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COMPLETELY CONTRACTIVE HILBERT MODULES AND PARROTT'S EXAMPLE

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1. Introduction

In two earlier papers [9, 10] the present author together with Sastry studied certain finite dimensional Hilbert modules $\mathbf{C}_{\mathbf{N}}^{n+1}$ over the function algebra $A(\Omega)$ for Ω a domain in $\mathbf{C}^m$. This paper is a continuation of that work and provides partial answers to some of the questions raised in [10] for the poly disk algebra. While most of the terminology and notations are from the two papers [9, 10] and will be used without any further apology, we point out in Remark 3.8 that the contractive module $\mathbf{C}_{\mathbf{N}}^{2n}$ (respectively completely contractive) gives rise to a matricially normed $2m$-dimensional vector space and a contractive (respectively completely contractive) linear map on it and conversely.

In the two papers cited above the main result showed that a contractive module $\mathbf{C}^{n+1}$ over the ball algebra $\mathcal{A}(\mathbf{B}^m)$ is completely bounded by $\sqrt{m}$ and examples were given to show that the bound is attained. This, in particular shows that for $m \geqq 2$, contractive modules are not necessarily completely contractive over the ball algebra. However, for the poly disk algebra $\mathcal{A}(\mathbf{D}^m)$, we know via Ando's theorem [2] that every contractive module over $\mathcal{A}\left(\mathbf{D}^2\right)$ is completely contractive while Parrott [11] provides an example of a contractive module over $\mathcal{A}\left(\mathbf{D}^3\right)$ which is not completely contractive. As Paulsen [12] points out, it would be good to know the difference in the internal structure of $\mathcal{A}\left(\mathbf{D}^2\right)$ and $\mathcal{A}\left(\mathbf{D}^3\right)$ that leads to this situation, see 4.4 for a partial answer. Our approach is to actually work out Parrot's example using the notion of complete contractivity rather than dilation, these notions are of course equivalent [cf. 12]. The methods of [9, 10], seem to work well in the context of the ball algebra but the actual computations over the poly disk algebra seem to be very messy. In fact, we are not able to produce an example of a contractive module over $\mathcal{A}\left(\mathbf{D}^3\right)$ which is not completely contractive within the class of the very simple modules considering in [9, 10], however see Remark 4.8. Therefore, we are forced to consider slightly more general module action than those of

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[9, 10]. This necessitates generalising many of the previous results to this new setup. Whenever the proof of a natural generalisation of results from [9, 10] is routine, we omit it. In this more general setup, apart from being able to show that there exists a contractive module over $\mathcal{A}\left(\mathbf{D}^3\right)$ which is not even 2-contractive (Theorem 4.7), we show that such phenomenon occurs in dual pairs, that is there also exists a contractive module over $\mathcal{A}\left(l^1(3)_1\right)$ which is not completely contractive (Theorem 4.1). Another interesting fact is: In the example of a contractive module over $\mathcal{A}\left(\mathbf{D}^3\right)$ in Section 4.6, we use only linear maps in $\mathcal{A}\left(\mathbf{D}^3\right) \otimes \mathcal{M}_2$ to detect that it fails to be completely contractive. Lastly following suggestions of Vern Paulsen, we show in Section 4.3, how our methods can be used to answer a question of Loebl [8].

To keep this work as self-contained as possible we have given more details than would seem necessary. In the rest of this section we give basic definitions. In Section 2, we show that most of the results in [10] can be modified to fit into the present context. The main new ideas are contained in Sections 3 and 4. The following definitions and terminology can be found in many places (cf. [5, 9]).

1.1. Definition. A Hilbert module $\mathcal{H}$ over a (not necessarily complete) complex algebra $\mathcal{A}$ consists of a complex Hilbert space $\mathcal{H}$ together with a continuous map $(a, f) \to a \cdot f$ from $\mathcal{A} \times \mathcal{H}$ to $\mathcal{H}$ satisfying the following conditions:

For a, b in $\mathcal{A}$, h, h_i in $\mathcal{H}$ and α, β in $\mathbf{C}$

(i) $1 \cdot h = h$,
(ii) $a \cdot (b \cdot h) = (a \cdot b) \cdot h$,
(iii) $(a + b) \cdot h = a \cdot h + b \cdot h$,
(iv) $a \cdot (\alpha h_1 + \beta h_2) = \alpha (a \cdot h_1) + \beta (a \cdot h_2)$.

The Hilbert module is *bounded* if there exists a constant K such that

$$\|a \cdot h\|_{\mathcal{H}} \leqq K \|a\|_{\mathcal{A}} \|h\|_{\mathcal{H}}$$

for all a in $\mathcal{A}$ and h in $\mathcal{H}$, and is *contractive* if $K \leqq 1$.

1.2. For any region Ω in $\mathbf{C}^m$, let $\mathcal{A}(\Omega)$ denote the closure of the polynomial algebra $\mathcal{P}(\Omega)$ with respect to the supremum norm on $\overline{\Omega}$ the closure of the region Ω. Throughout this paper we will assume that

(i) Ω is a bounded open neighbourhood of the origin in $\mathbf{C}^m$, and
(ii) Ω is convex and balanced.

We note that (i) and (ii) imply that Ω is polynomially convex [6, p.67] and so, by Oka's theorem [6, p.84], $\mathcal{A}(\Omega)$ contains all functions that are holomorphic in a neighbourhood of $\overline{\Omega}$.

