathscr{M}$ satisfies the decomposition formula given by Equation (3.3).

**Example 3.3.2.** Let $\mathscr{H} = H^{\lambda}(\mathbb{D}^3)$ and for $\epsilon > 0$, consider the submodules $\mathscr{M}_1, \mathscr{M}_2$ of $\mathscr{H}$, where

$$\mathscr{M}_1 = \{f \in \mathscr{H} : f = 0 \text{ on } z_1 - \epsilon z_2 = 0\}$$

and

$$\mathscr{M}_2 = \{f \in \mathscr{H} : f = 0 \text{ on } z_3 = 0\}.$$

Now, suppose $f(z_1, z_2, z_3) = \sum_{i,j,k \in \mathbb{N}\cup\{0\}} \hat{f}(i,j,k) z_1^i z_2^j z_3^k$, for all $z_1, z_2, z_3 \in \mathbb{D}$ and assume that $f \in \mathscr{M}_1$. Then we have

$$f(\epsilon z_2, z_2, z_3) = \sum_{i,j,k \geq 0} \epsilon^i \hat{f}(i,j,k) z_2^{i+j} z_3^k = 0.$$

As a result,

$$\begin{aligned}
f(z_1, z_2, z_3) &= \sum_{j,k \geq 0} \hat{f}(0,j,k) z_2^j z_3^k + \sum_{i \geq 1, j,k \geq 0} \hat{f}(i,j,k)\big(\epsilon z_2 + (z_1 - \epsilon z_2)\big)^i z_2^j z_3^k \\
&= \sum_{j,k \geq 0} \hat{f}(0,j,k) z_2^j z_3^k + \sum_{i \geq 1, j,k \geq 0} \hat{f}(i,j,k)\Big(\epsilon^i z_2^i + i\epsilon^{i-1} z_2^{i-1}(z_1 - \epsilon z_2) + \cdots + (z_1 - \epsilon z_2)^i\Big) z_2^j z_3^k \\
&= f(\epsilon z_2, z_2, z_3) + \sum_{i \geq 1, j,k \geq 0} \hat{f}(i,j,k)\Big(i\epsilon^{i-1} z_2^{i-1}(z_1 - \epsilon z_2) + \cdots + (z_1 - \epsilon z_2)^i\Big) z_2^j z_3^k \\
&= \sum_{n=1}^{\infty} \sum_{i \geq 1, i+j+k=n} \hat{f}(i,j,k)\Big(i\epsilon^{i-1} z_2^{i-1}(z_1 - \epsilon z_2) + \cdots + (z_1 - \epsilon z_2)^i\Big) z_2^j z_3^k \\
&= \sum_{n=1}^{\infty} p_n(z_1, z_2, z_3).
\end{aligned}$$

Clearly, for each $n \geq 1$, $p_n$ is a homogeneous polynomial of degree $n$ and $z_1 - \epsilon z_2$ divides $p_n$. Also, from degree arguments it follows that $p_n(z_1, z_2, z_3) = \sum_{i+j+k=n} \hat{f}(i,j,k) z_1^i z_2^j z_3^k$. Thus, if we define $s_l = \sum_{n=1}^{l} p_n$, for $l = 1, 2, \ldots$, then $s_l$ converges to $f$ in $\mathscr{H}$ proving $\mathscr{M}_1$ to be the closure of the principal ideal generated by $z_1 - \epsilon z_2$ in $\mathscr{H}$. Similarly, one can show that $\mathscr{M}_2$ is the closure of the ideal generated by $z_3$.

Next, consider the submodule $\mathscr{M}$ of $\mathscr{H}$ which is the closure of the polynomial ideal generated by $z_1 - \epsilon z_2$ and $z_3$. Then it can be easily checked that $\mathscr{M} = \overline{\mathscr{M}_1 + \mathscr{M}_2}$. We will show that the reproducing kernel $K$ of $\mathscr{M}$ satisfies Equation (3.3).

Choose an arbitrary element $f$ in $\mathscr{M}^{\perp}$. Since $f \perp \mathscr{M}_2$, we have $f(z_1, z_2, z_3) = \sum_{i,j \geq 0} \hat{f}(i,j,0) z_1^i z_2^j$. Consequently, $\mathscr{M}^{\perp}$ can be considered as a subspace of $H^{(\lambda_1,\lambda_2)}(\mathbb{D}^2) \ominus < \{z_1 - \epsilon z_2\} >$. Now, following the calculations done in [14, Section 2.1] we obtain that $H^{(\lambda_1,\lambda_2)}(\mathbb{D}^2) \ominus < \{z_1 - \epsilon z_2\} >$ is the closed linear span of the set $\{e_k : k \in \mathbb{N} \cup \{0\}\}$, where, for $z_1, z_2 \in \mathbb{D}$,

$$e_k(z_1, z_2) = \sum_{i+j=n, i,j \in \mathbb{N}\cup\{0\}} \frac{z_1^i z_2^j}{\binom{\lambda_1}{i}\binom{\lambda_2}{j}}.$$

Define $\varepsilon_k : \mathbb{D}^3 \to \mathbb{C}$ as $\varepsilon_k(z_1, z_2, z_3) = e_k(z_1, z_2)$. Clearly, $\mathcal{M}^\perp = \bigvee\{\varepsilon_k : k \in \mathbb{N} \cup \{0\}\}$. As a result, we have

$$\begin{aligned}
K(z,w) &= K_{\mathcal{H}}(z,w) - K_{\mathcal{M}^\perp}(z,w)\\
&= \left( \sum_{i,j\geq 0} \frac{z_1^i z_2^j \bar{w}_1^i \bar{w}_2^j}{\binom{\lambda_1}{i}\binom{\lambda_2}{j}} + \sum_{i,j\geq 0, k\geq 1} \frac{z_1^i z_2^j z_3^k \bar{w}_1^i \bar{w}_2^j \bar{w}_3^k}{\binom{\lambda_1}{i}\binom{\lambda_2}{j}\binom{\lambda_3}{k}} \right) - K_{\left(H^{(\lambda_1,\lambda_2)}(\mathbb{D}^2)\,\ominus\,<\{z_1-\epsilon z_2\}>\right)}(z,w)\\
&= \left( K_{H^{(\lambda_1,\lambda_2)}(\mathbb{D}^2)}(z,w) + z_3\bar{w}_3 \sum_{i,j,k\geq 0} \frac{z_1^i z_2^j z_3^k \bar{w}_1^i \bar{w}_2^j \bar{w}_3^k}{\binom{\lambda_1}{i}\binom{\lambda_2}{j}\binom{\lambda_3}{k+1}} \right) - K_{\left(H^{(\lambda_1,\lambda_2)}(\mathbb{D}^2)\,\ominus\,<\{z_1-\epsilon z_2\}>\right)}(z,w)\\
&= K_{(<\{z_1-\epsilon z_2\}>)}(z,w) + z_3\bar{w}_3\chi_3(z,w)\\
&= (z_1-\epsilon z_2)(\bar{w}_1 - \epsilon\bar{w}_2)\chi_{(<\{z_1-\epsilon z_2\}>)}(z,w) + z_3\bar{w}_3\chi_3(z,w),
\end{aligned}$$

where $K_{\mathcal{N}}$ denotes the reproducing kernel of the Hilbert space $\mathcal{N}$. For $z, w \in \mathbb{D}^3$, if $z_1 - \epsilon z_2 = 0$ and $w_3 = 0$, then, following previous equalities, it is clear that $K(z,w) = 0$. But according to Proposition 3.2.2, this is one of the sufficient conditions for the decomposition formula given by Equation (3.2) which proves our claim.

So far we have assumed that $\mathcal{M}_i$ is the maximal set of functions vanishing on a closed subset $\mathcal{Z}_i \subseteq \Omega$, for all $i = 1, \ldots, n$. In the following class of examples we will show that the decomposition formula is valid even when $\mathcal{M}_i$ is assumed to be the closure of a principal ideal in $\mathcal{H}$ which is not necessarily maximal.

**Example 3.3.3.** Let $\mathcal{H} = H^\lambda(\mathbb{D}^m)$ and for $i = 1,2,3$, consider the subspace $\mathcal{M}_i$ of $\mathcal{H}$ which is the closure of the principal ideal generated by $z_i^{r_i}$, where $m \geq 3$ and $r_1, r_2, r_3 \in \mathbb{N}$. Then we have

$$\mathcal{M}_i = \bigvee\{e_\alpha : \alpha_i \geq r_i, \alpha_j \geq 0 \text{ for } j \neq i\},$$

where $\alpha = (\alpha_1, \ldots, \alpha_m) \in \left(\mathbb{N} \cup \{0\}\right)^m$ is a multi-index and $e_\alpha(z) = \binom{\lambda}{\alpha} z^\alpha$, for all $z \in \mathbb{D}^m$. Next, consider the subspace which is of the form $\overline{<\{z_1^{r_1}, z_2^{r_2}, z_2^{r_2}\}>}$(closure in $\mathcal{H}$). Then $\mathcal{M} = \overline{\mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3}$. Similarly, one can check that $\overline{\mathcal{M}_2 + \mathcal{M}_3} = \overline{<\{z_2^{r_2}, z_3^{r_3}\}>}$. As a result,

$$\overline{\mathcal{M}_2 + \mathcal{M}_3} = \bigvee\{e_\alpha : \text{either } \alpha_2 \in \{0, \ldots, r_2 - 1\}, \alpha_3 \geq r_3 \text{ or } \alpha_2 \geq r_2, \alpha_3 \geq 0 \text{ and } \alpha_j \geq 0 \text{ for } j \notin \{2,3\}\}$$

which gives

$$\begin{aligned}
&\mathcal{M}_1 \cap \overline{\mathcal{M}_2 + \mathcal{M}_3}\\
&= \bigvee\{e_\alpha : \alpha_1 \geq r_1, \text{either } \alpha_2 \in \{0, \ldots, r_2-1\}, \alpha_3 \geq r_3 \text{ or } \alpha_2 \geq r_2, \alpha_3 \geq 0 \text{ and } \alpha_j \geq 0 \text{ for } j \notin \{1,2,3\}\}.
\end{aligned}$$

Thus, $\mathcal{M}_1 \cap \overline{\mathcal{M}_2 + \mathcal{M}_3} = \overline{<\{z_1^{r_1}, z_2^{r_2}, z_2^{r_2}\}>}$. On the other hand,

$$\mathcal{M}_{12} := \mathcal{M}_1 \cap \mathcal{M}_2 = \bigvee\{e_\alpha : \alpha_1 \geq r_1, \alpha_2 \geq r_2, \alpha_j \geq 0 \text{ for } j \notin \{1,2\}\} = \overline{<\{z_1^{r_1} z_2^{r_2}\}>}$$

and similarly, $\mathcal{M}_{13} = \overline{<\{z_1^{r_1} z_3^{r_3}\}>}$. So, $\overline{\mathcal{M}_{12} + \mathcal{M}_{13}} = \overline{<\{z_1^{r_1} z_2^{r_2}, z_1^{r_1} z_3^{r_3}\}>}$ which proves that $\mathcal{M}_1 \cap \overline{\mathcal{M}_2 + \mathcal{M}_3} = \overline{\mathcal{M}_{12} + \mathcal{M}_{13}}$. Now, observe the following:

1. $\mathcal{M}_1 \ominus (\mathcal{M}_{12} + \mathcal{M}_{13}) = \bigvee\{e_\alpha : \alpha_1 \geq r_1, \alpha_2 \in \{0,\dots,r_2-1\}, \alpha_3 \in \{0,\dots,r_3-1\}, \alpha_j \geq 0 \text{ for } j \notin \{1,2,3\}\}$ which is clearly orthogonal to $\overline{\mathcal{M}_2 + \mathcal{M}_3}$;

2. $\mathcal{M}_2 \ominus \mathcal{M}_{23} = \bigvee\{e_\alpha : \alpha_2 \geq r_2, \alpha_3 \in \{0,\dots,r_3-1\}, \alpha_j \geq 0 \text{ for } j \notin \{2,3\}\}$ and hence is orthogonal to $\mathcal{M}_3$;

3. $\mathcal{M}_{12} \cap \mathcal{M}_{13} = \mathcal{M}_1 \cap \mathcal{M}_2 \cap \mathcal{M}_3 = \bigvee\{e_\alpha : \alpha_1 \geq r_1, \alpha_2 \geq r_2, \alpha_3 \geq r_3, \alpha_j \geq 0 \text{ for } j \notin \{1,2,3\}\}$;

4. If we denote $\mathcal{M}_1 \cap \mathcal{M}_2 \cap \mathcal{M}_3$ as $\mathcal{M}_{123}$, then $\mathcal{M}_{12} \ominus \mathcal{M}_{123} = \bigvee\{e_\alpha : \alpha_1 \geq r_1, \alpha_2 \geq r_2, \alpha_3 \in \{0,\dots,r_3-1\}, \alpha_j \geq 0 \text{ for } j \notin \{1,2,3\}\}$ and is orthogonal to $\mathcal{M}_{13}$.

(1) implies that $\big(\mathcal{M}_1 \ominus (\mathcal{M}_{12} + \mathcal{M}_{13})\big) \perp \big(\overline{\mathcal{M}_2 + \mathcal{M}_3} \ominus (\mathcal{M}_{12} + \mathcal{M}_{13})\big)$. From Lemma 3.2.1 this is equivalent to the fact that $\mathcal{M} \ominus \mathcal{M}_1 = \overline{\mathcal{M}_2 + \mathcal{M}_3} \ominus (\mathcal{M}_{12} + \mathcal{M}_{13})$. Thus, for all $z, w \in \Omega$,

$$K(z,w) = K_1(z,w) + K_{\overline{\mathcal{M}_2+\mathcal{M}_3}}(z,w) - K_{\overline{\mathcal{M}_{12}+\mathcal{M}_{13}}}(z,w).$$

Similarly, (2) and (4) implies that $(\mathcal{M}_2 \ominus \mathcal{M}_{23}) \perp (\mathcal{M}_3 \ominus \mathcal{M}_{23})$ and $(\mathcal{M}_{12} \ominus \mathcal{M}_{123}) \perp (\mathcal{M}_{13} \ominus \mathcal{M}_{123})$, respectively. So, applying Lemma 3.2.1 we obtain that

$$K_{\overline{\mathcal{M}_2+\mathcal{M}_3}}(z,w) = K_2(z,w) + K_3(z,w) - K_{23}(z,w)$$

and

$$K_{\overline{\mathcal{M}_{12}+\mathcal{M}_{13}}}(z,w) = K_{12}(z,w) + K_{13}(z,w) - K_{123}(z,w),$$

for all $z, w \in \Omega$. Finally, if we plug the expressions of $K_{\overline{\mathcal{M}_2+\mathcal{M}_3}}$ and $K_{\overline{\mathcal{M}_{12}+\mathcal{M}_{13}}}$ in the expression of $K$, then it becomes apparent that $K$ satisfies the decomposition formula as desired.

Generalizing these arguments we can prove the following: Let $2 \leq n \leq m$, $i_1, \dots, i_n \in \{1,\dots,m\}$, $r_1,\dots,r_n \in \mathbb{N}$ and consider the subspace $\mathcal{M}$ of $\mathcal{H}$ given by the closure of the polynomial ideal generated by the set $\{z_{i_1}^{r_1}, \dots, z_{i_n}^{r_n}\}$. Then the reproducing kernel $K$ of $\mathcal{M}$ satisfies the decomposition formula given by Equation (3.3).

**Remark 3.3.4.** Let $\mathcal{M}_1, \mathcal{M}_2$ and $\mathcal{M}$ be submodules of the Hardy module $H^2(\mathbb{D}^2)$ obtained by taking closure of the polynomial ideals generated by $\{z_1 z_2\}$, $\{z_1 - z_2\}$ and $\{z_1 z_2, z_1 - z_2\}$ respectively. In this example, the reproducing kernel $K$ of $\mathcal{M}$ does not admit a decomposition of the form prescribed in Equation (3.2). This can be seen through the following steps:

1. For each $w \in \mathbb{D}^2$, $K_1(\cdot, w) \in \mathcal{M}_1 \subseteq \{f \in H^2(\mathbb{D}^2) : f(z) = 0 \text{ on } z_1 z_2 = 0\}$. So, if $z \in \mathbb{D}^2$ and $z_1 z_2 = 0$, then $K_1(z,w) = 0$.

2. Since $\mathcal{M}_{12} \subseteq \mathcal{M}_1$, $K_{12}(z,w) = 0$.

3. $K_2(z,w) = \overline{K_2(w,z)}$, $z, w \in \mathbb{D}^2$. So $K_2(z,w) = 0$ when $w_1 - w_2 = 0, w \in \mathbb{D}^2$.

4. If $z, w \in \mathbb{D}^2$, $z_1 z_2 = 0$, $w_1 - w_2 = 0$ and $K = K_1 + K_2 - K_{12}$, then $K(z,w) = 0$.

5. On the other hand, from direct computation we get

$$K(z,w) = \frac{(z_1 - z_2)(\bar{w}_1 - \bar{w}_2)}{2} + \sum_{i,j \geq 0, i+j \geq 2} z_1^i z_2^j \bar{w}_1^i \bar{w}_2^j.$$

   In particular, $K\big((\frac{1}{2},0),(\frac{1}{2},\frac{1}{2})\big) = \sum_{i=2}^{\infty} \big(\frac{1}{4}\big)^i = \frac{1}{12} \neq 0$ which contradicts the outcome mentioned in step 4.

## 3.4 Applications

### 3.4.1 An explicit description of the joint kernel on the singular set

**Proposition 3.4.1.** *Let $\mathscr{H} \subseteq \mathscr{O}(\Omega)$ be an analytic Hilbert module for some bounded domain $\Omega$ in $\mathbb{C}^m$ and $\mathscr{I}$ be a polynomial ideal in $\mathbb{C}[z_1,\dots,z_m]$ with generators $\{p_1,\dots,p_t\}$. Also, let $\mathscr{M}$ be the closure of $\mathscr{I}$ in $\mathscr{H}$. Assume that*

i) *$Z(p_1,\dots,p_t) \cap \Omega$ is a submanifold of codimension $t (\geq 2)$,*

ii) *$p_i, p_j$ are relatively prime for $i \neq j$, $1 \leq i,j \leq t$.*

*Then there exists anti-holomorphic maps $F_1,\dots,F_t : Z(p_1,\dots,p_t) \cap \Omega \to \mathscr{M}$ which satisfy conditions 1 to 4 of Theorem 2.1.4. For $i = 1,\dots,t$, let $\mathscr{I}_i$ be the principal ideal generated by $p_i$ and set $\mathscr{M}_i$ to be the closure of $\mathscr{I}_i$ in $\mathscr{H}$ with reproducing kernel $K_i$. Suppose $K$ admits a decomposition as in Equation (3.3). Then $F_i(v) = M_{p_i}\chi_i(\cdot,v)$ for all $v \in Z(p_1,\dots,p_t) \cap \Omega$, where $\chi_i$ is taken from the equation $K_i(z,w) = p_i(z)\overline{p_i(w)}\chi_i(z,w)$, $z,w \in \Omega$.*

As a prerequisite, we will first prove the following two lemmas.

**Lemma 3.4.2.** *Suppose $k \leq t$ and $i_1,\dots,i_k \in \{1,\dots,t\}$. Then an arbitrary element of $\mathscr{M}_{i_1} \cap \dots \cap \mathscr{M}_{i_k}$ is divisible by $p_{i_1} \cdots p_{i_k}$ in $\mathscr{O}(\Omega)$.*

*Proof.* Firstly, choose an arbitrary element $m \in \mathscr{M}_{12} = \mathscr{M}_1 \cap \mathscr{M}_2$. Then $m = p_1 h_1 = p_2 h_2$, for some holomorphic functions $h_1, h_2 : \Omega \to \mathbb{C}$. Clearly, $p_2$ divides $p_1 h_1$. We will show that $p_2$ divides $h_1$, i.e., there exists a holomorphic function $g_2 : \Omega \to \mathbb{C}$ such that $h_1 = p_2 g_2$ on $\Omega$.

It is enough to show that for each $x \in \Omega$, there exists a neighbourhood $U_x$ of $x$ in $\Omega$, a holomorphic function $g_2^{U_x} : U_x \to \mathbb{C}$ such that $h_1 = p_2 g_2^{U_x}$ on $U_x$. To see this, take $x, y \in \Omega$, $x \neq y$ such that $U_x \cap U_y$ is non-empty. Then, on $U_x \cap U_y$, $p_2 g_2^{U_x} = p_2 g_2^{U_y}$ which implies $g_2^{U_x} = g_2^{U_y}$, by the identity theorem. So, if we define $g_2 : \Omega \to \mathbb{C}$ as $g_2|_{U_x} = g_2^{U_x}$, then it is well-defined, holomorphic on $\Omega$ and satisfies $h_1 = p_2 g_2$ in $\mathscr{O}(\Omega)$.

