UNITARY INVARIANTS FOR HILBERT MODULES OF FINITE RANK

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ABSTRACT. A refined notion of curvature for a linear system of Hermitian vector spaces, in the sense of Grothendieck, leads to the unitary classification of a large class of analytic Hilbert modules. Specifically, we study Hilbert sub-modules, for which the localizations are of finite (but not constant) dimension, of an analytic function space with a reproducing kernel. The correspondence between analytic Hilbert modules of constant rank and holomorphic Hermitian bundles on domains of $\mathbb{C}^n$ due to Cowen and Douglas, as well as a natural analytic localization technique derived from the Hochschild cohomology of topological algebras play a major role in the proofs. A series of concrete computations, inspired by representation theory of linear groups, illustrate the abstract concepts of the paper.

1. PRELIMINARIES AND MAIN RESULTS

Without aiming at completeness, the rather lengthy introduction below recalls some of the main concepts of Cowen-Douglas theory, a localization technique in topological homology and aspects of complex Hermitian geometry as they interlace in the unitary classification of analytic Hilbert modules. The ideas invoked in the present work have evolved and converged from quite distinct sources for at least half a century.

One of the basic problem in the study of a Hilbert module $\mathcal{H}$ over the ring of polynomials $\mathbb{C}[\underline{z}] := \mathbb{C}[z_1, \ldots, z_m]$ (or equivalently $\mathcal{O}(\mathbb{C}^m)$ module) is to find unitary invariants (cf. [18, 4]) for $\mathcal{H}$. It is not always possible to find invariants that are complete and yet easy to compute. There are very few instances where a set of complete invariants have been identified. Examples are Hilbert modules over continuous functions (spectral theory of normal operator), contractive modules over the disc algebra (model theory for contractive operator) and Hilbert modules in the class $\mathrm{B}_n(\Omega)$ for a bounded domain $\Omega \subseteq \mathbb{C}^m$ (adjoint of multiplication operators on reproducing kernel Hilbert spaces). In this paper, we study Hilbert modules consisting of holomorphic functions on some bounded domain possessing a reproducing kernel. Our methods apply, in particular, to submodules of Hilbert modules in $\mathrm{B}_1(\Omega)$.

1.1. The algebraic and analytic framework.

The class $\mathrm{B}_n(\Omega)$ was introduced in [5, 6] and an alternative approach was outlined in [7]. The definition of this class given below is clearly equivalent to the one in [5, Definition 1.2] and [7, Definition 1.1].

Definition 1.1. A Hilbert module $\mathcal{H}$ over the polynomial ring $\mathbb{C}[\underline{z}]$ is said to be in the class $\mathrm{B}_n(\Omega)$, $n \in \mathbb{N}$, if

(const) $\dim \mathcal{H}/\mathfrak{m}_w\mathcal{H} = n < \infty$ for all $w \in \Omega$;
(span) $\cap_{w\in\Omega}\mathfrak{m}_w\mathcal{H} = 0$,

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where $\mathfrak{m}_w$ denotes the maximal ideal in $\mathbb{C}[\underline{z}]$ at w.

Recall that if $\mathfrak{m}_w\mathcal{H}$ has finite codimension then $\mathfrak{m}_w\mathcal{H}$ is a closed subspace of $\mathcal{H}$. Throughout this paper we call $\dim \mathcal{H}/\mathfrak{m}_w\mathcal{H}$ *the rank* of the analytic module at the point w. For any Hilbert module $\mathcal{H}$ in $\mathrm{B}_n(\Omega)$, the analytic localization $\mathcal{O}\hat{\otimes}_{\mathcal{O}(\mathbb{C}^m)}\mathcal{H}$ is a locally free module when restricted to Ω, see for details [21]. Let us denote in short

$$\hat{\mathcal{H}} := \mathcal{O}\hat{\otimes}_{\mathcal{O}(\mathbb{C}^m)}\mathcal{H}\big|_{\Omega},$$

and let $E_{\mathcal{H}} = \hat{\mathcal{H}}|_{\Omega}$ be the associated holomorphic vector bundle. Fix $w \in \Omega$. The minimal projective resolution of the maximal ideal at the point w is given by the Koszul complex $K_{\bullet}(z - w, \mathcal{H})$, where $K_p(z-w,\mathcal{H}) = \mathcal{H} \otimes \wedge^p(\mathbb{C}^m)$ and the connecting maps $\delta_p(w) : K_p \to K_{p-1}$ are defined, using the standard basis vectors e_i, $1 \le i \le m$ for $\mathbb{C}^m$, by

$$\delta_p(w)(fe_{i_1} \wedge \ldots \wedge e_{i_p}) = \sum_{j=1}^{p}(-1)^{j-1}(z_j - w_j)\cdot fe_{i_1} \wedge \ldots \wedge \hat{e}_{i_j} \wedge \ldots \wedge e_{i_p}.$$

Here, $z_i \cdot f$ is the module multiplication. In particular $\delta_1(w) : \mathcal{H} \oplus \ldots \oplus \mathcal{H} \to \mathcal{H}$ is defined by $(f_1,\ldots,f_m) \mapsto \sum_{j=1}^m (M_j - w_j)f_j$, where M_i is the operator $M_j : f \mapsto z_j \cdot f$, for $1 \le j \le m$ and $f \in \mathcal{H}$. The 0-th homology group of the complex, $H_0(K_{\bullet}(z-w,\mathcal{H}))$ is same as $\mathcal{H}/\mathfrak{m}_w\mathcal{H}$. For $w \in \Omega$, the map $\delta_1(w)$ induces a map localized at w,

$$K_1(z-w,\hat{\mathcal{H}}_w) \xrightarrow{\delta_{1w}(w)} K_0(z-w,\hat{\mathcal{H}}_w).$$

Then $\hat{\mathcal{H}}_w = \operatorname{coker}\ \delta_{1w}(w)$ is a locally free $\mathcal{O}_w$ module and the fiber of the associated holomorphic vector bundle $E_{\mathcal{H}}$ is given by

$$E_{\mathcal{H},w} = \hat{\mathcal{H}}_w \otimes_{\mathcal{O}_w} \mathcal{O}_w/\mathfrak{m}_w\mathcal{O}_w.$$

We identify $E^*_{\mathcal{H},w}$ with $\ker\ \delta_1(w)^*$. Thus $E^*_{\mathcal{H}}$ is a Hermitian holomorphic vector bundle on $\Omega^* := \{\bar{z} : z \in \Omega\}$. Let $D_{\mathbf{M}^*}$ be the commuting m-tuple $({M_1}^*,\ldots,{M_m}^*)$ from $\mathcal{H}$ to $\mathcal{H}\oplus\ldots\oplus\mathcal{H}$. Clearly $\delta_1(w)^* = D_{(\mathbf{M}-w)^*}$ and $\ker\ \delta_1(w)^* = \ker D_{(\mathbf{M}-w)^*} = \cap_{j=1}^m \ker(M_j - w_j)^*$ for $w \in \Omega$.

Let $\mathrm{Gr}(\mathcal{H},n)$ be the rank n Grassmanian on the Hilbert module $\mathcal{H}$. The map $\Gamma : \Omega^* \to \mathrm{Gr}(\mathcal{H},n)$ defined by $\bar{w} \mapsto \ker D_{(\mathbf{M}-w)^*}$ is shown to be holomorphic in [5]. The pull-back of the canonical vector bundle on $\mathrm{Gr}(\mathcal{H},n)$ under Γ is then the holomorphic Hermitian vector bundle $E^*_{\mathcal{H}}$ on the open set Ω^*. One of the main theorems of [5] states that isomorphic Hilbert modules correspond to equivalent vector bundles and vice-versa. Examples of these are Hardy and the Bergman modules over the ball and the poly-disc in $\mathbb{C}^m$.

1.2. Submodules of Hilbert modules possessing a reproducing kernel. Let $\mathcal{H}$ be a Hilbert module in $\mathrm{B}_1(\Omega)$ possessing a non-degenerate reproducing kernel $K(z,w)$, that is, $K(w,w) \neq 0$, $w \in \Omega$. We will often write K_w for the function $K(\cdot,w)$. Then $E^*_{\mathcal{H}} \cong \mathcal{O}_{\Omega^*}$, that is, the associate holomorphic vector bundle is trivial, with K_w as a non-vanishing global section. For modules in $\mathrm{B}_1(\Omega)$, the curvature of the vector bundle $E^*_{\mathcal{H}}$ is a complete invariant. However, in many natural examples of submodules of Hilbert modules from the class $\mathrm{B}_1(\Omega)$, the dimension of the joint kernel does not remain constant. For instance, in the case of $H^2_0(\mathbb{D}^2) := \{f \in H^2(\mathbb{D}^2) : f(0) = 0\}$ (cf. [11]), we have

$$\dim\ker D_{(\mathbf{M}-w)^*} = \dim H^2_0(\mathbb{D}^2) \otimes_{\mathbb{C}[z_1,z_2]} \mathbb{C}_w = \begin{cases} 1 & \text{if } w \neq (0,0) \\ 2 & \text{if } w = (0,0). \end{cases}$$

Here $\mathbb{C}_w$ is the one dimensional module over the polynomial ring $\mathbb{C}[z_1,z_2]$, where the module action is given by the map $(f,\lambda) \mapsto f(w)\lambda$ for $f \in \mathcal{C}_2$ and $\lambda \in \mathbb{C}_w \cong \mathbb{C}$.

In examples like the one given above, the map $\bar{w} \mapsto \ker D_{(\mathbf{M}-w)^*}$ is not holomorphic on all of $\mathbb{D}^2$ but only on $\mathbb{D}^2 \setminus \{(0,0)\}$. However, we recall that the map $w \mapsto \dim(\mathcal{M}/\mathfrak{m}_w\mathcal{M})$ is upper semi-continuous and the jump locus, which is the set $\Omega \setminus \{w : \dim(\mathcal{M}/\mathfrak{m}_w\mathcal{M}) = \text{constant}\}$, is an analytic set. In this paper, we begin a systematic study of a class of submodules of kernel Hilbert modules (over the polynomial ring $\mathbb{C}[\underline{z}]$) in $\mathrm{B}_1(\Omega)$ characterized by requiring that $\dim(\mathcal{M}/\mathfrak{m}_w\mathcal{M})$, $w \in \Omega$, is finite.

Definition 1.2. A Hilbert module $\mathcal{M}$ over the polynomial ring $\mathbb{C}[\underline{z}]$ is said to be in the class $\mathfrak{B}_1(\Omega)$ if

(rk) possess a reproducing kernel K (we don't rule out the possibility: $K(w,w) = 0$ for w in some closed subset X of Ω) and
(fin) The dimension of $\mathcal{M}/\mathfrak{m}_w\mathcal{M}$ is finite for all $w \in \Omega$.

The following Lemma isolates a large class of elements from $\mathfrak{B}_1(\Omega)$ which belong to $\mathrm{B}_1(\Omega_0)$ for some open subset $\Omega_0 \subseteq \Omega$.

Lemma 1.3. *Suppose $\mathcal{M} \in \mathfrak{B}_1(\Omega)$ is the closure of a polynomial ideal $\mathcal{I}$. Then $\mathcal{M}$ is in $\mathrm{B}_1(\Omega)$ if the ideal $\mathcal{I}$ is singly generated while if it is generated by the polynomials $p_1, p_2, \ldots, p_t$, then $\mathcal{M}$ is in $\mathrm{B}_1(\Omega \setminus X)$ for $X = \cap_{i=1}^t \{z : p_i(z) = 0\} \cap \Omega$.*

Proof. The proof is a refinement of the argument given in [14, pp. 285]. Let γ_w be any eigenvector at w for the adjoint of the module multiplication, that is, $M_p^*\gamma_w = \overline{p(w)}\gamma_w$ for $p \in \mathbb{C}[\underline{z}]$.

First, assume that the module $\mathcal{M}$ is generated by the single polynomial, say p. In this case, $K(z,w) = p(z)\chi(z,w)\overline{p(w)}$ for some positive definite kernel χ on all of Ω. Set $K_1(z,w) = p(z)\chi(z,w)$ and note that $K_1(\cdot,w)$ is a non-zero eigenvector at $w \in \Omega$. We have

$$\begin{aligned} \langle pq, \gamma_w\rangle = \langle p, M_q^*\gamma_w\rangle = \langle p, \overline{q(w)}\gamma_w\rangle &= q(w)\langle p, \gamma_w\rangle \\ &= \frac{\langle pq, K(\cdot,w)\rangle\langle p,\gamma_w\rangle}{p(w)} = \langle pq, \overline{\langle p,\gamma_w\rangle}K_1(\cdot,w)\rangle. \end{aligned}$$

Since vectors of the form $\{pq : q \in \mathbb{C}[\underline{z}]\}$ are dense in $\mathcal{M}$, it follows that $\gamma_w = \overline{\langle p,\gamma_w\rangle}K_1(\cdot,w)$ and the proof is complete in this case.