The Hilbert $\mathcal{P}(\Omega)$ module structure on the Hilbert space $\mathcal{H}$ determines and is completely determined by a commuting m-tuple $\mathbf{T} = (T_1, \dots, T_m)$ of bounded operators on $\mathcal{H}$ defined by

$$T_i h = z_i h$$

for h in $\mathcal{H}$ and $1 \leqq i \leqq m$. If $\mathcal{H}$ is a bounded (respectively contractive) $\mathcal{P}(\Omega)$ module then the module map extends to $\mathcal{A}(\Omega)$ and we write $\mathcal{H}_{\mathbf{T}}$ for this bounded Hilbert $\mathcal{A}(\Omega)$-module. As explained in [9, Section 1.2] the notion of $\mathbf{T}$ admitting $\overline{\Omega}$ as a k-spectral set is equivalent to $\mathcal{H}_{\mathbf{T}}$ being bounded.

1.3. For any function algebra $\mathcal{A}$ and an integer $k \geqq 1$, let $\mathcal{M}_k(\mathcal{A}) \cong \mathcal{A} \otimes \otimes \mathcal{M}_k(\mathbf{C})$ denote the algebra of $k \times k$ matrices with entries from $\mathcal{A}$. Here for $F = (f_{ij})$ in $\mathcal{M}_k(\mathcal{A})$, the norm $\|F\|$ of F is defined by

$$\|F\| = \sup\left\{\|(f_{ij}(z))\| : z \in \mathbf{M}\right\},$$

where $\mathbf{M}$ is the maximal ideal space for $\mathcal{A}$. We note that for $\mathcal{A} = \mathcal{A}(\Omega)$, the maximal ideal space can be identified with Ω [6, Theorem 1.2, p.67] and thus

$$\|F\| = \sup\left\{\|(f_{ij}(z))\| : z \in \Omega\right\}.$$

1.4. Definition. If $\mathcal{H}$ is a bounded Hilbert $\mathcal{A}$-module, then $\mathcal{H} \otimes \mathbf{C}^k$ is a bounded $\mathcal{M}_k(\mathcal{A})$-module. For each k let n_k denote the smallest bound for $\mathcal{H} \otimes \mathbf{C}_k$. Then the Hilbert $\mathcal{A}$-module is *completely bounded if*

$$n_\infty = \lim_{k \to \infty} n_k < \infty$$

and is *completely contractive* if $n_\infty \leqq 1$.

1.5. In the following, $l^p(n)$ stands for the vector space $\mathbf{C}^n$ with the usual l^p-norm and $(X)_1$ will denote the open unit ball of the Banach space X. For T a linear operator on $l^2(n)$ and ω any complex number; define the operator $N(T,\omega)$ on $l^2(n) \oplus l^2(n) \cong l^2(2n)$ by

$$N(T,\omega) = \begin{bmatrix} \omega I_n & T \\ \mathbf{0} & \omega I_n \end{bmatrix}.$$

Now, for $\omega = (\omega_1, \dots, \omega_m)$ in Ω, consider the pairwise commuting m-tuple of operators

$$\mathbf{N}(\mathbf{T},\omega) = \left(N(T_1,\omega_1), \dots, N(T_m,\omega_m)\right).$$

The central object of study is the Hilbert $\mathcal{A}(\Omega)$-module $l^2(2n)_{\mathbf{N}}$ and to determine when it is contractive (respectively completely contractive). We write $\mathbf{N}$ for $\mathbf{N}(\mathbf{T},\omega)$ when the meaning is clear from the context

2. The functional calculus

In this section we establish that the evaluation map $p \to p(\mathbf{N})$ on $\mathcal{P}(\Omega)$ extends continuously to $H(\omega)$, the algebra of germs of holomorphic functions at ω. This fact will be necessary in proving Lemma 3.3 in the next section.

2.1. LEMMA. *For S, T in $\mathcal{L}\left(l^2(n)\right)$ and λ, μ in $\mathbf{C}$*

(i) $N(S,\lambda)N(T,\mu) = N(\lambda T + \mu S, \lambda\mu)$,

(ii) $\|N(\lambda,\mu)\|^2 = \frac{1}{2}\left\{|\lambda|^2 + 2|\mu|^2 + |\lambda|\sqrt{|\lambda|^2 + 4|\mu|^2}\right\}$,

(iii) $\|N(S,\lambda)\| = \|N(\|S\|, |\lambda|)\|$.

PROOF. (i) and (ii) are straightforward. To prove (iii), note that

$$\det\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det D \det\left(A - BD^{-1}C\right)$$

and

$$N(S,\lambda)N(S,\lambda)^* = |\lambda|^2 I_{2n} + \begin{bmatrix} SS^* & \overline{\lambda}S \\ \lambda S^* & \mathbf{0} \end{bmatrix}.$$

For $x \in \mathbf{C}$, we have

$$\det\begin{bmatrix} SS^* - xI_n & \overline{\lambda}S \\ \lambda S^* & -xI_n \end{bmatrix} = (-x)^n \det\left(SS^* - xI_n + x^{-1}|\lambda|^2 SS^*\right) =$$

$$= (-x)^n \det\left(SS^*\left(1 + |\lambda|^2 x^{-1}\right) - xI_n\right) =$$

$$= (-x)^n\left(1 + |\lambda|^2 x^{-1}\right)^n \det\left(SS^* - \frac{x}{1 + |\lambda|^2 x^{-1}} I_n\right).$$