Observe that $\Omega = \big(\Omega \setminus Z(p_2)\big) \cup \big(Z(p_2) \setminus Z(p_1,p_2)\big) \cup Z(p_1,p_2)$. It is clear that for any $x \in \Omega \setminus Z(p_2)$, $U_x = \Omega \setminus Z(p_2)$ and $g_2^{U_x} = h/p_2$. Now, choose an arbitrary point $x \in \big(Z(p_2) \setminus Z(p_1,p_2)\big)$.

Since $(p_2)_x$ divides $(p_1h_1)_x$ in $\mathcal{O}_x$ and $(p_1)_x$ is an unit, $(p_2)_x$ divides $(h_1)_x$. So, $p_2$ divides $h_1$ in some neighbourhood $U_x$ of $x$. Finally, let $x \in Z(p_1, p_2)$. If $(p_1)_x, (p_2)_x$ are relatively prime in $\mathcal{O}_x$, then, from $(p_2)_x$ divides $(p_1h_1)_x$ we can also conclude that $(p_2)_x$ divides $(h_1)_x$. To see this, consider an irreducible factorization of $(p_2)_x$ in $\mathcal{O}_x$ given by

$$(p_2)_x = (a_1)_x^{n_1} \cdots (a_k)_x^{n_k},$$

where $(a_1)_x, \dots, (a_k)_x$ are distinct irreducible factors of $(p_2)_x$, $n_1, \dots, n_k \in \mathbb{N}$. Since $(p_1)_x, (p_2)_x$ are relatively prime, none of them is a factor of $(p_1)_x$. So, $(a_l)_x$ divides $(h_1)_x$, for each $l, 1 \leq l, \leq k$. Now, $(p_2)_x$ divides $(p_1h_1)_x$ implies $(a_l)_x^{n_l}$ divides $(p_1h_1)_x$, or equivalently, $(a_l)_x^{n_l-1}$ divides $(p_1)_x(h/a_1)_x$. If $n_l > 1$, then this gives $(a_l)_x$ divides $(p_1)_x(h/a_1)_x$ which implies $(a_l)_x$ divides $(h/a_1)_x$. Thus, $(a_l)_x^2$ divides $(h_1)_x$. Proceeding similarly, we obtain that $(a_l)_x^{n_l}$ divides $(h_1)_x$, for each $l, 1 \leq l, \leq k$. Since the least common multiple of $\{(a_1)_x^{n_1}, \dots, (a_k)_x^{n_k}\}$ is their product $(p_2)_x$, from the previous line it follows that $(p_2)_x$ divides $(h_1)_x$ in $\mathcal{O}_x$.

Now, we will show that $(p_1)_x, (p_2)_x$ are relatively prime for any $x \in Z(p_1, p_2)$. Consider the biholomorphism $\Phi$ on $\mathbb{C}^m$ given by $\Phi(z_1, \dots, z_m) = (z_1 - x_1, \dots, z_m - x_m)$. Then, it is clear that $\Phi(x) = 0$ and $p_1 \circ \Phi^{-1}, p_2 \circ \Phi^{-1}$ are polynomials. Also, note that relatively prime is invariant under biholomorphism. This means

i) $p_1, p_2$ are relatively prime in $\mathbb{C}[z_1, \dots, z_m]$ if and only if $p_1 \circ \Phi^{-1}, p_2 \circ \Phi^{-1}$ are relatively prime in $\mathbb{C}[z_1, \dots, z_m]$;

ii) $(p_1)_x, (p_2)_x$ are relatively prime in $\mathcal{O}_x$ if and only if $(p_1 \circ \Phi^{-1})_0, (p_2 \circ \Phi^{-1})_0$ are relatively prime in $\mathcal{O}_0$.

So, without loss of generality, we can assume that $x = 0$ and it is enough to show that $p_1, p_2$ are relatively prime in $\mathbb{C}[z_1, \dots, z_m]$ implies $(p_1)_0, (p_2)_0$ are relatively prime in $\mathcal{O}_0$.

Identify $\mathbb{C}[z_1, \dots, z_m]$ as $\mathbb{C}[z_1, \dots, z_{m-1}][z_m]$. The latter is the collection of all polynomials of the form

$$b_0 + b_1 z_m + b_2 z_m^2 + \cdots + b_k z_m^k,$$

where $k \in \mathbb{N} \cup \{0\}$, $b_0, \dots, b_k \in \mathbb{C}[z_1, \dots, z_{m-1}]$. Since $p_1, p_2$ are relatively prime in $\mathbb{C}[z_1, \dots, z_m]$, they do not have any common irreducible factors in $\mathbb{C}[z_1, \dots, z_{m-1}][z_m]$. By Gauss' lemma any irreducible element in $\mathbb{C}[z_1, \dots, z_m]$ whose degree in $z_m$ variable is strictly positive, is irreducible in $Q[z_1, \dots, z_{m-1}][z_m]$, where $Q[z_1, \dots, z_{m-1}]$ is the quotient field of $\mathbb{C}[z_1, \dots, z_{m-1}]$ and $Q[z_1, \dots, z_m][z_m]$ consists of polynomials in $z_m$ with coefficients in $Q[z_1, \dots, z_{m-1}]$. Consequently, $p_1, p_2$ are relatively prime as elements in $Q[z_1, \dots, z_{m-1}][z_m]$. It can be checked that $Q[z_1, \dots, z_{m-1}][z_m]$ is a PID. So, there exists $r_1, r_2 \in Q[z_1, \dots, z_{m-1}][z_m]$ such that

$$p_1 r_1 + p_2 r_2 = 1,$$

or equivalently, there exists $q_1, q_2, q \in \mathbb{C}[z_1, \ldots, z_m]$ such that $q$ is independent of $z_m$, not identically 0 and we have the equality

$$p_1 q_1 + p_2 q_2 = q.$$

This clearly implies that $(p_1)_0, (p_2)_0$ are relatively prime $\mathcal{O}_{\mathbb{C}^{m-1},0}[z_m]$. Finally, from [24, Lemma II.B.5] it follows that $(p_1)_0, (p_2)_0$ are relatively prime in $\mathcal{O}_0$ which proves our claim. As a result, $p_1 p_2$ divides $m$ in $\mathcal{O}(\Omega)$.

Next, let $m$ be an arbitrary element of $\mathcal{M}_{i_1} \cap \mathcal{M}_{i_2} \cap \mathcal{M}_{i_3} = \mathcal{M}_{i_1 i_2} \cap \mathcal{M}_{i_3}$. Then, from the previous claim it follows that both $p_{i_1} p_{i_2}$ and $p_{i_3}$ divides $m$. Now, if we choose $p_1 = p_{i_1} p_{i_2}$ and $p_2 = p_{i_3}$, then, following similar arguments as above we obtain that $p_{i_1} p_{i_2} p_{i_3}$ divides $m$ in $\mathcal{O}(\Omega)$. Proceeding similarly, the general case follows. □

**Lemma 3.4.3.** *Let $K_{i_1 \ldots i_k}$ be the reproducing kernel of $\mathcal{M}_{i_1} \cap \cdots \cap \mathcal{M}_{i_k}$. Then $K_{i_1 \ldots i_k}(z, w) = (p_{i_1} \cdots p_{i_k})(z)\overline{(p_{i_1} \cdots p_{i_k})(w)}\chi_{i_1 \ldots i_k}(z, w)$, for some sesquianalytic function $\chi_{i_1 \ldots i_k} : \Omega \times \Omega \to \mathbb{C}$.*

*Proof.* Let $\{e_n : n \in \mathbb{N} \cup \{0\}\}$ be an orthonormal basis of $\mathcal{M}_{i_1} \cap \cdots \cap \mathcal{M}_{i_k}$. Then, by Lemma 3.4.2, we have $e_n(z) = (p_{i_1} \cdots p_{i_k})(z)\epsilon_n(z)$, for $\epsilon_n \in \mathcal{O}(\Omega)$, $n \in \mathbb{N} \cup \{0\}$. Define

$$f_n(z, w) = e_n(z)\overline{e_n(w)} \text{ and } f(z, w) = (p_{i_1} \cdots p_{i_k})(z)\overline{(p_{i_1} \cdots p_{i_k})(w)},$$

for all $(z, w) \in \Omega \times \Omega$. Then, it is clear that $f$ divides $f_n$ for all $n$ and hence $\operatorname{ord}_{(z,w)} f \le \operatorname{ord}_{(z,w)} f_n$, where $\operatorname{ord}_{(z,w)} f, \operatorname{ord}_{(z,w)} f_n$ denotes the total order of $f, f_n$ at the point $(z, w)$ as defined in [24, Page 8].

Now, consider two arbitrary multi-indices $\alpha, \beta \in \left(\mathbb{N} \cup \{0\}\right)^m$ such that $|\alpha + \beta| = |\alpha| + |\beta| < \operatorname{ord}_{(z,w)} f$. Then $|\alpha + \beta| < \operatorname{ord}_{(z,w)} f_n$ and hence $\partial^\alpha \bar{\partial}^\beta f_n(z, w) = 0$, for all $z, w \in \Omega$. Moreover, we have

$$\partial^\alpha \bar{\partial}^\beta K_{i_1 \ldots i_k}(z, w) = \sum_{n \ge 0} \partial^\alpha e_n(z)\overline{\partial^\beta e_n(w)} = \sum_{n \ge 0} \partial^\alpha \bar{\partial}^\beta f_n(z, w) = 0.$$

Consequently, $\operatorname{ord}_{(z,w)} f \le \operatorname{ord}_{(z,w)} K_{i_1 \ldots i_k}$, for all $(z, w) \in \Omega \times \Omega$ and the lemma follows from [9, Section 1.5]. □

Now we will prove Proposition 3.4.1.

*Proof.* From Equation (3.3), for each $w \in \Omega$, we have the following:

$$\begin{aligned}
&K(\cdot, w) \\
&= \sum_{i=1}^t K_i(\cdot, w) - \sum_{i<j, i,j=1}^t K_{ij}(\cdot, w) + \cdots + (-1)^{t-1} K_{1\ldots t}(\cdot, w) \\
&= \Big(K_1(\cdot, w) - \sum_{i=2}^t K_{1i}(\cdot, w) + \cdots + (-1)^{k-1} \sum_{i_2 < \ldots < i_k, i_2, \ldots, i_k = 2}^t K_{1 i_2 \ldots i_k}(\cdot, w) + \cdots + (-1)^{n-1} K_{1\ldots t}(\cdot, w)\Big)
\end{aligned}$$

$$
\begin{aligned}
&+\Big(K_2(\cdot,w)-\sum_{i=3}^{t}K_{2i}(\cdot,w)+\cdots+(-1)^{t-2}K_{2\ldots t}(\cdot,w)\Big)+\cdots+K_t(\cdot,w)\\
&=\overline{p_1(w)}\Big(M_{p_1}\chi_1(\cdot,w)-\sum_{i=2}^{t}\overline{p_i(w)}M_{p_1p_i}\chi_{1i}(\cdot,w)+\cdots+(-1)^{n-1}\overline{(p_2\cdots p_t)(w)}M_{p_1\cdots p_t}\chi_{1\ldots t}(\cdot,w)\Big)\\
&+\overline{p_2(w)}\Big(M_{p_2}\chi_2(\cdot,w)-\sum_{i=3}^{t}\overline{p_i(w)}M_{p_2p_i}\chi_{2i}(\cdot,w)+\cdots+(-1)^{n-2}\overline{(p_3\cdots p_t)(w)}M_{p_2\cdots p_t}\chi_{2\ldots t}(\cdot,w)\Big)\\
&+\cdots+\overline{p_t(w)}M_{p_t}\chi_t(\cdot,w)\\
&=\overline{p_1(w)}G_1(w)+\overline{p_2(w)}G_2(w)+\cdots\overline{p_t(w)}G_t(w).
\end{aligned}
$$

Clearly, $G_1,\ldots,G_t:\Omega\to\mathcal{M}$ are anti-holomorphic maps and $G_j(v)=M_{p_j}\chi_j(\cdot,v)$, for all $v\in V(\mathcal{M})$, $j=1,\ldots,t$. Consequently, applying condition 2 of Theorem 2.1.4 we obtain that $F_j(v)=G_j(v)=M_{p_j}\chi_j(\cdot,v)$ which proves the proposition. □

From the proof given above, it follows that under the additional hypotheses mentioned in Proposition 3.4.1, $\Omega_w=\Omega$, for all $w\in V(\mathcal{M})$, $Z=V(\mathcal{M})$ and we have the bundles $\tilde{E}_1,\tilde{E},\tilde{E}_2$ on $Z\times\mathbb{P}^{t-1}$. Fix $i\in\{1,\ldots,t\}$ and consider the set $Z\times\{\pi(0,\ldots,0,1,0,\ldots,0)\}\subseteq\tilde{U}_i\subseteq Z\times\mathbb{P}^{t-1}$, where 1 is at the $i$-th position. If we restrict $\tilde{E}_1,\tilde{E}$ and $\tilde{E}_2$ on this set and denote them by $\tilde{E}_1^{Z,i},\tilde{E}^{Z,i}$ and $\tilde{E}_2^{Z,i}$ respectively, then $0\to\tilde{E}_1^{Z,i}\to\tilde{E}^{Z,i}\to\tilde{E}_2^{Z,i}\to 0$ can be thought as an exact sequence of Hermitian, anti-holomorphic bundles on $Z$. When $i=1$, these bundles can be described as follows:

$$\tilde{E}_1^{Z,1}=\bigsqcup_{v\in Z}<\big\{s_1(v,\pi(1,0,\ldots,0))\big\}>=\bigsqcup_{v\in Z}<\{F_1(v)\}>=\bigsqcup_{v\in Z}<\{M_{p_1}\chi_1(\cdot,v)\}>,$$

$$\tilde{E}^{Z,1}=\bigsqcup_{v\in Z}<\big\{s_1(v,\pi(1,0,\ldots,0)),F_2(v),\ldots,F_t(v)\big\}>=\bigsqcup_{v\in Z}<\{M_{p_1}\chi_1(\cdot,v),\ldots,M_{p_t}\chi_t(\cdot,v)\}>,$$

$$\tilde{E}_2^{Z,1}=\bigsqcup_{v\in Z}<\big\{[F_2(v)],\ldots,[F_t(v)]\big\}>=\bigsqcup_{v\in Z}<\big\{[M_{p_2}\chi_2(\cdot,v)],\ldots,[M_{p_t}\chi_t(\cdot,v)]\big\}>.$$

**Corollary 3.4.4.** *Let $(\Omega,\mathcal{H},\mathcal{I},\mathcal{M}),(\Omega,\mathcal{H}',\mathcal{I},\mathcal{M}')$ be as above. Then we obtain two exact sequences of Hermitian, anti-holomorphic bundles on $Z$, namely, $0\to\tilde{E}_1^{Z,i}\to\tilde{E}^{Z,i}\to\tilde{E}_2^{Z,i}\to 0$ and $0\to\tilde{E}'^{Z,i}_1\to\tilde{E}'^{Z,i}\to\tilde{E}'^{Z,i}_2\to 0$. If $L:\mathcal{M}\to\mathcal{M}'$ is an unitary module map, then $\tilde{E}_1^{Z,i},\tilde{E}'^{Z,i}_1$; $\tilde{E}^{Z,i},\tilde{E}'^{Z,i}$ and $\tilde{E}_2^{Z,i},\tilde{E}'^{Z,i}_2$ are equivalent.*

*Proof.* From Theorem 3.1.4 it follows that $L$ induces isometric bundle maps between $\tilde{E}_1,\tilde{E}_1'$; $\tilde{E},\tilde{E}'$ and $\tilde{E}_2,\tilde{E}_2'$. If we restrict these maps on $Z\times\{\pi(0,\ldots,0,1,0,\ldots,0)\}$, we will obtain the equivalence between each pair of bundles given above on $Z$. □

For any $n\in\mathbb{N}$, $2\le n<m$, consider an $n$-tuple $I=(i_1,\ldots,i_n)$, where $i_p\neq i_q$ for $p\neq q$, $p,q\in\{1,\ldots,n\}$ and $\{i_1,\ldots,i_n\}\subseteq\{1,\ldots,m\}$. Next, consider an $m$-tuple $\lambda_I=(\lambda_1,\ldots,\lambda_m)\in\mathbb{R}_+^m$

corresponding to $I$, where $\lambda_i = 1$, for all $i \in \{i_1, \dots, i_n\}$. For any such $\lambda_I$, define $H_I^{\lambda_I}(\mathbb{D}^m)$ to be the submodule of $H^{\lambda_I}(\mathbb{D}^m)$ given by

$$H_I^{\lambda_I}(\mathbb{D}^m) := \{f \in H^{\lambda_I}(\mathbb{D}^m) : f = 0 \text{ on } z_{i_1} = \cdots = z_{i_n} = 0\}.$$

Then, as an application of Corollary 3.4.4, we can prove the following.

**Proposition 3.4.5.** *For any two $m$-tuples $\lambda_I, \lambda_I^{'}$ as above, $H_I^{\lambda_I}(\mathbb{D}^m)$ is unitarily equivalent to $H_I^{\lambda_I^{'}}(\mathbb{D}^m)$ as Hilbert modules if and only if $\lambda_I = \lambda_I^{'}$.*

*Proof.* The if part is trivial. To prove the converse part let us denote $H_I^{\lambda_I}(\mathbb{D}^m), H_I^{\lambda_I^{'}}(\mathbb{D}^m)$ by $\mathcal{M}, \mathcal{M}'$ respectively. Then it can be checked that both $\mathcal{M}, \mathcal{M}'$ are the closure of the polynomial ideal $\mathcal{I}$ generated by $\{z_{i_1}, \dots, z_{i_n}\}$(with respect to two different inner products). As a result, both of them satisfy the hypotheses of Theorem 2.1.4. Next, for each $r, 1 \le r \le n$, define

$$H_r^{\lambda_I}(\mathbb{D}^m) = \{f \in H^{\lambda_I}(\mathbb{D}^m) : f = 0 \text{ on } z_{i_r} = 0\}$$

and

$$H_r^{\lambda_I^{'}}(\mathbb{D}^m) = \{g \in H^{\lambda_I^{'}}(\mathbb{D}^m) : g = 0 \text{ on } z_{i_r} = 0\}.$$

If we denote them by $\mathcal{M}_r, \mathcal{M}_r^{'}$ respectively, then we obtain the following:

a) Both $\mathcal{M}_r, \mathcal{M}_r^{'}$ are of the form $\overline{<\{z_{i_r}\}>}$ in their respective spaces.

b) $\mathcal{M} = \overline{\mathcal{M}_1 + \cdots + \mathcal{M}_n}$ and $\mathcal{M}' = \overline{\mathcal{M}_1^{'} + \cdots + \mathcal{M}_n^{'}}$.

c) The reproducing kernels of $\mathcal{M}, \mathcal{M}'$ satisfy the decomposition formula given by Equation (3.3).