Now, assume that $p_1, \ldots, p_t$ is a set of generators for the ideal $\mathcal{I}$. Then for $w \notin X$, there exist a $k \in \{1, \ldots, t\}$ such that $p_k(w) \neq 0$. We note that for any i, $1 \leq m$,

$$p_k(w)\langle p_i, \gamma_w\rangle = \langle p_i, M_{p_k}^*\gamma_w\rangle = \langle p_ip_k, \gamma_w\rangle = \langle p_k, M_{p_i}^*\gamma_w\rangle = p_i(w)\langle p_k, \gamma_w\rangle.$$

Therefore we have

$$\begin{aligned} \langle\sum_{i=1}^t p_iq_i, \gamma_w\rangle = \sum_{i=1}^t\langle p_iq_i, \gamma_w\rangle = \sum_{i=1}^t\langle p_i, M_{q_i}^*\gamma_w\rangle &= \sum_{i=1}^t q_i(w)\langle p_i, \gamma_w\rangle \\ &= \sum_{i=1}^t\langle p_iq_i, \frac{\overline{\langle p_k,\gamma_w\rangle}K(\cdot,w)}{\overline{p_k(w)}}\rangle. \end{aligned}$$

Let $c(w) = \frac{\langle p_k,\gamma_w\rangle}{p_k(w)}$. Hence

$$\sum_{i=1}^t\langle p_iq_i, \gamma_w\rangle = \langle\sum_{i=1}^t p_iq_i, \overline{c(w)}K(\cdot,w)\rangle.$$

Since vectors of the form $\{\sum_{i=1}^t p_iq_i : q_i \in \mathbb{C}[\underline{z}], 1 \leq i \leq t\}$ are dense in $\mathcal{M}$, it follows that $\gamma_w = \overline{c(w)}K(\cdot,w)$ completing the proof of the second half. ∎

1.3. The sheaf construction. From the work of [5, 6], it is known that invariants for holomorphic Hermitian bundles are not easy to compute. We show how to do this for a large family of examples. It then becomes clear that to find easily computable invariants, we must look elsewhere. Using techniques from commutative algebra and complex analytic geometry, in the framework of Hilbert modules, we have obtained some new invariants.

Let us consider a Hilbert module $\mathcal{M}$ in the class $\mathfrak{B}_1(\Omega)$ which is a submodule of some Hilbert module $\mathcal{H}$ in $\mathrm{B}_1(\Omega)$, possessing a nondegenerate reproducing kernel K. Clearly then we have the following module map

$$\mathcal{O}\hat{\otimes}_{\mathcal{O}(\mathbb{C}^m)}\mathcal{M} \longrightarrow \mathcal{O}\hat{\otimes}_{\mathcal{O}(\mathbb{C}^m)}\mathcal{H} \cong \mathcal{O}_\Omega. \tag{1.1}$$

Let $\mathcal{S}^{\mathcal{M}}$ denotes the range of the composition map in the above equation. Then the stalk of $\mathcal{S}^{\mathcal{M}}$ at $w \in \Omega$ is given by $\{(f_1)_w\mathcal{O}_w + \cdots + (f_n)_w\mathcal{O}_w : f_1, \ldots, f_n \in \mathcal{M}\}$

Motivated by the above construction and the analogy with the correspondence of a vector bundle with a locally free sheaf [38], we construct a sheaf $\mathcal{S}^{\mathcal{M}}$ for the Hilbert module $\mathcal{M}$ over the polynomial ring $\mathbb{C}[\underline{z}]$, in the class $\mathfrak{B}_1(\Omega)$. The sheaf $\mathcal{S}^{\mathcal{M}}$ is the subsheaf of the sheaf of holomorphic functions $\mathcal{O}_\Omega$ whose stalk $\mathcal{S}^{\mathcal{M}}_w$ at $w \in \Omega$ is

$$\big\{(f_1)_w\mathcal{O}_w + \cdots + (f_n)_w\mathcal{O}_w : f_1, \ldots, f_n \in \mathcal{M}\big\},$$

or equivalently,

$$\mathcal{S}^{\mathcal{M}}(U) = \Big\{\sum_{i=1}^n \big(f_{i|U}\big)g_i : f_i \in \mathcal{M}, g_i \in \mathcal{O}(U)\Big\}$$

for U open in Ω.

Following the proof of [4, Theorem 2.3.3], which is a consequence of the well known Cartan Theorems A and B, it is not hard to see that if $\mathcal{M}$ is any module in $\mathfrak{B}_1(\Omega)$ with a finite set of generators $\{f_1, \ldots, f_t\}$, then for any $f \in \mathcal{S}^{\mathcal{M}}$ we have

$$f = f_1g_1 + \cdots + f_tg_t \tag{1.2}$$

for some $g_1, \ldots, g_t \in \mathcal{O}(\Omega)$. Consequently, if $\mathcal{M}_1$ and $\mathcal{M}_2$ are isomorphic modules in $\mathfrak{B}_1(\Omega)$ which are finitely generated, then $\mathcal{S}^{\mathcal{M}_1}$ and $\mathcal{S}^{\mathcal{M}_2}$ are isomorphic as modules. This isomorphism is implemented by extending the map given on the generators of $\mathcal{M}_1$ to the module $\mathcal{S}^{\mathcal{M}_1}$. It is easily seen to be well-defined using (1.2).

It is clear that if the Hilbert module $\mathcal{M}$ is in the class $\mathrm{B}_1(\Omega)$, then the sheaf $\mathcal{S}^{\mathcal{M}}$ is locally free. Also, if the Hilbert module is taken to be the maximal set of functions vanishing on an analytic hyper-surface $\mathcal{Z}$, then the sheaf $S^{\mathcal{M}}$ coincides with the ideal sheaf $\mathcal{I}_{\mathcal{Z}}(\Omega)$ and therefore it is coherent (cf.[25]). However, much more is true

Proposition 1.4. *For any Hilbert module $\mathcal{M}$ in $\mathfrak{B}_1(\Omega)$, the sheaf $\mathcal{S}^{\mathcal{M}}$ is coherent.*

Proof. The sheaf $S^{\mathcal{M}}$ is generated by the family $\{f : f \in \mathcal{M}\}$ of global sections of the sheaf $\mathcal{O}(\Omega)$. Let J be a finite subset of $\mathcal{M}$ and $S^{\mathcal{M}}_J \subseteq \mathcal{O}(\Omega)$ be the subsheaf generated by the sections $f,\ f \in J$. It follows (see [26, Corollary 9, page. 130]) that $S^{\mathcal{M}}_J$ is coherent. The family $\{S^{\mathcal{M}}_J : J \text{ is a finite subset of } \mathcal{M}\}$ is increasingly filtered, that is, for any two finite subset I and J of $\mathcal{M}$, the union $I \cup J$ is again a finite subset of $\mathcal{M}$ and $S^{\mathcal{M}}_I \cup S^{\mathcal{M}}_J \subset S^{\mathcal{M}}_{I\cup J}$. Also, clearly $S^{\mathcal{M}} = \bigcup_J S^{\mathcal{M}}_J$. Using Noether's lemma [25, page. 111] which says that every increasingly filtered family must be stationary, we conclude that the sheaf $S^{\mathcal{M}}$ is coherent. □

For $w \in \Omega$, the coherence of $\mathcal{S}^{\mathcal{M}}$ ensures the existence of $m, n \in \mathbb{N}$ and an open neighborhood U of w such that

$$(\mathcal{O}^m)_{|U} \to (\mathcal{O}^n)_{|U} \to (\mathcal{S}^{\mathcal{M}})_{|U} \to 0$$

is an exact sequence. Thus

$$\left\{ \left(\mathcal{S}_w^{\mathcal{M}} / \mathfrak{m}_w \mathcal{S}_w^{\mathcal{M}} \right)^* : w \in \Omega \right\}$$

defines a holomorphic linear space on Ω (cf. [22, 1.8 (p. 54)]). Although, we have not used this correspondence in any essential manner, we expect it to be a useful tool in the investigation of some of the questions we raise here.

The coherence of the sheaf $\mathcal{S}^{\mathcal{M}}$ implies, in particular, that the stalk $(\mathcal{S}^{\mathcal{M}})_w$ at $w \in \Omega$ is generated by a finite number of elements $g_1, \ldots, g_d$ from $\mathcal{O}(\Omega)$. If K is the reproducing kernel for $\mathcal{M}$ and $w_0 \in \Omega$ is a fixed but arbitrary point, then for w in a small neighborhood Ω_0 of w_0, we obtain the following decomposition theorem.

Theorem 1.5. *Suppose g_i^0, $1 \le i \le d$, be a minimal set of generators for the stalk $\mathcal{S}_{w_0}^{\mathcal{M}}$. Then*

(i) *there exists a open neighborhood Ω_0 of w_0 such that*

$$K(\cdot, w) := K_w = g_1^0(w) K_w^{(1)} + \cdots + g_n^0(w) K_w^{(d)},\ w \in \Omega_0$$

for some choice of anti-holomorphic functions $K^{(1)}, \ldots, K^{(d)} : \Omega_0 \to \mathcal{M}$,

(ii) *the vectors $K_w^{(i)}$, $1 \le i \le d$, are linearly independent in $\mathcal{M}$ for w in some small neighborhood of w_0,*

(iii) *the vectors $\{K_{w_0}^{(i)} \mid 1 \le i \le d\}$ are uniquely determined by these generators $g_1^0, \ldots, g_d^0$,*

(iv) *the linear span of the set of vectors $\{K_{w_0}^{(i)} \mid 1 \le i \le d\}$ in $\mathcal{M}$ is independent of the generators $g_1^0, \ldots, g_d^0$, and*

(v) *$M_p^* K_{w_0}^{(i)} = \overline{p(w_0)} K_{w_0}^{(i)}$ for all i, $1 \le i \le d$, where M_p denotes the module multiplication by the polynomial p.*

For simplicity, we have stated the decomposition theorem for Hilbert modules consisting of holomorphic functions taking values in $\mathbb{C}$. However, all the tools that we use for the proof work equally well in the case of vector valued holomorphic functions. Consequently, it is not hard to see that the theorem remains valid in this more general set-up.

1.4. Gleason's property and privilege. It is evident from the above theorem that the dimension of the joint kernel of the adjoint of the multiplication operator $D_{\mathbf{M}^*}$ at a point w_0 is greater or equal to the number of minimal generators of the stalk $\mathcal{S}_{w_0}^{\mathcal{M}}$ at $w_0 \in \Omega$, that is,

$$\dim \mathcal{M}/(\mathfrak{m}_{w_0}\mathcal{M}) \ge \dim \mathcal{S}_{w_0}^{\mathcal{M}} / \mathfrak{m}_{w_0} \mathcal{S}_{w_0}^{\mathcal{M}}. \tag{1.3}$$

It would be interesting to produce a Hilbert module $\mathcal{M}$ for which the inequality of (1.3) is strict. Leaving aside this question, for the moment, we go on to identify several classes of Hilbert modules for which we have equality in (1.3).

A Hilbert module $\mathcal{M}$ over the polynomial ring $\mathbb{C}[\underline{z}]$ is said to be an *analytic Hilbert module* (cf. [4]) if we assume that

(rk) it consists of holomorphic functions on a bounded domain $\Omega \subseteq \mathbb{C}^m$ and possesses a reproducing kernel K,

(dense) the polynomial ring $\mathbb{C}[\underline{z}]$ is dense in it,

(vp) the set of virtual points $\{w \in \mathbb{C}^m : p \mapsto p(w),\ p \in \mathbb{C}[\underline{z}],\ \text{extends continuously to } \mathcal{M}\}$, is Ω.

We apply Lemma 1.3 to analytic Hilbert modules, which are singly generated by the constant function 1, to conclude that they must be in $\mathrm{B}_1(\Omega)$. Evidently, in this case, we have equality in (1.3). However, we have equality in many more cases. For example, suppose $\mathcal{I}$ is a polynomial ideal and $[\mathcal{I}]$ is the closure of $\mathcal{I}$ in some analytic Hilbert module $\mathcal{M}$. Then for $[\mathcal{I}]$, we have equality in (1.3) as well.