Thus, the maximum eigenvalue of $\begin{bmatrix} SS^* & \overline{\lambda}S \\ \lambda S^* & \mathbf{0} \end{bmatrix}$ is

$$\frac{\|S\|}{2}\left(\|S\| + \sqrt{\|S\|^2 + 4|\lambda|^2}\right).$$

By the spectral mapping theorem, the maximum eigenvalue of $N(S,\lambda)N(S,\lambda)^*$ is

$$\frac{1}{2}\left\{2|\lambda|^2 + \|S\|^2 + \|S\|\sqrt{\|S\|^2 + 4|\lambda|^2}\right\}.$$

Using (ii) to compute the norm of $N(S,\lambda)$ we verify that (iii) is correct.

2.2. LEMMA. *Let $\mathcal{A}$ be a complex algebra, $\Theta : \mathcal{A} \to \mathbf{C}$ be a continuous algebra homomorphism and $\varphi : \mathcal{A} \to \mathcal{L}\left(l^2(n)\right)$ be a continuous linear map such that*

$$\varphi(ab) = \Theta(a)\varphi(b) + \Theta(b)\varphi(a).$$

Then the map $a \to N(\varphi(a), \Theta(a))$ is a continuous algebra homomorphism from $\mathcal{A}$ to $\mathcal{L}\left(l^2(n)\right)$.

PROOF. The continuity follows from (ii) and (iii) of the previous Lemma.

As in [9] Lemma 2.1 (iii) and Lemma 2.2 yield the following proposition.

2.3. PROPOSITION. *For f in $H(\omega)$ let $\nabla f(\omega)$ be the vector $(a_1, \ldots, a_m)$. Then the map $\rho : f \to \mathbf{N}\left(a_1T_1 + \ldots + a_mT_m, f(\omega)\right)$ is a continuous algebra homomorphism from $H(\omega)$ to $\mathcal{L}\left(l^2(n)\right)$ coinciding with the evaluation map $p \to p(\mathbf{N}(\mathbf{T}, \omega))$ on $\mathcal{P}(\Omega)$.*

2.4. Since the map ρ extends the evaluation map on $\mathcal{P}(\Omega)$ it follows that $\rho \otimes I_k$ is also a continuous algebra homomorphism of $\mathcal{A}(\Omega) \otimes \mathcal{M}_k$ coinciding with the evaluation map on $\mathcal{P}(\Omega) \otimes \mathcal{M}_k$.

Let X, Y be finite dimensional normed linear spaces and Ω be an open subset of X. A function $f : \Omega \subseteqq X \to Y$ is said to be holomorphic if the Frechet derivative of f at ω exists as a complex linear map from X to Y. Let $I = (i_1, \ldots, i_m)$ denote a multi-index of length $|I| = i_1 + \ldots + i_m$ and let e_k denote the multi-index with a one in the k^{th} position and zero elsewhere. Let $P : \Omega \to \mathcal{M}_k$ be a polynomial matrix valued function, that is, $P(z) = (p_{ij}(z))$, where each p_{ij} is a polynomial function in m-variables. Then we can write

$$P(z) = \sum P_I(z - \omega)^I$$

where each P_I is a scalar $k \times k$ matrix. Now it is easy to verify that the derivative $DP(\omega)$ of P at ω is

$$DP(\omega) = (P_{e_1}, \ldots, P_{e_m})$$

which acts on a vector $\mathbf{v} = (v_1, \ldots, v_m)$ by

$$DP(\omega) \cdot \mathbf{v} = v_1 P_{e_1} + \ldots + v_m P_{e_m}.$$

However, for notational convenience we always write $DP(\omega)$ for $(P_{e_1}, \ldots, P_{e_m})$. We introduce a pairing between two m-tuples of operators $\mathbf{S}$ and $\mathbf{T}$ as follows:

$$\langle \mathbf{S}, \mathbf{T} \rangle = S_1 \otimes T_1 + \ldots + S_m \otimes T_m$$

where the matrix for $A \otimes B$ is just $((a_{ij}b))$. In this notation, we have

$$(\rho \otimes I_k)(F) \sim \begin{bmatrix} I_n \otimes F(\omega) & \langle DF(\omega), \mathbf{N} \rangle \\ \mathbf{0} & I_n \otimes F(\omega) \end{bmatrix},$$

where $\sim$ indicates that the matrix on the right is obtained from the one on the left after elementary row and/or column operations.

3. Characterization of completely contractive modules

The main result in this section says that to determine when $\|\rho \otimes I_k\| \leqq \leqq 1$, it is enough to consider those functions which vanish at a fixed but arbitrary point of Ω. However to prove this we need, as in [10], the following result of Douglas, Muhly and Pearcy [4, Proposition 2.2].