Let $K_{i_r}$ be the reproducing kernel of $\mathcal{M}_r$. Then, for all $z, w \in \Omega$,

$$K_{i_r}(z,w) = \sum_{\alpha:\alpha_{i_r} \ge 1} \binom{\lambda_I}{\alpha} z^\alpha \bar{w}^\alpha = z_{i_r}\bar{w}_{i_r}\Big(\sum_\alpha c_\alpha z^\alpha \bar{w}^\alpha\Big),$$

where,

$$c_\alpha = \prod_{i:i \neq i_r} \binom{\lambda_i}{\alpha_i}\binom{\lambda_{i_r}}{\alpha_{i_r}+1} = \prod_{i:i \notin \{i_1,\dots,i_n\}} \binom{\lambda_i}{\alpha_i}.$$

This is because $\lambda_i = 1$, for $i \in \{i_1, \dots, i_n\}$. Consequently, $\chi_{i_1}(z,w) = \cdots = \chi_{i_n}(z,w) = \chi(z,w)$, for all $z, w \in \Omega$. Similarly, we have

$$\chi_{i_1}'(z,w) = \cdots = \chi_{i_n}'(z,w) = \chi'(z,w) = \sum_\alpha c_\alpha^{'} z^\alpha \bar{w}^\alpha = \sum_\alpha \Big(\prod_{i:i \notin \{i_1,\dots,i_n\}} \binom{\lambda_i^{'}}{\alpha_i}\Big) z^\alpha \bar{w}^\alpha.$$

Since $\chi, \chi' : \Omega \times \Omega \to \mathbb{C}$ are positive definite functions, they induce reproducing kernel Hilbert modules $\mathscr{H}_\chi, \mathscr{H}'_{\chi'}$ respectively. Moreover, it can be checked that the operator given by the pointwise multiplication of the function $z_{i_r}$ is an unitary module map from $\mathscr{H}_\chi$ to $\mathscr{M}_r$ (similarly, from $\mathscr{H}'_{\chi'}$ to $\mathscr{M}_r^{'}$). If we denote them by $M_{z_{i_r}}, M_{z_{i_r}}^{'}$ respectively, then, following Corollary 3.4.4 we obtain that the bundles $\bigsqcup_{w \in V(\mathscr{M})} < \{M_{z_{i_r}} \chi(\cdot, w)\} >$ and $\bigsqcup_{w \in V(\mathscr{M}')} < \{M_{z_{i_r}}^{'} \chi'(\cdot, w)\} >$ are equivalent for all $r = 1, \ldots, n$. Thus, for all $w \in V(\mathscr{M}) = V(\mathscr{M}') = \{w \in \mathbb{D}^m : w_{i_1} = \cdots = w_{i_n} = 0\}$, $i \in \big(\{1, \ldots, m\} \backslash \{i_1, \ldots, i_n\}\big)$, we have

$$\partial\bar{\partial} \log \chi(w, w) = \partial\bar{\partial} \log \chi'(w, w),$$

as $(1,1)$ forms on $V(\mathscr{M})$, which implies

$$\partial_{w_i}\bar{\partial}_{w_i} \log \chi(w, w) = \partial_{w_i}\bar{\partial}_{w_i} \log \chi'(w, w),$$

or equivalently,

$$\begin{aligned} &\frac{\chi(w, w) \cdot \partial_{w_i}\bar{\partial}_{w_i} \chi(w, w) - \partial_{w_i}\chi(w, w) \cdot \bar{\partial}_{w_i}\chi(w, w)}{\big(\chi(w, w)\big)^2} \\ &= \frac{\chi'(w, w) \cdot \partial_{w_i}\bar{\partial}_{w_i} \chi'(w, w) - \partial_{w_i}\chi'(w, w) \cdot \bar{\partial}_{w_i}\chi'(w, w)}{\big(\chi'(w, w)\big)^2}. \end{aligned}$$

Finally, if we evaluate the previous equation at $w = 0$ using the explicit descriptions of $\chi, \chi'$ given above, we will obtain $c_{(0,\ldots,1,0,\ldots,0)} = c_{(0,\ldots,1,0,\ldots,0)}^{'}$, where 1 is at the $i$-th position. This gives $\lambda_i = \lambda_i^{'}$, for all $i$ outside the set $\{i_1, \ldots, i_n\}$, proving the proposition. □

### 3.4.2 An alternate for the Curto-Salinas bundle on the blow up space

Let $\Omega$ be a bounded domain in $\mathbb{C}^m$. $\mathscr{M}, \mathscr{M}' \in \mathfrak{B}_1(\Omega)$ and $\mathfrak{L}(\mathscr{M}), \mathfrak{L}(\mathscr{M}')$ are the associated line bundles on the blow-up space which are obtained from a decomposition of the reproducing kernels given in [3, Section 3.1]. Then, in [3, Theorem 5.1] it has been shown that the equivalence of these bundles completely determines the equivalence of $\mathscr{M}, \mathscr{M}'$ as Hilbert modules. In the following discussion, with additional hypotheses on $\mathscr{M}, \mathscr{M}'$, we will associate alternate line bundles with these modules which also completely determine their equivalence. Finally, we will work out certain examples where the latter approach might be computationally simpler.

Let $\mathscr{H} \subseteq \mathscr{O}(\Omega)$ be an analytic Hilbert module and $\mathscr{I}$ be a polynomial ideal generated by $\{p_1, \ldots, p_t\}$ that satisfies the following:

i) $Z(p_1, \ldots, p_t) \cap \Omega$ is a submanifold of codimension $t (\geq 2)$,

ii) $p_i, p_j$ are relatively prime for $i \neq j, 1 \leq i, j \leq t$ and

iii) $\mathcal{M}_i = \{f \in \mathcal{H} : f = 0 \text{ on } Z(p_i)\}$, where $\mathcal{M}_i$ is the closure of the principal ideal generated by $p_i$ in $\mathcal{H}$.

Now, set $\mathcal{M} = [\mathcal{I}]$(closure in $\mathcal{H}$) and assume that the reproducing kernel $K$ of $\mathcal{M}$ satisfies Equation (3.3). Then, following the proof of Proposition 3.4.1 and the blow-up construction mentioned before, we canonically obtain a line bundle $\hat{E}_1$ on $\hat{\Omega}$, where

$$\hat{\Omega} = \big\{(w, \pi(u)) \in \Omega \times \mathbb{P}^{t-1} : u_i p_j(w) - u_j p_i(w) = 0, i, j = 1, \ldots, t\big\} \tag{3.5}$$

and $\pi : \mathbb{C}^t \backslash \{0\} \to \mathbb{P}^{t-1}$ is the canonical projection map. This leads us to the following theorem.

**Theorem 3.4.6.** *Let $\mathcal{H}, \mathcal{H}' \subseteq \mathcal{O}(\Omega)$ be analytic Hilbert modules. With $\mathcal{I}$ as above, let $\mathcal{M}, \mathcal{M}'$ be the closure of $\mathcal{I}$ in $\mathcal{H}, \mathcal{H}'$ respectively. Then $\mathcal{M}, \mathcal{M}'$ are equivalent as Hilbert modules if and only if the respective line bundles $\hat{E}_1, \hat{E}_1^{'}$ are equivalent as Hermitian, anti-holomorphic bundles on $\hat{\Omega}$.*

*Proof.* Let us write $\hat{\Omega} = \cup_{i=1}^{t} \hat{U}_i$, where, for $i = 1, \ldots, t$, $\hat{U}_i := (\Omega \times \{u_i \neq 0\}) \cap \hat{\Omega}$. Then in particular, we have

$$\hat{U}_1 = \big\{(w, \pi(1, \theta_2^1, \ldots, \theta_t^1)) : p_j(w) = \theta_j^1 p_1(w), w \in \Omega, \theta_j^1 \in \mathbb{C}, j = 2, \ldots, t\big\}.$$

If we denote $\Theta = (1, \theta_2^1, \ldots, \theta_t^1)$, then, on $\hat{U}_1$,

$$\hat{E}_1 = \bigsqcup_{(w,\pi(\Theta)) \in \hat{U}_1} < \{s_1(w, \pi(\Theta))\} > \text{ and } \hat{E}_1^{'} = \bigsqcup_{(w,\pi(\Theta)) \in \hat{U}_1} < \{s_1'(w, \pi(\Theta))\} >,$$

where $s_1(w, \pi(\Theta)) = G_1(w) + \bar{\theta}_2^1 G_2(w) + \cdots + \bar{\theta}_t^1 G_t(w)$, $s_1'(w, \pi(\Theta)) = G_1^{'}(w) + \bar{\theta}_2^1 G_2^{'}(w) + \cdots + \bar{\theta}_t^1 G_t^{'}(w)$ and for each $j \in \{1, \ldots, t\}$, $G_j, G_j^{'}$ have the same form as described in the proof of Proposition 3.4.1.

Given any unitary module map $L : \mathcal{M} \to \mathcal{M}'$, it is enough to show that $\hat{E}_1, \hat{E}_1^{'}$ are equivalent on $\hat{U}_1$. Since $\operatorname{codim} V(\mathcal{M}) = \operatorname{codim} V(\mathcal{M}') \geq 2$, by Hartog's Extension Theorem [26, Page 198] we can find a non-vanishing holomorphic function $a : \Omega \to \mathbb{C}$ such that $L(m) = am$, for all $m \in \mathcal{M}$. Now, choose an arbitrary element $m_i \in \mathcal{M}_i$, for $i \in \{1, \ldots, t\}$. Then $L(m_i) \in \mathcal{M}'$ and it vanishes on $Z(p_i)$. As a result, $L(\mathcal{M}_i) \subseteq \mathcal{M}_i^{'}$. Similarly, with $L^{-1}$ we have $L^{-1}(\mathcal{M}_i^{'}) \subseteq \mathcal{M}_i$, or equivalently, $\mathcal{M}_i^{'} \subseteq L(\mathcal{M}_i)$. Thus, $L(\mathcal{M}_i) = \mathcal{M}_i^{'}$, for all $i$. More generally, for any $k \leq t$, $i_1, \ldots, i_k \in \{1, \ldots, t\}$,

$$L(\mathcal{M}_{i_1} \cap \cdots \cap \mathcal{M}_{i_k}) = \mathcal{M}_{i_1}^{'} \cap \cdots \cap \mathcal{M}_{i_k}^{'},$$

which follows from the bijectivity of $L$. Let $m_{i_1 \ldots i_k}'$ be an arbitrary element in $\mathcal{M}_{i_1}^{'} \cap \cdots \cap \mathcal{M}_{i_k}^{'}$. Then, applying Lemma 3.4.2 we obtain $m_{i_1 \ldots i_k}^{'} = M_{p_{i_1} \ldots p_{i_k}}^{'}(m_{i_1 \ldots i_k}^{\chi'})$, for some holomorphic function $m_{i_1 \ldots i_k}^{\chi'} : \Omega \to \mathbb{C}$, where $M_{p_{i_1} \ldots p_{i_k}}^{'}$ is the multiplication operator by the function $p_{i_1} \cdots p_{i_k}$

on the Hilbert space $\mathcal{H}_{\chi'_{i_1\ldots i_k}}$ that is generated by the positive definite function $\chi'_{i_1\ldots i_k} : \Omega \times \Omega \to \mathbb{C}$. As a result, for all $w \in \Omega$,

$$\begin{aligned}
\Big\langle L\big(M_{p_{i_1}\ldots p_{i_k}} \chi_{i_1\ldots i_k}(\cdot, w)\big), m'_{i_1\ldots i_k}\Big\rangle_{\mathcal{M}'} &= \Big\langle M_{p_{i_1}\ldots p_{i_k}} \chi_{i_1\ldots i_k}(\cdot, w), L^*\big(m'_{i_1\ldots i_k}\big)\Big\rangle_{\mathcal{M}} \\
&= \Big\langle M_{p_{i_1}\ldots p_{i_k}} \chi_{i_1\ldots i_k}(\cdot, w), L^{-1}\big(m'_{i_1\ldots i_k}\big)\Big\rangle_{\mathcal{M}} \\
&= \Big\langle M_{p_{i_1}\ldots p_{i_k}} \chi_{i_1\ldots i_k}(\cdot, w), \big(m'_{i_1\ldots i_k}/a\big)\Big\rangle_{\mathcal{M}} \\
&= \Big\langle M_{p_{i_1}\ldots p_{i_k}} \chi_{i_1\ldots i_k}(\cdot, w), M_{p_{i_1}\ldots p_{i_k}}\big(m^{\chi'}_{i_1\ldots i_k}/a\big)\Big\rangle_{\mathcal{M}} \\
&= \Big\langle \chi_{i_1\ldots i_k}(\cdot, w), \big(m^{\chi'}_{i_1\ldots i_k}/a\big)\Big\rangle_{\mathcal{H}_{\chi_{i_1\ldots i_k}}} \\
&= \frac{1}{\overline{a(w)}}\Big\langle \chi'_{i_1\ldots i_k}(\cdot, w), m^{\chi'}_{i_1\ldots i_k}\Big\rangle_{\mathcal{H}_{\chi'_{i_1\ldots i_k}}} \\
&= \frac{1}{\overline{a(w)}}\Big\langle M'_{p_{i_1}\ldots p_{i_k}} \chi'_{i_1\ldots i_k}(\cdot, w), m'_{i_1\ldots i_k}\Big\rangle_{\mathcal{M}'},
\end{aligned}$$

which implies $LG_j(w) = \frac{G'_j(w)}{\overline{a(w)}}$, $j = 1, \ldots, t$. Thus, we obatin $Ls_1(w, \pi(\Theta)) = \frac{s'_1(w,\pi(\Theta))}{\overline{a(w)}}$ proving that $L$ induces an isometric bundle map between $\hat{E}_1$ and $\hat{E}'_1$ on $\hat{U}_1$.

To prove the converse, first observe that

a) $\big((\Omega \setminus V(\mathcal{M})) \times \mathbb{P}^{t-1}\big) \cap \hat{\Omega} = \big\{\big(w, \pi\big(p_1(w), \ldots, p_t(w)\big)\big) : w \in \Omega \setminus V(\mathcal{M})\big\}$ and

b) on this set, $\hat{E}_1 = \bigsqcup_{(w,\pi(u))} < \{K(\cdot, w)\} >$, $\hat{E}'_1 = \bigsqcup_{(w,\pi(u))} < \{K'(\cdot, w)\} >$, where $K, K'$ are the reproducing kernel of $\mathcal{M}, \mathcal{M}'$ respectively.

To see this, take an arbitrary element $(w, \pi(u)) \in \big((\Omega \setminus V(\mathcal{M})) \times \mathbb{P}^{t-1}\big) \cap \hat{\Omega}$. Then $u \neq 0$ and there exists some $j \in \{1, \ldots, t\}$ such that $p_j(w) \neq 0$. Without loss of generality, assume that $p_1(w) \neq 0$. Then, from Equation (3.5) it follows that $u_j = \frac{u_1 p_j(w)}{p_1(w)}$, for all $j = 2, \ldots, t$. Thus, $u_1 \neq 0$(otherwise, it contradicts that $u \neq 0$) and

$$\begin{aligned}
\pi(u_1, u_2, \ldots, u_t) &= \pi\Big(u_1, \frac{u_1 p_2(w)}{p_1(w)}, \ldots, \frac{u_1 p_t(w)}{p_1(w)}\Big) \\
&= \pi\Big(1, \frac{p_2(w)}{p_1(w)}, \ldots, \frac{p_t(w)}{p_1(w)}\Big) \\
&= \pi\big(p_1(w), p_2(w) \ldots, p_t(w)\big),
\end{aligned}$$

which proves a). To prove b), observe that $(w, \pi(u))$ as chosen above, is in $\hat{U}_1$. As a result,

$$\begin{aligned}
s_1\Big(w, \pi\Big(1, \frac{u_2}{u_1}, \ldots, \frac{u_t}{u_1}\Big)\Big) &= s_1\Big(w, \pi\Big(1, \frac{p_2(w)}{p_1(w)}, \ldots, \frac{p_t(w)}{p_1(w)}\Big)\Big) \\
&= G_1(w) + \frac{\overline{p_2(w)}}{\overline{p_1(w)}} G_2(w) + \cdots + \frac{\overline{p_t(w)}}{\overline{p_1(w)}} G_t(w) = \frac{K(\cdot, w)}{\overline{p_1(w)}}.
\end{aligned}$$

Since $p_1(w) \neq 0$, $(\hat{E}_1)_{(w,\pi(u))} =< \{K(\cdot,w)/\overline{p_1(w)}\} >=< \{K(\cdot,w)\} >$. Similarly, we can show that $(\hat{E}_1^{'})_{(w,\pi(u))} =< \{K'(\cdot,w)\} >$, which proves b)

Next, consider the holomorphic map $F : (\Omega \setminus V(\mathscr{M})) \to \big((\Omega \setminus V(\mathscr{M})) \times \mathbb{P}^{t-1}\big) \cap \hat{\Omega}$ given by $F(w) = \big(w, \pi\big(p_1(w),\ldots,p_t(w)\big)\big)$. Then

$$F^*\hat{E}_1 = \bigsqcup_{w\in(\Omega - V(\mathscr{M}))} < \{K(\cdot,w)\} > \quad \text{and} \quad F^*\hat{E}_1^{'} = \bigsqcup_{w\in(\Omega - V(\mathscr{M}))} < \{K'(\cdot,w)\} > .$$

Now, if $\hat{E}_1, \hat{E}_1^{'}$ are equivalent on $\hat{\Omega}$, they are equivalent on $\big((\Omega \setminus V(\mathscr{M}))\times\mathbb{P}^{t-1}\big)\cap\hat{\Omega}$. Consequently, $F^*\hat{E}_1, F^*\hat{E}_1^{'}$ are equivalent on $\Omega\setminus V(\mathscr{M})$ following the discussion above. Since both are line bundles, there exists a nonvanishing holomorphic function $\varphi : (\Omega \setminus V(\mathscr{M})) \to \mathbb{C}$ such that $K(z,w) = \varphi(z)K'(z,w)\overline{\varphi(w)}$, for all $z, w \in (\Omega\setminus V(\mathscr{M}))$. But $K, K'$ are sharp on $\Omega\setminus V(\mathscr{M})$. So, from [12, Theorem 3.7, Lemma 3.9], we obtain that $\mathscr{M}, \mathscr{M}'$ are equivalent as Hilbert modules. □

As an application of Theorem 3.4.6, we will now consider the following class of examples. Let

$$H^{(\lambda,\mu)}(\mathbb{D}^2) := \Big\{ f \in \mathscr{O}(\mathbb{D}^2) : \text{if } f(z_1,z_2) = \sum_{i,j\geq 0} a_{ij} z_1^i z_2^j, \text{ then } \sum_{i,j\geq 0} \frac{|a_{ij}|^2}{\binom{\lambda}{i}\binom{\mu}{j}} < \infty \Big\}$$

with the inner product $\langle f, g\rangle = \sum_{i,j\geq 0} \frac{a_{ij}\bar{b}_{ij}}{\binom{\lambda}{i}\binom{\mu}{j}}$, where $\lambda,\mu > 0$ and $g(z_1,z_2) = \sum_{i,j\geq 0} b_{ij} z_1^i z_2^j$. Then it is clear that $H^{(\lambda,\mu)}(\mathbb{D}^2)$ is a Hilbert module over $\mathbb{C}[z_1,z_2]$. Consider the submodule $H_0^{(\lambda,\mu)}(\mathbb{D}^2)$ given by

$$H_0^{(\lambda,\mu)}(\mathbb{D}^2) = \{f \in H^{(\lambda,\mu)}(\mathbb{D}^2) : f = 0 \text{ at } (0,0)\}.$$

**Proposition 3.4.7.** *For $\lambda,\lambda',\mu,\mu' > 0$, $H_0^{(\lambda,\mu)}(\mathbb{D}^2)$ is equivalent to $H_0^{(\lambda',\mu')}(\mathbb{D}^2)$ as Hilbert modules if and only if $\lambda = \lambda'$ and $\mu = \mu'$.*

*Proof.* The if part is trivial. To prove the only if part let us denote $H_0^{(\lambda,\mu)}(\mathbb{D}^2), H_0^{(\lambda',\mu')}(\mathbb{D}^2)$ as $\mathscr{M}, \mathscr{M}'$ respectively. Then it can be checked that both $\mathscr{M}, \mathscr{M}'$ are the closure of the polynomial ideal $\mathscr{I}$ generated by $z_1, z_2$, in their respective ambient spaces. Furthermore, $\mathscr{I}$ satisfies conditions i), ii) and iii) previously mentioned. If we denote the reproducing kernel of $\mathscr{M}$ as $K$, then

$$\begin{aligned} K(z,w) &= \frac{1}{(1-z_1\bar{w}_1)^\lambda(1-z_2\bar{w}_2)^\mu} - 1 \\ &= \sum_{i,j\geq 0, i+j\geq 1} \binom{\lambda}{i}\binom{\mu}{j} z_1^i z_2^j \bar{w}_1^i \bar{w}_2^j. \end{aligned}$$

So, if $z_1 = 0$ and $w_2 = 0$, then $K(z,w) = 0$. Similarly, $K'(z,w) = 0$ for such $z, w \in \mathbb{D}^2$. As a result, from Proposition 3.2.2, it follows that both $K, K'$ satisfy Equation (3.3). Now, applying Theorem 3.4.6 we obtain that $\hat{E}_1$ and $\hat{E}_1^{'}$ are equivalent on $\hat{\mathbb{D}}^2$, where

$$\hat{\mathbb{D}}^2 = \{(w,\pi(u)) \in \mathbb{D}^2 \times \mathbb{P}^1 : u_1 w_2 - u_2 w_1 = 0\} = \hat{U}_1 \cup \hat{U}_2.$$

In particular, they are equivalent on $\hat{U}_1 = \{(w, \pi(1,\theta_1)) : w_2 = \theta_1 w_1, w \in \mathbb{D}^2, \theta_1 \in \mathbb{C}\}$. Next, observe that on $\hat{U}_1$,

$$\hat{E}_1 = \bigsqcup_{(w,\pi(1,\theta_1))\in\hat{U}_1} < \{s_1(w,\pi(1,\theta_1))\} >= \bigsqcup_{(w,\pi(1,\theta_1))\in\hat{U}_1} < \{G_1(w) + \bar{\theta}_1 G_2(w)\} >$$

and

$$\hat{E}_1^{'} = \bigsqcup_{(w,\pi(1,\theta_1))\in\hat{U}_1} < \{G_1^{'}(w) + \bar{\theta}_1 G_2^{'}(w)\} >,$$

where $G_1(w) = \big(M_{z_1}\chi_1(\cdot,w) - \bar{w}_2 M_{z_1z_2}\chi_{12}(\cdot,w)\big)$, $G_1^{'}(w) = \big(M_{z_1}^{'}\chi_1^{\prime}(\cdot,w) - \bar{w}_2 M_{z_1z_2}^{'}\chi_{12}^{\prime}(\cdot,w)\big)$, $G_2(w) = M_{z_2}\chi_2(\cdot,w)$ and $G_2^{'}(w) = M_{z_2}^{'}\chi_2^{\prime}(\cdot,w)$. For any polynomial $p$, $M_p, M_p^{'}$ represents the point-wise multiplication operator by the function $p$ on a suitable Hilbert space. If $K_1, K_2, K_{12}$ are the reproducing kernels for the submodules of $H^{(\lambda,\mu)}(\mathbb{D}^2)$ generated by $z_1, z_2$ and $z_1z_2$ respectively, then it follows that

1. $\chi_1(z,w) = \sum_{i,j\geq 0}\binom{\lambda}{i+1}\binom{\mu}{j} z_1^i z_2^j \bar{w}_1^i \bar{w}_2^j$,

2. $\chi_2(z,w) = \sum_{i,j\geq 0}\binom{\lambda}{i}\binom{\mu}{j+1} z_1^i z_2^j \bar{w}_1^i \bar{w}_2^j$ and

3. $\chi_{12}(z,w) = \sum_{i,j\geq 0}\binom{\lambda}{i+1}\binom{\mu}{j+1} z_1^i z_2^j \bar{w}_1^i \bar{w}_2^j$,

for all $z, w \in \mathbb{D}^2$. Similarly, we can derive the expressions for $\chi_1^{\prime}, \chi_2^{\prime}$ and $\chi_{12}^{\prime}$.

