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ORIGINAL RESEARCH

KRP and his imprimitivity theorem

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I first met Professor Parthasarathy when he was visiting Sambalpur university where I was a postgraduate student. He then stopped over at Bhubaneswar on his way to Delhi. This provided me an opportunity to talk to him in a somewhat informal atmosphere. After several years, I joined the Indian Statistical Institute Kolkata in 1986. Soon after, Professor Parthasarathy invited me for a short visit to the Delhi Center of the Indian Statistical Institute. He was, of course, well known for his very meticulously prepared lectures that he delivered with great clarity. So, when I was asked to give a lecture with Professor Parthasarathy in the audience, I was very nervous. I remember to this date how during my lecture he was making several suggestions for picking better notation among a myriad of other things. I am sure this and many other friendly tips over the years has made me rethink my own approach to both teaching and research.

In the early nineties, together with my colleague Bhaskar Bagchi, I was trying to understand the Wallach set: Let $K : X \times X \to \mathbb{C}$ be a positive definite kernel defined on a set X, that is, the $n \times n$ matrix $\big(K(x_j, x_k)\big)_{j,k=1}^{n}$ is positive definite for all subsets $\{x_1, \dots, x_n\}$ of X and all $n \in \mathbb{N}$. The Wallach set of the pair (X, K) for any bounded domain X in $\mathbb{C}^n$ is the set

$$\{\lambda > 0 \mid K^{\lambda} \text{is positive definite}\},$$

where K is assumed to be holomorphic in the first variable and anti-holomorphic in the second. Moreover, K^{λ} is defined by first defining $K(w, w)^{\lambda}$ for any $\lambda > 0$ and then defining $K(z, w)^{\lambda}$ by polarizing the power series of the real analytic function $K(w, w)^{\lambda}$ in a neighbourhood of the set $\{(w, \bar{w}) \mid w \in X\}$. In [3], topics closely related to the Wallach set are discussed. Therefore, I thought it would be great if KRP (by now, like everybody else, I have switched to addressing Professor Parthasarathy by the more familiar name of KRP) can visit us at ISI Bangalore and give a few lectures on positive definite kernels. To my delight, when I checked with him, he happily agreed and delivered a series of mesmerizing lectures on positive definite kernels. He left his very detailed and complete lecture notes with me. Although, he never said it, I think, the idea was for me to convert his carefully prepared handwritten notes to a more formal set of lecture notes or a book. It is entirely my misfortune that I never got around to actually doing it.

A week long conference, "Mathematical Foundations of Quantum Mechanics" at IISER Kolkata in the year 2010 provided another opportunity for me to talk to KRP at length. After my lecture on imprimitivity in this conference, he said that I should learn Quantum Mechanics. We used to take long walks in the evening around the

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Dedicated to the memory of K. R. Parthasarathy

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campus. During these long walks, he made it a point to patiently explain some of the basic principles of Quantum Mechanics to someone who had absolutely no idea about the subject. Among other things, he recommended that I get hold of a copy of “PCT, Spin and Statistics, and all that" and read it. Following his advice, of course, I bought the book promptly but I can’t say I have been able to read much of it. Nevertheless let me attempt to describe a version of the imprimitivity theorem due to KRP [1] that is both deep, like many of his other theorems, and is at the confluence of the broad themes of Representation theory and Quantum mechanics.

1 States

We assume all Hilbert spaces are complex and separable and all operators are bounded. Replace a Borel σ-algebra by the lattice $\mathcal{P}(\mathcal{H})$ of projections on a Hilbert space $\mathcal{H}$ and a Borel measure by a function,

$$\mu : \mathcal{P}(\mathcal{H}) \to [0, 1],\ \text{satisfying}\ \mu(0) = 0\ \text{and}\ \mu(I) = 1;\ \mu\Big(\bigvee_{i=1}^{\infty} P_i\Big) = \sum_{i=1}^{\infty} \mu(P_i)$$

whenever $P_i P_j = 0$ for every $i \neq j$. The map μ is called a *state* on $\mathcal{P}(\mathcal{H})$. Examples are easy to construct: Given a unit vector u in $\mathcal{H}$ define $\mu_u : \mathcal{P}(\mathcal{H}) \to [0, 1]$ by setting $\mu_u(P) = \langle Pu, u\rangle$. Are there other states? In general, we have the following theorem due to Gleason.