3.1. LEMMA. *For $i = 1,2$ let T_i be a contraction on a Hilbert space $\mathcal{H}_i$ and let X be an operator mapping $\mathcal{H}_2$ into $\mathcal{H}_i$. A necessary and sufficient condition for the operator on $\mathcal{H}_1 \oplus \mathcal{H}_2$ defined by the matrix $\begin{bmatrix} T_1 & X \\ \mathbf{0} & T_2 \end{bmatrix}$ to be a contraction is that there exist a contraction C mapping $\mathcal{H}_2$ into $\mathcal{H}_1$ such that*

$$X = (I_{\mathcal{H}_1} - T_1 T_1^*)^{\frac{1}{2}} C (I_{\mathcal{H}_2} - T_2^* T_2)^{\frac{1}{2}}.$$

Again as in [10], we need some results about biholomorphic automorphisms of the unit ball in $\mathcal{M}_k$, which can be found in Harris [7, Theorem 2]. We collect the results we will need in the following.

3.2. LEMMA. *For each B in the unit ball $(\mathcal{M}_k)_1$ of $\mathcal{M}_k$, the Mobius transformation*

$$\varphi_B(A) = (I - BB^*)^{-\frac{1}{2}}(A + B)(I + B^*A)^{-1}(I - B^*B)^{\frac{1}{2}}$$

is a biholomorphic mapping of $(\mathcal{M}_k)_1$ onto itself with $\varphi_B(0) = B$. Moreover,

$$\varphi_B^{-1} = \varphi_{-B}, \quad \varphi_B(A)^* = \varphi_{B^*}(A^*), \quad \|\varphi_B(A)\| \leqq \varphi_{\|B\|}(\|A\|)$$

and

$$D\varphi_B(A)\mathbf{C} = (I - BB^*)^{\frac{1}{2}}(I + AB^*)^{-1}C(I + B^*A)^{-1}(I - B^*B)^{\frac{1}{2}}.$$

Now we are ready to prove the main result of this section. While this lemma is similar to Lemma 3.2 in [9] and the lemma in [10], in the present situation, some extra care is necessary for the proof.

3.3. LEMMA. *If $\|F(\mathrm{N})\| \leqq 1$ for all F in $\mathcal{M}_k(\mathrm{Hol}(\overline{\Omega}))$ with $\|F\|_\infty \leqq 1$ and $F(w) = \mathbf{0}$, then $\|G(\mathrm{N})\| \leqq 1$ for all G in $\mathcal{M}_k(\mathrm{Hol}(\overline{\Omega}))$ with $(\|G)\|_\infty \leqq \leqq 1$.*

PROOF. Any G in $\mathcal{M}_k(\mathrm{Hol}(\overline{\Omega}))$ of norm less than or equal to one, maps Ω into $(\mathcal{M}_k)_1$. In particular, for ω in Ω, $\|G(\omega)\| < 1$ we can form the Mobius transformation $\varphi_{-G(\omega)}$ of $(\mathcal{M}_k)_1$. Consider the map $\varphi_{-G(\omega)} \circ G$, which maps ω onto zero. Thus,

$$1 \geqq \left\| \left(\varphi_{-G(\omega)} \circ G\right)(\mathbf{N}) \right\| = \left\| \begin{bmatrix} \mathbf{0} & \langle D\left(\varphi_{-G(\omega)} \circ G\right)(\omega), \mathbf{T} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \right\|.$$

However,

$$\left\langle D\left(\varphi_{-G(\omega)} \circ G\right), \mathbf{T} \right\rangle = \left\langle D\varphi_{-G(\omega)}(G(\omega)) DG(\omega), \mathbf{T} \right\rangle =$$

$$= (I - G(\omega)G(\omega)^*)^{1/2} \left\langle DG(\omega), \mathbf{T} \right\rangle (I - G(\omega)^* G(\omega))^{-1/2}$$

by Lemma 3.2. On the other hand,

$$(I_{nk} - G(\omega)G(\omega)^* \otimes I_n)^{-1/2} \langle DG(\omega), \mathbf{T} \rangle (I_{nk} - G(\omega)^* G(\omega) \otimes I_n)^{-1/2} =$$

$$= \left((I_k - G(\omega)G(\omega)^*)^{-1/2} \otimes I_n \right) \langle DG(\omega), \mathbf{T} \rangle \cdot$$

$$\cdot \left((I_k - G(\omega)^* G(\omega))^{-1/2} \otimes I_n \right) =$$

$$= \left\langle (I_k - G(\omega)G(\omega)^*)^{-1/2} DG(\omega) (I_k - G(\omega)^* G(\omega))^{-1/2}, \mathbf{T} \right\rangle$$

and Lemma 3.1 implies that

$$G(\mathbf{N}) = \begin{bmatrix} G(\omega) \otimes I_n & \langle DG(\omega), \mathbf{T} \rangle \\ 0 & G(\omega) \otimes I_n \end{bmatrix}$$

has norm at most one.

3.4. The hypothesis on Ω guarantees that it can be realized as the unit ball in $\mathbf{C}^m$ with respect to a suitable norm $\|\cdot\|_\Omega$ on $\mathbf{C}^m$. In the following, the norm of a map between two normed linear spaces is always understood to be the usual operator norm. The following definition is an adaptation of Definition 1.2 [10].