Let us again consider a Hilbert module $\mathcal{M}$ in the class $\mathfrak{B}_1(\Omega)$ which is a submodule of some Hilbert module $\mathcal{H}$ in $\mathrm{B}_1(\Omega)$, possessing a nondegenerate reproducing kernel K. We note that the module map

$$\mathcal{O}\hat{\otimes}_{\mathcal{O}(\mathbb{C}^m)}\mathcal{M} \longrightarrow \mathcal{S}^{\mathcal{M}}$$

induced from (1.1) is surjective. This naturally defines a map

$$\mathcal{M}/\mathfrak{m}_{w_0}\mathcal{M} \cong \mathcal{O}_{w_0}/\mathfrak{m}_{w_0}\mathcal{O}_{w_0} \otimes \mathcal{M} \longrightarrow \mathcal{S}^{\mathcal{M}}_{w_0}/\mathfrak{m}_{w_0}\mathcal{S}^{\mathcal{M}}_{w_0}$$

for $w \in \Omega$. The map given above can be constructed similarly for any Hilbert module $\mathcal{M} \in \mathfrak{B}_1(\Omega)$. The question of equality in (1.3) is same as the question of whether this map is an isomorphism and can be interpreted as a global factorization problem. To be more specific, we say that the module $\mathcal{M} \in \mathfrak{B}_1(\Omega)$ possesses *Gleason's property at a point* $w_0 \in \Omega$ if for every element $f \in \mathcal{M}$ vanishing at w_0 there are $f_1, ..., f_m \in \mathcal{M}$ such that $f = \sum_{i=1}^m (z_i - w_{0i}) f_i$.

Proposition 1.6. *The Hilbert module* $\mathcal{M}$ *has Gleason's property at* w_0 *if and only if*

$$\dim \mathcal{M}/\mathfrak{m}_{w_0}\mathcal{M} = \dim \mathcal{S}^{\mathcal{M}}_{w_0}/\mathfrak{m}_{w_0}\mathcal{S}^{\mathcal{M}}_{w_0}.$$

We note the following corollary.

Corollary 1.7. *For an analytic Hilbert module and its submodules which arises as closure of an ideal in the polynomial ring* $\mathbb{C}[\underline{z}]$*, Gleason's problem is solvable.*

It is well known that Gleason's probelm is solvable in the space of all analytic functions, that is, assuming that the domain Ω is pseudo-convex, it follows that for any $f \in \mathcal{M}$ with $f(w_0) = 0$, we have

$$f = \sum_{i=1}^m (z_i - w_{0i}) f_i,\ f_i \in \mathcal{O}(\Omega).$$

see for instance [28, Theorem 7.2.9] or [21]. Thus Gleason's problem asks that the functions f_i can be chosen from the Hilbert module $\mathcal{M}$.

This is a special case of a more general division problem for Hilbert modules. To fix ideas, we consider the following setting: let $\mathcal{M}$ be an analytic Hilbert module with the domain Ω disjoint of its essential spectrum, let $A \in M_{p,q}(\mathcal{O}(\overline{\Omega}))$ be a matrix of analytic functions defined in a neighborhood of $\overline{\Omega}$, where p, q are positive integers, and let $f \in \mathcal{M}^p$. GIven a solution $u \in \mathcal{O}(\Omega)^q$ to the linear equation $Au = f$, is it true that $u \in \mathcal{M}^q$? Numerous "hard analysis" questions, such as problems of moduli, or Corona Problem, can be put into this framework.

We study below this very division problem in conjunction with an earlier work of the third author [33] dealing with the "disc" algebra $\mathcal{A}(\Omega)$ instead of Hilbert modules, and within the general concept of "privilege" introduced by Douady more than forty years ago [9, 10].

Below we only focus on the case of Bergman space. Specifically, the $\mathcal{A}(\Omega)$-module $\mathcal{N} = \mathrm{coker}(A : \mathcal{M} \otimes_{\mathbb{C}} \mathbb{C}^p \longrightarrow \mathcal{M} \otimes_{\mathbb{C}} \mathbb{C}^q)$ is called *privileged with respect to the module* $\mathcal{M}$ if it is a Hilbert module in the quotient metric and there exists a resolution

$$0 \to \mathcal{M} \otimes_{\mathbb{C}} \mathbb{C}^{n_p} \xrightarrow{d_p} \cdots \to \mathcal{M} \otimes_{\mathbb{C}} \mathbb{C}^{n_1} \xrightarrow{d_1} \mathcal{M} \otimes_{\mathbb{C}} \mathbb{C}^{n_0} \to \mathcal{N} \to 0, \tag{1.4}$$

where $d_q \in M_{n_{q+1},n_q}(\mathcal{A}(\Omega))$. Note that implicitly in the statement is assumed that the range of the operator A is closed at the level of the Hilbert module $\mathcal{M}$.

An affirmative answer to the division problem is equivalent to the question of "privilege" in case of the Bergman module on a strictly convex bounded domain Ω with smooth boundary.

Theorem 1.8. *Let* $\Omega \subset \mathbb{C}^m$ *be a strictly convex domain with smooth boundary, let* p, q *be positive integers and let* $A \in M_{p,q}(\mathcal{A}(\Omega))$ *be a matrix of analytic functions belonging to the disk algebra of* Ω*. The following assertions are equivalent:*

(a) *The analytic module* $\operatorname{coker}(A : L^2_a(\Omega)^p \longrightarrow L^2_a(\Omega)^q)$ *is privileged with respect to the Bergman space;*
(b) *The function* $\zeta \mapsto \operatorname{rank} A(\zeta), \quad \zeta \in \partial\Omega$, *is constant;*
(c) *Let* $f \in L^2_a(\Omega)^q$. *The equation* $Au = f$ *has a solution* $u \in L^2_a(\Omega)^p$ *if and only if it has a solution* $u \in \mathcal{O}(\Omega)^p$.

While we have stated our results for the Bergman module, they remain true for the Hardy space $H^2(\partial\Omega)$, that is, the closure of entire functions in the L^2-space with respect to the surface area measure supported on $\partial\Omega$. Also, the results remain true for the Bergman or Hardy spaces of a poly-domain $\Omega = \Omega_1 \times \cdots \times \Omega_d$, where $\Omega_j \subset \mathbb{C},\ 1 \le j \le d$, are convex bounded domains with smooth boundary in $\mathbb{C}$. For these Hilbert modules, the notion of the sheaf model from the earlier work of [29, 30] coincides with the sheaf model described here.

1.5. The Hermitian structure. It follows, from the Lemma 1.3 that $H^2_0(\mathbb{D}^2)$ is in $\mathrm{B}_1(\mathbb{D}^2 \setminus \{(0,0)\})$. Thus the machinery of [5, 7] applies here. But explicit calculation of unitary invariants are somewhat difficult. As was pointed out in [18], the dimension of the localization $H^2_0(\mathbb{D}^2) \otimes_{\mathbb{C}[z_1,z_2]} \mathbb{C}_w$, $w \in \mathbb{D}^2$ is an invariant of the module $H^2_0(\mathbb{D}^2)$. Therefore, it may not be desirable to exclude the point $(0,0)$ altogether in any attempt to study the module $H^2_0(\mathbb{D}^2)$. Fortunately, implicit in the proof of Theorem 2.2 in [7], there is a construction which makes it possible to write down invariants on all of $\mathbb{D}^2$. This theorem assumes only that the module multiplication has closed range as in Definition 1.1. Therefore, it plays a significant role in the study of the class of Hilbert modules $\mathfrak{B}_1(\Omega)$.

We also note, from Theorem 1.5, that the map $\Gamma_K : \Omega_0^* \to \mathrm{Gr}(\mathcal{M}, d)$ defined by $\Gamma_K(\bar{w}) = (K_w^{(1)}, \ldots, K_w^{(d)})$ is holomorphic. The pull-back of the canonical bundle on $Gr(\mathcal{M}, d)$ under Γ_K then defines a holomorphic Hermitian vector bundle on the open set Ω_0^*. Unfortunately, the decomposition of the reproducing kernel given in Theorem 1.5 is not canonical except when the stalk is singly generated. In this special case, the holomorphic Hermitian bundle obtained in this manner is indeed canonical. However, in general, it is not clear if this vector bundle contains any useful information. Suppose we have equality in (1.3) for a Hilbert module $\mathcal{M}$. Then it is possible to obtain a canonical decomposition following [7], which leads in the same manner as above, to the construction of a Hermitian holomorphic vector bundle in a neighborhood of each point $w \in \Omega$.

For any fixed but arbitrary $w_0 \in \Omega$ and a small enough neighborhood Ω_0 of w_0, the proof of Theorem 2.2 from [7] shows the existence of a holomorphic function $P_{w_0} : \Omega_0 \to \mathcal{L}(\mathcal{M})$ with the property that the operator P_{w_0} restricted to the subspace $\ker D_{(\mathbf{M}-w_0)^*}$ is invertible. The range of P_{w_0} can then be seen to be equal to the kernel of the operator $\mathbb{P}_0 D_{(\mathbf{M}-w)^*}$, where $\mathbb{P}_0$ is the orthogonal projection onto $ranD_{(\mathbf{M}-w_0)^*}$.

Lemma 1.9. *The dimension of* $\ker \mathbb{P}_0 D_{(\mathbf{M}-w)^*}$ *is constant in a suitably small neighborhood of* $w_0 \in \Omega$, *say* Ω_0.

Let $\{e_0, \ldots, e_k\}$ be a basis for $\ker(\mathbf{M} - w_0)^*$. Since P_{w_0} is holomorphic on Ω_0, it follows that $\gamma_1(w) := P_{w_0}(w)e_1, \ldots, \gamma_k(w) := P_{w_0}(w)e_k$ are holomorphic on Ω_0. Thus $\Gamma : \Omega_0 \to Gr(\mathcal{M}, k)$, given by $\Gamma(w) = \ker \mathbb{P}_0 D_{(\mathbf{M}-w)^*}$, defines a holomorphic Hermitian vector bundle $\mathcal{P}_0$ on Ω_0 of rank k corresponding to the Hilbert module $\mathcal{M}$.

Theorem 1.10. *If any two Hilbert modules* $\mathcal{M}$ *and* $\tilde{\mathcal{M}}$ *from* $\mathfrak{B}_1(\Omega)$ *are isomorphic via an unitary module map, then the corresponding holomorphic Hermitian vector bundles* $\mathcal{P}_0$ *and* $\tilde{\mathcal{P}}_0$ *on* Ω_0^* *are equivalent.*

1.6. Organization. We now describe the organization of the paper. In Section 2, we prove the decomposition theorem 1.5. The equivalence of the Gleason property with the equality in (1.3) is shown in Section 3. At the end of this Section, we give a simple proof of a conjecture from [17] for smooth points. This was first proved in [19]. In Section 4, we prove that the Bergman module is privileged. In Section 5, we show that the sheaf model of [29, 30] coincides with the one proposed here if the Hilbert modules are assumed to be privileged. Finally in Section 6, we construct a Hermitian holomorphic vector bundle following [7]. We show how to extract invariants for the Hilbert module from this vector bundle. These invariants are not easy to compute in general, but we provide explicit computations for a class of examples in Section 7.1 and in Section 7.2, we give detailed calculations of an invariant which is somewhat easier to compute.

1.7. Index of notations

$\mathbb{C}[\underline{z}]$	the polynomial ring $\mathbb{C}[z_1,\ldots,z_m]$ of m- complex variables
$\mathfrak{m}_w$	maximal ideal of $\mathbb{C}[\underline{z}]$ at the point $w\in\mathbb{C}^m$
Ω	a bounded domain in $\mathbb{C}^m$
Ω^*	$\{\bar z : z\in\Omega\}$
$\mathbb{D}^m$	the unit polydisc in $\mathbb{C}^m$
M_i	module multiplication by the co-ordinate function z_i, $1\le i\le m$
M_i^*	adjoint of M_i
$D_{(\mathbf{M}-w)^*}$	the operator $\mathcal{M}\to\mathcal{M}\oplus\ldots\oplus\mathcal{M}$ defined by $f\mapsto((M_j-w_j)^*f)_{j=1}^m$
$\hat{\mathcal{H}}$	the analytic localization $\mathcal{O}\hat\otimes_{\mathcal{O}(\mathbb{C}^m)}\mathcal{H}$ of the Hilbert module $\mathcal{H}$
$B_n(\Omega)$	Cowen-Douglas class of operators, $n\ge 1$
$\partial^\alpha,\bar\partial^\alpha$	$\partial^\alpha=\frac{\partial^{\vert\alpha\vert}}{\partial z_1^{\alpha_1}\cdots z_m^{\alpha_m}},\bar\partial^\alpha=\frac{\partial^{\vert\alpha\vert}}{\partial\bar z_1^{\alpha_1}\cdots\bar z_m^{\alpha_m}}$ for $\alpha=(\alpha_1,\ldots,\alpha_m)\in\mathbb{Z}^+\times\cdots\times\mathbb{Z}^+$ and $\vert\alpha\vert=\sum_{i=1}^m\alpha_i$
$q(D)$	the differential operator $\sum_\alpha a_\alpha\partial^\alpha$ corresponding to the polynomial $q=\sum_\alpha a_\alpha z^\alpha$
$\mathcal{S}^{\mathcal{M}}$	the analytic submodule of $\mathcal{O}_\Omega$, corresponding to the Hilbert module $\mathcal{M}\in\mathfrak{B}(\Omega)$
$K(z,w)$	a reproducing kernel
$E(w)$	the evaluation functional (the linear functional induced by $K(\cdot,w)$)
$\Vert\cdot\Vert_{\bar\Delta(0;r)}$	supremum norm
$\Vert\cdot\Vert_2$	L^2 norm with respect to the volume measure
$\mathbb{V}_w(\mathcal{F})$	the characteristic space at w, which is $\{q\in\mathbb{C}[\underline{z}]:q(D)f\big\vert_w=0 \text{ for all } f\in\mathcal{F}\}$ for some set $\mathcal{F}$ of holomorphic functions
$[\mathcal{I}]$	the closure of the polynomial ideal $\mathcal{I}\subseteq\mathcal{M}$ in some Hilbert module $\mathcal{M}$
$\mathcal{A}(\Omega)$	the "disk" algebra $\mathcal{O}(\Omega)\cap C(\overline{\Omega})$ over Ω
$\mathcal{O}(\overline{\Omega})$	the space of germs of analytic functions defined in a neighborhood of $\overline{\Omega}$
$\mathbb{P}_0$	orthogonal projection onto ran $D_{(\mathbf{M}-w_0)^*}$
$\mathcal{P}_w$	$\ker\mathbb{P}_0D_{(\mathbf{M}-w)^*}$ for $w\in\Omega$