Theorem 1.1 *(Gleason) Any state μ must be of the form $\mu(P) = \mathrm{tr}(PT)$ for some non-negative operator T on a Hilbert space $\mathcal{H}$, $\dim(\mathcal{H}) \geqslant 3$, with $\mathrm{tr}(T) = 1$.*

Since T is a non-negative operator with trace 1 there exists an orthonormal set of eigenvectors $\{u_j : j = 1, 2, \cdots\}$ of T with $Tu_j = \lambda_j u_j$, $\lambda_j \geq 0$, $\sum_{j=1}^{\infty} \lambda_j = 1$, such that

$$\mathrm{Tr}\,(PT) = \sum_{j=1}^{\infty} \lambda_j \left\langle Pu_j, u_j\right\rangle.$$

Consequently, $\{\mu_u \mid \|u\| = 1, u \in \mathcal{H}\}$ are the extreme points of the convex set consisting of all states. These are called *pure* states. For a pure state μ_u, we have $\mu_u(u) = \mu_u(cu)$ for any c in the unit circle $\mathbb{T}$. We can therefore identify pure states with elements of the Projective Hilbert space $P(\mathcal{H})$ obtained by identifying any two unit vectors u and v in $\mathcal{H}$ if $u = \alpha v$ for some $\alpha \in \mathbb{T}$.

Suppose that $\Gamma : \mathcal{P}(\mathcal{H}) \to \mathcal{P}(\mathcal{H})$ is a one to one onto map satisfying

(i) $\Gamma(0) = 0, \Gamma(I) = I$;
(ii) $\Gamma\left(\bigvee_j P_j\right) = \bigvee_j \Gamma\left(P_j\right), \Gamma\left(\bigwedge_j P_j\right) = \bigwedge_j \Gamma\left(P_j\right)$ for every sequence $\{P_j\}$ in $\mathcal{P}(\mathcal{H})$,
(iii) $\Gamma(I - P) = I - \Gamma(P)$.

Then Γ is called an automorphism of $\mathcal{P}(\mathcal{H})$. All such automorphisms constitute a group under composition. Let $\mathrm{Aut}\,\mathcal{P}(\mathcal{H})$ denote this group. Evidently, if U is a unitary operator on $\mathcal{H}$, then the map $\Gamma_U : \mathcal{P}(\mathcal{H}) \to \mathcal{P}(\mathcal{H})$ defined by

$$\Gamma_U(P) = UPU^{-1},\ P \in \mathcal{P}(\mathcal{H})$$

is an automorphism. Are there other automorphisms? Wigner’s theorem says that every automorphism Γ of $\mathcal{P}(\mathcal{H})$ is induced in this manner by a unitary or antiunitary operator (UA operator in short), namely, a map $V : \mathcal{H} \to \mathcal{H}$ that is onto, $V(u + v) = Vu + Vv$ for all $u, v \in \mathcal{H}$, $V(cu) = \bar{c}Vu$, $c \in \mathbb{C}$ and $u \in \mathcal{H}$ and $\langle Vu, Vv\rangle = \langle v, u\rangle$ for all $u, v \in \mathcal{H}$.

Theorem 1.2 *(Wigner) Let $\mathcal{H}$ be a Hilbert space with $\dim(\mathcal{H}) \geqslant 3$. Then to every automorphism Γ of $\mathcal{P}(\mathcal{H})$ there corresponds a unitary or antiunitary operator U satisfying*

$$\Gamma(P) = UPU^{-1} \quad \textit{for all } P \in \mathcal{P}(\mathcal{H}).$$

If V is another unitary or antiunitary operator satisfying the identity $\Gamma(P) = VPV^{-1}$ for all P then there exists $c \in \mathbb{T}$ such that $V = cU$.

Complete self contained proof of Gleason’s theorem as well as the theorem of Wigner is in [4].