Now, observe that $\hat{U}_1 \cap \{\theta_1 = 0\} = \big(\{w_2 = 0\} \cap \mathbb{D}^2\big) \times \{\pi(1,0)\}$. So, $\{w_2 = 0\} \cap \mathbb{D}^2$ can be considered as subset of $\hat{U}_1$ through the holomorphic map $f$ given by $f(w) = (w, \pi(1,0))$. Also, it is clear that

$$f^*\hat{E}_1 = \bigsqcup_{w_1\in\mathbb{D}} < \big\{M_{z_1}\chi_1\big(\cdot,(w_1,0)\big)\big\} > \quad \text{and} \quad f^*\hat{E}_1^{'} = \bigsqcup_{w_1\in\mathbb{D}} < \big\{M_{z_1}^{'}\chi_1^{\prime}\big(\cdot,(w_1,0)\big)\big\} > .$$

Since $\hat{E}_1, \hat{E}_1^{'}$ are equivalent on $\hat{U}_1$, $f^*\hat{E}_1, f^*\hat{E}_1^{'}$ are equivalent on $\{w_2 = 0\} \cap \mathbb{D}^2$. As a result,

$$\partial\bar{\partial}\log\chi_1\big((w_1,0),(w_1,0)\big) = \partial\bar{\partial}\log\chi_1^{\prime}\big((w_1,0),(w_1,0)\big)$$

as (1,1) forms on $\mathbb{D}$. But this is equivalent to

$$\frac{\partial^2}{\partial w_1 \partial\bar{w}_1}\log\chi_1\big((w_1,0),(w_1,0)\big) = \frac{\partial^2}{\partial w_1 \partial\bar{w}_1}\log\chi_1^{\prime}\big((w_1,0),(w_1,0)\big),$$

which implies

$$\frac{\chi_1\big((w_1,0),(w_1,0)\big)\frac{\partial^2}{\partial w_1\partial\bar{w}_1}\chi_1\big((w_1,0),(w_1,0)\big) - \frac{\partial}{\partial w_1}\chi_1\big((w_1,0),(w_1,0)\big)\frac{\partial}{\partial\bar{w}_1}\chi_1\big((w_1,0),(w_1,0)\big)}{\Big(\chi_1\big((w_1,0),(w_1,0)\big)\Big)^2}$$

$$= \frac{\chi'\big((w_1,0),(w_1,0)\big)\frac{\partial^2}{\partial w_1 \partial \bar{w}_1}\chi_1'\big((w_1,0),(w_1,0)\big) - \frac{\partial}{\partial w_1}\chi_1'\big((w_1,0),(w_1,0)\big)\frac{\partial}{\partial \bar{w}_1}\chi_1'\big((w_1,0),(w_1,0)\big)}{\Big(\chi_1'\big((w_1,0),(w_1,0)\big)\Big)^2}.$$

If we plug the expressions of $\chi_1, \chi_1'$ in this equation and put $w_1 = 0$, we will get $\frac{\binom{\lambda}{1}\binom{\lambda}{2}}{\binom{\lambda}{1}^2} = \frac{\binom{\lambda'}{1}\binom{\lambda'}{2}}{\binom{\lambda'}{1}^2}$ which gives $\lambda = \lambda'$. Finally, using the fact that $\hat{E}_1, \hat{E}_1'$ are equivalent on $\hat{U}_2$ and following similar arguments as above, we will obtain that $\mu = \mu'$. □

### 3.4.3 A submodule invariant and its relation with the quotient module

In this section, for each $i = 1, \dots, t$, let $\mathscr{H}, \mathscr{I}, \mathscr{I}_i, \mathscr{M}, \mathscr{M}_i$ be as in Proposition 3.4.1. Consider the module map $M_{p_1} : \mathscr{H} \to \mathscr{M}_1$. This induces the map

$$M_{p_1} \otimes_{\mathbb{C}[\underline{z}]} 1_w : \mathscr{H} \otimes_{\mathbb{C}[\underline{z}]} \mathbb{C}_w \to \mathscr{M}_1 \otimes_{\mathbb{C}[\underline{z}]} \mathbb{C}_w,$$

for each $w \in \Omega$, where $\mathbb{C}_w$ is the one dimensional module over $\mathbb{C}[\underline{z}]$ given by $p \cdot \lambda := p(w)\lambda$, $p \in \mathbb{C}[z]$, $\lambda \in \mathbb{C}$. It is easily checked that for each $w \in \Omega$,

$$\mathscr{H} \otimes_{\mathbb{C}[\underline{z}]} \mathbb{C}_w =< \{K_{\mathscr{H}}(\cdot, w)\} > \otimes\ \mathbb{C} \text{ and } \mathscr{M}_1 \otimes_{\mathbb{C}[\underline{z}]} \mathbb{C}_w =< \{M_{p_1}\chi_1(\cdot, w)\} > \otimes\ \mathbb{C},$$

where $K_{\mathscr{H}}$ is the reproducing kernel of $\mathscr{H}$. So, we obtain a canonical map between $< \{K_{\mathscr{H}}(\cdot, w)\} >$ and $< \{M_{p_1}\chi_1(\cdot, w)\} >$ which we will denote by $M_{p_1}(w)$. Now, define

$$\mathscr{K}_{M_{p_1}}(w) = \partial_w \bar{\partial}_w \log \|M_{p_1}(w)\|^2. \tag{3.6}$$

Note that if $\|M_{p_1}(w)\|^2$ vanishes on some subset of $\Omega$, the right hand side of the Equation (3.6) will be thought of as a (1,1) current. Since $M_{p_1} : \mathscr{H} \to \mathscr{M}_1$ has a dense range, it follows from [15, Lemma 1] that $\mathscr{K}_{M_{p_1}}$ is a (1,1) form on $\Omega$ and for each $w \in \Omega$,

$$\mathscr{K}_{M_{p_1}}(w) = \mathscr{K}_{E_{\mathscr{H}}}(w) - \mathscr{K}_{E_{\mathscr{M}_1}}(w) = \partial_w \bar{\partial}_w \log K_{\mathscr{H}}(w, w) - \partial_w \bar{\partial}_w \log \chi_1(w, w). \tag{3.7}$$

Here $E_{\mathscr{H}}, E_{\mathscr{M}_1}$ are line bundles on $\Omega$ given by $\bigsqcup_{w \in \Omega} < \{K(\cdot, w)\} >$, $\bigsqcup_{w \in \Omega} < \{M_{p_1}\chi_1(\cdot, w)\} >$ respectively and $\mathscr{K}_{E_{\mathscr{H}}}, \mathscr{K}_{E_{\mathscr{M}_1}}$ are the associated curvatures with respect to the Hermitian metrics induced by $\mathscr{H}$.

For $Z = V(\mathscr{M})$, consider the canonical inclusion map $i_Z : Z \to \Omega$. Then, on $Z$, we have the following equality of (1,1) forms

$$i_Z^* \mathscr{K}_{E_{\mathscr{M}_1}} + i_Z^* \mathscr{K}_{M_{p_1}} = i_Z^* \mathscr{K}_{E_{\mathscr{H}}}, \tag{3.8}$$

where $i_Z^* \mathscr{K}_{M_{p_1}}, i_Z^* \mathscr{K}_{E_{\mathscr{H}}}$ and $i_Z^* \mathscr{K}_{E_{\mathscr{M}_1}}$ are pullbacks of $\mathscr{K}_{M_{p_1}}$, $\mathscr{K}_{E_{\mathscr{H}}}$ and $\mathscr{K}_{E_{\mathscr{M}_1}}$, respectively, by $i_Z$. Now, we have the following lemma.

**Lemma 3.4.8.** *$i_Z^*\mathcal{K}_{E_{\mathcal{M}_1}} = \mathcal{K}_{\tilde{E}_1^{Z,1}}$ as (1,1) forms on $Z$, where $\mathcal{K}_{\tilde{E}_1^{Z,1}}$ is the canonical curvature of $\tilde{E}_1^{Z,1}$ with respect to the Hermitian norm $\tilde{N}_1^{Z,1}$ induced by the norm in the module $\mathcal{H}$.*

*Proof.* Choose an arbitrary point $p \in Z$. Since $Z$ is a submanifold of codimension $t$, there exists a neighbourhood $W_p$ of $p$ in $\Omega$, an open set $O_p \subseteq \mathbb{C}^{m-t}$ containing 0 and an immersion $\phi_p : O_p \to W_p$ such that $\phi_p(0) = p$, $\phi_p(O_p) = W_p \cap Z$ and $\phi_p : O_p \to W_p \cap Z$ is a homeomorphism. As a result, $W_p \cap Z$ becomes a co-ordinate chart of $Z$ around $p$. If $v = (v_1, \ldots, v_{m-t})$ is the co-ordinate representation of an arbitrary point in $O_p$, then any point in $W_p \cap Z$ can be denoted as $\phi_p(v)$. Now, observe that for any $v \in O_p$,

$$\mathcal{K}_{\tilde{E}_1^{Z,1}}(\phi_p(v)) = \partial_v\bar{\partial}_v \log \|M_{p_1}\chi_1(\cdot, \phi_p(v))\|^2_{\mathcal{H}} = \partial_v\bar{\partial}_v \log \chi_1(\phi_p(v), \phi_p(v)).$$

On the other hand, $\mathcal{K}_{E_{\mathcal{M}_1}} = \partial_w\bar{\partial}_w \log\chi_1(\cdot,\cdot)$ on $W_p$. So, on $i_Z^{-1}(W_p) = W_p \cap Z$,

$$i_Z^*\mathcal{K}_{E_{\mathcal{M}_1}} = i_Z^*\partial_w\bar{\partial}_w \log\chi_1(\cdot,\cdot) = \partial_v\bar{\partial}_v i_Z^* \log\chi_1(\cdot,\cdot) = \partial_v\bar{\partial}_v \log\chi_1\big(i_Z(\cdot), i_Z(\cdot)\big).$$

In particular, for any $v \in O_p$,

$$i_Z^*\mathcal{K}_{E_{\mathcal{M}_1}}(\phi_p(v)) = \partial_v\bar{\partial}_v \log\chi_1\big(i_Z(\phi_p(v)), i_Z(\phi_p(v))\big) = \partial_v\bar{\partial}_v \log\chi_1(\phi_p(v), \phi_p(v)) = \mathcal{K}_{\tilde{E}_1^{Z,1}}(\phi_p(v)).$$

Thus, $i_Z^*\mathcal{K}_{E_{\mathcal{M}_1}} = \mathcal{K}_{\tilde{E}_1^{Z,1}}$ as (1,1) forms on $W_p \cap Z$. The Lemma then follows from the arbitrariness of $p$. □

As a corollary of the Lemma 3.4.8, from Equation (3.8) we get

$$\mathcal{K}_{\tilde{E}_1^{Z,1}} + i_Z^*\mathcal{K}_{M_{p_1}} = i_Z^*\mathcal{K}_{E_{\mathcal{H}}}. \tag{3.9}$$

If $\mathcal{M}_q$ denotes the quotient module of $\mathcal{M}$ in $\mathcal{H}$, then for each $w \in \Omega$,

$$\mathcal{M}_q \otimes_{\mathbb{C}[\underline{z}]} \mathbb{C}_w = \Big(\bigcap_{p \in \mathbb{C}[\underline{z}]} \ker\big(P_{\mathcal{M}_q} M_p^*|_{\mathcal{M}_q} - \overline{p(w)} I_{\mathcal{M}_q}\big)\Big) \otimes \mathbb{C} = \begin{cases} <\{K_{\mathcal{H}}(\cdot, w)\}> \otimes \mathbb{C} & \text{if } w \in Z, \\ \{0\} & \text{otherwise.} \end{cases}$$

Thus, $\mathcal{M}_q$ canonically induces a line bundle $E_{\mathcal{M}_q} := \bigsqcup_{w \in Z} <\{K_{\mathcal{H}}(\cdot, w)\}>$ on $Z$. Its canonical curvature $\mathcal{K}_{E_{\mathcal{M}_q}}$ is a (1,1) form on $Z$ and following a similar proof as given in Lemma 3.4.8 we can show that $i_Z^*\mathcal{K}_{E_{\mathcal{H}}} = \mathcal{K}_{E_{\mathcal{M}_q}}$. Plugging this equality in the Equation (3.9) we finally get the following equality of (1,1) forms on $Z$:

$$\mathcal{K}_{\tilde{E}_1^{Z,1}} + i_Z^*\mathcal{K}_{M_{p_1}} = \mathcal{K}_{E_{\mathcal{M}_q}}. \tag{3.10}$$

From Corollary 3.4.4 it follows that $\mathcal{K}_{\tilde{E}_1^{Z,1}} = (-2\pi i)c_1(\tilde{E}_1^{Z,1}, \tilde{N}_1^{Z,1})$ is a geometric invariant for $\mathcal{M}$. On the other hand, it is clear that $\mathcal{K}_{E_{\mathcal{M}_q}}$ is a geometric invariant for $\mathcal{M}_q$. So, from Equation (3.10) it follows that $\mathcal{K}_{\tilde{E}_1^{Z,1}} + i_Z^*\mathcal{K}_{M_{p_1}}$ is a geometric invariant for $\mathcal{M}_q$. Summarizing these steps, in general, we obtain the following proposition.

**Proposition 3.4.9.** *Let $\mathscr{H},\mathscr{I},\mathscr{I}_i,\mathscr{M},\mathscr{M}_i$, $i=1,\ldots,t$, be as above. Consider the set*

$$Z\times\{\pi(0,\ldots,0,1,0,\ldots,0)\}\subseteq\tilde{U}_i\subseteq Z\times\mathbb{P}^{t-1},$$

*where 1 is at the $i$-th position and define $\tilde{E}_1^{Z,i}$ by restricting $\tilde{E}_1$ on this set. For each $i\in\{1,\ldots,t\}$, we have*

*(a) $\mathscr{K}_{\tilde{E}_1^{Z,i}}$ is an invariant for $\mathscr{M}$;*

*(b) $\mathscr{K}_{\tilde{E}_1^{Z,i}}+i_Z^*\mathscr{K}_{M_{p_i}}=\mathscr{K}_{E_{\mathscr{M}_q}}$ as (1,1) forms on $Z$. In particular, $\mathscr{K}_{\tilde{E}_1^{Z,i}}+i_Z^*\mathscr{K}_{M_{p_i}}$ is a geometric invariant for $\mathscr{M}_q$.*

Now, consider the example where $\mathscr{H}=H^2(\mathbb{D}^m)$, $m\in\mathbb{N},m\geq 2$, $\mathscr{M}=\overline{<\{z_1,\ldots,z_t\}>}$ and for each $i=1,\ldots,t$, $\mathscr{M}_i=\overline{<\{z_i\}>}$. In this case, $M_{z_i}:\mathscr{H}\to\mathscr{M}_i$ is an unitary map and hence, for each $w\in\mathbb{D}^m$, $\mathscr{M}_i\otimes_{\mathbb{C}[\underline{z}]}\mathbb{C}_w=<\{M_{z_i}K_{\mathscr{H}}(\cdot,w)\}>\otimes\mathbb{C}$. From Equation (3.7) it would then follow that $\mathscr{K}_{M_{z_i}}(w)=0$, for each $w\in\Omega$ resulting $i_Z^*\mathscr{K}_{M_{z_i}}=0$ as a (1,1) form on $Z$. Consequently, by Proposition 3.4.9 we obtain that $\mathscr{K}_{\tilde{E}_1^{Z,i}}$ is a geometric invariant for both $\mathscr{M}$ and $\mathscr{M}_q$.

# Chapter 4

# Some complete invariants of a subclass of similar submodules

## 4.1 Analytic sets and the Lelong-Poincaré formula

Let $\Omega$ be a bounded domain in $\mathbb{C}^m$ and $\mathcal{Z}$ be an analytic set in $\Omega$. We define the dimension of $\mathcal{Z}$ at an arbitrary point $x \in \mathcal{Z}$ by the number

$$\dim_x \mathcal{Z} = \limsup_{z \to x, z \in \text{reg}\mathcal{Z}} \dim_z \mathcal{Z},$$

where reg$\mathcal{Z}$ denotes the regular points. Finally, the dimension of $\mathcal{Z}$ is defined as the maximum of its dimensions at all points, i.e.,

$$\dim \mathcal{Z} := \max_{z \in \mathcal{Z}} \dim_z \mathcal{Z} = \max_{z \in \text{reg}\mathcal{Z}} \dim_z \mathcal{Z}.$$

The codimension of an analytic set is, by definition, equal to $m - \dim \mathcal{Z}$.

For any $k \in \mathbb{N}$, $\mathcal{Z}$ is said to be a pure $k$-dimensional analytic set if $\dim_x \mathcal{Z} = k$ for all $x \in \mathcal{Z}$. As an example, consider the set $Z(z_1 z_2) := \{(z_1, z_2) \in \mathbb{C}^2 : z_1 z_2 = 0\}$. One can check that this is an analytic set of pure dimension 1. On the other hand, the set $z_1 z_2 = 0 = z_1 z_3$ in $\mathbb{C}^3$ has dimension 2 at all points of the complex plane $\mathbb{C}_{23}$ and dimension 1 at the points of the punctured one-dimensional plane $\mathbb{C}_1 - \{0\}$. If $\mathcal{Z}$ is a pure $k$-dimensional, analytic subset(an analytic set which is also closed) in $\Omega$, it defines a $(m-k, m-k)$ current on $\Omega$ given by

$$[\mathcal{Z}](\alpha) := \int_{\text{reg}\mathcal{Z}} i^* \alpha,$$

where $\alpha$ is any compactly supported, smooth, $(k,k)$ form(or test form) on $\Omega$, $i : \text{reg}\mathcal{Z} \to \Omega$ is the canonical inclusion map and $i^*\alpha$ is the pullback of $\alpha$ by $i$ which is a smooth, $(k,k)$ form on reg$\mathcal{Z}$ [13, Theorem III.2.7]. Note that supp($i^*\alpha$) is not necessarily compact in reg$\mathcal{Z}$. So, the main difficulty is to show that the integral is well defined in the following sense:

Let $\{U_n\}_{n\in\mathbb{N}}$ be a countable, locally finite open cover of reg$\mathcal{Z}$ consisting of co-ordinate charts and $\{\rho_n\}_{n\in\mathbb{N}}$ be a partition of unity subordinate to this cover. Then $i^*\alpha = \sum_n \rho_n(i^*\alpha)$ on reg$\mathcal{Z}$ and $\int_{U_n}\rho_n(i^*\alpha)$ is well defined for each $n\in\mathbb{N}$. Now, if the sum $\sum_n\int_{U_n}\rho_n(i^*\alpha)$ is convergent for any such choice of $\{U_n\}$ and $\{\rho_n\}$, then we'll define

$$\int_{\mathrm{reg}\mathcal{Z}} i^*\alpha = \sum_n \int_{U_n}\rho_n(i^*\alpha).$$

Finally, it can be checked that the right hand side is independent of any such choice of $\{U_n\}$ and $\{\rho_n\}$ which makes this integral well defined.