2. The proof of the decomposition theorem

Proof of Theorem 1.5. For simplicity of notation, we assume, without loss of generality, that $0=w_0\in\Omega$. Let $\{e_n\}_{n=0}^\infty$ be a orthonormal basis for $\mathcal{M}$. Let $\{e_n^*\}_{n=0}^\infty$ be the corresponding dual basis of $\mathcal{M}^*$, that is, $e_n^*(e_j)=\delta_{nj}$, Kronecker delta, $n,j\in\mathbb{N}\cup\{0\}$. Let $E(z)$ be the evaluation functional at the point $z\in\Omega$. Clearly $E(z)\in\mathcal{M}^*$, as $\mathcal{M}$ posses a reproducing kernel K. So $E(z)=\sum_{n=0}^\infty a_n(z)e_n^*$. Now,

$$e_n(z)=E(z)e_n=\{\sum_{k=0}^\infty a_k(z)e_k^*\}e_n=\sum_{k=0}^\infty a_k(z)e_k^*(e_n)=\sum_{k=0}^\infty a_k(z)\delta_{kn}=a_n(z).$$

It follows from [26, Theorem 2, page. 82] that for every element f in $\mathcal{S}_0^{\mathcal{M}}$, and therefore in particular for every e_n, we have

$$e_n(z) = \sum_{i=1}^{d} g_i^0(z) h_i^{(n)}(z),\ z \in \Delta(0;r)$$

for some holomorphic functions $h_i^{(n)}$ defined on the closed polydisk $\bar{\Delta}(0;r) \subseteq \Omega$. Furthermore, they can be chosen with the bound $\| h_i^{(n)} \|_{\infty,\bar{\Delta}(0;r)} \leq C \| e_n \|_{\infty,\bar{\Delta}(0;r)}$ for some positive constant C independent of n. Although, the decomposition is not necessarily with respect to the standard coordinate system at 0, we will be using only a point wise estimate. Consequently, in the equation given above, we have chosen not to emphasize the change of variable involved and we have,

$$E(z) = \sum_{n=0}^{\infty}\{\sum_{i=1}^{d} g_i^0(z) h_i^{(n)}(z)\} e_n^* = \sum_{i=1}^{d} g_i^0(z)\{\sum_{n=0}^{\infty} h_i^{(n)}(z) e_n^*\}.$$

Setting $H_i(z)$ to be the sum $\sum_{n=0}^{\infty} h_i^{(n)}(z) e_n^*$, we can write $E(z) = \sum_{i=1}^{d} H_i(z) g_i^0(z)$ on $\Delta(0;r)$. For the proof of part (i), we need to show that $H_i(z) \in \mathcal{M}^*$ where $z \in \Delta(0;r)$. Or, equivalently, we have to show that $\sum_{n=0}^{\infty} |h_i^{(n)}(z)|^2 < \infty$ for each $z \in \Delta(0;r)$. First, using the estimate on $h_i^{(n)}$, we have

$$|h_i^{(n)}(z)| \leq \| h_i^{(n)} \|_{\bar{\Delta}(0;r)} \leq C \| e_n \|_{\bar{\Delta}(0;r)}.$$

We prove below the inequality $\sum_{n=0}^{\infty} \| e_n \|^2_{\infty\bar{\Delta}(0;r)} < \infty$, completing the proof of part (i). We prove, more generally, that for $f \in \mathcal{M}$,

$$\| f \|_{\bar{\Delta}(0;r)} \leq C' \| f \|_{2,\bar{\Delta}(0;r)}, \tag{2.1}$$

where $\| \,.\, \|_2$ denotes the L^2 norm with respect to the volume measure on $\bar{\Delta}(0;r)$ It is evident from the proof that the constant C' may be chosen to be independent of the functions f.

Any function f holomorphic on Ω belongs to the Bergman space $L_a^2(\Delta(0;r+\varepsilon))$ as long as $\Delta(0;r+\varepsilon) \subseteq \Omega$. We can surely pick $\varepsilon > 0$ small enough to ensure $\Delta(0;r+\varepsilon) \subseteq \Omega$. Let B be the Bergman kernel of the Bergman space $L_a^2(\Delta(0;r+\varepsilon))$. Thus we have

$$|\ f(w)\ | = |\ \langle f, B(\cdot,w)\rangle\ | \leq\ \| f \|_{2,\Delta(0;r+\varepsilon)} B(w,w)^{\frac{1}{2}},\ w \in \Delta(0;r+\varepsilon).$$

Since the function $B(w,w)$ is bounded on compact subsets of $\Delta(0;r+\varepsilon)$, it follows that $C'^2 := \sup\{B(w,w) : w \in \bar{\Delta}(0;r)\}$ is finite. We therefore see that

$$\| f \|_{\bar{\Delta}(0;r)} = \sup\{|\ f(w)\ | : w \in \bar{\Delta}(0;r)\} \leq C' \| f \|_{2,\Delta(0;r+\varepsilon)}.$$

Since $\varepsilon > 0$ can be chosen arbitrarily close to 0, we infer the inequality (2.1).

The inequality (2.1) implies, in particular, that

$$\sum_{n=0}^{\infty} \| e_n \|^2_{\bar{\Delta}(0;r)} \leq C' \sum_{n=0}^{\infty} \int_{\bar{\Delta}(0;r)} |\ e_n(z)\ |^2 dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_m \wedge d\bar{z}_m.$$

We have shown that the evaluation functional $E(z) \in \mathcal{M}^*$ is of the form $\sum_{n=0}^{\infty} e_n(z) e_n^*$ and hence the function $G(z) := \sum_{n=0}^{\infty} |e_n(z)|^2$ is finite for each $z \in \Omega$. The sequence of positive continuous functions $G_k(z) := \sum_{n=0}^k |e_n(z)|^2$ converges uniformly to G on $\bar\Delta(0;r)$. To see this, we note that

$$\begin{aligned}\| G_k - G \|^2{}_{\bar\Delta(0;r)} &\leq C'^2 \int_{\bar\Delta(0;r)} |G_k(z) - G(z)|^2 dz_1 \wedge d\bar z_1 \wedge \cdots \wedge dz_m \wedge d\bar z_m \\ &\leq C'^2 \int_{\bar\Delta(0;r)} \{ \sum_{n=k+1}^{\infty} \mid e_n(z) \mid^2 \}^2 dz_1 \wedge d\bar z_1 \wedge \cdots \wedge dz_m \wedge d\bar z_m,\end{aligned}$$

which tends to 0 as $k \to \infty$. So, by monotone convergence theorem, we can interchange the integral and the infinite sum to conclude

$$\sum_{n=0}^{\infty} \| e_n \|^2_{\bar\Delta(0;r)} \leq C \int_{\bar\Delta(0;r)} \sum_{n=0}^{\infty} \mid e_n(z) \mid^2 dz_1 \wedge d\bar z_1 \wedge \cdots \wedge dz_m \wedge d\bar z_m < \infty$$

as G is a continuous function on $\bar\Delta(0;r)$. This shows that

$$\sum_{n=0}^{\infty} \Big| \, h_i^{(n)}(z) \, \Big|^2 \leq K \sum_{n=0}^{\infty} \| e_n \|^2_{\bar\Delta(0;r)} < \infty.$$

Hence $H_i(z) \in \mathcal{M}^*$.

Now, for each i, $1 \leq i \leq d$, there exist $K_w^{(i)} \in \mathcal{M}$ such that $H_i(w)f = \langle f, K_w^{(i)} \rangle$. Let us set $K_i(z,w) := K_w^{(i)}(z)$. The function K_i is holomorphic in the first variable and anti-holomorphic in the second by definition. For $w \in \Delta(0;r)$, we have $\langle f, K(\cdot,w) \rangle = E(w)f = \sum_{i=1}^d g_i^0(w) H_i(w) f = \sum_{i=1}^d \langle f, \bar g_i^0(w) K_i(\cdot,w) \rangle = \langle f, \sum_{i=1}^d \bar g_i^0(w) K_i(\cdot,w) \rangle$ for all $f \in \mathcal{M}$. Hence $K(z,w) = \sum_{i=1}^d \bar g_i^0(w) K_i(z,w)$ and (i) is proved.

To prove the statement in (ii), at 0, we have to show that whenever there exist complex numbers $\alpha_1, \ldots, \alpha_d$ such that $\sum_{i=1}^d \alpha_i K_i(z,0) = 0$, then $\alpha_i = 0$ for all i. We assume, on the contrary, that there exists some $i \in 1, \ldots, d$ such that $\alpha_i \neq 0$. Without loss of generality, we assume $\alpha_1 \neq 0$, then $K_1(z,0) = \sum_{i=2}^d \beta_i K_i(z,0)$ where $\beta_i = \frac{\alpha_i}{\alpha_1}, 2 \leq i \leq d$. This shows that $K_1(z,w) - \sum_{i=2}^d \beta_i K_i(z,w)$ has a zero at $w = 0$. From [28, Theorem 7.2.9], it follows that

$$K_1(z,w) - \sum_{i=2}^d \beta_i K_i(z,w) = \sum_{j=1}^m \bar w_j G_j(z,w)$$

for some function $G_j : \Omega \times \Delta(0;r) \to \mathbb{C}, 1 \leq j \leq m$, which is holomorphic in the first and antiholomorphic in the second variable. So, we can write

$$\begin{aligned} K(z,w) &= \sum_{i=1}^d \bar g_i^0(w) K_i(z,w) = \bar g_1^0(w) K_1(z,w) + \sum_{i=2}^d \bar g_i^0(w) K_i(z,w) \\ &= \bar g_1^0(w) \{ \sum_{i=2}^d \beta_i K_i(z,w) + \sum_{j=1}^m \bar w_j G_j(z,w) \} + \sum_{i=2}^d \bar g_i^0(w) K_i(z,w) \\ &= \sum_{i=2}^d (\bar g_i^0(w) + \beta_i \bar g_1^0(w)) K_i(z,w) + \sum_{j=1}^m \bar w_j \bar g_1^0(w) G_j(z,w). \end{aligned}$$

For $f \in \mathcal{M}$ and $w \in \Delta(0;r)$, we have

$$f(w) = \langle f, K(\cdot,w) \rangle = \sum_{i=2}^d (g_i^0(w) + \bar\beta_i g_1^0(w)) \langle f, K_i(z,w) \rangle + g_1^0(w) \langle f, \sum_{j=1}^m \bar w_j G_j(z,w) \rangle.$$

We note that $\langle f, \sum_{j=1}^m \bar{w}_j G_j(z,w)\rangle$ is a holomorphic function in w which vanishes at $w=0$ It then follows that $\langle f, \sum_{j=1}^m \bar{w}_j G_j(z,w) = \sum_{j=1}^m w_j \tilde{G}_j(w)$ for some holomorphic functions $\tilde{G}_j$, $1 \leq j \leq m$ on $\Delta(0;r)$. Therefore, we have

$$f(w) = \sum_{i=2}^d (g_i^0(w) + \bar{\beta}_i g_1^0(w))\langle f, K_i(z,w)\rangle + g_1^0(w)\sum_{j=1}^m w_j g_1^0(w)\tilde{G}_j(w).$$

Since the sheaf $\mathcal{S}^{\mathcal{M}}|_{\Delta(0;r)}$ is generated by the Hilbert module $\mathcal{M}$, it follows that the set $\{g_2^0 + \bar{\beta}_2 g_1^0, \ldots, g_d^0 + \bar{\beta}_d g_1^0, z_1 g_1^0, \ldots, z_m g_1^0\}$ also generates $\mathcal{S}^{\mathcal{M}}|_{\Delta(0;r)}$. In particular, they generate the stalk at 0. This, we claim, is a contradiction. Suppose $A \subset \mathcal{S}_0^{\mathcal{M}}$ is generated by germs of the functions $g_2^0 + \bar{\beta}_2 g_1^0, \ldots, g_d^0 + \bar{\beta}_d g_1^0$. Let $\mathfrak{m}(\mathcal{O}_0)$ denotes the the only maximal ideal of the local ring $\mathcal{O}_0$, consisting of the germs of functions vanishing at 0. Then it follows that

$$\mathfrak{m}(\mathcal{O}_0)\{\mathcal{S}_0^{\mathcal{M}}/A\} = \mathcal{S}_0^{\mathcal{M}}/A.$$

Using Nakayama's lemma (cf. [36, p.57]), we see that $\mathcal{S}_0^{\mathcal{M}}/A = 0$, that is, $\mathcal{S}_0^{\mathcal{M}} = A$. This contradicts the minimality of the generators of the stalk at 0 completing the proof of first half of (ii).