3.5. DEFINITION. For ω in Ω, define

$$\mathbf{D}_{\mathcal{M}_k}\Omega(\omega) = \left\{ DF(\omega) \in \mathcal{L}\left((\mathbf{C}^m, \|\cdot\|_\Omega), \mathcal{M}_k \right) : F \in \operatorname{Hol}(\overline{\Omega}),\ \|F\|_\infty \leqq 1 \right\}.$$

The m-tuple $\mathbf{N}(\mathbf{T}, \omega)$determines a linear map

$$\rho_{\mathbf{N}}^{(k)} : \mathcal{L}\left((C^m, \|\cdot\|_\Omega),\ \mathcal{M}_k \right) \to \mathcal{L}\left(\mathbf{C}^{kn}, \mathbf{C}^{kn} \right)$$

defined by

$$\rho_{\mathbf{N}}^{(k)}(P_1,\ldots,P_m) = P_1 \otimes T_1 + \ldots + P_m \otimes T_m = \langle \mathbf{P},\mathbf{T}\rangle$$

We set

$$M_\Omega^k(\mathbf{N}(\mathbf{T},\omega)) = \sup\left\{\|\rho_{\mathbf{N}}^{(k)}(\mathbf{P})\| : \mathbf{P} \in \mathbf{D}_{\mathcal{M}_k}\Omega(\omega)\right\}$$

and

$$M_\Omega^c(\mathbf{N}(\mathbf{T},\omega)) = \sup\left\{M_\Omega^k(\mathbf{N}(\mathbf{T},\omega)) : k \in \mathbf{N}\right\}.$$

In what follows, when $k=1$, we will write $\rho_{\mathbf{N}}$ and $M_\Omega(\mathbf{N}(\mathbf{T},\omega))$ instead of $\rho_{\mathbf{N}}^{(1)}$ and $M_\Omega^1(\mathbf{N}(\mathbf{T},\omega))$. The map $\rho_{\mathbf{N}}$ is essentially the map ρ of Proposition 2.3. In view of Lemma 3.3, it is straightforward to prove the following theorem.

3.6. THEOREM. *$l^2(2n)_{\mathbf{N}}$ is a completely contractive $\mathcal{A}(\Omega)$-module if and only if $M_\Omega^c(\mathbf{N}(\mathbf{T},\omega)) \leqq 1$.*

Parts (a) and (b) of the following theorem are identical to Theorem 1.9 in [10]. However, part (c) and (d) are slightly different in view of the fact that we are using a more general module action than the previous set up. Also note that neither part (a) nor part (c) of the following theorem is very useful unless we assume Ω admits a transitive group of biholomorphic automorphisms.

3.7. THEOREM. *Let ω be in Ω and assume that there exists a biholomorphic automorphism Θ_ω of Ω such that $\Theta_\omega(\omega) = \mathbf{0}$, and let $D\Theta_k$ and $D\Theta^k$ be the k^{th} column and k^{th} row respectively of the derivative $D\Theta_\omega(\omega)$. Then,*

(a) $\mathbf{D}_{\mathcal{M}_k}\Omega(\omega) = \mathbf{D}_{\mathcal{M}_k}\Omega(\mathbf{0}) \cdot D\Theta_\omega(\omega) = \{(DP(0)\cdot D\Theta_1,\ldots,DP(0)\cdot D\Theta_m) : DP(0) \in \mathbf{D}_{\mathcal{M}_k}\Omega(\mathbf{0})\}$,

(b) $\mathbf{D}_{\mathcal{M}_k}\Omega(\mathbf{0}) = \{\mathbf{P} \in \mathcal{L}((\mathbf{C}^m, \|\cdot\|_\Omega), \mathcal{M}_k) : \|\mathbf{P}\| \leqq 1\}$,

(c) $M_\Omega^k(\mathbf{N}(\mathbf{T},\omega)) = M_\Omega^k(\mathbf{N}(D\Theta_\omega(\omega)\cdot\mathbf{T},\mathbf{0})) = M_\Omega^k(\mathbf{N}(D\Theta^1\cdot\mathbf{T},\ldots, D\Theta^m\cdot\mathbf{T};\,\mathbf{0}))$.

(d) $M_\Omega^k(\mathbf{N}(\mathbf{T},\mathbf{0})) = \sup\{\|\langle\mathbf{P},\mathbf{T}\rangle\| : \mathbf{P} \in \mathbf{D}_{\mathcal{M}_k}\Omega(\mathbf{0})\}$.

3.8. REMARK. Note that for $k=1$, $\mathbf{D}_{\mathcal{M}_1}\Omega(\mathbf{0}) = \left\{\mathbf{P} \in \mathcal{L}\left((\mathbf{C}^m, \|\cdot\|_\Omega), \mathbf{C}\right) : \|\mathbf{P}\| \leqq 1\right\}$. In other words, if $\|\cdot\|_{\Omega^*}$ denotes the norm on $\mathbf{C}^m$ that is dual to $\|\cdot\|_\Omega$ then $\mathbf{D}_{\mathcal{M}_1}\Omega(\mathbf{0})$ can be identified with $(\mathbf{C}^m, \|\cdot\|_{\Omega^*})_1$, which we write as Ω^*. Consequently, $M_\Omega(\mathbf{N}(\mathbf{T},\mathbf{0}))$ is less than or equal to one if and only if $\|\rho_{\mathbf{N}}\| \leqq 1$, that is, $\|z_1T_1 + \ldots + z_mT_m\| \leqq 1$ for all $(z_1,\ldots,z_m)$ in Ω^*, that is, $(T_1,\ldots,T_m)$ is in $D_{\mathcal{M}_1}\Omega^*(\mathbf{0})$.