However, if supp($\alpha$) is a compact subset of $\Omega\setminus$sng$\mathcal{Z}$, supp($i^*\alpha$) is compact in reg$\mathcal{Z}$ and hence the integral $\int_{\mathrm{reg}\mathcal{Z}} i^*\alpha$ is well defined, where sng$\mathcal{Z}$ denotes the singular points in $\mathcal{Z}$ and equals to $\mathcal{Z}\setminus$reg$\mathcal{Z}$. To see this, observe that

$$\mathrm{supp}(\alpha)\cap\mathcal{Z} = \mathrm{supp}(\alpha)\cap(\mathrm{reg}\mathcal{Z}\cup\mathrm{sng}\mathcal{Z}) = \mathrm{supp}(\alpha)\cap\mathrm{reg}\mathcal{Z}.$$

Since $\mathcal{Z}$ is closed and supp($\alpha$) is compact in $\Omega$, supp($\alpha$) $\cap$ reg$\mathcal{Z}$ = $i^{-1}$(supp($\alpha$)) is a compact subset in reg$\mathcal{Z}$. Finally, the result follows from the fact that supp($i^*\alpha$) $\subseteq i^{-1}$(supp($\alpha$)).

**Definition 4.1.1.** A holomorphic function $g:\Omega\to\mathbb{C}$ is said to be a defining function of the subset $\mathcal{Z}\subseteq\Omega$ if

i) $\mathcal{Z} = \{z\in\Omega : g(z) = 0\}$, and

ii) for any open subset $U$ of $\Omega$, if $f\in\mathcal{O}(U)$ vanishes on $\mathcal{Z}\cap U$, then $f = gh$ on $U$ for some holomorphic function $h:U\to\mathbb{C}$.

For any $x\in\mathcal{Z}$, if there exists an open subset $U_x\subseteq\Omega$ and a holomorphic function $g_x:U_x\to\mathbb{C}$ such that $g_x$ is a defining function of $\mathcal{Z}\cap U_x\subseteq U_x$, then we say that $g_x$ is a defining function of $\mathcal{Z}$ near $x$. From [9, Proposition 1.2.9] it follows that if $\mathcal{Z}$ is an analytic hypersurface in $\Omega$ and $x\in\mathcal{Z}$, then $\mathcal{Z}$ has a defining function near $x$. Recall that an analytic hypersurface is a pure $m-1$ diemensional analytic set in $\Omega$.

In what follows, we say that a holomorphic function $g:\Omega\to\mathbb{C}$ defines $\mathcal{Z}$ if $g$ is a defining function of $\mathcal{Z}$ in the sense of Definition 4.1.1

Let $f\in\mathcal{O}(\Omega)$ and $\mathcal{Z}$ be an analytic hypersurface in $\Omega$. For any point $x\in\mathcal{Z}$, the order $\mathrm{ord}_{\mathcal{Z},x}f$ of $f$ along $\mathcal{Z}$ at $x$ is defined to be the largest integer $k$ such that in the local ring $\mathcal{O}_x$ we have

$$(f)_x = (g)_x^k(h)_x,$$

where $g$ is the defining function of $\mathcal{Z}$ near $x$. With this definition we'll prove the following.

**Lemma 4.1.2.** *Let $\mathcal{Z}$ be an analytic hypersurface in $\Omega$ which is irreducible, that is, $\mathcal{Z}$ cannot be written as the union of two analytic hypersurfaces in $\Omega$ distinct from $\mathcal{Z}$. Then, for any $f \in \mathcal{O}(\Omega)$, the integer valued map $\phi$ on $\mathcal{Z}$ given by $\phi(x) = \mathrm{ord}_{\mathcal{Z},x} f$ is constant.*

*Proof.* Firstly, we will show that $\phi$ is constant on reg$\mathcal{Z}$ which, from [9, Section 5.3], is a connected $m-1$ dimensional manifold in $\Omega$. Choose an arbitrary point $x \in \mathrm{reg}\mathcal{Z}$. Then there exists a neighbourhood $U_x$ of $x$ in $\Omega$, a holomorphic function $g : U_x \to \mathbb{C}$ such that $g$ defines $\mathcal{Z} \cap U_x \subseteq U_x$. Since reg$\mathcal{Z}$ is an open subset of $\mathcal{Z}$, $g$ defines reg$\mathcal{Z}$ near $x$. But any complex submanifold of codimension 1 can be locally defined by an analytic submersion. So, in $\mathcal{O}_x$, $(g)_x$ is the germ of an analytic submersion at $x$ modulo a unit. As a result, $(g)_x$ is irreducible. To see this, observe that $g$ is a submersion at $x$ or equivalently, the total order of $g$ at $x$, namely $\mathrm{ord}_x g$ equals 1. If we have $(g)_x = (g_1)_x (g_2)_x$ in $\mathcal{O}_x$, then

$$\mathrm{ord}_x g = \mathrm{ord}_x g_1 + \mathrm{ord}_x g_2 = 1$$

which implies either $\mathrm{ord}_x g_1$ or $\mathrm{ord}_x g_2$ equals 0. Thus, one of the factors of $(g)_x$ is a unit.

From the definition of $\mathrm{ord}_{\mathcal{Z},x} f$, we have $(f)_x = (g)_x^{\mathrm{ord}_{\mathcal{Z},x} f} (h)_x$ in $\mathcal{O}_x$, where $(g)_x$ does not divide $(h)_x$. Since $(g)_x$ is irreducible, this means $(g)_x$ and $(h)_x$ are relatively prime in $\mathcal{O}_x$. Following [25, Proposition 1.1.35] we can find a neighbourhood $V_x$ of $x$ in $U_x$ such that

a) $f = g^{\mathrm{ord}_{\mathcal{Z},x} f} h$ on $V_x$,

b) for all $y \in V_x$, $(g)_y$ and $(h)_y$ are relatively prime.

Now, pick an arbitrary point $y \in V_x \cap \mathrm{reg}\mathcal{Z}$. Clearly, $(f)_y = (g)_y^{\mathrm{ord}_{\mathcal{Z},x} f} (h)_y$ in $\mathcal{O}_y$ and $(g)_y$, $(h)_y$ are relatively prime. Clearly, $g$ is a defining function of reg$\mathcal{Z}$ near $y$ and following similar arguments as above, we obtain that $(g)_y$ is irreducible. As a result, $(g)_y$ does not divide $(h)_y$ which implies $\mathrm{ord}_{\mathcal{Z},y} f = \mathrm{ord}_{\mathcal{Z},x} f$. Thus, $\phi$ is locally constant on the connected set reg$\mathcal{Z}$ proving our first claim.

Next, pick an arbitrary point $x \in \mathrm{sng}\mathcal{Z}$. We will show that $\mathrm{ord}_{\mathcal{Z},x} f = c$, where $c$ is the constant value of $\phi$ on reg$\mathcal{Z}$. Let $g$ be a defining function of $\mathcal{Z}$ near $x$. Then we can find a holomorphic function $h$ in a neighbourhood of $x$ such that $(f)_x = (g)_x^{\mathrm{ord}_{\mathcal{Z},x} f} (h)_x$ in $\mathcal{O}_x$ and $(g)_x$ does not divide $(h)_x$. Since $\mathcal{O}_x$ is an UFD, we have

$$(g)_x = (g_1)_x^{n_1} \cdots (g_k)_x^{n_k}$$

in $\mathcal{O}_x$, where $n_1, \ldots, n_k \in \mathbb{N}$, $g_1, \ldots, g_k$ are holomorphic functions defined near $x$ with the property that $(g_1)_x, \ldots, (g_k)_x$ are distinct irreducible factors of $g_x$ in $\mathcal{O}_x$. Observe that $n_1 = \cdots = n_k = 1$. Otherwise, $g_1 \cdots g_k$ vanishes on $\mathcal{Z}$ near $x$ and $g$ does not divide $g_1 \cdots g_k$ in any neighbourhood of $x$ contradicting that $g$ is a defining function of $\mathcal{Z}$ near $x$.

Since $(g)_x$ does not divide $(h)_x$, one of its irreducible factors does not divide $(h)_x$. This is because distinct irreducible factors of $(g)_x$ are relatively prime. If all irreducible factors of $(g)_x$ divide $(h)_x$, then their least common multiple(lcm) divides $(h)_x$. But the lcm of relatively prime elements is their product which is $(g)_x$ in this case. Without loss of generality, let $(g_1)_x$ does not divide $(h)_x$. Then they are relatively prime. As a result, there exists a neighbourhood $V_x$ of $x$ in $\Omega$ such that

i) $(g_1)_y$ and $(h)_y$ are relatively prime for all $y \in V_x$,

ii) $(g_i)_y, (g_j)_y$ are relatively prime for all $y \in V_x$, $i, j \in \{1, \ldots, k\}$, $i \neq j$,

iii) $f = g^{\mathrm{ord}_{\mathcal{Z},x} f} h$ on $V_x$ and

iv) $g = g_1 \cdots g_k$ on $V_x$.

Now, we claim the following: There exists a regular point of $\mathcal{Z}$ in $Z(g_1) \cap V_x$.
If possible, let $(Z(g_1) \cap V_x) \cap \mathrm{reg}\mathcal{Z}$ be empty. Then

$$\mathrm{reg}(\mathcal{Z} \cap V_x) = \mathrm{reg}(Z(g) \cap V_x) = \cup_{i=2}^{k} \mathrm{reg}(Z(g_i) \cap V_x),$$

whereas $Z(g) \cap V_x = \cup_{i=1}^{k} (Z(g_i) \cap V_x)$. Taking closure in $V_x$, we get

$$Z(g) \cap V_x = \overline{\mathrm{reg}(Z(g) \cap V_x)} = \cup_{i=2}^{k} \overline{\mathrm{reg}(Z(g_i) \cap V_x)} = \cup_{i=2}^{k} (Z(g_i) \cap V_x).$$

Thus, $Z(g_1) \cap V_x \subseteq \cup_{i=2}^{k} (Z(g_i) \cap V_x)$ and from [25, Corollary 1.1.19] we get $(g_1)_x$ divides $(g_2)_x \cdots (g_k)_x$ in $\mathcal{O}_x$. Since $(g_1)_x$ is an irreducible element of an UFD, it is a prime element and hence there exists $j \in \{2, \ldots, k\}$ such that $(g_1)_x$ divides $(g_j)_x$ in $\mathcal{O}_x$. But this contradicts the fact that $(g_1)_x, \ldots, (g_k)_x$ are distinct irreducible factors of $(g)_x$.

Choose a point $y \in (Z(g_1) \cap V_x) \cap \mathrm{reg}\mathcal{Z}$. From condition iii) we have $(f)_y = (g)_y^{\mathrm{ord}_{\mathcal{Z},x} f} (h)_y$ in $\mathcal{O}_y$. Moreover, from condition i) and iv) it follows that $(g)_y$ does not divide $(h)_y$. This implies $\mathrm{ord}_{\mathcal{Z},x} f = \mathrm{ord}_{\mathcal{Z},y} f = c$ which proves the lemma. □

The constant attained in Lemma 4.1.2 is called the order of $f$ along $\mathcal{Z}$ and is denoted by $\mathrm{ord}_{\mathcal{Z}} f$. Now, we will state the *Lelong-Poincaré* formula.

**Theorem 4.1.3.** *Suppose $f$ is a holomorphic function on $\Omega$ which is not identically zero. Then $\log|f|$ is a plurisubharmonic function and it is pluriharmonic outside the zero set $Z(f)$ of $f$. Moreover, as (1,1) currents on $\Omega$,*

$$\frac{i}{2\pi} \partial\bar{\partial} \log|f|^2 = \sum_i m_i [V_i],$$

*where $V_i$ is an irreducible component of $Z(f)$ and $m_i = \mathrm{ord}_{V_i} f$, for $i$ in some locally finite set.*

Note that in the left hand side we have taken the distributional derivatives of $\log|f|^2$. This function is not differentiable around any point in $Z(f)$. However, it is locally integrable on $\Omega$ and hence can be considered as a distribution on $\Omega$. So, its distributional derivatives exist. The sum in the right hand side is locally finite because the collection of all irreducible components of an analytic subset in $\Omega$ is locally finite [9, Theorem 5.4]. Also, for any holomorphic function $f:\Omega\to\mathbb{C}$ one can check that $Z(f)$ is an analytic subset of $\Omega$ with pure dimension $m-1$. So, for each $i$, the current $[V_i]$ is well defined.

## 4.2 Support of currents defined by a holomorphic function

Let $T$ be a $(k,k)$ current on $\Omega$ and $k\le m$. The support of $T$, supp$(T)$ is defined as the smallest closed subset of $\Omega$ such that $T$ vanishes on $\Omega\setminus\text{supp}(T)$, i.e., $T(\alpha)=0$ for any smooth $(m-k,m-k)$ form $\alpha$ on $\Omega$ with compact support in $\Omega\setminus\text{supp}(T)$. This leads us to the following lemma.

**Lemma 4.2.1.** *For any holomorphic function* $f:\Omega\to\mathbb{C}$, $\text{supp}\big([Z(f)]\big)=Z(f)$.

*Proof.* From the definition of support it follows that

$$\Omega\setminus Z(f)\subseteq\Omega\setminus\text{supp}\big([Z(f)]\big)\quad\text{or equivalently,}\quad\text{supp}\big([Z(f)]\big)\subseteq Z(f).$$

To prove the converse part, note that it is enough to show $\text{reg}Z(f)\subseteq\text{supp}\big([Z(f)]\big)$. Fix a point $x\in\text{reg}Z(f)$ and choose any neighbourhood $U_x$ of $x$ in $\Omega$. We want to show that there exists a $(m-1,m-1)$ test form $\alpha$ on $\Omega$ such that $\text{supp}(\alpha)\subseteq U_x$ and $[Z(f)](\alpha)\neq 0$.

Since $\text{reg}Z(f)$ is a complex manifold of dimension $m-1$, we can find a neighbourhood $V_x$ of $x$ in $\Omega$, a biholomorphic map $\psi:V_x\to\psi(V_x)$ such that

a) $\psi(x)=0$ and

b) $\psi\big(V_x\cap\text{reg}Z(f)\big)=\{z_1=0\}\cap\psi(V_x)$.

Consequently, $\Big(V_x\cap\text{reg}Z(f),P_1\circ\psi|_{V_x\cap\text{reg}Z(f)}\Big)$ is a chart of $\text{reg}Z(f)$ around $x$, where $P_1:\mathbb{C}^m\to\mathbb{C}^{m-1}$ is the map given by $P_1(z_1,z')=z'$, $(z_1,z')\in\mathbb{C}^m$. Furthermore, we can assume that $V_x\subseteq\Omega\setminus\text{sng}Z(f)$. This is because $\text{sng}Z(f)$ is a closed subset of $\Omega$ and $x\notin\text{sng}Z(f)$. If we write $\psi=(\psi_1,\dots,\psi_m)$, then $P_1\circ\psi|_{V_x\cap\text{reg}Z(f)}=\big(\psi_2|_{V_x\cap\text{reg}Z(f)},\dots,\psi_m|_{V_x\cap\text{reg}Z(f)}\big)$. Now, choose a smooth function $\zeta_x:\Omega\to\mathbb{C}$ such that

i) $\zeta_x\ge 0$ on $\Omega$,

ii) $\zeta_x\equiv 1$ on a compact set $K_x$ containing $x$ in $U_x\cap V_x$ with non-empty interior and

iii) $\operatorname{supp}(\zeta_x) \subseteq U_x \cap V_x$.

Finally, define $\alpha$ as follows:

$$\alpha = \begin{cases} \zeta_x (d\psi_2 \wedge d\bar{\psi}_2) \wedge \ldots \wedge (d\psi_m \wedge d\bar{\psi}_m) & \text{on } V_x \\ 0 & \text{outside } V_x. \end{cases}$$

Clearly, $\operatorname{supp}(\alpha) \subseteq U_x \cap V_x$. Also, since $\operatorname{reg} Z(f)$ is a complex manifold, any atlas is orientable. If $i_f : \operatorname{reg} Z(f) \to \Omega$ is the canonical inclusion, then

$$\begin{aligned}
[Z(f)](\alpha) &= \int_{\operatorname{reg} Z(f)} i_f^* \alpha \\
&= \int_{V_x \cap \operatorname{reg} Z(f)} i_f^* \alpha \\
&= \int_{V_x \cap \operatorname{reg} Z(f)} \zeta_x (d\psi_2 \wedge d\bar{\psi}_2) \wedge \ldots \wedge (d\psi_m \wedge d\bar{\psi}_m) \\
&= \int_{(P_1 \circ \psi)(V_x \cap \operatorname{reg} Z(f))} \Big( \big( P_1 \circ \psi|_{V_x \cap \operatorname{reg} Z(f)} \big)^{-1} \Big)^* \zeta_x (d\psi_2 \wedge d\bar{\psi}_2) \wedge \ldots \wedge (d\psi_m \wedge d\bar{\psi}_m) \\
&= \int_{(z_2, \ldots, z_m) : (0, z_2, \ldots, z_m) \in \psi(V_x)} \big( \psi^{-1}|_{\{z_1 = 0\} \cap \psi(V_x)} \circ i_0 \big)^* \zeta_x (d\psi_2 \wedge d\bar{\psi}_2) \wedge \ldots \wedge (d\psi_m \wedge d\bar{\psi}_m) \\
&= \int_{(z_2, \ldots, z_m) : (0, z_2, \ldots, z_m) \in \psi(V_x)} (\zeta_x \circ \psi^{-1} \circ i_0)(dz_2 \wedge d\bar{z}_2) \wedge \ldots \wedge (dz_m \wedge d\bar{z}_m),
\end{aligned}$$

where $i_0 : \mathbb{C}^{m-1} \to \mathbb{C}^m$ is the holomorphic map given by $i_0(z_2, \ldots, z_m) = (0, z_2, \ldots, z_m)$. The second equality is true because $\operatorname{supp}(i_f^* \alpha) \subseteq i_f^{-1}(\operatorname{supp}(\alpha)) \subseteq V_x \cap \operatorname{reg} Z(f)$. Finally, in the right hand side we have the integration of the function $\zeta_x \circ \psi^{-1} \circ i_0$ over an open subset of $\mathbb{C}^{m-1}$ with respect to the Lebesgue measure. Since the function $\zeta_x \circ \psi^{-1} \circ i_0$ is non-negative over this open subset and strictly positive over a subdomain, the resulting integration is nonzero. This proves $\operatorname{reg} Z(f) \subseteq \operatorname{supp}\big([Z(f)]\big)$ which implies $Z(f) \subseteq \operatorname{supp}\big([Z(f)]\big)$. □

The next two results generalize Lemma 4.2.1.