To prove the slightly stronger statement, namely, the independence of the vectors $K_{w_0}^{(i)}$, $1 \leq i \leq d$ in a small neighborhood of 0, consider the Grammian $\big(\!\big(\langle K_w^{(i)}, K_w^{(j)}\rangle\big)\!\big)_{i,j=1}^d$. The determinant of this Grammian is nonzero at 0. Therefore it remains non-zero in a suitably small neighborhood of 0 since it is a real analytic function on Ω_0. Consequently, the vectors $K_w^{(i)}$, $i = 1, \ldots, d$ are linearly independent for all w in this neighborhood.

To prove the statement in (iii), that is, to prove that $K_0^{(i)}$ are uniquely determined by the generators g_i^0 , $1 \leq i \leq d$. We will let g_i^0 denote the germ of g_i^0 at 0 as well. The uniqueness of the set of vectors $K_0^{(i)}$ is clearly the same as the uniqueness of the set of linear functionals H_i, $1 \leq i \leq d$. Thus enough to show if $\sum_{i=1}^d g_i^0(z)\{H_i(z) - \tilde{H}_i(z)\} = 0$ for some choice of $\tilde{H}_i$, $1 \leq i \leq d$, then $(H_i - \tilde{H}_i)(0) = 0$. But then we have $\sum_{n=0}^\infty \sum_{i=1}^d g_i^0(z)\{h_i^n(z) - \tilde{h}_i^n(z)\}e_n^* = 0$. Hence, for each n

$$\sum_{i=1}^d g_i^0(z)\{h_i^n(z) - \tilde{h}_i^n(z)\} = 0.$$

Fix n and let $\alpha_i(z) = h_i^n(z) - \tilde{h}_i^n(z)$. In this notation, $\sum_{i=1}^d g_i^0(z)\alpha_i(z) = 0$. Now we claim that $\alpha_i(0) = 0$ for all $i \in \{1, \ldots, d\}$. If not, we may assume $\alpha_1(0) \neq 0$. Then the germ of α_1 at 0 is a unit in ${}_m\mathcal{O}$. Hence we can write, in ${}_m\mathcal{O}$,

$$g_1^0 = -(\sum_{i=2}^d g_i^0 \alpha_{i0}){\alpha_{10}}^{-1},$$

where α_{i0} denotes the germs of the analytic functions α_i at 0, $1 \leq i \leq d$. This is a contradiction, as $g_1^0, \ldots, g_d^0$ is a minimal set of generators of the stalk $\mathcal{S}_0^{\mathcal{M}}$ by hypothesis. As a result, $h_i^n(0) = \tilde{h}_i^n(0)$ for all $i \in \{1, \ldots, d\}$ and $n \in \mathbb{N} \cup \{0\}$. Then $H_i(0) = \tilde{H}_i(0)$ for all $i \in \{1, \ldots, d\}$. This completes the proof of (iii).

To prove the statement in (iv), let $\{g_1^0, \ldots, g_d^0\}$ and $\{\tilde{g}_1^0, \ldots, \tilde{g}_d^0\}$ be two sets of generators for $\mathcal{S}_0^{\mathcal{M}}$ both of which are minimal. Let $K^{(i)}$ and $\tilde{K}^{(i)}$, $1 \leq i \leq d$, be the corresponding vectors that appear in the decomposition of the reproducing kernel K as in (i). It is enough to show that

$$\operatorname{span}_{\mathbb{C}}\{K_i(z,0) : 1 \leq i \leq d\} = \operatorname{span}_{\mathbb{C}}\{\tilde{K}_i(z,0) : 1 \leq i \leq d\}.$$

There exists holomorphic functions ϕ_{ij}, $1 \leq i,j \leq d$, in a small enough neighborhood of 0 such that $\tilde{g}_i^0 = \sum_{j=1}^d \phi_{ij} g_j^0$. It now follows that

$$\begin{aligned} K(z,w) &= \sum_{i=1}^d \bar{\tilde{g}}_i^0(w)\tilde{K}_i(z,w) = \sum_{i=1}^d (\sum_{j=1}^d \bar{\phi}_{ij}(w)\bar{g}_j^0(w))\tilde{K}_i(z,w) \\ &= \sum_{j=1}^d \bar{g}_j^0(w)(\sum_{i=1}^d \bar{\phi}_{ij}(w)\tilde{K}_i(z,w)) \end{aligned}$$

for w possibly from an even smaller neighborhood of 0. But $K(z,w) = \sum_{j=1}^d \bar{g}_j^0(w)K_j(z,w)$ and uniqueness at the point 0 implies that

$$K_j(z,0) = \sum_{i=1}^d \bar{\phi}_{ij}(0)\tilde{K}_i(z,0)$$

for $1 \leq j \leq d$. So, we have $\mathrm{span}_{\mathbb{C}}\{K_i(z,0) : 1 \leq i \leq d\} \subseteq \mathrm{span}_{\mathbb{C}}\{\tilde{K}_i(z,0) : 1 \leq i \leq d\}$. Writing g_j^0 in terms of $\tilde{g}_i^0$, we get the other inclusion.

Finally, to prove the statement in (v), let us apply ${M_j}^*$ to both sides of the decomposition of the reproducing kernel K given in part (i) to obtain $\bar{w}_j K(z,w) = \sum_{i=1}^d \bar{g}_i^0(w){M_j}^*K_i(z,w)$. Substituting K from the first equation, we get $\sum_{i=1}^d \bar{g}_i^0(w)(M_j - w_j)^*K_i(z,w) = 0$. Let $F_{ij}(z,w) = (M_j - w_j)^*K_i(z,w)$. For a fixed but arbitrary $z_0 \in \Omega$, consider the equation $\sum_{i=1}^d \bar{g}_i^0(w)F_{ij}(z_0,w) = 0$. Suppose there exists $k, 1 \leq k \leq d$ such that $F_{kj}(z_0,0) \neq 0$. Then

$$g_k^0 = \{\overline{F_{kj}(z_0,\cdot)}_0\}^{-1} \sum_{i=1, i\neq k}^d g_i^0 \overline{F_{ij}(z_0,\cdot)}_0.$$

This is a contradiction. Therefore $F_{ij}(z_0,0) = 0$, $1 \leq i \leq d$, and for all $z_0 \in \Omega$. So ${M_j}^*K_i(z,0) = 0$, $1 \leq i \leq d$, $1 \leq j \leq m$. This completes the proof of the theorem. ☐

Remark 2.1. Let $\mathcal{I}$ be an ideal in the polynomial ring $\mathbb{C}[\underline{z}]$. Suppose $\mathcal{M} \supset \mathcal{I}$ and that $\mathcal{I}$ is dense in $\mathcal{M}$. Let $\{p_i \in \mathbb{C}[\underline{z}] : 1 \leq i \leq t\}$ be a minimal set of generators for the ideal $\mathcal{I}$. Let $V(\mathcal{I})$ be the zero variety of the ideal $\mathcal{I}$. If $w \notin V(\mathcal{I})$, then $\mathcal{S}_w^{\mathcal{M}} = {}_m\mathcal{O}_w$. Although $p_1, \ldots, p_t$ generate the stalk at every point, they are not necessarily a minimal set of generators. For example, let $\mathcal{I} =< z_1(1+z_1), z_1(1-z_2), z_2^2 > \subset \mathbb{C}[z_1,z_2]$. The functions $z_1(1+z_1), z_1(1-z_2), z_2^2$ form a minimal set of generators for the ideal $\mathcal{I}$. Since $1+z_1$ and $1-z_2$ are units in ${}_2\mathcal{O}_0$, it follows that the functions z_1 and z_2^2 form a minimal set of generators for the stalk $\mathcal{S}_0^{\mathcal{M}}$.

3. The joint kernel at w_0 and the stalk $\mathcal{S}_{w_0}^{\mathcal{M}}$

Let $g_1^0, \ldots, g_d^0$ be a minimal set of generators for the stalk $\mathcal{S}_{w_0}^{\mathcal{M}}$ as before. For $f \in \mathcal{S}_{w_0}^{\mathcal{M}}$, we can find holomorphic functions f_i, $1 \leq i \leq d$ on some small open neighborhood U of w_0 such that $f = \sum_{i=1}^d g_i^0 f_i$ on U. We write

$$f = \sum_{i=1}^d g_i^0 f_i = \sum_{i=1}^d g_i^0\{f_i - f_i(w_0)\} + \sum_{i=1}^d g_i^0 f_i(w_0).$$

on U. Let $\mathfrak{m}(\mathcal{O}_{w_0})$ be the maximal ideal (consisting of the germs of holomorphic functions vanishing at the point w_0) in the local ring $\mathcal{O}_{w_0}$ and $\mathfrak{m}(\mathcal{O}_{w_0})\mathcal{S}_{w_0}^{\mathcal{M}} = \mathfrak{m}_{w_0}\mathcal{S}_{w_0}^{\mathcal{M}}$. Thus the linear span of the equivalence classes $[g_1^0], \ldots, [g_d^0]$ is the quotient module $\mathcal{S}_{w_0}^{\mathcal{M}}/\mathfrak{m}_{w_0}\mathcal{S}_{w_0}^{\mathcal{M}}$. Therefore we have

$$\dim \mathcal{S}_{w_0}^{\mathcal{M}}/\mathfrak{m}_{w_0}\mathcal{S}_{w_0}^{\mathcal{M}} \leq d.$$

It turns out that the elements $[g_1^0], \ldots, [g_d^0]$ in the quotient module are linearly independent. Then $\dim \mathcal{S}_{w_0}^{\mathcal{M}}/\mathfrak{m}_{w_0}\mathcal{S}_{w_0}^{\mathcal{M}} = d$. To prove the linear independence, let us consider the equation $\sum_{i=1}^d \alpha_i[g_i^0] = 0$ for some complex numbers α_i, $1 \le i \le d$, or equivalently, $\sum_{i=1}^d \alpha_i g_i^0 \in \mathfrak{m}(\mathcal{O}_w)S_w^{\mathcal{M}}$. Thus there exists holomorphic functions f_i, $1 \le i \le d$, on a small enough neighborhood of w_0 and vanishing at w_0 such that $\sum_{i=1}^d(\alpha_i - f_i)g_i = 0$. Now suppose $\alpha_k \neq 0$ for some $k, 1 \le k \le d$. Then we can write $g_k^0 = -\sum_{i\neq k}(\alpha_k - f_k)_0^{-1}(\alpha_i - f_i)_0 g_i^0$ which is a contradiction. From the decomposition theorem 1.5, it follows that

$$\dim \cap_{j=1}^m \ker(M_j - w_{0j})^* \ge \sharp\{minimal\ generators\ for\ S_{w_0}^{\mathcal{M}}\} = \dim \mathcal{S}_{w_0}^{\mathcal{M}}/\mathfrak{m}_{w_0}\mathcal{S}_{w_0}^{\mathcal{M}}. \tag{3.1}$$

We will impose natural conditions on the Hilbert module $\mathcal{M}$, always assumed to be in the class $\mathfrak{B}_1(\Omega)$, so as to ensure equality in (1.3) which is also the inequality given above. Let $V(\mathcal{M}) := \{w \in \Omega : f(w) = 0, \text{ for all } f \in \mathcal{M}\}$. Then for $w_0 \notin V(\mathcal{M})$, the number of minimal generators for the stalk at w_0 is one, in fact, $S_{w_0}^{\mathcal{M}} = {}_m\mathcal{O}_{w_0}$. Also, $\dim \ker D_{(\mathbf{M}-w_0)^*} = 1$ since the joint kernel at w_0 is spanned by the Kernel function $K(\cdot, w_0)$ of $\mathcal{M}$ for $w_0 \notin V(\mathcal{M})$. Therefore, outside the zero set, we have equality in (1.3). For a large class of Hilbert modules, we will show, even on the zero set, that the reverse inequality is valid. For instance, for Hilbert modules of rank 1 over $\mathbb{C}[\underline{z}]$, we have equality everywhere. This is easy to see:

$$1 \ge \dim \mathcal{M} \otimes_{\mathcal{C}_m} \mathbb{C}_{w_0} = \dim \cap_{j=1}^m \ker(M_j - w_{0j})^* \ge \dim \mathcal{S}_{w_0}^{\mathcal{M}}/\mathfrak{m}_{w_0}\mathcal{S}_{w_0}^{\mathcal{M}} \ge 1.$$