1.1 Projective unitary antiunitary representations

All unitary and antiunitary operators on $\mathcal{H}$ form a group $\mathcal{UA}(\mathcal{H})$. The product of two antiunitary operators is unitary. The product of a unitary and an antiunitary operator is antiunitary. The group $\mathcal{U}(\mathcal{H})$ of all unitary operators is a normal open subgroup of $\mathcal{UA}(\mathcal{H})$ and the quotient $\mathcal{UA}(\mathcal{H})/\mathcal{U}(\mathcal{H})$ consists of two elements. Let $\pi(\mathcal{H})$ denote the compact subgroup $\{cI, |c| = 1\}$. Then $\pi(\mathcal{H})$ is the centre of $\mathcal{UA}(\mathcal{H})$. Wigner's theorem implies that there is a group isomorphism between Aut $\mathcal{P}(\mathcal{H})$ and the quotient group $\tilde{\mathscr{U}}(\mathcal{H}) := \mathcal{UA}(\mathcal{H})/\pi(\mathcal{H})$. The group $\mathcal{UA}(\mathcal{H})$ with the weak topology (equivalently, the strong topology) inherited by it, is shown in [1, page. 308] to be a complete and separable metric group. When endowed with the quotient topology, $\tilde{\mathscr{U}}(\mathcal{H})$ becomes a separable metric group. Moreover, [1, Lemma 2.3] implies that it is actually a complete and separable metric group. Let

$$^{\sim} : \mathcal{UA}(\mathcal{H}) \to \tilde{\mathscr{U}}(\mathcal{H})$$

be the canonical quotient homomorphism. Thus we may topologise Aut $\mathcal{P}(\mathcal{H})$ by giving it the quotient topology of $\tilde{\mathscr{U}}(\mathcal{H})$ through Wigner's isomorphism. This makes Aut $\mathcal{P}(\mathcal{H})$ a complete and separable metric group. A sequence $\{\Gamma_n\}$ in Aut $\mathcal{P}(\mathcal{H})$ converges to an automorphism Γ if the weak limit, as $n \to \infty$, of $\Gamma_n(P)$ is $\Gamma(P)$ for every $P \in \mathcal{P}(\mathcal{H})$. Moreover, there exists a Borel cross-section for $^{\sim}$, namely, a one to one Borel map $\eta : \tilde{\mathscr{U}}(\mathcal{H}) \to \mathcal{UA}(\mathcal{H})$ such that $\eta(U^{\sim})^{\sim} = U^{\sim}$, see [1, Corollary 2.2].

Let G denote a locally compact second countable group equipped with the natural Borel structure compatible with the topology. Also, for the sake of brevity, we write $\tilde{\mathscr{U}}$ instead of $\tilde{\mathscr{U}}(\mathcal{H})$. As before, it is equipped with the quotient topology.

A Borel homomorphism from G into $\tilde{\mathscr{U}}$ is called a *projective unitary antiunitary representation* or simply a PUA representation of G in $\mathcal{H}$.

A well-known theorem due to Mackey (cf. [2, Theorem 2,2]) states that if G is a locally compact second countable group and H is a separable metric group, and $\pi : G \to H$ is a Borel homomorphism from G into H, then π is continuous. Since $\tilde{\mathscr{U}}$ is a separable metric group, it follows that the map $g \mapsto \pi(U_g)$ is continuous. Thus, any PUA representation of G is continuous, see [1, Lemma 3.1].

1.2 Multipliers

The lifting of a projective unitary representation to a multiplier representation is well-known. In the paper [1], first, how to lift PUA representations to multiplier representations (see below) is discussed. This is necessarily more complicated since both unitary and antiunitary representations are involved. Secondly, the imprimitivity theorem due to Mackey, originally proved only for *projective unitary representations* is now proved for PUA representations. Let me conclude by providing some details briefly of the imprimitivity theorem of KRP following [1].