**Lemma 4.2.2.** *For any pure $k$-dimensional analytic subset $\mathcal{Z}$ of $\Omega$, $\operatorname{supp}\big([\mathcal{Z}]\big) = \mathcal{Z}$.*

*Proof.* From the definition of support we obtain that $\operatorname{supp}\big([\mathcal{Z}]\big) \subseteq \mathcal{Z}$. To prove $\operatorname{supp}\big([\mathcal{Z}]\big) \supseteq \mathcal{Z}$, take an arbitrary element $x \in \operatorname{reg} \mathcal{Z}$. Following similar arguments given in Lemma 4.2.1 we can show that for any open set $U_x$ of $x$ in $\Omega$, there exists a $(m-k, m-k)$ test form $\alpha$ on $\Omega$ such that $\operatorname{supp}(\alpha) \subseteq U_x$ and $[\mathcal{Z}](\alpha) \neq 0$. This shows $\operatorname{reg} \mathcal{Z} \subseteq \operatorname{supp}\big([\mathcal{Z}]\big)$. But $\operatorname{supp}\big([\mathcal{Z}]\big)$ is closed in $\Omega$ and $\operatorname{reg} \mathcal{Z}$ is dense in $\mathcal{Z}$. Thus, $\mathcal{Z} = \overline{(\operatorname{reg} \mathcal{Z})} \subseteq \operatorname{supp}\big([\mathcal{Z}]\big)$. □

**Lemma 4.2.3.** *For any non-zero holomorphic function $f : \Omega \to \mathbb{C}$, $\operatorname{supp}(\partial \bar{\partial} \log |f|^2) = Z(f)$.*

*Proof.* If $\{V_i : i \in I\}$ is the collection of all irreducible components of $Z(f)$, then from Theorem 4.1.3 it is clear that

$$\operatorname{supp}(\partial\bar\partial \log|f|^2) = \operatorname{supp}\Big(\sum_i m_i[V_i]\Big),$$

where $m_i = \operatorname{ord}_{V_i} f$. Take any $(m-1, m-1)$ current $\alpha$ such that $\operatorname{supp}(\alpha)$ is a compact subset of $\Omega \setminus Z(f) = \Omega \setminus (\cup_{i\in I} V_i)$. Then $\big(\sum_i m_i[V_i]\big)(\alpha) = \sum_i m_i\big([V_i](\alpha)\big) = 0$ which proves

$$\operatorname{supp}\Big(\sum_i m_i[V_i]\Big) \subseteq Z(f).$$

To prove the reverse inclusion, it is enough to show that $\operatorname{reg}Z(f) \subseteq \operatorname{supp}\big(\sum_i m_i[V_i]\big)$. From [9, Theorem 5.4] it follows that any irreducible component $V_i$ of $Z(f)$ is of the form $\overline{S}_i$(closure in $\Omega$), where $S_i$ is a connected component of $\operatorname{reg}Z(f)$. Firstly, we claim that $\operatorname{reg}Z(f) = \cup_{i\in I} S_i$. Clearly, $\operatorname{reg}Z(f) \supseteq \cup_{i\in I} S_i$. Now, if possible let $\operatorname{reg}Z(f) \nsubseteq \cup_{i\in I} S_i$. Then there exists a connected component $S$ of $\operatorname{reg}Z(f)$ which is not of the form $S_i$ for any $i \in I$. This means $\overline{S}$ is not of the form $V_i$ for any $i \in I$. But $\overline{S}$ is an irreducible component of $Z(f)$ [9, Lemma 5.4]. This contradicts that any irreducible component of $Z(f)$ is of the form $V_i$.

Now, take an arbitrary point $x \in \operatorname{reg}Z(f)$ and let $W_x$ be an open set in $\Omega$ containing $x$. Then $x \in S_i$ for some $i \in I$. We'll show that $x \notin \cup_{j\neq i} V_j$. If possible, let $x \in \cup_{j\neq i} V_j$. Since the collection $\{V_i\}_{i\in I}$ is locally finite, $x$ has a neighbourhood $V_x$ in $W_x$ which intersects finitely many elements in the collection. Suppose all the elements in $\{V_i\}_{i\in I}$ that intersects $V_x$ are given by $\{V_i, V_{j_1}, \ldots, V_{j_k}\}$. Then we have

$$Z(f)_x = (\cup_{j\in I} V_j)_x = (V_i)_x \cup (V_{j_1})_x \cup \cdots \cup (V_{j_k})_x$$

as analytic germs. In other words, $Z(f)_x$ is reducible. But $\operatorname{reg}Z(f)$ is a complex submanifold of codimension 1. So, $Z(f)$ is defined by an analytic submersion $g$ near $x$. As a result, $Z(f)_x = Z(g)_x$. Also, the total order $\operatorname{ord}_x g$ of $g$ at $x$ equals 1 which implies $g_x$ is irreducible. Consequently, by [25, Lemma 1.1.28] we obtain that $Z(g)_x$ is irreducible which is a contradiction.

Since $V_x \cap (\cup_{j\neq i} V_j) = V_x \cap (V_{j_1} \cup \cdots \cup V_{j_k})$, it is a closed subset in $V_x$ which does not contain $x$. So, there exists a neighbourhood $U_x$ of $x$ in $V_x$ such that $U_x \cap (\cup_{j\neq i} V_j)$ is an empty set. Shrinking $U_x$ if necessary, we can also assume that $U_x \subseteq \Omega \setminus \operatorname{sng}Z(f)$. Consequently,

$$U_x \cap V_i = U_x \cap Z(f) = U_x \cap \operatorname{reg}Z(f)$$

is a $(m-1)$-dimensional submanifold of $\Omega$ which means $x$ is a regular point of $V_i$. Now, similar to the proof of Lemma 4.2.2, we can find a $(m-1, m-1)$ test form $\alpha$ on $\Omega$ such that $\operatorname{supp}(\alpha) \subseteq U_x \subseteq W_x$ and $[V_i](\alpha) \neq 0$. This gives

$$\Big(\sum_{j\in I} m_j[V_j]\Big)(\alpha) = \sum_{j\in I} m_j\Big([V_j](\alpha)\Big) = m_i[V_i](\alpha) \neq 0.$$

Since $x$ and $W_x$ are arbitrarily chosen, we can conclude that $\operatorname{reg}Z(f) \subseteq \operatorname{supp}\big(\sum_i m_i[V_i]\big)$ which proves the desired result. □

## 4.3 Description of various submodules and their complete invariants

A Hilbert module $\mathscr{H}$ over the polynomial ring $\mathbb{C}[z]$ is said to be an analytic Hilbert module if it satisfies the following properties:

a) it consists of holomorphic functions over a bounded domain $\Omega \subseteq \mathbb{C}^m$ and possesses a reproducing kernel $K_{\mathscr{H}}$;

b) the polynomial ring $\mathbb{C}[\underline{z}]$ is dense in it.

We begin by proving a useful Lemma.

**Lemma 4.3.1.** *Let $\mathscr{H}, \mathscr{H}' \subseteq \mathscr{O}(\Omega)$ be analytic Hilbert modules for a bounded domain $\Omega \subseteq \mathbb{C}^m$. Suppose there is a $L : \mathscr{H} \to \mathscr{H}'$ is a bijective module map. Then there exists a non-vanishing holomorphic function $a : \Omega \to \mathbb{C}$ such that $L(h) = ah$ for all $h \in \mathscr{H}$.*

*Proof.* Firstly, we'll show that $L^*\big(\cap_{i=1}^m \ker(M_{z_i}^{\mathscr{H}'} - w_i)^*\big) = \cap_{i=1}^m \ker(M_{z_i}^{\mathscr{H}} - w_i)^*$. Choose an arbitrary element $h'$ in $\cap_{i=1}^m \ker(M_{z_i}^{\mathscr{H}'} - w_i)^*$. Since $L$ is a module map, we have

$$\big((M_{z_i}^{\mathscr{H}})^* \circ L^*\big)(h') = \big(L^* \circ (M_{z_i}^{\mathscr{H}'})^*\big)(h') = L^*(\bar{w}_i h') = \bar{w}_i L^*(h'),$$

for all $i$, $1 \le i \le m$. This proves $L^*\big(\cap_{i=1}^m \ker(M_{z_i}^{\mathscr{H}'} - w_i)^*\big) \subseteq \cap_{i=1}^m \ker(M_{z_i}^{\mathscr{H}} - w_i)^*$. Similarly, one can show that $(L^{-1})^*\big(\cap_{i=1}^m \ker(M_{z_i}^{\mathscr{H}} - w_i)^*\big) \subseteq \cap_{i=1}^m \ker(M_{z_i}^{\mathscr{H}'} - w_i)^*$ or equivalently, $\cap_{i=1}^m \ker(M_{z_i}^{\mathscr{H}} - w_i)^* \subseteq L^*\big(\cap_{i=1}^m \ker(M_{z_i}^{\mathscr{H}} - w_i)^*\big)$ which proves the claim.

Since $\mathscr{H}, \mathscr{H}'$ are analytic Hilbert modules, their reproducing kernels $K_{\mathscr{H}}, K_{\mathscr{H}'}$ are sharp. As a result,

$$< \{L^* K_{\mathscr{H}'}(\cdot, w)\} > = < \{K_{\mathscr{H}}(\cdot, w)\} >,$$

for all $w \in \Omega$. This means $w \mapsto L^* K_{\mathscr{H}'}(\cdot, w), w \mapsto K_{\mathscr{H}}(\cdot, w)$ are two anti-holomorphic frames of the line bundle $\bigsqcup_{w \in \Omega} < \{K_{\mathscr{H}}(\cdot, w)\} >$. Consequently, there exists a holomorphic function $a : \Omega \to \mathbb{C}$ such that $L^* K_{\mathscr{H}'}(\cdot, w) = \overline{a(w)} K_{\mathscr{H}}(\cdot, w)$. Similarly, $(L^{-1})^* K_{\mathscr{H}}(\cdot, w) = \overline{b(w)} K_{\mathscr{H}'}(\cdot, w)$, for some holomorphic function $b : \Omega \to \mathbb{C}$. Combining these we get

$$K_{\mathscr{H}'}(\cdot, w) = (L^{-1})^* L^* K_{\mathscr{H}'}(\cdot, w) = \overline{a(w)}(L^{-1})^* K_{\mathscr{H}}(\cdot, w) = \overline{a(w) b(w)} K_{\mathscr{H}'}(\cdot, w)$$

which implies

$$K_{\mathscr{H}'}(\cdot, w)\big(1 - \overline{a(w) b(w)}\big) = 0,$$

for all $w \in \Omega$. Since $1 \in \mathscr{H}'$, $K_{\mathscr{H}'}(w, w) \neq 0$ which gives $a(w)b(w) = 1$ for all $w \in \Omega$. This proves that $a$ is non-vanishing on $\Omega$. Finally, for any $z \in \Omega$, $h \in \mathscr{H}$,

$$(Lh)(z) = \langle Lh, K_{\mathscr{H}'}(\cdot, z)\rangle_{\mathscr{H}'} = \langle h, L^* K_{\mathscr{H}'}(\cdot, z)\rangle_{\mathscr{H}} = a(z) h(z)$$

which proves the lemma. $\square$

Let $\mathscr{H}$ be an analytic Hilbert module in $\mathscr{O}(\Omega)$, $\mathscr{M}$ be a submodule of $\mathscr{H}$ given by $\mathscr{M} = \{f \in \mathscr{H} : f = 0 \text{ on } \mathscr{Z}\}$, where $\mathscr{Z}$ is an analytic hypersurface in $\Omega$ and $i_{\mathscr{M}} : \mathscr{M} \to \mathscr{H}$ be the canonical inclusion map. Clearly, $i_{\mathscr{M}}$ is a module map. So, for each $w \in \Omega$, it induces the map

$$i_{\mathscr{M}} \otimes_{\mathbb{C}[\underline{z}]} 1_w : \mathscr{M} \otimes_{\mathbb{C}[\underline{z}]} \mathbb{C}_w \to \mathscr{H} \otimes_{\mathbb{C}[\underline{z}]} \mathbb{C}_w,$$

where $\mathbb{C}_w$ is the one dimensional module over $\mathbb{C}[\underline{z}]$ given by, $p \cdot \lambda := p(w)\lambda$, $p \in \mathbb{C}[\underline{z}]$, $\lambda \in \mathbb{C}$. If we assume that $\varphi$ is a global defining function of $\mathscr{Z}$ in $\Omega$, $\varphi \in \mathscr{H}$ and $\text{rank}(\mathscr{M}) = 1$, then for all $w \in \Omega$,

$$\mathscr{M} \otimes_{\mathbb{C}[\underline{z}]} \mathbb{C}_w = < \{M_\varphi \chi(\cdot, w)\} > \otimes\, \mathbb{C} \quad \text{and} \quad \mathscr{H} \otimes_{\mathbb{C}[\underline{z}]} \mathbb{C}_w = < \{K_{\mathscr{H}}(\cdot, w)\} > \otimes\, \mathbb{C},$$

where $K_{\mathscr{M}}(z, w) = \varphi(z)\overline{\varphi(w)}\chi(z, w)$, $K_{\mathscr{H}}, K_{\mathscr{M}}$ are the reproducing kernels of $\mathscr{H}, \mathscr{M}$ respectively. Thus, we canonically obtain a map $i_{\mathscr{M}}(w) : < \{M_\varphi \chi(\cdot, w)\} > \to < \{K_{\mathscr{H}}(\cdot, w)\} >$ given by

$$i_{\mathscr{M}}(w) := P_{<\{K_{\mathscr{H}}(\cdot,w)\}>}\, i_{\mathscr{M}}|_{<\{M_\varphi \chi(\cdot,w)\}>},$$

where $P_{<\{K_{\mathscr{H}}(\cdot,w)\}>} : \mathscr{H} \to < \{K_{\mathscr{H}}(\cdot, w)\} >$ is the canonical projection map on the vector space $< \{K_{\mathscr{H}}(\cdot, w)\} >$ for all $w \in \Omega$. Define,

$$\mathscr{K}_{i_{\mathscr{M}}}(w) = \partial\bar{\partial} \log \| i_{\mathscr{M}}(w) \|^2.$$

Note that $i_{\mathscr{M}}(w)$ vanishes at every point $w \in \mathscr{Z}$ and the right hand side is thought of as a (1,1) current on $\Omega$. Finally, consider the line bundles $\bigsqcup_{w\in\Omega} < \{M_\varphi \chi(\cdot, w)\}$ and $\bigsqcup_{w\in\Omega} < \{K_{\mathscr{H}}(\cdot, w)\} >$. If we denote their curvatures as $\mathscr{K}_{\mathscr{M}}$ and $\mathscr{K}_{\mathscr{H}}$ respectively, then from [14, Theorem 1.4] we obtain that

$$\mathscr{K}_{i_{\mathscr{M}}} - \mathscr{K}_{\mathscr{M}} + \mathscr{K}_{\mathscr{H}} = \partial\bar{\partial} \log |\varphi|^2$$

as (1,1) currents on $\Omega$. This gives us the following theorem.

### 4.3.1 The first main theorem

**Theorem 4.3.2.** *Let $\Omega$ be a bounded domain in $\mathbb{C}^m$ and $\mathscr{H}, \mathscr{H}'$ be analytic Hilbert modules in $\mathscr{O}(\Omega)$. Suppose $\varphi, \psi : \Omega \to \mathbb{C}$ are holomorphic functions that define $Z(\varphi) := \{z \in \Omega : \varphi(z) = 0\}$, $Z(\psi) := \{z \in \Omega : \varphi(z) = 0\}$, respectively. Set $\mathscr{M} = \{f \in \mathscr{H} : f = 0 \text{ on } Z(\varphi)\}$, $\mathscr{M}' = \{g \in \mathscr{H}' : g = 0 \text{ on } Z(\psi)\}$. Assume that $\varphi \in \mathscr{H}, \psi \in \mathscr{H}'$ and that $\text{rank}(\mathscr{M}) = \text{rank}(\mathscr{M}') = 1$. If $L : \mathscr{H} \to \mathscr{H}'$ is a bijective module map, then the following are equivalent*

*a)* $L(\mathscr{M}) = \mathscr{M}'$;

*b)* $Z(\varphi) = Z(\psi)$;

*c)* $\mathscr{K}_{i_{\mathscr{M}}} - \mathscr{K}_{\mathscr{M}} + \mathscr{K}_{\mathscr{H}} = \mathscr{K}_{i_{\mathscr{M}'}} - \mathscr{K}_{\mathscr{M}'} + \mathscr{K}_{\mathscr{H}'}$ *as (1,1) currents on* $\Omega$.

*Proof.* a)⇔b) Since $L : \mathscr{H} \to \mathscr{H}'$ is a bijective module map, there exists a non-vanishing holomorphic function $a : \Omega \to \mathbb{C}$ such that $L(h) = ah$, for all $h \in \mathscr{H}$. Observe that $\varphi \in \mathscr{M}$. So, $L(\mathscr{M}) = \mathscr{M}'$ implies $L(\varphi) = a\varphi \in \mathscr{M}'$. As a result, $a\varphi$ vanishes on $Z(\psi)$ or equivalently, $Z(a\varphi) = Z(\varphi) \supseteq Z(\psi)$. Similarly, from $L^{-1}$ we'll obtain $Z(\psi) \supseteq Z(\varphi)$ proving $Z(\varphi) = Z(\psi)$.

For the converse part, take an arbitrary element $m \in \mathscr{M} \subseteq \mathscr{H}$. Since $m$ vanishes on $Z(\varphi) = Z(\psi)$ and $L(m) = am$, $L(m)$ vanishes on $Z(\psi)$. Moreover, $L(m) \in \mathscr{H}'$. This implies $L(m) \in \mathscr{M}'$ which proves $L(\mathscr{M}) \subseteq \mathscr{M}'$. Similarly, from $L^{-1}$ one can show that $L^{-1}(\mathscr{M}') \subseteq \mathscr{M}$ or equivalently, $\mathscr{M}' \subseteq L(\mathscr{M})$. Thus, $L(\mathscr{M}) = \mathscr{M}'$.

b)⇔c) Since both $\varphi$ and $\psi$ are defining functions of the analytic hypersurface $Z(\varphi) = Z(\psi)$ on $\Omega$, there exists a non-vanishing holomorphic function $f : \Omega \to \mathbb{C}$ such that $\varphi = f\psi$ on $\Omega$. Also, from the discussion prior to this theorem it follows that

$$\mathcal{K}_{i_{\mathscr{M}}} - \mathcal{K}_{\mathscr{M}} + \mathcal{K}_{\mathscr{H}} = \partial\bar{\partial}\log|\varphi|^2$$

and

$$\mathcal{K}_{i_{\mathscr{M}'}} - \mathcal{K}_{\mathscr{M}'} + \mathcal{K}_{\mathscr{H}'} = \partial\bar{\partial}\log|\psi|^2.$$

Next, we claim that $\partial\bar{\partial}\log|\varphi|^2 = \partial\bar{\partial}\log|f\psi|^2 = \partial\bar{\partial}\log|f|^2 + \partial\bar{\partial}\log|\psi|^2$ as (1,1) currents on $\Omega$. This follows from the sequence of observations given below.

1) From Theorem 4.1.3 it follows that for any holomorphic function $u : \Omega \to \mathbb{C}$, $\log|u|^2$ is a plurisubharmonic function on $\Omega$.

2) Every plurisubharmonic function on $\Omega$ is locally integrable on $\Omega$.

3) $\log|\varphi|^2 = \log|f\psi|^2 = \log|f|^2 + \log|\psi|^2$ as locally integrable functions on $\Omega$.

4) Since the space of locally integrable functions can be embedded in the space of distributions, we have the equality $\log|\varphi|^2 = \log|f\psi|^2 = \log|f|^2 + \log|\psi|^2$ as distributions, or equivalently, as (0,0) currents on $\Omega$.

5) Applying the operators $\partial$ and $\bar{\partial}$ on the space of (0,0) currents and using their linearity, we obtain $\partial\bar{\partial}\log|\varphi|^2 = \partial\bar{\partial}\log|f\psi|^2 = \partial\bar{\partial}\log|f|^2 + \partial\bar{\partial}\log|\psi|^2$ as (1,1) currents on $\Omega$.

But $f$ is non-vanishing, so $\partial\bar{\partial}\log|f|^2 = 0$. So, $\partial\bar{\partial}\log|\varphi|^2 = \partial\bar{\partial}\log|\psi|^2$ as (1,1) currents on $\Omega$ which proves part c)

Conversely, let $\mathcal{K}_{i_{\mathscr{M}}} - \mathcal{K}_{\mathscr{M}} + \mathcal{K}_{\mathscr{H}} = \mathcal{K}_{i_{\mathscr{M}'}} - \mathcal{K}_{\mathscr{M}'} + \mathcal{K}_{\mathscr{H}'}$ as (1,1) currents on $\Omega$. Following previous arguments, this gives $\partial\bar{\partial}\log|\varphi|^2 = \partial\bar{\partial}\log|\psi|^2$. Finally, applying Lemma 4.2.3 we obtain that $Z(\varphi) = Z(\psi)$ which proves b). $\square$

**Example 4.3.3.** Let $\mathscr{H}$ denote the Hardy module $H^2(\mathbb{D}^2)$. Then the set $\{e_{i,j} : i, j \in \mathbb{N} \cup \{0\}\}$ is an orthonormal basis of $\mathscr{H}$, where $e_{i,j}(z_1, z_2) = z_1^i z_2^j$, for any $z_1, z_2 \in \mathbb{D}$. Define $M_{z_1}^{\mathscr{H}}, M_{z_2}^{\mathscr{H}}$ to be the pointwise multiplication operators on $\mathscr{H}$ by $z_1, z_2$ respectively. Then, one can easily check that both these operators are unilateral shifts with weight sequence 1.