To understand the more general case, consider the map $i_w : \mathcal{M} \longrightarrow \mathcal{M}_w$ defined by $f \mapsto f_w$, where f_w is the germ of the function f at w. Clearly, this map is a vector space isomorphism onto its image. The linear space $\mathcal{M}^{(w)} := \sum_{j=1}^m (z_j - w_j)\mathcal{M} = \mathfrak{m}_w\mathcal{M}$ is closed since $\mathcal{M}$ is assumed to be in $\mathfrak{B}_1(\Omega)$. Then the map $f \mapsto f_w$ restricted to $\mathcal{M}^{(w)}$ is a linear isomorphism from $\mathcal{M}^{(w)}$ to $(\mathcal{M}^{(w)})_w$. Consider

$$\mathcal{M} \xrightarrow{i_w} \mathcal{S}_w^{\mathcal{M}} \xrightarrow{\pi} \mathcal{S}_w^{\mathcal{M}}/\mathfrak{m}(\mathcal{O}_w)\mathcal{S}_w^{\mathcal{M}},$$

where π is the quotient map. Now we have a map $\psi : \mathcal{M}_w/(\mathcal{M}^{(w)})_w \longrightarrow \mathcal{S}_w^{\mathcal{M}}/\{\mathfrak{m}(\mathcal{O}_w)\mathcal{S}_w^{\mathcal{M}}\}$ which is well defined because $(\mathcal{M}^{(w)})_w \subseteq \mathcal{M}_w \cap \mathfrak{m}(\mathcal{O}_w)\mathcal{S}_w^{\mathcal{M}}$. Whenever ψ can be shown to be one-one, equality in (1.3) is forced. To see this, note that $\mathcal{M} \ominus \mathcal{M}^{(w)} \cong \mathcal{M}/\mathcal{M}^{(w)}$ and

$$\cap_{j=1}^m \ker(M_j - w_j)^* = \cap_{j=1}^m \{\operatorname{ran}(M_j - w_j)\}^\perp = \mathcal{M} \ominus \sum_{j=1}^m (z_j - w_j)\mathcal{M} = \mathcal{M} \ominus \mathcal{M}^{(w)}.$$

Hence

$$d \le \dim \cap_{j=1}^m \ker(M_j - w_j)^* = \dim \mathcal{M}/\mathcal{M}^{(w)} \le \dim \mathcal{S}_w^{\mathcal{M}}/\mathfrak{m}(\mathcal{O}_w)\mathcal{S}_w^{\mathcal{M}} = d. \tag{3.2}$$

Suppose $\psi(f) = 0$ for some $f \in \mathcal{M}$. Then $f_w \in \mathfrak{m}(\mathcal{O}_w)\mathcal{S}_w^{\mathcal{M}}$ and consequently, $f = \sum_{i=1}^m (z_i - w_i)f_i$ for holomorphic functions f_i, $1 \le i \le m$, on some small open set U. The main question is if the functions f_i, $1 \le i \le m$, can be chosen from the Hilbert module $\mathcal{M}$. We isolate below, a class of Hilbert modules for which this question has an affirmative answer.

Let $\mathcal{H}$ be a Hilbert module over the polynomial ring $\mathbb{C}[\underline{z}]$ in the class $\mathrm{B}_1(\Omega)$. Pick, for each $w \in \Omega$, a $\mathbb{C}$ - linear subspace $\mathbb{V}_w$ of the polynomial ring $\mathbb{C}[\underline{z}]$ with the property that it is invariant under the action of the partial differential operators $\{\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_m}\}$ (see [4]). Set

$$\mathcal{M}(w) = \{f \in \mathcal{H} : q(D)f|_w = 0 \text{ for all } q \in \mathbb{V}_w\}.$$

For $f \in \mathcal{M}(w)$ and $q \in \mathbb{V}_w$,

$$q(D)(z_j f)|_w = w_j q(D)f|_w + \frac{\partial q}{\partial z_j}(D)f|_w = 0.$$

Now, the assumption on $\mathbb{V}_w$ ensure that $\mathcal{M}(w)$ is a module. We consider below, the class of (non-trivial) Hilbert modules which are of the form $\mathcal{M} := \bigcap_{w\in\Omega}\mathcal{M}(w)$. It is easy to see that

$$w \notin V(\mathcal{M}) \text{ if and only if } \mathbb{V}_w = \{0\} \text{ if and only if } \mathcal{M}(w) = \mathcal{H}.$$

Therefore, $\mathcal{M} = \bigcap_{w\in V(\mathcal{M})}\mathcal{M}(w)$. Let $\mathbb{V}_w(\mathcal{M}) := \{q \in \mathbb{C}[\underline{z}] : q(D)f\big|_w = 0 \text{ for all } f \in \mathcal{M}\}$. We note that $\mathbb{V}_w(\mathcal{M}) = \mathbb{V}_w$. Fix a point in $V(\mathcal{M})$, say w_0. Consider $\tilde{\mathbb{V}}_{w_0}(\mathcal{M}) = \{q \in \mathbb{C}[\underline{z}] : \frac{\partial q}{\partial z_i} \in \mathbb{V}_{w_0}(\mathcal{M}),\ 1 \le i \le m\}$. For $w \in V(\mathcal{M})$, let

$$\mathbb{V}_w^{w_0}(\mathcal{M}) = \begin{cases} \mathbb{V}_w(\mathcal{M}) & \text{if } w \neq w_0 \\ \tilde{\mathbb{V}}_{w_0}(\mathcal{M}) & \text{if } w = w_0. \end{cases}$$

Now, define $\mathcal{M}^{w_0}(w)$ to be the submodule (of $\mathcal{H}$) corresponding to the family of the $\mathbb{C}$-linear subspaces $\mathbb{V}_w^{w_0}(\mathcal{M})$ and let $\mathcal{M}^{w_0} = \bigcap_{w\in V(\mathcal{M})}\mathcal{M}^{w_0}(w)$. So we have $\mathbb{V}_w(\mathcal{M}^{w_0}) = \mathbb{V}_w^{w_0}(\mathcal{M})$. For $f \in \mathcal{M}^{(w_0)}$, we have $f = \sum_{j=1}^m (z_j - w_{0j})f_j$, for some choice of $f_1, \ldots, f_m \in \mathcal{M}$. Now for any $q \in \mathbb{C}[\underline{z}]$, following [4], we have

$$q(D)f = \sum_{i=1}^m q(D)\{(z_j - w_{0j})f_j\} = \sum_{i=1}^m \{(z_j - w_{0j})q(D)f_j + \frac{\partial q}{\partial z_j}(D)f_j\}. \tag{3.3}$$

For $w \in V(\mathcal{M})$ and $f \in \mathcal{M}^{(w_0)}$, it follows from the definitions that

$$q(D)f\big|_w = \begin{cases} \sum_{i=1}^m \{(w_j - w_{0j})q(D)f_j|_w + \frac{\partial q}{\partial z_j}(D)f_j|_w\} = 0 & q \in \mathbb{V}_w^{w_0},\ w \neq w_0 \\ \sum_{i=1}^m \{\frac{\partial q}{\partial z_j}(D)f_j|_{w_0}\} = 0 & q \in \mathbb{V}_{w_0}^{w_0},\ w = w_0. \end{cases}$$

Thus $f \in \mathcal{M}^{(w_0)}$ implies that $f \in \mathcal{M}^{w_0}(w)$ for each $w \in V(\mathcal{M})$. Hence $\mathcal{M}^{(w_0)} \subseteq \mathcal{M}^{w_0}$. Now we describe the Gleason property for $\mathcal{M}$ at a point w_0.

Definition 3.1. We say that $\mathcal{M} := \cap_{w\in V(\mathcal{M})}\mathcal{M}(w)$ has the Gleason property at a point $w_0 \in V(\mathcal{M})$ if $\mathcal{M}^{w_0} = \mathcal{M}^{(w_0)}$.

Analogous to the definition of $\mathbb{V}_{w_0}(\mathcal{M})$ for a Hilbert module $\mathcal{M}$, we define the space $\mathbb{V}_{w_0}(\mathcal{S}_{w_0}^{\mathcal{M}}) = \{q \in \mathbb{C}[\underline{z}] : q(D)f\big|_{w_0} = 0,\ f_{w_0} \in \mathcal{S}_{w_0}^{\mathcal{M}}\}$. It will be useful to record the relation between $\mathbb{V}_{w_0}(\mathcal{M})$ and $\mathbb{V}_{w_0}(\mathcal{S}_{w_0}^{\mathcal{M}})$ in a separate lemma.

Lemma 3.2. *For any Hilbert module in $\mathfrak{B}_1(\Omega)$ and $w_0 \in \Omega$, we have $\mathbb{V}_{w_0}(\mathcal{M}) = \mathbb{V}_{w_0}(\mathcal{S}_{w_0}^{\mathcal{M}})$.*

Proof. We note that the inclusion $\mathbb{V}_{w_0}(\mathcal{S}_{w_0}^{\mathcal{M}}) \subseteq \mathbb{V}_{w_0}(\mathcal{M})$ follows from $\mathcal{M}_{w_0} \subseteq \mathcal{S}_{w_0}^{\mathcal{M}}$. To prove the reverse inclusion, we need to show that $q(D)h|_{w_0} = 0$ for $h \in \mathcal{S}_{w_0}^{\mathcal{M}}$, for all $q \in \mathbb{V}_{w_0}(\mathcal{M})$. Since $h \in \mathcal{S}_{w_0}^{\mathcal{M}}$, we can find functions $f_1, \ldots, f_n \in \mathcal{M}$ and $g_1, \ldots, g_n \in \mathcal{O}_{w_0}$ such that $h = \sum_{i=1}^n f_i g_i$ in some small open neighborhood of w_0. Therefore, it is enough to show that $q(D)(fg)|_{w_0} = 0$ for $f \in \mathcal{M}$, g holomorphic in a neighborhood, say U_{w_0} of w_0, and $q \in \mathbb{V}_{w_0}(\mathcal{M})$. We can choose U_{w_0} to be a small enough polydisk such that $g = \sum_\alpha a_\alpha(z - w_0)^\alpha,\ z \in U_{w_0}$. We then see that $q(D)(fg) = \sum_\alpha a_\alpha q(D)\{(z - w_0)^\alpha f\}$ for $z \in U_{w_0}$. Clearly, $(z - w_0)^\alpha f$ belongs to $\mathcal{M}$ whenever $f \in \mathcal{M}$. Hence $q(D)\{(z - w_0)^\alpha f\}|_{w_0} = 0$ and we have $q(D)(fg)|_{w_0} = 0$ completing the proof of $\mathbb{V}_{w_0}(\mathcal{M}) \subseteq \mathbb{V}_{w_0}(\mathcal{S}_{w_0}^{\mathcal{M}})$.. □

We will show that we have equality in (1.3) for all Hilbert modules with the Gleason property.

Proof of proposition 1.6. We first show that $\ker(\pi \circ i_{w_0}) = \mathcal{M}^{w_0}$. Showing $\ker(\pi \circ i_{w_0}) \subseteq \mathcal{M}^{w_0}$ is same as showing $\mathcal{M}_{w_0} \cap \mathfrak{m}_{w_0}\mathcal{S}_{w_0}^{\mathcal{M}} \subseteq (\mathcal{M}^{w_0})_{w_0}$. We claim that

$$\mathbb{V}_{w_0}(\mathfrak{m}_{w_0}\mathcal{S}_{w_0}^{\mathcal{M}}) = \mathbb{V}_{w_0}^{w_0}(\mathcal{M}). \tag{3.4}$$

If $f \in \mathfrak{m}_{w_0}\mathcal{S}^{\mathcal{M}}_{w_0}$, then there exists $f_j \in \mathcal{S}^{\mathcal{M}}_{w_0}$ such that $f = \sum_{i=1}^m (z_j - w_{0j})f_j$. From equation (3.3), we have

$$q \in \mathbb{V}_{w_0}(\mathfrak{m}_{w_0}\mathcal{S}^{\mathcal{M}}_{w_0}) \text{ if and only if } \frac{\partial q}{\partial z_j} \in \mathbb{V}_{w_0}(\mathcal{S}^{\mathcal{M}}_{w_0}) = \mathbb{V}_{w_0}(\mathcal{M}) \text{ for all } j, 1 \le j \le m.$$

Now, from lemma 3.2, we see that $\frac{\partial q}{\partial z_j} \in \mathbb{V}_{w_0}(\mathcal{M})$ $1 \le j \le m$, if and only if $q \in \mathbb{V}^{w_0}_{w_0}(\mathcal{M})$, which proves our claim. So for $f \in \mathcal{M}$, if $f_{w_0} \in \mathfrak{m}_{w_0}\mathcal{S}^{\mathcal{M}}_{w_0}$, then $f \in \mathcal{M}^{w_0}(w)$ for all $w \in V(\mathcal{M})$. Hence $f \in \mathcal{M}^{w_0}$ and as a result, we have $\mathcal{M}_{w_0} \cap \mathfrak{m}_{w_0}\mathcal{S}^{\mathcal{M}}_{w_0} \subseteq (\mathcal{M}^{w_0})_{w_0}$.