Suppose that $g \to U_g{}^{\sim}$ is a PUA representation of G. Making use of the cross section η, construct a Borel map $g \to \eta\left(U_g{}^{\sim}\right)$ from G into $\mathcal{UA}(\mathcal{H})$. Since $\eta\left(U_g{}^{\sim}\right)^{\sim} = U_g{}^{\sim}$, it follows that $U_g = \eta\left(U_g{}^{\sim}\right)$ without loss of generality. Then $g \to U_g$ is a Borel map and for any two elements $g_1, g_2 \in G$, $\left(U_{g_1}U_{g_2}\right)^{\sim} = U^{\sim}_{g_1g_2}$. Hence there exists a complex number $\sigma(g_1, g_2) \in \mathbb{T}$ such that

$$U_{g_1}U_{g_2} = \sigma(g_1, g_2)\, U_{g_1g_2} \text{ for all } g_1, g_2 \in G. \tag{1.1}$$

Assume that $U_e = I$, where e is the identity element of G. Then

$$\sigma(e, g) = \sigma(g, e) = 1 \quad \text{for all } g \in G. \tag{1.2}$$

Computing $U_{g_1}U_{g_2}U_{g_3}$ in two different ways, as $U_{g_1}\left(U_{g_2}U_{g_3}\right)$ and $\left(U_{g_1}U_{g_2}\right)U_{g_3}$, it is shown (see [1, Equation (3.3)]) that

$$\sigma(g_1, g_2)\,\sigma(g_1g_2, g_3) = \begin{cases} \sigma(g_1, g_2g_3)\,\sigma(g_2, g_3) & \text{if } g_1 \in G^+ \\ \sigma(g_1, g_2g_3)\,\bar{\sigma}(g_2, g_3) & \text{if } g_1 \in G^-, \end{cases} \tag{1.3}$$

where the set G^+ is the open and closed normal subgroup $\left\{g : U_g^\sim \text{ is unitary modulo } \pi(\mathcal{H})\right\}$, see [1, Lemma 3.1], and $G^- := \left\{g : U_g^\sim \text{ is antiunitary modulo } \pi(\mathcal{H})\right\}$.

A Borel function σ defined on $G \times G$ and taking values in $\mathbb{T}$ is called a multiplier if it satisfies Equations (1.2) and (1.3). A Borel map $g \to U_g$ from G into $\mathcal{UA}(\mathcal{H})$ is called a multiplier representation if there exists a multiplier σ such that Equation (1.3) is satisfied. When G^- is the empty set, that is, ${U_g}^\sim$ is a projective unitary representation, Equations (1.2) and (1.3) coincide with the usual multiplier identities, see [2, page 2].

Theorem 1.3 *(Theorem 3.1, [1]) Let G be a locally compact second countable group and $g \to U_g^\sim$ be a PUA representation of G. Then there exists a multiplier representation $g \to V_g$ of G such that $V_g^\sim = U_g^\sim$ for all $g \in G$. Conversely every multiplier representation $g \to V_g$ of G determines a PUA representation $g \to V_g^\sim$ of G.*

1.3 Imprimitivity

We first recall Mackey's imprimitivity theorem and then describe the non-trivial generalization of this theorem obtained by KRP.

Let G be a locally compact second countable group and X be a locally compact G - space, that is, there is a map $\alpha : G \times X \to X$, such that for a fixed $g \in G$, the map $x \to \alpha_g(x)$, $\alpha_g(x) := \alpha(g, x)$ is bijective and continuous on X, moreover, $g \to \alpha_g$ is a homomorphism. The action of G on X is said to be *transitive* if for every pair x_1, x_2 in X, there is a $g \in G$ such that $g \cdot x_1 = x_2$, $g \cdot x := \alpha(g, x)$. Let $H \subseteq G$ be a closed subgroup and let $X := G/H$ be the space of cosets: $\{gH \mid g \in G\}$. Equipped with the action of G by left multiplication: $g'(gH) := \big(g'g\big) H$, $g', g \in G$, the coset space X is a transitive G- space.

Let $(X, \mathcal{B})$ be the Borel measurable space, and note that each $g \in G$ defines a continuous map on X by our assumption. Given a σ-finite measure μ on X, define the *push-forward* $g_*\mu$ of the measure μ by the requirement

$$(g_*\mu)(A) := \mu\big(g \cdot A\big),\;\; g \cdot A := \{g^{-1} \cdot s \mid s \in A\}, A \in \mathcal{B}.$$

The measure μ on X is said to be invariant if $g_*\mu = \mu$ and *quasi-invariant* if $g_*\mu$ is equivalent (mutually absolutely continuous) to μ for all $g \in G$. There is a quasi-invariant measure uniquely determined modulo equivalence on X, see page 313 of [1].