Next, for each $n \in \mathbb{N} \cup \{0\}$, define the sequence $v_n$ of positive real numbers as follows:

$$v_0 = 1, v_1 = \frac{1}{4} \text{ and } v_n = \frac{v_{n-1}\left(1 + \frac{1}{n-1}\right)^2}{\left(1 + \frac{1}{n}\right)^2}, n \geq 2.$$

Then the set $\mathscr{H}' = \left\{f \in \mathscr{O}(\mathbb{D}^2) : \text{if } f(z_1, z_2) = \sum_{i,j \geq 0} \hat{f}(i, j) z_1^i z_2^j, \text{ then } \sum_{i,j \geq 0} \frac{|\hat{f}(i,j)|^2}{v_i v_j} < \infty\right\}$ with the inner product $\langle f, g \rangle_{\mathscr{H}'} := \sum_{i,j \geq 0} \frac{\hat{f}(i,j)\overline{\hat{g}(i,j)}}{v_i v_j}$ is a Hilbert space consists of holomorphic functions on $\mathbb{D}^2$. Clearly, the set $\{e'_{i,j} : i, j \in \mathbb{N} \cup \{0\}\}$ is an orthonormal basis of $\mathscr{H}'$, where $e'_{i,j}(z_1, z_2) = \sqrt{v_i v_j} z_1^i z_2^j$. One can check that for each $w \in \mathbb{D}$, the sum $\sum_{n \geq 0} v_n (|w|^2)^n$ is convergent. This implies $\sum_{i,j \geq 0} |e'_{i,j}(w_1, w_2)|^2$ is convergent. As a result, $\mathscr{H}'$ is a reproducing kernel Hilbert space with kernel $K_{\mathscr{H}'}(z, w) = \sum_{i,j \geq 0} e'_{i,j}(z)\overline{e'_{i,j}(w)} = \sum_{i,j \geq 0} v_i v_j z_1^i z_2^j \bar{w}_1^i \bar{w}_2^j$, for $z = (z_1, z_2), w = (w_1, w_2) \in \mathbb{D}^2$. Observe that $\mathscr{H}'$ can be considered as a Hilbert module over $\mathbb{C}[z_1, z_2]$ with module action being pointwise multiplication. Also, from the construction of $\mathscr{H}'$ it is clear that $\mathbb{C}[z_1, z_2]$ is a dense subset of $\mathscr{H}'$. If $M_{z_i}^{\mathscr{H}'}, i = 1, 2$ denotes the pointwise multiplication operator on $\mathscr{H}'$ by $z_i$, then

$$M_{z_1}^{\mathscr{H}'}(e'_{i,j}) = \sqrt{v_i v_j} z_1^{i+1} z_2^j = \frac{\left(1 + \frac{1}{i+1}\right)}{\left(1 + \frac{1}{i}\right)} e'_{i+1,j} = w_i e'_{i+1,j}$$

and similarly, $M_{z_2}^{\mathscr{H}'}(e'_{i,j}) = w_j e'_{i,j+1}$, for $i, j \in \mathbb{N} \cup \{0\}$, where $w_0 = 2$ and $w_n = \frac{\left(1+\frac{1}{n+1}\right)}{\left(1+\frac{1}{n}\right)}$, for $n \geq 1$. Thus, $M_{z_1}^{\mathscr{H}'}, M_{z_2}^{\mathscr{H}'}$ are unilateral shifts with weight sequences $(w_i)_{i \geq 0}$ and $(w_j)_{j \geq 0}$, respectively.

Define the linear operator $L : \mathscr{H} \to \mathscr{H}'$ as $L(e_{i,j}) := \alpha_i \alpha_j e'_{i,j}$, where $\alpha_0 = 1$ and for each $n \in \mathbb{N}$, $\alpha_n = 1 + \frac{1}{n}$. Since the set $\{\alpha_n : n \in \mathbb{N} \cup \{0\}\}$ is bounded, $L$ is a bounded linear operator. It is also invertible with $L^{-1}(e'_{i,j}) = \frac{e_{i,j}}{\alpha_i \alpha_j}$. Now, we claim that $L$ is a module map. To see this, note that

$$L \circ M_{z_1}^{\mathscr{H}}(e_{i,j}) = L(e_{i+1,j}) = \alpha_{i+1} \alpha_j e'_{i+1,j}.$$

On the other hand,

$$M_{z_1}^{\mathscr{H}'} \circ L(e_{i,j}) = \alpha_i \alpha_j M_{z_1}^{\mathscr{H}'}(e'_{i,j}) = \alpha_i \alpha_j w_i e'_{i+1,j} = \alpha_{i+1} \alpha_j e'_{i+1,j}.$$

Thus, $L \circ M_{z_1}^{\mathscr{H}} = M_{z_1}^{\mathscr{H}'} \circ L$. Similarly, we have $L \circ M_{z_2}^{\mathscr{H}} = M_{z_2}^{\mathscr{H}'} \circ L$ which proves our claim. Thus, the analytic Hilbert modules $\mathscr{H}$ and $\mathscr{H}'$ are similar via the bijective module map L.

Now, define $\mathscr{M} = \{f \in \mathscr{H} : f = 0 \text{ on } \{z_1 = 0\}\}, \mathscr{M}' = \{g \in \mathscr{H}' : g = 0 \text{ on } Z(\psi)\}$, where $\psi = z_1 h$, for some non-vanishing holomorphic function $h : \mathbb{D}^2 \to \mathbb{C}$. Clearly, $\mathscr{M}, \mathscr{M}'$ are submodules of $\mathcal{H}, \mathcal{H}'$, respectively. Then, from Theorem 4.3.2, it follows that these submodules are also similar via the operator $L$.

### 4.3.2 The second main theorem

In the previous theorem, both $\mathcal{M}, \mathcal{M}'$ are the maximal set of functions in $\mathcal{H}, \mathcal{H}'$ that vanishes on $Z(\varphi), Z(\psi)$, respectively. Now, we'll extend the theorem for non-maximal sets when $\mathcal{M}, \mathcal{M}'$ are assumed to be the closure of principal ideals.

**Theorem 4.3.4.** *Let $\Omega$ be a bounded domain in $\mathbb{C}^m$ and $\mathcal{H}, \mathcal{H}' \subseteq \mathcal{O}(\Omega)$ be analytic Hilbert modules. Assume that $\mathcal{I}, \mathcal{I}'$ are two principal ideals in $\mathbb{C}[z_1, \ldots, z_m]$ generated by $p, p'$ respectively, and that the zero set of each irreducible components of $p, p'$ have non-empty intersection with $\Omega$. Set $\mathcal{M}, \mathcal{M}'$ to be the closure of the polynomial ideals $\mathcal{I}, \mathcal{I}'$ in $\mathcal{H}, \mathcal{H}'$, respectively. If $L : \mathcal{H} \to \mathcal{H}'$ is a bijective module map, then the following are equivalent:*

*a)* $L(\mathcal{M}) = \mathcal{M}'$;

*b)* $\mathcal{K}_{i_{\mathcal{M}}} - \mathcal{K}_{\mathcal{M}} + \mathcal{K}_{\mathcal{H}} = \mathcal{K}_{i_{\mathcal{M}'}} - \mathcal{K}_{\mathcal{M}'} + \mathcal{K}_{\mathcal{H}'}$ *as (1,1) currents on* $\Omega$;

*c)* $\mathcal{I} = \mathcal{I}'$.

*Proof.* Proof of a)⇒b) follows from [15, Theorem 1]. To prove b)⇒c), first observe that

$$\mathcal{K}_{i_{\mathcal{M}}} - \mathcal{K}_{\mathcal{M}} + \mathcal{K}_{\mathcal{H}} = \partial\bar{\partial}\log|p|^2 \text{ and } \mathcal{K}_{i_{\mathcal{M}'}} - \mathcal{K}_{\mathcal{M}'} + \mathcal{K}_{\mathcal{H}'} = \partial\bar{\partial}\log|p'|^2.$$

So, b) implies $\partial\bar{\partial}\log|p|^2 = \partial\bar{\partial}\log|p'|^2$ as (1,1) currents on $\Omega$. As a result, from Lemma 4.2.3 we obtain that $Z(p) \cap \Omega = Z(p') \cap \Omega$. Since $\mathbb{C}[z_1, \ldots, z_m]$ is an UFD, we have

$$p = a_1^{m_1} \cdots a_k^{m_k} \text{ and } p' = b_1^{n_1} \cdots b_l^{n_l},$$

where $m_1, \ldots, m_k, n_1, \ldots, n_l \in \mathbb{N}$ and $\{a_1, \ldots, a_k\}, \{b_1, \ldots, b_l\}$ are the collection of distinct irreducible factors of $p, p'$ respectively. Now, we claim the following:

i) $k = l$,

ii) $a_i = b_{\sigma(i)}$ in $\mathbb{C}[z_1, \ldots, z_m]$ modulo an unit, where $1 \le i \le k$ and $\sigma$ is a permutation on the set $\{1, \ldots, k\}$.

To see this, observe that for each $i$, $1 \le i \le k$, $a_i$ divides $p$. This gives $Z(a_i) \subseteq Z(p)$ which implies $Z(a_i) \cap \Omega \subseteq Z(p) \cap \Omega = Z(p') \cap \Omega$. As a result, $Z(a_i) \cap Z(p') \supseteq Z(a_i) \cap Z(p') \cap \Omega = Z(a_i) \cap \Omega$ i.e. $Z(a_i) \cap Z(p')$ contains a non-empty, open subset of $Z(a_i)$. But $a_i$ is an irreducible element in the UFD $\mathbb{C}[z_1, \ldots, z_m]$. So, it is a prime element or equivalently, the principal ideal generated by

$a_i$ is a prime ideal. If we denote this ideal by $<\{a_i\}>$, then, from Hilbert's Nullstellensatz, it follows that $I\big(Z(a_i)\big) =<\{a_i\}>$, where, for any subset $X$ of $\mathbb{C}^m$,

$$I(X) := \{q \in \mathbb{C}[z_1,\dots,z_m] : q = 0 \text{ on } X\}.$$

Thus, $I\big(Z(a_i)\big)$ is a prime ideal in $\mathbb{C}[z_1,\dots,z_m]$ and hence $Z(a_i)$ is irreducible [23, Proposition 1.5.1]. Consequently, from [9, Corollary 5.3.2], this implies $Z(a_i) \subseteq Z(p')$. As a result, $p' \in I\big(Z(a_i)\big) = <\{a_i\}>$ which proves $a_i$ divides $p'$. Thus, all irreducible factors of $p$ is an irreducible factor of $p'$. Similarly, one can show that every irreducible factor of $p'$ is an irreducible factor of $p$ which proves the claim.

Relabelling $n_1,\dots,n_k$ if necessary, we get

$$p = a_1^{m_1}\cdots a_k^{m_k} \text{ and } p' = a_1^{n_1}\cdots a_k^{n_k} (\text{modulo a unit}).$$

Next, we claim that either $m_i = n_i$, for all $i \in \{1,\dots,k\}$. If possible, without loss of generality assume that $m_i > n_i$, for $1 \le i \le k_1$, $n_i > m_i$, for $k_1+1 \le i \le k_2$ and $m_i = n_i$, for $k_2+1 \le i \le k$, where $k_1, k_2$ are positive integers satisfying $k_1 < k_2 < k$. We have

$$\partial\bar\partial \log|p|^2 = \sum_{i=1}^{k} m_i \partial\bar\partial \log|a_i|^2 = \sum_{i=1}^{k} n_i \partial\bar\partial \log|a_i|^2 = \partial\bar\partial \log|p'|^2$$

as (1,1) currents on $\Omega$. This implies

$$\sum_{i=1}^{k_1}(m_i - n_i)\partial\bar\partial \log|a_i|^2 = \sum_{i=k_1+1}^{k_2}(n_i - m_i)\partial\bar\partial \log|a_i|^2,$$

or equivalently,

$$\partial\bar\partial \log\big|a_1^{m_1-n_1}\cdots a_{k_1}^{m_{k_1}-n_{k_1}}\big|^2 = \partial\bar\partial \log\big|a_{k_1+1}^{n_{k_1+1}-m_{k_1+1}}\cdots a_{k_2}^{n_{k_2}-m_{k_2}}\big|^2.$$

Now, applying Lemma 4.2.3 we obtain that

$$Z\big(a_1^{m_1-n_1}\cdots a_{k_1}^{m_{k_1}-n_{k_1}}\big)\cap\Omega = Z\big(a_{k_1+1}^{n_{k_1+1}-m_{k_1+1}}\cdots a_{k_2}^{n_{k_2}-m_{k_2}}\big)\cap\Omega,$$

or equivalently,

$$Z(a_1\cdots a_{k_1})\cap\Omega = Z(a_{k_1+1}\cdots a_{k_2})\cap\Omega.$$

Thus, $Z(a_1)\cap\Omega \subseteq Z(a_{k_1+1}\cdots a_{k_2})\cap\Omega$. As a result, $Z(a_1)\cap Z(a_{k_1+1}\cdots a_{k_2})$ contains $Z(a_1)\cap Z(a_{k_1+1}\cdots a_{k_2})\cap\Omega = Z(a_1)\cap\Omega$. Since $Z(a_1)$ is irreducible, from [9, Corollary 5.3.2] it follows that $Z(a_1) \subseteq Z(a_{k_1+1}\cdots a_{k_2})$. This means $a_{k_1+1}\cdots a_{k_2} \in I\big(Z(a_1)\big) =<\{a_1\}>$, or equivalently, $a_1$ divides $a_{k_1+1}\cdots a_{k_2}$. But $a_1$ is an irreducible element in the UFD $\mathbb{C}[z_1,\dots,z_m]$. Consequently, we have $a_1$ divides $a_i$ for some $i$, $k_1+1 \le i \le k_2$. This is a contradiction to the fact that $a_1$ and $a_i$ are distinct irreducible factors of $p$ for all $i \in \{k_1+1,\dots,k_2\}$, proving our claim.

Thus, $p, p'$ are associates in $\mathbb{C}[z_1,\dots,z_m]$. So, it clearly follows that the respective principal ideals are equal, i.e., $\mathscr{I} = \mathscr{I}'$.

Now, we'll prove c)⇒a). Since $L : \mathscr{H} \to \mathscr{H}'$ is a bijective module map, there exists a non-vanishing holomorphic function $a : \Omega \to \mathbb{C}$ such that $L(h) = ah$, for all $h \in \mathscr{H}$. So, $L(1) = a \in \mathscr{H}'$. Moreover, for any $q \in \mathbb{C}[z_1,\dots,z_m]$, $L(pq) = apq = p(aq) \in M_p(\mathscr{H}') \subseteq \mathscr{M}'$. To check that $M_p(\mathscr{H}') \subseteq \mathscr{M}'$, take an arbitrary element $h' \in \mathscr{H}'$. Since polynomials are dense in $\mathscr{H}'$, there exists a sequence $p_n \in \mathbb{C}[z_1,\dots,z_m]$ such that $p_n$ converges to $h'$ in $\mathscr{H}'$. This implies $M_p(p_n) \in \mathscr{I} = \mathscr{I}'$ converges to $M_p(h')$ in $\mathscr{H}'$. But $M_p(h')$ is an arbitrary element of $M_p(\mathscr{H}')$. So, $M_p(\mathscr{H}')$ is a subset of the closure of $\mathscr{I}'$ in $\mathscr{H}'$ which is $\mathscr{M}'$. This shows $L(\mathscr{I}) \subseteq \mathscr{M}'$. Since $\mathscr{I}$ is dense in $\mathscr{M}$, for any $m \in \mathscr{M}$, we can find a sequence $p_n \in \mathscr{I}$ such that $p_n$ converges to $m$ in $\mathscr{H}$. So, $L(p_n) \in \mathscr{M}'$ converges to $L(m)$ in $\mathscr{H}'$. Thus, $L(\mathscr{M}) \subseteq \mathscr{M}'$. Similarly, one can show that $L^{-1}(M') \subseteq \mathscr{M}$ or equivalently, $\mathscr{M}' \subseteq L(\mathscr{M})$. This proves a). □

**Example 4.3.5.** Let $\mathscr{H}, \mathscr{H}'$ be the Hilbert modules as given in Example 4.3.3 and let $L$ be the bijective module map between them. Take an arbitrary polynomial $p \in \mathbb{C}[z_1, z_2]$ and let $\mathscr{I}$ be the principal ideal generated by $p$. If $\mathscr{M}, \mathscr{M}'$ denote the closure of $\mathscr{I}$ in $\mathscr{H}, \mathscr{H}'$ respectively, then, from Theorem 4.3.4 it follows that $L$ takes $\mathscr{M}$ to $\mathscr{M}'$ and vice versa.

### 4.3.3 An important example

However, it is important to note that the alternating sum given in Theorem 4.3.2 and 4.3.4 is not a complete invariant for similarity. To see this, consider the analytic Hilbert modules $\mathscr{H} = H^2(\mathbb{D}^2)$ and $\mathscr{H}' = H^{(2,1)}(\mathbb{D}^2)$, where

$$H^{(2,1)}(\mathbb{D}^2) = \Big\{ f \in \mathscr{O}(\mathbb{D}^2) : \text{if } f = \sum_{i,j\ge 0} f(i,j) z_1^i z_2^j, \text{ then } \sum_{i,j\ge 0} \frac{|f(i,j)|^2}{i+1} < \infty \Big\}$$

with the inner product given by $\langle f, g\rangle_{\mathscr{H}'} := \sum_{i,j\ge 0} \frac{f(i,j)\overline{g(i,j)}}{i+1}$. Then the submodules $\mathscr{M}, \mathscr{M}'$ are defined as the closure of the principal ideal $<\{z_1\}>$ in $\mathscr{H}, \mathscr{H}'$ respectively. Since

$$K_{\mathscr{H}}(z,w) = \frac{1}{(1 - z_1\bar{w}_1)(1 - z_2\bar{w}_2)} = \sum_{i,j\ge 0} z_1^i z_2^j \bar{w}_1^i \bar{w}_2^j$$

and

$$K_{\mathscr{H}'}(z,w) = \frac{1}{(1 - z_1\bar{w}_1)^2(1 - z_2\bar{w}_2)} = \sum_{i,j\ge 0} (i+1) z_1^i z_2^j \bar{w}_1^i \bar{w}_2^j,$$

for all $z, w \in \Omega$, we have $K_{\mathscr{M}}(z,w) = z_1\bar{w}_1\chi(z,w)$ and $K_{\mathscr{M}'}(z,w) = z_1\bar{w}_1\chi'(z,w)$, where $\chi(z,w) = \sum_{i,j\ge 0} z_1^i z_2^j \bar{w}_1^i \bar{w}_2^j = K_{\mathscr{H}}(z,w)$, $\chi'(z,w) = \sum_{i,j\ge 0}(i+2) z_1^i z_2^j \bar{w}_1^i \bar{w}_2^j$. It can be checked that $\chi, \chi' : \Omega\times\Omega \to \mathbb{C}$ are non-vanishing, sesquianalytic, positive definite functions. So, they induce reproducing kernel Hilbert modules $\mathscr{H}_\chi$, $\mathscr{H}'_{\chi'}$ on $\mathbb{D}^2$ where the polynomials are dense. Moreover, we obtain that the multiplication by $z_1$ is an unitary module map from $\mathscr{H}_\chi$, $\mathscr{H}'_{\chi'}$ to $\mathscr{M},\mathscr{M}'$ and we'll denote them by $M_{z_1}$, $M'_{z_1}$, respectively. Finally, observe that $\mathscr{H}_\chi = H^2(\mathbb{D}^2)$ and $\mathscr{H}'_{\chi'}$ can be described by the set

$$\Big\{f \in \mathscr{O}(\mathbb{D}^2) : \text{if } f = \sum_{i,j\geq 0} f(i,j) z_1^i z_2^j, \text{ then } \sum_{i,j\geq 0} \frac{|f(i,j)|^2}{i+2} < \infty\Big\}$$

and the inner product $\langle f, g\rangle_{\mathscr{H}'_{\chi'}} := \sum_{i,j\geq 0} \frac{f(i,j)\overline{g(i,j)}}{i+2}$. If $\{e_{i,j}\}$, $\{e'_{i,j}\}$ are their orthonormal bases given by $e_{i,j}(z) = z_1^i z_2^j, e'_{i,j}(z) = \sqrt{(i+2)} z_1^i z_2^j$, for all $z = (z_1, z_2) \in \mathbb{D}^2$, then

$$M_{z_1}^{\chi}(e_{i,j}) = e_{i+1,j} \text{ and } M_{z_1}^{\chi'}(e'_{i,j}) = \sqrt{\frac{i+2}{i+3}} e'_{i+1,j},$$

i.e. $M_{z_1}^{\chi}, M_{z_1}^{\chi'}$ can be considered as weighted unilateral shifts with weight sequences $w_i = 1$ and $w'_i = \sqrt{\frac{i+2}{i+3}}$, where $M_{z_1}^{\chi}, M_{z_1}^{\chi'}$ are multiplication by $z_1$ operator on $\mathscr{H}_\chi, \mathscr{H}'_{\chi'}$ respectively. Now, if possible, let $\mathscr{M},\mathscr{M}'$ are similar via a bijective module map $L$. Then $L_\chi := (M'_{z_1})^* \circ L \circ M_{z_1}$ is a bijective module map between $\mathscr{H}_\chi$ and $\mathscr{H}'_{\chi'}$ which satisfies $L_\chi \circ M_{z_1}^{\chi} = M_{z_1}^{\chi'} \circ L_\chi$. As a result, there exists positive constants $C_1, C_2$ such that

$$0 < C_1 \leq |w_0 \cdots w_n| / |w'_0 \cdots w'_n| \leq C_2,$$

for all $n \in \mathbb{N} \cup \{0\}$ [28, Theorem 2]. But in our case $|w_0 \cdots w_n|/|w'_0 \cdots w'_n| = \sqrt{\frac{3\cdot 4 \cdots (n+3)}{2\cdot 3 \cdots (n+2)}} = \sqrt{\frac{n+3}{2}}$ which is not bounded above. This is a contradiction to the previous statement which means $\mathscr{M},\mathscr{M}'$ are not similar. On the other hand, it can be easily checked that

$$\mathscr{K}_{i_{\mathscr{M}}} - \mathscr{K}_{\mathscr{M}} + \mathscr{K}_{\mathscr{H}} = \mathscr{K}_{i_{\mathscr{M}'}} - \mathscr{K}_{\mathscr{M}'} + \mathscr{K}_{\mathscr{H}'} = \partial\bar{\partial} \log |z_1|^2.$$

Thus, the alternating sums coincide even though the submodules are non-similar.