Now let $f \in \mathcal{M}^{w_0}$. From (3.4) it follows that

$$f \in \{g \in \mathcal{O}_{w_0} : q(D)g\big|_{w_0} = 0 \text{ for all } q \in \mathbb{V}_{w_0}(\mathfrak{m}_{w_0}\mathcal{S}^{\mathcal{M}}_{w_0})\}.$$

Then from [4, Prposotion 2.3.1] we have $f \in \mathfrak{m}_{w_0}\mathcal{S}^{\mathcal{M}}_{w_0}$. Therefore $f \in \ker(\pi \circ i_{w_0}$ and $\ker(\pi \circ i_{w_0}) = \mathcal{M}^{w_0}$.

Next we show that the map $\pi \circ i_{w_0}$ is onto. Let $\sum_{i=1}^n f_i g_i \in \mathcal{S}^{\mathcal{M}}_{w_0}$, where $f_i \in \mathcal{M}$ and g_i's are holomorphic function in some neighborhood of w_0, $1 \le i \le n$. We need to show that there exist $f \in \mathcal{M}$ such that the class $[f]$ is equal to $[\sum_{i=1}^n f_i g_i]$ in $\mathcal{S}^{\mathcal{M}}_{w_0}/\mathfrak{m}_{w_0}\mathcal{S}^{\mathcal{M}}_{w_0}$. Let us take $f = \sum_{i=1}^n f_i g_i(w_0)$. Then

$$\sum_{i=1}^n f_i g_i - f = \sum_{i=1}^n f_i\{g_i - g_i(w_0)\} \in \mathfrak{m}_{w_0}\mathcal{S}^{\mathcal{M}}_{w_0}.$$

This completes the proof of surjectivity.

Suppose Gleason property holds for $\mathcal{M}$ at w_0. Since $\mathcal{M}^{(w_0)} \subseteq \ker(\pi \circ i_{w_0})$, and we have just shown that $\ker(\pi \circ i) = \mathcal{M}^{w_0}$, it follows from the Gleason property at w_0 that we have the equality $\ker(\pi \circ i_{w_0}) = \mathcal{M}^{(w_0)}$. We recall then that the map $\psi : \mathcal{M}/\mathcal{M}^{(w_0)} \longrightarrow \mathcal{S}^{\mathcal{M}}_{w_0}/\{\mathfrak{m}_{w_0}\mathcal{S}^{\mathcal{M}}_{w_0}\}$ is one to one. The equality in (1.3) is established using the equations (3.1) and (3.2).

Now suppose equality holds in (1.3). From the above, it is clear that $\mathcal{M}/\mathcal{M}^{w_0}$ is isomorphic to $\mathcal{S}^{\mathcal{M}}_{w_0}/\mathfrak{m}_{w_0}\mathcal{S}^{\mathcal{M}}_{w_0}$. Thus

$$\dim \mathcal{M}/\mathcal{M}^{w_0} = \dim \mathcal{M}/\mathcal{M}^{(w_0)}.$$

But as $\mathcal{M}^{(w_0)} \subseteq \mathcal{M}^{w_0}$, we have $\mathcal{M}^{(w_0)} = \mathcal{M}^{w_0}$ and hence Gleason property holds for $\mathcal{M}$ at w_0. □

A class of examples of Hilbert spaces satisfying Gleason property can be found in [23]. It was shown in [23] that Gleason property holds for analytic Hilbert module. However it is not entirely clear if it continues to hold for submodules of analytic Hilbert module. We will show here, never the less, we have equality in (1.3). Let $\mathcal{M}$ be a submodule of an analytic Hilbert module over $\mathbb{C}[\underline{z}]$. Assume that $\mathcal{M}$ is a closure of an ideal $\mathcal{I} \subseteq \mathbb{C}[\underline{z}]$. From [4, 19], we note that

$$\dim \cap_{j=1}^m \ker(M_j - w_{0j})^* = \dim \mathcal{I}/\mathfrak{m}_{w_0}\mathcal{I}.$$

Therefore from (3.1) we have

$$\dim \mathcal{I}/\mathfrak{m}_{w_0}\mathcal{I} \ge \dim \mathcal{S}^{\mathcal{M}}_{w_0}/\mathfrak{m}_{w_0}\mathcal{S}^{\mathcal{M}}_{w_0}.$$

So we need to prove the reverse inequality. Fix a point $w_0 \in \Omega$. Consider the map

$$\mathcal{I} \xrightarrow{i_{w_0}} \mathcal{S}^{\mathcal{M}}_{w_0} \xrightarrow{\pi} \mathcal{S}^{\mathcal{M}}_{w_0}/\mathfrak{m}_{w_0}\mathcal{S}^{\mathcal{M}}_{w_0}.$$

We will show that $\ker(\pi \circ i_{w_0}) = \mathfrak{m}_{w_0}\mathcal{I}$. Let $V(\mathcal{I})$ denote the zero set of the ideal $\mathcal{I}$ and $\mathbb{V}_w(\mathcal{I})$ be its characteristic space at w. We begin by proving that the characteristic space of the ideal coincides with that of corresponding Hilbert module.

Lemma 3.3. *Assume that* $\mathcal{M} = [\mathcal{I}]$. *Then* $\mathbb{V}_{w_0}(\mathcal{I}) = \mathbb{V}_{w_0}(\mathcal{M})$ *for* $w_0 \in \Omega$.

Proof. Clearly $\mathbb{V}_{w_0}(\mathcal{I}) \supseteq \mathbb{V}_{w_0}(\mathcal{M})$, so we prove $\mathbb{V}_{w_0}(\mathcal{I}) \subseteq \mathbb{V}_{w_0}(\mathcal{M})$. For $q \in \mathbb{V}_{w_0}(\mathcal{I})$ and $f \in \mathcal{M}$, we show that $q(D)f|_{w_0} = 0$. Now, for each $f \in \mathcal{M}$, there exists a sequence of polynomial $p_n \in \mathcal{I}$ such that $p_n \to f$ in the Hilbert space norm. Recall that if K is the reproducing kernel for $\mathcal{M}$, then

$$(\partial^\alpha f)(w) \;=\; \langle f, \bar{\partial}^\alpha K(\cdot, w)\rangle, \text{ for } \alpha \in \mathbb{Z}_m^+,\ w \in \Omega,\ f \in \mathcal{M} \tag{3.5}$$

For $w \in \Omega$ and a compact neighborhood C of w, we have

$$\begin{aligned} & |q(D)p_n(w) - q(D)f(w)| \\ = \ & |\langle p_n - f, q(\bar{D})K(\cdot, w)\rangle| \leq \| p_n - f \|_{\mathcal{M}} \| q(\bar{D})K(\cdot, w) \|_{\mathcal{M}} \\ \leq \ & \| p_n - f \|_{\mathcal{M}} \sup_{w \in C} \| q(\bar{D})K(\cdot, w) \|_{\mathcal{M}} . \end{aligned}$$

So, in particular, $q(D)p_n\big|_{w_0} \longrightarrow q(D)f\big|_{w_0}$ as $n \longrightarrow \infty$. Since $q(D)p_n\big|_{w_0} = 0$ for all n, it follows that $q(D)f\big|_{w_0} = 0$. Hence $q \in \mathbb{V}_{w_0}(\mathcal{M})$ and we are done. $\square$

Now let $\mathcal{J} = \mathfrak{m}_{w_0}\mathcal{I}$. Recall [19, Proposition 2.3] that $V(\mathcal{J}) \backslash V(\mathcal{I}) := \{w \in \mathbb{C}^m : \mathbb{V}_w(\mathcal{I}) \subsetneq \mathbb{V}_w(\mathcal{J})\} = \{w_0\}$. Here we will explicitly write down the characteristic space.

Lemma 3.4. *For $w \in \mathbb{C}^m$, $\mathbb{V}_w(\mathcal{J}) = \mathbb{V}_w^{w_0}(\mathcal{I})$. Here $\mathbb{V}_w^{w_0}(\mathcal{I}) = \begin{cases} \mathbb{V}_w(\mathcal{I}), & w \neq w_0; \\ \tilde{\mathbb{V}}_{w_0}(\mathcal{I}), & w = w_0 \end{cases}$, and $\tilde{\mathbb{V}}_{w_0}(\mathcal{I}) = \{q \in \mathbb{C}[\underline{z}] : \frac{\partial q}{\partial z_i} \in \mathbb{V}_{w_0}(\mathcal{I}),\ 1 \leq i \leq m\}$.*

Proof. Since $\mathcal{J} \subset \mathcal{I}$, we have $\mathbb{V}_w(\mathcal{I}) \subseteq \mathbb{V}_w(\mathcal{J})$ for all $w \in \mathbb{C}^m$. Now let $w \neq w_0$. For $f \in \mathcal{I}$ and $q \in \mathbb{V}_w(\mathcal{J})$, we show that $q(D)f\big|_w = 0$ which implies q must be in $\mathbb{V}_w(\mathcal{I})$.

Note that for any $k \in \mathbb{N}$ and $j \in \{1, \ldots, m\}$, $q(D)\{(z_j - w_{0j})^k f\}\big|_w = 0$ as $(z_j - w_{0j})^k f \in \mathcal{J}$. This implies $\sum_{l=0}^k (w_j - w_{0j})^l \binom{k}{l} \frac{\partial^{k-l} q}{\partial z_j^{k-l}}(D)f\big|_w = 0$. Hence we have

$$(w_j - w_{0j})^k q(D)f\big|_w = (-1)^k \frac{\partial^k q}{\partial z_j^k}(D)f\big|_w \text{ for all } k \in \mathbb{N} \text{ and } j \in \{1, \ldots, m\}.$$

So, if $w \neq w_0$, then there exists $i \in \{1, \ldots, m\}$ such that $w_i \neq w_{0i}$. Therefore, by choosing k large enough with respect to the degree of q, we can ensure $(w_i - w_{0i})^k q(D)f\big|_w = 0$. Thus $q(D)f\big|_w = 0$. For $w = w_0$, we have

$q \in \mathbb{V}_{w_0}(\mathcal{J})$ if and only if $q(D)\{(z_j - w_{0j})f\}\big|_{w_0} = 0$ for all $f \in \mathcal{I}$ and $j \in \{1, \ldots, m\}$ if and only if $\frac{\partial q}{\partial z_j}(D)f\big|_{w_0} = 0$ for all $f \in \mathcal{I}$ and $j \in \{1, \ldots, m\}$ if and only if $q \in \mathbb{V}_{w_0}(\mathcal{J})$ if and only if $\frac{\partial q}{\partial z_j} \in \mathbb{V}_{w_0}(\mathcal{I})$ for all $j \in \{1, \ldots, m\}$ if and only if $q \in \tilde{\mathbb{V}}_{w_0}(\mathcal{I})$.

This completes the proof of the lemma. $\square$

We have shown that $\mathbb{V}_{w_0}(\mathcal{I}) = \mathbb{V}_{w_0}(\mathcal{M}) = \mathbb{V}_{w_0}(\mathcal{S}_{w_0}^{\mathcal{M}})$. The next Lemma provides a relationship between the characteristic space of $\mathcal{J}$ at the point w_0 and the sheaf $\mathcal{S}_{w_0}^{\mathcal{M}}$.

Lemma 3.5. $\mathbb{V}_{w_0}(\mathcal{J}) = \mathbb{V}_{w_0}(\mathfrak{m}(\mathcal{O}_{w_0})\mathcal{S}_{w_0}^{\mathcal{M}})$.