If G is second countable, then there is a Borel cross-section $p : G/H \to G$, that is, a Borel subset $B \subset G$ that meets each coset of H in exactly one point. Thus, each $g \in G$ can be written uniquely as $g = g_1 g_0$ with $g_0 \in H$ and $g_1 \in B$, see page 315 of [1].

A spectral measure, or a projection valued measure, defined on X is a projection valued map $P : \mathcal{B} \to \mathcal{P}(\mathcal{H})$ such that $P(X) = I$ and $P(\cup E_k) = \sum_{k=1}^\infty P(E_k)$ for any disjoint collection of sets E_k, $k = 1, 2, \ldots,$ in $\mathcal{B}$, where the convergence is in the strong operator topology.

A system of imprimitivity $(\mathcal{H}, U_g, P(E))$ introduced by Mackey consists of a *projective unitary representation* U of a second countable locally compact group G on a Hilbert space $\mathcal{H}$ and a regular $\mathcal{H}$-projection-valued measure P on X such that

$$U(g)P(E)U(g)^{-1} = P(g \cdot E) \tag{1.4}$$

for all $g \in G$ and every Borel subset E of X.

The imprimitivity theorem of Mackey (*involving only projective unitary representations*) has two parts: Firstly, any transitive imprimitivity $(\mathcal{H}, U_g, P(E))$ is equivalent to a canonical imprimitivity, where $\mathcal{H} = L^2(X, \mu, \mathcal{H}_n)$, U is a projective unitary representation on $L^2(X, \mu, \mathcal{H}_n)$, that is,

$$\big(U(g)h\big)(x) = c(g, x)(g \cdot h)(x),\;\; h \in L^2(X, \mu, \mathcal{H}_n),\;\; g \in G, (g \cdot h)(x) = h(g \cdot x),$$

where $c : G \times X \to \mathcal{U}(\mathcal{H}_n)$ is a Borel map taking values in the group of unitary operators acting on the Hilbert space $\mathcal{H}_n$ of dimension n. For U to be a homomorphism, the function c must be a cocycle. The spectral measure P is defined, via the functional calculus, by setting $P(E) = M_{\mathbb{1}_E}$, $E \in \mathcal{B}$ and $\mathbb{1}_E$ is the characteristic function of E. Here M_f denotes the multiplication by f, $f \in L^\infty(X, \mu, \mathcal{H}_n)$ on $L^2(X, \mu, \mathcal{H}_n)$. Secondly, the imprimitivity theorem asserts that such a multiplier representation is induced from a unitary representation of the subgroup H acting on the Hilbert space $\mathcal{H}_n$.

In the generalization of Mackey's imprimitivity theorem obtained by KRP, the projective unitary representation is replaced by a PUA (projective unitary antiunitary) representation. An automorphism of the lattice of projections induces a map on the state space and that along with Wigner's theorem discussed in the beginning not only justifies such a generalization but makes it indispensable. However, there are new complications arising from the decomposition of the group $G = G^+ \biguplus G^-$. As we have seen, the multiplier identities are a lot more complicated.

Obtaining a canonical form of the imprimitivity when both projective unitary and antiunitary (PUA) representations are involved is the first non-trivial step in the generalization of Mackey's imprimitivity theorem to the case of PUA representations. Let me reproduce below how KRP achieves this in [1], where his $L_2(\mu, n)$ stands for what we have called $L^2(X, \mu, \mathcal{H}_n)$.

"In the space $L_2(\mu, n)$, the complex conjugation which maps $\boldsymbol{f}$ to $\overline{\boldsymbol{f}}$ is a canonical antiunitary operator. By the discussion in §2, it follows that every antiunitary operator is the product of a unitary operator and this conjugation. Making use of this fact and following the arguments of Mackey [2] one can prove the following lemma.