Theorem 4.3.4 proves a version of the Rigidity Theorem in the part a)⇒c). When $\mathscr{H} = \mathscr{H}'$, it also says the following: If $\mathscr{M},\mathscr{M}'$ are two distinct submodules of $\mathscr{H}$ as given in Theorem 4.3.4, then they are not similar via a bijective module map $L$ on $\mathscr{H}$. Now, we'll generalize Theorem 4.3.2 for the submodules consisting of functions that vanishes to order greater that 1 on the zero set.

### 4.3.4 Generalization of the first main theorem: higher order case

**Theorem 4.3.6.** *Let $\Omega$ be a bounded domain in $\mathbb{C}^m$ and $\mathscr{H},\mathscr{H}' \subseteq \mathscr{O}(\Omega)$ be analytic Hilbert modules. Suppose $\varphi, \psi : \Omega \to \mathbb{C}$ are holomorphic functions that define $Z(\varphi) := \{z \in \Omega : \varphi(z) = 0\}$, $Z(\psi) := \{z \in \Omega : \varphi(z) = 0\}$, respectively. Assume that $Z(\varphi), Z(\psi)$ are irreducible analytic subsets* *in* $\Omega$ *and set* $\mathcal{M} = \{f \in \mathcal{H} : \mathrm{ord}_{Z(\varphi)} f \geq k\}$, $\mathcal{M}' = \{g \in \mathcal{H}' : \mathrm{ord}_{Z(\psi)} g \geq k'\}$, *where* $k, k' \geq 1$ *are two positive integers. Furthermore, assume that* $\varphi^k \in \mathcal{H}, \psi^{k'} \in \mathcal{H}'$ *and* $\mathrm{rank}(\mathcal{M}) = \mathrm{rank}(\mathcal{M}') = 1$. *If there exists a bijective module map* $L : \mathcal{H} \to \mathcal{H}'$, *then the following are equivalent.*

*a)* $L(\mathcal{M}) = \mathcal{M}'$;

*b)* $Z(\varphi) = Z(\psi)$, $k = k'$;

*c)* $\mathcal{K}_{i_{\mathcal{M}}} - \mathcal{K}_{\mathcal{M}} + \mathcal{K}_{\mathcal{H}} = \mathcal{K}_{i_{\mathcal{M}'}} - \mathcal{K}_{\mathcal{M}'} + \mathcal{K}_{\mathcal{H}'}$ *as (1,1) currents on* $\Omega$.

(Note: Lemma 4.1.2 ensures that the submodules $\mathcal{M}, \mathcal{M}'$ are well defined. )

*Proof.* a)⇔b) Since $L : \mathcal{H} \to \mathcal{H}'$ is a bijective module map, there exists a non-vanishing holomorphic function $a : \Omega \to \mathbb{C}$ such that $L(h) = ah$, for all $h \in \mathcal{H}$. Observe that $\varphi^k \in \mathcal{M}$. So, $L(\mathcal{M}) = \mathcal{M}'$ implies $L(\varphi^k) = a\varphi^k \in \mathcal{M}'$. Thus, $Z(a\varphi^k) = Z(\varphi^k) = Z(\varphi) \supseteq Z(\psi)$. From similar arguments with $L^{-1}$ it follows that $Z(\psi) \supseteq Z(\varphi)$. As a result, $Z(\varphi) = Z(\psi)$ and hence $\varphi, \psi$ defines the same set. Consequently, there exists a non-vanishing, holomorphic function $f : \Omega \to \mathbb{C}$ such that $\varphi = f\psi$ on $\Omega$. Now, if possible, without loss of generality assume that $k < k'$. Since $a\varphi^k \in \mathcal{M}'$, $\psi^{k'}$ divides $a\varphi^k = af^k\psi^k$. This implies $\psi^{k'-k}$ divides $af^k$. But this is absurd which proves $k = k'$.

For the converse part, take an arbitrary element $m \in \mathcal{M}$. Then $m = \varphi^k \tilde{m}$, for some $\tilde{m} \in \mathcal{O}(\Omega)$. As a result, $L(m) = am = a\varphi^k\tilde{m}$ is an element in $\mathcal{H}'$. Since $Z(\varphi) = Z(\psi), k = k'$, $\mathrm{ord}_{Z(\psi)}(a\varphi^k\tilde{m}) \geq k'$ and hence $L(m) \in \mathcal{M}'$. Thus, $L(\mathcal{M}) \subseteq \mathcal{M}'$. Similarly, we can show that $L^{-1}(\mathcal{M}') \subseteq \mathcal{M}$. Consequently, $L(\mathcal{M}) = \mathcal{M}'$ and b)⇒a) part is proved.

b)⇔c) Similar to the case when $k = k' = 1$, one can show that

$$\mathcal{K}_{i_{\mathcal{M}}} - \mathcal{K}_{\mathcal{M}} + \mathcal{K}_{\mathcal{H}} = \partial\bar{\partial}\log|\varphi^k|^2 \text{ and } \mathcal{K}_{i_{\mathcal{M}'}} - \mathcal{K}_{\mathcal{M}'} + \mathcal{K}_{\mathcal{H}'} = \partial\bar{\partial}\log|\psi^{k'}|^2 \tag{4.1}$$

as (1,1) currents on $\Omega$. Now, b) implies $\varphi = f\psi$, for some non-vanishing holomorphic function $f \neq 0$ in $\mathcal{O}(\Omega)$. Following similar arguments as given in the proof of Theorem 4.3.2, we get

$$\partial\bar{\partial}\log|\varphi^k|^2 = k\partial\bar{\partial}\log|\varphi|^2 = k\partial\bar{\partial}\log|f\psi|^2 = k\partial\bar{\partial}\log|f|^2 + k\partial\bar{\partial}\log|\psi|^2 = \partial\bar{\partial}\log|\psi^k|^2. \tag{4.2}$$

The last equality is true because $\partial\bar{\partial}\log|f|^2 = 0$ for any non-vanishing holomorphic function $f : \Omega \to \mathbb{C}$. Since $k = k'$, Equation (4.1) and (4.2) proves c).

Conversely, observe that c) implies $\partial\bar{\partial}\log|\varphi^k|^2 = \partial\bar{\partial}\log|\psi^{k'}|^2$ as (1,1) currents on $\Omega$. Applying Lemma 4.2.3, this gives $Z(\varphi^k) = Z(\psi^{k'})$ or equivalently, $Z(\varphi) = Z(\psi)$. As a result, $\varphi = f\psi$ for a non-vanishing holomorphic function $f$ on $\Omega$. From Equation (4.2) this means

$$\partial\bar{\partial}\log|\psi^k|^2 = \partial\bar{\partial}\log|\psi^{k'}|^2 \Leftrightarrow k\partial\bar{\partial}\log|\psi|^2 = k'\partial\bar{\partial}\log|\psi^k|^2 \Leftrightarrow (k-k')\partial\bar{\partial}\log|\psi|^2 = 0.$$

If $k \neq k'$, then from Lemma 4.2.3 the last equality will imply that $Z(\psi)$ is an empty set which is a contradiction. Thus, c) implies $Z(\varphi) = Z(\psi)$, $k = k'$. □

When $\mathscr{H} = \mathscr{H}'$ and $\mathscr{M}, \mathscr{M}'$ satisfies the hypotheses of Theorem 4.3.6, we have the following: If there exists a bijective module map $L$ on $\mathscr{H}$ such that $L(\mathscr{M}) = \mathscr{M}'$, then $\mathscr{M} = \mathscr{M}'$. Finally, we'll generalize Theorem 4.3.2 where the zero set has codimension greater than 1.

### 4.3.5 Generalization of the first main theorem: higher codimension case

**Theorem 4.3.7.** *Let $\Omega$ be a bounded domain in $\mathbb{C}^m$ and $\mathscr{H}, \mathscr{H}' \subseteq \mathscr{O}(\Omega)$ be analytic Hilbert modules. Also, let $\varphi := (\varphi_1, \ldots, \varphi_r), \psi := (\psi_1, \ldots, \psi_r)$ be holomorphic maps from $\Omega$ to $\mathbb{C}^r$ satisfying the following three conditions.*

*i) For each $i$, $1 \le i \le r$, $\varphi_i \in \mathscr{H}, \psi_i \in \mathscr{H}'$ and they define $Z(\varphi_i), Z(\psi_i)$ respectively;*

*ii) $Z(\varphi), Z(\psi)$ are complete intersections of $\varphi, \psi$ respectively, where $Z(\varphi) := Z(\varphi_1) \cap \cdots \cap Z(\varphi_r)$ and $Z(\psi) := Z(\psi_1) \cap \cdots \cap Z(\psi_r)$;*

*iii) $Z(\varphi_i), Z(\psi_i), Z(\varphi), Z(\psi)$ are connected subsets of $\Omega$ for all $i = 1, \ldots, r$.*

*For $i = 1, \ldots, r$ set $\mathscr{M}_i = \{f \in \mathscr{H} : f = 0 \text{ on } Z(\varphi_i)\}$, $\mathscr{M} = \{f \in \mathscr{H} : f = 0 \text{ on } Z(\varphi)\}$, $\mathscr{M}_i^{'} = \{g \in \mathscr{H}' : g = 0 \text{ on } Z(\psi_i)\}$, $\mathscr{M}' = \{g \in \mathscr{H}' : g = 0 \text{ on } Z(\psi)\}$. Assume that $\operatorname{rank}(\mathscr{M}_i) = \operatorname{rank}(\mathscr{M}_i^{'}) = 1$. If $L : \mathscr{H} \to \mathscr{H}'$ is a bijective module map, then the following are equivalent:*

*a) $L(\mathscr{M}) = \mathscr{M}'$;*

*b) $Z(\varphi) = Z(\psi)$;*

*c) $\bigwedge_{i=1}^r \left(\mathscr{K}_{X_i} - \mathscr{K}_{\mathscr{M}_i} + \mathscr{K}_{\mathscr{H}}\right) = \bigwedge_{i=1}^r \left(\mathscr{K}_{X_i^{'}} - \mathscr{K}_{\mathscr{M}_i^{'}} + \mathscr{K}_{\mathscr{H}'}\right)$ as $(r,r)$ currents on $\Omega$, where $X_i : \mathscr{M}_i \to \mathscr{H}, X_i^{'} : \mathscr{M}_i^{'} \to \mathscr{H}'$ are the canonical inclusion maps for $i = 1, \ldots, r$.*

*Proof.* a)⇔b) Let $a$ be the non-vanishing holomorphic function from $\Omega$ to $\mathbb{C}$ such that $L(h) = ah$, for all $h \in \mathscr{H}$. Since $\varphi_i \in \mathscr{M}$, $L(\mathscr{M}) = \mathscr{M}'$, $L(\varphi_i) = a\varphi_i$ vanishes on $Z(\psi)$, for all $i = 1, \ldots, r$. Thus, $Z(\varphi_i) \supseteq Z(\psi)$ for all $i$ which implies $Z(\varphi) = \cap_{i=1}^r Z(\varphi_i) \supseteq Z(\psi)$. Similarly, we'll get $Z(\psi) \supseteq Z(\varphi)$ which proves b).

Conversely, take an arbitrary element $m \in \mathscr{M}$. Then $m$ vanishes on $Z(\varphi) = Z(\psi)$. As a result, $L(m) = am$ is an element in $\mathscr{H}'$ which also vanishes on $Z(\psi)$. Thus, $L(\mathscr{M}) \subseteq \mathscr{M}'$. Similarly, we'll get $L^{-1}(\mathscr{M}') \subseteq \mathscr{M}$, or equivalently, $\mathscr{M}' \subseteq L(\mathscr{M})$ which finally gives $L(\mathscr{M}) = \mathscr{M}'$.

b)⇔c) From the definition of the fundamental class and Lemma 4.2.1 it follows that $Z(\varphi) = Z(\psi)$ if and only if $[Z(\varphi)] = [Z(\psi)]$ as $(r,r)$ currents on $\Omega$. Now, $Z(\varphi), Z(\psi)$ are connected submanifolds in $\Omega$ of codimension $r$ and hence they are irreducible as analytic subsets of $\Omega$. As a result, [13, Proposition III.4.12] gives

$$[Z(\varphi_1)] \wedge \cdots \wedge [Z(\varphi_r)] = [Z(\psi_1)] \wedge \cdots \wedge [Z(\psi_r)].$$

Finally, from the discussions prior to Theorem 4.3.2 we see that

$$\mathcal{K}_{X_i} - \mathcal{K}_{\mathcal{M}_i} + \mathcal{K}_{\mathcal{H}} = \partial\bar{\partial}\log|\varphi_i|^2 \text{ and } \mathcal{K}_{X_i'} - \mathcal{K}_{\mathcal{M}_i'} + \mathcal{K}_{\mathcal{H}'} = \partial\bar{\partial}\log|\psi_i|^2,$$

for all $i$, $1 \le i \le r$. The result then follows from *Lelong-Poincaré* formula. □

When $\mathcal{H} = \mathcal{H}'$, we can prove a similar corollary as we have for Theorem 4.3.4 and 4.3.6. In fact, we have a stronger result in this case. To see this, assume that $\mathcal{M}$ is generated by $\varphi_1, \dots, \varphi_r$, $\mathcal{M}'$ is generated by $\psi_1, \dots, \psi_r$ and $\varphi_i, \psi_i$ are polynomials. Then, from [3, Lemma 1.11], it follows that the reproducing kernels of $\mathcal{M}, \mathcal{M}'$ are sharp on $\Omega \setminus Z(\varphi), \Omega \setminus Z(\psi)$, respectively. As a result, we have the following.

**Corollary 4.3.8.** *Let $\mathcal{M}$ and $\mathcal{M}'$ be submodules of an analytic Hilbert module $\mathcal{H}$, as described above. If there exists a bijective module map L from $\mathcal{M}$ onto $\mathcal{M}'$, then*

*i) $\mathcal{M} = \mathcal{M}'$ and*

*ii) $\wedge_{i=1}^r \left(\mathcal{K}_{X_i} - \mathcal{K}_{\mathcal{M}_i} + \mathcal{K}_{\mathcal{H}}\right) = \wedge_{i=1}^r \left(\mathcal{K}_{X_i'} - \mathcal{K}_{\mathcal{M}_i'} + \mathcal{K}_{\mathcal{H}}\right)$ as $(r, r)$ currents on $\Omega$.*

*Proof.* Following similar arguments as given in Lemma 4.3.1 we obtain that for all $w \in \Omega$,

$$L^*\left(\cap_{i=1}^m \ker(M_{z_i} - w_i)^*\right) = \cap_{i=1}^m \ker(M_{z_i}^{'} - w_i)^*,$$

where $M_{z_i}, M_{z_i}^{'}$ are pointwise multiplication by $z_i$, for $i = 1, \dots, r$. But the kernels of $\mathcal{M}, \mathcal{M}'$ are sharp outside their respective zero sets. So,

$$< \{L^* K_{\mathcal{M}'}(\cdot, w)\} >=< \{K_{\mathcal{M}}(\cdot, w)\} >,$$

for all $w \in \Omega \setminus \big(Z(\varphi) \cup Z(\psi)\big)$. As a result, there exists a non-vanishing holomorphic function $a : \Omega \setminus \big(Z(\varphi) \cup Z(\psi)\big) \to \mathbb{C}$ such that $L^* K_{\mathcal{M}'}(\cdot, w) = \overline{a(w)} K_{\mathcal{M}}(\cdot, w)$. But codimension of $Z(\varphi) \cup Z(\psi)$ is greater or equal to 2. So, by Hartog's theorem [26, Page 198] $a$ extends as a non-vanishing holomorphic function on $\Omega$. Thus, $L^* K_{\mathcal{M}'}(\cdot, w) = \overline{a(w)} K_{\mathcal{M}}(\cdot, w)$ for all $w \in \Omega$ which means $L(m) = am$ for all $m \in \mathcal{M}$. Finally, following the proof of Theorem 4.3.7, a)⇒b) we obtain that $Z(\varphi) = Z(\psi)$ which proves $\mathcal{M} = \mathcal{M}'$. The equality of the $(r, r)$ currents follows from Theorem 4.3.7, b)⇒c). □

Consider the analytic Hilbert modules $\mathcal{H} = H^2(\mathbb{D}^2)$ and $\mathcal{H}' = H^{(2,1)}(\mathbb{D}^2)$. Next, define the submodules $\mathcal{M}, \mathcal{M}'$ by the closure of the polynomial ideal $< \{z_1, z_2\} >$ in $\mathcal{H}, \mathcal{H}'$ respectively. Also, for each $i = 1, 2$, define the submodules $\mathcal{M}_i, \mathcal{M}_i^{'}$ by the closure of the ideal $< \{z_i\} >$. Then it can be checked that $\mathcal{M}_i = \{f \in \mathcal{H} : f = 0 \text{ on } z_i = 0\}$ and $\mathcal{M}_i^{'} = \{g \in \mathcal{H}' : g = 0 \text{ on } z_i = 0\}$. If possible, suppose there exists a bijective module $L$ from $\mathcal{M}$ to $\mathcal{M}'$. From the proof of Corollary 4.3.8 this means $L(m) = am$ for all $m \in \mathcal{M}$, where $a : \Omega \to \mathbb{C}$ is a non-vanishing holomorphic function. As a result, $L(m_i) = am_i$, for $m_i \in \mathcal{M}_i, i = 1,2$ which means $L(\mathcal{M}_i) = \mathcal{M}_i^{'}$. But this is a contradiction because we already know that $\mathcal{M}_1$ and $\mathcal{M}_1^{'}$ are not similar from the discussions following Theorem 4.3.4. Thus, $\mathcal{M}$ and $\mathcal{M}'$ are not similar. On the other hand,

$$\bigwedge_{i=1}^{2}\left(\mathcal{K}_{X_i} - \mathcal{K}_{\mathcal{M}_i} + \mathcal{K}_{\mathcal{H}}\right) = \partial\bar{\partial}\log|z_1|^2 \wedge \partial\bar{\partial}\log|z_2|^2 = \left(\frac{2\pi}{i}\right)^2 \delta_{(0,0)} = \bigwedge_{i=1}^{2}\left(\mathcal{K}_{X_i'} - \mathcal{K}_{\mathcal{M}_i'} + \mathcal{K}_{\mathcal{H}'}\right),$$

where $\delta_{(0,0)}$ is the dirac-delta distribution at (0,0) considered as a (2,2) current on $\mathbb{D}^2$. Thus, the wedge of alternating sums is not a complete invariant for similarity.

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