PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 91, Number 1, May 1984

CURVATURE AND THE BACKWARD SHIFT OPERATORS

G. MISRA

ABSTRACT. Let φ_α be a Möbius transformation of the unit disk $\mathbf{D}$, $|\alpha| < 1$. We characterize all the operators T in $B_1(\mathbf{D})$ which are unitarily equivalent to $\varphi_\alpha(T)$ for all α with $|\alpha| < 1$, using curvature techniques.

0. Introduction. The backward shift operator U_+^* lies in the class $B_1(\mathbf{D})$, first introduced in Cowen and Douglas [**1**]. It is easy to compute the curvature $\mathcal{K}_{U_+^*}(\omega)$, which turns out to be $-(1-|\omega|^2)^{-2}$. For any operator T in $B_1(\mathbf{D})$ with $\|T\| \leq 1$, we have [**3**], $\mathcal{K}_T(\omega) \leq -(1-|\omega|^2)^{-2}$. This inequality is best possible over all of $\mathbf{D}$ since equality holds for $T = U_+^*$. Some time back R. G. Douglas asked if the inequality is best possible pointwise; that is, if $T \in B_1(\mathbf{D})$, $\|T\| \leq 1$ and $\mathcal{K}_T(\omega_0) = -(1-|\omega_0|^2)^{-2}$ for some ω_0 in $\mathbf{D}$, does it follow that T is unitarily equivalent to U_+^*?

In this note we obtain a characterization of those operators T in $B_1(\mathbf{D})$ that are unitarily equivalent to $\varphi_\alpha(T)$ for all α, where φ_α is a Möbius transformation of the disk, and answer the above problem in the negative.

1. The class $B_1(\mathbf{D})$ is defined as follows.

$$\begin{aligned} B_1(\mathbf{D}) = \{T \in \mathcal{L}(\mathcal{H}) :\ & \text{(i) } \mathbf{D} \subset \sigma(T), \\ & \text{(ii) } \textstyle\bigvee_{\omega \in \mathbf{D}} \ker(T-\omega) = \mathcal{H}, \\ & \text{(iii) } \operatorname{ran}(T-\omega) = \mathcal{H}, \\ & \text{(iv) } \dim\ker\,(T-\omega) = 1 \text{ for all } \omega \in \mathbf{D}\}. \end{aligned}$$

For each operator T in $B_1(\mathbf{D})$, such that $T(\gamma(\omega)) = \omega\gamma(\omega)$, it is possible to find a holomorphic family of eigenvectors $\gamma(\omega)$ on $\mathbf{D}$. Following Cowen and Douglas [**1**], we can define the curvature of an operator T in $B_1(\mathbf{D})$ to be

$$\mathcal{K}_T(\omega) = \frac{\partial^2}{\partial\omega\partial\overline{\omega}} \log \|\gamma(\omega)\|^{-2}.$$

Let $\varphi_\alpha(\omega) = (\alpha - \omega)(1 - \overline{\alpha}\omega)^{-1}$ be a Möbius transformation of the unit disk, $|\alpha| < 1$. Whenever $\|T\| \leq 1$, the operator $\varphi_\alpha(T)$ is well defined and a simple application of chain rule yields

$$\|\varphi_\alpha'(\omega)|^2 \mathcal{K}_{\varphi_\alpha(T)}(\varphi_\alpha(\omega)) = \mathcal{K}_T(\omega).$$

In particular if $T = U_+^*$, we obtain

$$\begin{aligned} \mathcal{K}_{\varphi_\alpha(U_+^*)}(\varphi_\alpha(\omega)) &= |\varphi_\alpha'(\omega)|^{-2}\mathcal{K}_{U_+^*}(\omega) = -|\varphi_\alpha'(\omega)|^{-2}(1-|\omega|^2)^{-2} \\ &= -(1-|\varphi_\alpha(\omega)|^2)^{-2} = \mathcal{K}_{U_+^*}(\varphi_\alpha(\omega)). \end{aligned}$$

Received by the editors June 28, 1983.
1980 *Mathematics Subject Classification*. Primary 47B37.

The Cowen-Douglas theorem, which states that two operators in $B_1(\mathbf{D})$ are unitarily equivalent if and only if their curvatures are equal, implies $\varphi_\alpha(U_+^*)$ is unitarily equivalent to U_+^* for all α. We can now ask ourselves, which other operators in $B_1(\mathbf{D})$ share this property.

PROPOSITION. *If T is in $B_1(\mathbf{D})$ and $\|T\| \leq 1$ then $\varphi_\alpha(T)$ is unitarily equivalent to T for all α if and only if*

$$\mathcal{K}_T(\omega) = -c(1 - |\omega|^2)^{-2},$$

for some constant $c \geq 1$.

PROOF. if $\mathcal{K}_T(\omega) = -c(1-|\omega|^2)^{-2}$, a calculation similar to the one above shows that $\varphi_\alpha(T)$ must be unitarily equivalent to T for all α.