Note that the inclusion $(\mathbf{C}^m, \|\cdot\|_{\Omega^*})$ in $\mathcal{A}(\Omega)$ via the map $\mathbf{z} \to l_{\mathbf{z}}$, where for $\mathbf{z}$ in $(\mathbf{C}^m, \|\cdot\|_{\Omega^*})$ and ω in $(\mathbf{C}^m, \|\cdot\|_{\Omega})$, $l_{\mathbf{z}}(\omega) = \sum_{j=1}^{n} \omega_j \bar{\mathbf{z}}_j$ is an isometry. We define, for each $[x_{ij}]$ in $(\mathbf{C}^m, \|\cdot\|_{\Omega^*}) \otimes \mathbf{M}_k$ the norm $\|[x_{ij}]\|$ using the inclusion map, which turns $(\mathbf{C}^m, \|\cdot\|_{\Omega^*})$ into a matricially normed space. (The definition and other related material is in [3].) Now, it is possible to talk of the *cb*-norm of the map $\rho_{\mathbf{N}} : (\mathbf{C}^m, \|\cdot\|_{\Omega^*}) \to \mathcal{M}_n$, we again refer the reader to [3] for this definition. It is easy to see, in view of Theorem 3.6, that studying completely bounded modules $\mathbf{C}^{2n}_{\mathbf{N}}$ over $\mathcal{A}(\Omega)$ is the same as studying completely bounded maps on the matrix normed space $(\mathbf{C}^m, \|\cdot\|_{\Omega^*})$. We will have more to say about this in Section 4.3.

4. Parrott's example and duality

The main theorem in this section is a duality result. As in the previous section, let Ω be the unit ball in $\mathbf{C}^m$ with respect to some norm $\|\cdot\|_{\Omega}$. Let $\|\cdot\|_{\Omega^*}$ be the dual norm and Ω^* be the unit ball in $\mathbf{C}^m$ with respect to the dual norm $\|\cdot\|_{\Omega^*}$.

4.1. THEOREM. *The following statements are equivalent:*

(i) *If $l^2(2n)_{\mathbf{N}(\mathbf{T},0)}$ is a contractive module over $\mathcal{A}(\Omega)$, then it is completely contractive.*

(ii) *If $l^2(2n)_{\mathbf{N}(\mathbf{T},0)}$ is a contractive module over $\mathcal{A}(\Omega^*)$, then it is completely contractive.*

PROOF. We prove (i) implies (ii). Note that by Remark 3.8, $l^2(2n)_{\mathbf{N}(\mathbf{T},0)}$ is contractive over $\mathcal{A}(\Omega^*)$, if and only if $\mathbf{T}$ is in $\mathbf{D}_{\mathcal{M}_n}(\Omega^*)^*(\mathbf{0})$. But $(\Omega^*)^*$ is equal to Ω so that the module $l^2(2n)_{\mathbf{N}(\mathbf{T},0)}$ is contractive over $\mathcal{A}(\Omega^*)$ if and only if $\mathbf{T}$ is in $\mathbf{D}_{\mathcal{M}_n}\Omega(\mathbf{0})$. By Theorem 3.7 (d), to show that $l^2(2n)_{\mathbf{N}(\mathbf{T},0)}$ is completely contractive, we have to establish for all k in $\mathbf{N}$

$$\|\langle \mathbf{P}, \mathbf{T}\rangle\| \leqq 1, \quad \text{for all} \quad \mathbf{P} \quad \text{in} \quad \mathbf{D}_{\mathcal{M}_k}\Omega^*(\mathbf{0}).$$

However, $\mathbf{P}$ in $\mathbf{D}_{\mathcal{M}_k}\Omega^*(\mathbf{0})$ is equivalent to saying $l^2(2k)_{\mathbf{N}(\mathbf{P},0)}$ is a contractive module over $\mathcal{A}(\Omega)$, again by Remark 3.8. But we are assuming any contractive module over $\mathcal{A}(\Omega)$ is completely contractive, so $l^2(2k)_{\mathbf{N}(\mathbf{P},0)}$ is completely contractive. Or equivalently, via Theorem 3.7(d), for all k in $\mathbf{N}$

$$\|\langle \mathbf{T}, \mathbf{P}\rangle\| \leqq 1, \quad \text{for all} \quad \mathbf{T} \quad \text{in} \quad \mathbf{D}_{\mathcal{M}_n}\Omega(\mathbf{0}).$$

Using the flip map to change the order of tensor products occurring in $\langle \mathbf{T}, \mathbf{P}\rangle$, we see that for all k and n in $\mathbf{N}$, we have

$$\|\langle \mathbf{P}, \mathbf{T}\rangle\| = \|\langle \mathbf{T}, \mathbf{P}\rangle\| \leqq 1,$$

for all $\mathbf{P}$ in $\mathbf{D}_{\mathcal{M}_n}\Omega^*(\mathbf{0})$ and all $\mathbf{T}$ in $\mathbf{D}_{\mathcal{M}_n}\Omega(\mathbf{0})$. This completes the proof of (i) implies (ii) and the other implication can be verified in a similar manner.