Lemma 4.2. Let $\left\{L_2(\mu, n), V_g, P^0(E)\right\}$ be an imprimitivity system for G on X. Let $G = G^+ \cup G^-$be the UA decomposition of G associated with the PUA representation $g \to V_g{}^{\sim}$. Then there exist functions $C(g, x)$ and $D(g, x)$ defined respectively on $G^+ \times X$ and $G^- \times X$ and taking values in the space of unitary operators in $\mathbb{C}^n$ such that

$$\begin{aligned}\left(V_g \boldsymbol{f}\right)(x) &= \left[\frac{d\mu}{d\mu^g}\left(g^{-1}x\right)\right]^{\frac{1}{2}} C\left(g, g^{-1}x\right) \boldsymbol{f}\left(g^{-1}x\right) \quad \text{if} \quad g \in G^+ \\ &= \left[\frac{d\mu}{d\mu^g}\left(g^{-1}x\right)\right]^{\frac{1}{2}} D\left(g, g^{-1}x\right) \overline{\boldsymbol{f}}\left(g^{-1}x\right) \quad \text{if} \quad g \in G^-\end{aligned}$$

where μ^g is the quasi invariant measure defined by the equation $\mu^g(E) = \mu(gE)$, $E \in \mathscr{B}_X$."

The second part of the Imprimitivity theorem is to show that the unitary representation U in any system of imprimitivity based on $X = G/H$ is a representation induced from a representation κ of the subgroup H, that is, the representation U is of the form

$$(U(g)f)(x) = \sqrt{\frac{d\mu(g \cdot x)}{d\mu(x)}}\kappa(h) f(g^{-1} \cdot x).$$

Here $h \in H$ is determined from the relation $g\, p(g^{-1} \cdot x) = p(x)h$, $x \in X$, where $p : G/H \to G$ is a Borel cross-section.

It is impossible to go through all the intricacies of the powerful generalization obtained by KRP of Mackey's imprimitivity theorem in a short article like this one. Therefore, I have decided to conclude by reproducing one of the main theorems of KRP from [1]. I hope that the reader will not have any difficulty with what is being said and in appreciating the depth of what is involved. I am sure this will be motivation enough to read the original work of KRP.

Theorem 1.4 *(Theorem 4.1 [1]) Let G be a locally compact second countable group, $H \subset G$ a closed subgroup and $X = G/H$ the homogeneous space of left cosets. Let $\left\{\mathscr{H}, U_g, P(E)\right\}$ be an imprimitivity system for G on X. Let $G = G^+ \cup G^-$be the UA decomposition of G with respect to PUA representation $g \to U_g, g \in G$. Suppose that G^+acts transitively on X and σ is the multiplier of the representation $g \to U_g$. Let γ be a one one Borel map from X into G^+such that $\pi\gamma$ is the identity map of X onto itself. Then there exists an equivalent imprimitivity system $\left\{L_2(\mu, n), V_g, P^0(E)\right\}$ where*

$$\left(V_g \boldsymbol{f}\right)(x) = \frac{\sigma\left(g, \gamma\left(g^{-1}x\right)\right)}{\sigma\left(\gamma(x), \gamma(x)^{-1} g \gamma\left(g^{-1}x\right)\right)} \left\{\left(\frac{d\mu}{d\mu^g}\right)\left(g^{-1}x\right)\right\}^{\frac{1}{2}} \cdot M_{\gamma(x)^{-1}\gamma(g^{-1}x)} \boldsymbol{f}\left(g^{-1}x\right),$$

μ is a quasi invariant measure, n is a finite or countable cardinal, $h \to M_h$ is a σ- representation of H and P^0 is the canonical projection valued measure on $\mathscr{B}_X$. This imprimitivity system is irreducible if and only if the σ representation $h \to M_h$ of H is irreducible.

In the statement of the theorem reproduced above from [1], (i) P^0 is the canonical projection valued measure on $L^2(\mu, n)$ as described at the bottom of page 313, [1], and (ii) a "σ- representation" is a multiplier representation with multiplier σ, Definition 3.3, [1].

The slight familiarity that I have with the terminology from mathematical physics is mostly from my conversations with KRP. This article, based on one of the papers of KRP that I have always admired, is dedicated to his fond memory.

Competing interests The author has no competing interests to declare.

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