Conversely, if T is unitarily equivalent to $\varphi_\alpha(T)$ for all α then we must have

$$\mathcal{K}_{\varphi_\alpha(T)}(\varphi_\alpha(\omega)) = |\varphi_\alpha'(\omega)|^2 \mathcal{K}_T(\omega) = \mathcal{K}_T(\varphi_\alpha(\omega)).$$

In particular, $|\varphi_\alpha'(0)|^{-2}\mathcal{K}_T(0) = \mathcal{K}_T(\alpha)$ so $\mathcal{K}_T(\alpha) = (|\alpha|^2 - 1)^{-2}\mathcal{K}_T(0)$ for all α in $\mathbf{D}$. Let c equal $\mathcal{K}_T(0)$, then $\mathcal{K}_T(\omega) \leq -(1 - |\omega|^2)^{-2}$ implies that $c \geq 1$.

Now, consider the weighted shift operator T with weights $\omega_n = (c_n/c_{n+1})^{1/2}$, where c_n is the nth coefficient in the generalized binomial expansion of $(1-|\omega|^2)^{-c}$ for a fixed real number c. The adjoint of T is in $B_1(\mathbf{D})$ (Seddighi [**4**]) and $\gamma(\omega) = (1 - |\omega|^2)^{-c}$ is a holomorphic family of eigenvectors for T^*. It is easy to compute

$$\mathcal{K}_{T^*}(\omega) = -c(1 - |\omega|^2)^{-2}.$$

When c is an even integer these operators can be identified with the adjoint of multiplication on the Hilbert space of square integrable holomorphic functions on $\mathbf{D}$ with respect to the measure $d\mu = (i/2)(1 - |\omega|^2)^{2-2q}\, d\omega \wedge d\bar{\omega}$ (cf. Kra [**2**, pp. 89 and 95]). Thus, we are able to idenify all of the operators that are unitarily equivalent to all their Möbius transforms $\varphi_\alpha(T)$.

It follows from the Proposition that if $T \in B_1(\mathbf{D})$ and $\|T\| \leq 1$, then the following two statements are equivalent.

(1) $\mathcal{K}_T(\omega_0) = -(1 - |\omega_0|^2)^{-2}$ for some ω_0 and $\varphi_\alpha(T)$ is unitarily equivalent to T for all α.

(2) T is unitarily equivalent to U_+^*.

However, $\mathcal{K}_T(\omega_0) = -(1-|\omega_0|^2)^{-2}$ does not necessarily imply that T is unitarily equivalent to U_+^* as we will show by means of an example.

Let T be a weighted shift operator with weights $\omega_0, \omega_1, \omega_2, \ldots$. We can consider T to be an ordinary shift on a weighted sequence space (Shields [**5**]) with weights $\beta(0), \beta(1), \ldots$. For $\omega \in \mathbf{D}$,

$$\gamma(\omega) = \left(\frac{1}{\beta(0)}, \frac{\omega}{\beta(1)}, \frac{\omega^2}{\beta(2)}, \ldots\right)$$

is an eigenvector for T^* and

$$\|\gamma(\omega)\|^2 = \sum_{n=0}^{\infty} \frac{|\omega|^{2n}}{|\beta(n)|^2}.$$

Assuming T^* is in $B_1(\mathbf{D})$ (Seddighi [**4**] determines when a weighted shift is in $B_1(\mathbf{D})$), we compute

$$\mathcal{K}_{T^*}(\omega) = -\left[\left(\sum_{n=0}^{\infty}(n+1)^2\frac{|\omega|^{2n}}{|\beta(n+1)|^2}\right)\left(\sum_{n=0}^{\infty}\frac{|\omega|^{2n}}{|\beta(n)|^2}\right)\right.$$
$$\left. - |\omega|^2\left(\sum_{n=0}^{\infty}(n+1)\frac{|\omega|^{2n}}{|\beta(n+1)|^2}\right)^2\right]\left[\sum_{n=0}^{\infty}\frac{|\omega|^{2n}}{|\beta(n)|^2}\right]^{-2}.$$

Putting $\omega = 0$, we see that

$$\mathcal{K}_{T^*}(0) = -|\beta(0)\beta(1)^{-1}|^2.$$

Now, let T be the weighted shift with weights $1, \frac{1}{2}, 1, 1, 1, \ldots$. It is easy to verify that $T^* \in B_1(\mathbf{D})$ and $\|T^*\| = 1$. Since $\beta(0) = 1$ and $\beta(1) = 1$, it follows that $\mathcal{K}_{T^*}(0) = -1$. Obviously T^* is not unitarily equivalent to U_+^*.

In fact, we can compute $h_{T^*}(\omega)$ explicitly for the weighted shift of our example and show that $h_{T^*}(\varphi_\alpha(\omega)) \not\equiv |\varphi_\alpha'(\omega)|h_{T^*}(\omega)$, therefore T is not unitarily equivalent to $\varphi_\alpha(T)$ for any α.

References

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Department of Mathematics, University of Georgia, Athens, Georgia 30602