4.2. COROLLARY. *If Ω admits a transitive group of biholomorphic automorphisms and $l^2(2)$ is a contractive module over $\mathcal{A}(\Omega)$ then $l^2(2)$ is completely contractive.*

PROOF. Note that in this case

$$\mathbf{N}(\mathbf{t},\omega)=\left(\begin{bmatrix}\omega_1 & t_1\\ 0 & \omega_1\end{bmatrix},\dots,\begin{bmatrix}\omega_m & t_m\\ 0 & \omega_m\end{bmatrix}\right).$$

If $\omega=0$ then $\mathbf{N}(\mathbf{t},\omega)$ is contractive if and only if $\mathbf{t}=(t_1,\dots,t_m)$ is in Ω and the proof is obvious. If $\omega\neq 0$ then $\mathbf{N}(\mathbf{t},\omega)$ is contractive if and only if $D\Theta_\omega(\omega)\cdot\mathbf{t}$ is in Ω. To check complete contractivity we have to verify that

$$M_\Omega^k(\mathbf{N}(\mathbf{t},\omega))=M_\Omega^k(\mathbf{N}(D\Theta_\omega(\omega)\cdot\mathbf{t},\mathbf{0}))=$$

$$=\sup\{|\langle D\Theta_\omega(\omega)\cdot\mathbf{t},\mathbf{P}\rangle|:\mathbf{P}\in\mathbf{D}_{\mathcal{M}_k}\Omega(0)\}\leqq 1,$$

but the last inequality is clearly true since $D\Theta_\omega(\omega)\cdot\mathbf{t}\in\Omega$ and the proof is complete.

We wish to point out in this connection that, while the above corollary is not very hard to prove, J. Agler [1] has shown by using a more refined form of Schwarz lemma that the same statement as in the previous corollary holds for arbitrary convex bounded subsets of $\mathbf{C}^n$. As a consequence he reproves a result from complex geometry, which says that for such domains the Caratheodory and the Kobayashi metric are the same.

4.3. REMARK. Vern Paulsen has shown me that the above theorem can be used to answer the following question.

Note first that $\rho_\mathbf{N}$ is a linear map from $(C^m,\|\ \|_{\Omega^*})$ into the matrix algebra $\mathcal{M}_n$. What we have defined as M_Ω^c is nothing else but the completely bounded norm of $\rho_\mathbf{N}$ provided we endow the normed space $(C^m,\|\ \|_{\Omega^*})$ with matrix norm structure as follows [cf. 3]. If X is a normed space, then the canonical inclusion of X into the continuous functions on the unit ball of its dual allows us to identify X with a subspace of a C^*-algebra and hence endows X with a matrix norm structure such that X is an operator space. Given a matricially normed Banach space X and linear maps $\rho: X\to\mathcal{L}(\mathcal{H})$, define

$$\alpha(X)=\sup\{\|\rho\|_{cb}:\|\rho\|\leqq 1\}.$$

One natural question is to determine matricially normed Banach spaces for which $\alpha(X)=1$. The affirmative answer to this question is equivalent to asserting that any contractive module $l^2(2n)_{\mathbf{N}(\mathbf{T},\mathbf{0})}$ over $\mathcal{A}(\Omega)$ is also completely contractive. This question for $l^1(2)$ was first raised in Loebl [8].

If we look at the finite dimensional vector space $(\mathbf{C}^m, \| \|_{\Omega^*})$, then the matrix norm structure it inherits from $\mathcal{A}(\Omega)$, as discussed in 3.8, is the same as the matrix norm structure defined above. Note that by Ando's theorem [2], $\alpha(l^\infty(2)) \leqq 1$ and examples can be given to show that the bound is attained, thus $\alpha(l^\infty(2)) = 1$. Ando's result together with the previous theorem imply that $\alpha(l^1(2)) = 1$.

4.4. REMARK. The example we wish to discuss here is like that of Parrott [11]. Our discussion is computational in nature and shows that there is a contractive Hilbert module over $\mathcal{A}(\mathbf{D}^3)$, which is not even 2-contractive. (For a discussion see [12, p.92]). By Theorem 4.2, it follows that the same is true for $\mathcal{A}\left(\left(l^1(3)\right)_1\right)$, that is, there is a contractive Hilbert module over $\mathcal{A}\left(\left(l^1(3)\right)_1\right)$ which is not 2-contractive. We hope this clarifies some of the mystery surrounding Parrott's example.

4.5. LEMMA. *The norm of the map* $V : l^\infty(2) \to l^2(2)$ *is*

$$\|V\| = \left(\|\mathbf{v}_1\|^2 + \|\mathbf{v}_2\|^2 + 2\,|\langle \mathbf{v}_1, \mathbf{v}_2\rangle|\right)^{1/2},$$

where $\mathbf{v}_1$ *and* $\mathbf{v}_2$ *are the columns of the matrix for* V.

PROOF. It is enough to note that

$$\left\| V \cdot \begin{bmatrix} 1 \\ e^{i\vartheta} \end{bmatrix} \right\| = \left\| \mathbf{v}_1 + e^{i\vartheta}\mathbf{v}_2 \right\|_2 =$$

$$= \|\mathbf{v}_1\|^2 + \|\mathbf{v}_2\|^2 + e^{-i\vartheta}\langle \mathbf{v}_1, \mathbf{v}_2\rangle + e^{i\vartheta}\langle \mathbf{v}_2, \mathbf{v}_1\rangle.$$

The result follows by choosing $\vartheta = \arg\langle \mathbf{v}_1, \mathbf{v}_2\rangle$.

However, the natural generalisation of this formula does not hold for $V : l^\infty(n) \to l^2(n)$ for $n > 2$. In fact, $\left(\|V\|_\infty^2\right)^2$ is, in general, strictly smaller than $\sum\limits_{i,j} |\langle \mathbf{v}_i, \mathbf{v}_j\rangle|$. This is what we exploit the following.

4.6. EXAMPLE. Let $\mathbf{v}_1 = (1,0)$, $\mathbf{v}_2 = \frac{1}{2}\left(1, \sqrt{3}\right)$ and $\mathbf{v}_3 = \left(1, -\sqrt{3}\right)$. It is easy to see that the map $\eta_1 : \left(1, e^{i\vartheta}, e^{i\varphi}\right) \to \mathbf{v}_1 + e^{i\vartheta}\mathbf{v}_2 + e^{i\varphi}\mathbf{v}_3$ from $l^\infty(3)$ to $l^2(2)$ has norm strictly smaller than $\sqrt{6}$. Similarly for $\mathbf{u}_1 = (0,1)$, $\mathbf{u}_2 = \frac{1}{2}\left(\sqrt{3}, -1\right)$ and $\mathbf{u}_3 = \frac{1}{2}\left(\sqrt{3}, 1\right)$, we see that the norm of the map $\eta_2 : \left(1, e^{i\vartheta}, e^{i\varphi}\right) \to \mathbf{u}_1 + e^{i\vartheta}\mathbf{u}_2 + e^{i\varphi}\mathbf{u}_3$ from $l^\infty(3)$ to $l^2(2)$ is strictly less than $\sqrt{6}$. In fact, with a little effort, one can show that each of these norms equals $\frac{3}{\sqrt{2}}$. Let U be the unitary matrix whose rows are $\mathbf{v}_2$ and $\mathbf{u}_2$. Similarly let V be the unitary matrix whose rows are $\mathbf{v}_3$ and $\mathbf{u}_3$. Now, consider the map

$$\rho_{(I,U,V)} : \left(1, e^{i\vartheta}, e^{i\varphi}\right) \to I + e^{i\vartheta}U + e^{i\varphi}V$$

and note that the operator norm of $\left(I+e^{i\vartheta}U+e^{i\varphi}V\right)$ is at most $\sqrt{\|\eta_1\|^2+\|\eta_2\|^2}$. However if at a fixed ϑ, φ either of the norms $\|\eta_1\|$ or $\|\eta_2\|$ is equal to $3/\sqrt{2}$ then the other one is strictly less than $3/\sqrt{2}$. So that the operator norm of $\left(I+e^{i\vartheta}U+e^{i\varphi}V\right)$ is strictly less than 3. Thus we have shown the form of the map

$$\left(1, e^{i\vartheta}, e^{i\varphi}\right) \to I+e^{i\vartheta}U+e^{i\varphi}V$$

from $l^\infty(3)$ to $\mathcal{L}\left(l^2(2)\right)$ is strictly less than 3.

4.7. THEOREM. *$l^2(4)_{\mathrm{N}((I,U,V),0)}$ is contractive but not completely contractive over $\mathcal{A}(\mathbf{D}^3)$.*

PROOF. To show that $l^2(4)_{\mathrm{N}}$ is contractive we have to establish — by Remark 3.8 — that

$$\|z_1I+z_2U+z_3V\| \leqq 1$$

for all $(z_1,z_2,z_3)\in (l^1(3))_1$. But the inequality holds since each of I, U and V is a contraction operator. Note that the above discussion implies that $(I,U,V)/\delta$, for some $\delta<3$, is in $\mathbf{D}_{\mathcal{M}_2}\mathbf{D}^3(\mathbf{0})$. To show that l^2_{N} is not completely contractive, we compute

$$\|\langle (I,U,V)/\delta,\ (I,U,V)\rangle\| =$$

$$= \delta^{-1}\|I\otimes I+U\otimes U+V\otimes V\| =$$

$$= \delta^{-1}\left\|\begin{bmatrix} i+\frac{1}{2}u+\frac{1}{2}v & \frac{\sqrt{3}}{2}(U-V)\\ \frac{\sqrt{3}}{2}(U+V) & I-\frac{1}{2}U+\frac{1}{2}V\end{bmatrix}\right\| =$$

$$= \frac{\sqrt{3}}{2}\delta^{-1}\left\|\begin{bmatrix} \sqrt{3} & 0 & 0 & \sqrt{3}\\ 1 & 1 & 0 & -1\\ 1 & 0 & 1 & -1\\ \sqrt{3} & 0 & 0 & \sqrt{3}\end{bmatrix}\right\| = 3\delta^{-1} > 1.$$

4.8. REMARK. We have not been able to decide, whether for operators of the form

$$N(V,\mathbf{0}) = (N(\mathbf{v}_1,\mathbf{0}),\ldots,N(\mathbf{v}_m,\mathbf{0}))$$

as in [9], where $\mathbf{v}_1,\ldots,\mathbf{v}_m$ are vectors in $\mathbf{C}^n$, contractivity implies complete contractivity over $\mathcal{A}(\mathbf{D}^m)$. Vern Paulsen has shown that in this case the complete bound for a contractive map can be at most K_G, the universal constant of Grothendieck.

Note added in proof (November 8, 1993). In the paper "Contractive homomorphisms and tensor product norms", written jointly with B. Bagchi we have obtained many results relating to Remark 4.8.

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