Proof. We have $\mathbb{V}_{w_0}(\mathfrak{m}(\mathcal{O}_{w_0})\mathcal{S}_{w_0}^{\mathcal{M}}) \subseteq \mathbb{V}_{w_0}(\mathcal{J})$. From the previous lemma, it follows that if $q \in \mathbb{V}_{w_0}(\mathcal{J})$, then $q \in \tilde{\mathbb{V}}_{w_0}(\mathcal{I})$, that is, $\frac{\partial q}{\partial z_j} \in \mathbb{V}_{w_0}(\mathcal{I}) = \mathbb{V}_{w_0}(\mathcal{S}_{w_0}^{\mathcal{M}})$ for all $j \in \{1, \ldots, m\}$. From (3.4), it follows that $q \in \mathbb{V}_{w_0}(\mathfrak{m}(\mathcal{O}_{w_0})\mathcal{S}_{w_0}^{\mathcal{M}})$. $\square$

Now, we have all the ingredients to prove that we must have equality in (1.3) for submodules of analytic Hilbert modules which are obtained as closure of some polynomial ideal.

Proposition 3.6. *Let $\mathcal{M} = [\mathcal{I}]$ be a submodule of an analytic Hilbert module over $\mathbb{C}[\underline{z}]$, where $\mathcal{I}$ is an ideal in the polynomial ring $\mathbb{C}[\underline{z}]$. Then*

$$\sharp\{\textit{minimal set of generators for } \mathcal{S}^{\mathcal{M}}_{w_0}\} = \dim \cap_{j=1}^{m} \ker(M_j - w_{0j})^*.$$

Proof. Let $p \in \mathcal{I}$ such that $\pi \circ i_{w_0}(p) = 0$, that is, $p_{w_0} \in \mathfrak{m}(\mathcal{O}_{w_0})\mathcal{S}^{\mathcal{M}}_{w_0}$. The preceding Lemma implies $q(D)p\big|_{w_0} = 0$ for all $q \in \mathbb{V}_{w_0}(\mathcal{J})$. So, $p \in \mathcal{J}^e_{w_0} := \{r \in \mathbb{C}[\underline{z}] : q(D)p\big|_{w_0} = 0, \text{ for all } q \in \mathbb{V}_{w_0}(\mathcal{J})\}$. Therefore, from [4, Corollary 2.1.2] we have $p \in \bigcap_{w \in \mathbb{C}^m} \mathcal{J}^e_w = \mathcal{J}$. Thus $\ker(\pi \circ i_{w_0}) = \mathcal{J} = \mathfrak{m}_{w_0}\mathcal{I}$. Then the map $\pi \circ i_{w_0} : \dim \mathcal{I}/\mathfrak{m}_{w_0}\mathcal{I} \to \dim \mathcal{S}^{\mathcal{M}}_{w_0}/\mathfrak{m}_{w_0}\mathcal{S}^{\mathcal{M}}_{w_0}$ is one-one and we have

$$\dim \mathcal{I}/\mathfrak{m}_{w_0}\mathcal{I} \leq \dim \mathcal{S}^{\mathcal{M}}_{w_0}/\mathfrak{m}_{w_0}\mathcal{S}^{\mathcal{M}}_{w_0}.$$

Therefore, we have equality in (1.3). ☐

Remark 3.7. Corollary 1.7 is immediate from the Theorem 1.6 and the proposition given above.

Remark 3.8. In the paper [19], it is proved that if $\mathcal{M}$ is a closure of an ideal in the polynomial ring and $w_0 \in V(\mathcal{I})$ is a smooth point then,

$$\dim \cap_{i=1}^{m} \ker(M_j - w_{0j})^* = \begin{cases} 1 & \text{for } w_0 \notin V(\mathcal{I}) \cap \Omega; \\ \text{codimension of } V(\mathcal{I}) & \text{for } w_0 \in V(\mathcal{I}) \cap \Omega. \end{cases}$$

This can be easily derived from the Proposition given above. In the course of the proof of the main theorem in [19], a change of variable argument is used to show that one may assume without of loss of generality that the stalk at w_0 is generated by the co-ordinate functions $z_1, \dots, z_r$, where r is the co-dimension of $V(\mathcal{I})$. Therefore, the number of minimal generators for the stalk at a smooth point is equal to the codimension of $V(\mathcal{I})$. It now follows from the Proposition that the dimension of the joint kernel at a smooth point is equal to the co-dimension of $V(\mathcal{I})$.

4. Bergman space privilege

Fix two positive integer p, q. The division problem asks if the solution $u \in \mathcal{O}(\Omega)^q$ to the linear equation $Au = f$ must belong to $L^2_a(\Omega)^q$ if $f \in L^2_a(\Omega)^p$ and the matrix $A \in M_{p,q}(\mathcal{O}(\overline{\Omega}))$ of analytic functions defined in a neighborhood of $\overline{\Omega}$ are given. Two independent steps are necessary to understand the nature of the Division problem.

First, the solution u may not be unique, simply due to the non-trivial relations among the columns of the matrix A. This difficulty is clarified by homological algebra: at the level of coherent analytic sheaves, $\mathfrak{N} = \operatorname{coker}(A : \mathcal{O}|^p_{\overline{\Omega}} \longrightarrow \mathcal{O}|^q_{\overline{\Omega}})$ admits a finite free resolution

$$0 \to \mathcal{O}|^{n_p}_{\overline{\Omega}} \xrightarrow{d_p} \cdots \to \mathcal{O}|^{n_1}_{\overline{\Omega}} \xrightarrow{d_1} \mathcal{O}|^{n_0}_{\overline{\Omega}} \to \mathfrak{N} \to 0, \tag{4.1}$$

where $n_1 = p, n_0 = q$ and $d_1 = A$. The existence of such a resolution is assured by the analogue of Hilbert syzygies theorem in the analytic context, see for instance [25].

The second step, of circumventing the non-existence of boundary values for Bergman space functions, is resolved by a canonical quantization method, that is, by passing to the algebra of Toeplitz operators with continuous symbol on $L^2_a(\Omega)$. We import below, from the well understood theory of Toeplitz operators on domains of $\mathbb{C}^m$, a crucial criterion for a matrix of Toepliz operators to be Fredholm (cf. [35, 37]).

Assume that the analytic matrix $A(z)$ is defined on a neighborhood of $\overline{\Omega}$. One proves by standard homological techniques that every free, finite type resolution of the analytic coherent sheaf $\mathfrak{N} = \operatorname{coker}(A : \mathcal{O}|^p_{\overline{\Omega}} \longrightarrow \mathcal{O}|^q_{\overline{\Omega}})$ induces at the level of the Bergman space $L^2_a(\Omega)$ an exact complex, see [9]. The similarity between the two resolutions given above are not accidental, as it will be revealed in the next theorem. After understanding the disc-algebra privilege on a strictly convex domain [33], the statement of Theorem 1.8 is not surprising.

Proof of Theorem 1.8. The proof is very similar to the one of the disk algebra case [33], and we only sketch below the main ideas. Assume that the resolution 1.4 exists and that the last arrow has closed range. The exactness at each degree of the resolution is equivalent to the invertibility of the Hodge operator:

$$d_k^* d_k + d_{k+1} d_{k+1}^* : L_a^2(\Omega)^{n_k} \longrightarrow L_a^2(\Omega)^{n_k}, \quad 1 \leq k \leq p,$$

where we put $d_{p+1} = 0$. To be more specific: the condition $\ker[d_k^* d_k + d_{k+1} d_{k+1}^*] = 0$ is equivalent to the exactness of the complex at stage k, implying hence that $\operatorname{ran}(d_{k+1})$ is closed. In addition, if the range of d_k is closed, then, and only then, the self-adjoint operator $d_k^* d_k + d_{k+1} d_{k+1}^*$ is invertible.

Since the boundary of Ω is smooth, the commutator $[T_f, T_g]$ of two Toeplitz operators acting on the Bergman space and with continuous symbols $f, g \in C(\overline{\Omega})$ is compact, see for details and terminology [3, 35, 37]. Consequently for every k, $d_k^* d_k + d_{k+1} d_{k+1}^*$ is, modulo compact operators, a $n_k \times n_k$ matrix of Toeplitz operators with symbol

$$d_k(z)^* d_k(z) + d_{k+1}(z) d_{k+1}(z)^*, \quad w \in \overline{\Omega},$$

where the adjoint is now taken with respect to the canonical inner product in $\mathbb{C}^{n_k}$. According to a main result of [3], or [37, 35], if the Toeplitz operator $d_k^* d_k + d_{k+1} d_{k+1}^*$ is Fredholm, then its matrix symbol is invertible. Hence

$$\ker[d_k(z)^* d_k(z) + d_{k+1}(z) d_{k+1}(z)^*] = 0, \quad 1 \leq k \leq p.$$

Thus, for every $z \in \partial\Omega$,

$$\operatorname{rank} A(z) = \dim \operatorname{coker}(d_1(w)) = n_0 - n_1 + n_2 - \ldots + (-1)^p n_p.$$

To prove the other implication, we rely on the disk algebra privilege criterion obtained in the note [33]. Namely, in view of Theorem 2.2 of [33], if the rank of the matrix $A(z)$ does not jump for z belonging to the boundary of Ω, then there exists a resolution of $\mathbf{N} = \operatorname{coker} A : \mathcal{A}(\Omega)^p \longrightarrow \mathcal{A}(\Omega)^q$ with free, finite type $\mathcal{A}(\Omega)$-modules:

$$0 \to \mathcal{A}(\Omega)^{n_p} \xrightarrow{d_p} \cdots \to \mathcal{A}(\Omega)^{n_1} \xrightarrow{d_1} \mathcal{A}(\Omega)^{n_0} \to \mathbf{N} \to 0. \tag{4.2}$$

As before, we denote $d_1 = A$. We have to prove that the induced complex (1.4), obtained after applying (4.2) the functor $\otimes_{\mathcal{A}(\Omega)} L_a^2(\Omega)$, remains exact and the boundary operator d_1 has closed range.

For this, we "glue" together local resolutions of coker$A$ with the aid of Cartan's lemma of invertible matrices, as originally explained in [10], or in [33]. For points close to the boundary of Ω, such a resolution exists by the local freeness assumption, while in the interior, in neighborhoods of the points where the rank of the matrix A may jump, they exist by Douady's privilege on polydiscs. This proves that the Hilbert analytic module $\mathcal{N} = \operatorname{coker}(A : L_a^2(\Omega)^p \longrightarrow L_a^2(\Omega)^q)$ is privileged with respect to the Bergman space.

As for assertion c), we simply remark that it is equivalent to the injectivity of the restriction map

$$\operatorname{coker}(A : L_a^2(\Omega)^p \longrightarrow L_a^2(\Omega)^q) \longrightarrow \operatorname{coker}(A : \mathcal{O}(\Omega)^p \longrightarrow \mathcal{O}(\Omega)^q).$$

The last co-kernel is always Hausdorff in the natural quotient topology as the global section space of a coherent analytic sheaf.

The only place in the proof where the convexity of Ω is needed, is to ensure that, if the resolution 1.4 exists, then the induced complex at the level of sheaf models

$$0 \to \widehat{L_a^2}(\Omega)^{n_p} \xrightarrow{d_p} \cdots \to \widehat{L_a^2}(\Omega)^{n_1} \xrightarrow{d_1} \widehat{L_a^2}(\Omega)^{n_0} \to \widehat{\mathcal{N}} \to 0,$$

is exact. For a proof see [33]. □

Remark 4.1. It is worth mentioning that for non-smooth domains Ω in $\mathbb{C}^m$ the above result is not true. For instance $\mathcal{A}(\Omega)$-privilege for a poly-domain Ω was fully characterized by Douady [10]. On the other hand, even for smooth boundaries, the privilege with respect to the Fréchet algebra $\mathcal{O}(\Omega) \cap C^\infty(\overline{\Omega})$ seems to be quite intricate and definitely different than the Bergman space or disk algebra privileges, as indicated by an observation of Amar [1].

Corollary 4.2. *Coker* $[(\varphi_1, \dots, \varphi_m) : L^2_a(\Omega)^m \to L^2_a(\Omega)^m]$ *is privileged if and only if the analytic functions* $(\varphi_1, \dots, \varphi_m)$ *have no common zero on the boundary.*

Proof. No common zero of the functions $\varphi_1, \dots, \varphi_m$ lies on the boundary of Ω. Therefore, the matrix $(\varphi_1, \dots, \varphi_m)$ is of full rank 1 on the boundary of Ω. □

For many semi-Fredholm Hilbert module such as the Hardy space on Ω, the result given above, remains true ([9, 10]).

Since the restriction to an open subset $\Omega_0 \subseteq \Omega$ does not change the equivalence class of a module in $\mathfrak{B}_1(\Omega)$, we can always assume, without loss of generality, that the domain Ω is pseudoconvex in our context. For $w_0 \in \Omega$, the m-tuple $(z_1 - w_{01}, \dots, z_m - w_{0m})$ has no common zero on the boundary of Ω. We have pointed out, in Section 1, that if for $f \in \mathcal{M}$ the equation $f = \sum_{i=1}^m (z_i - w_{0i}) f_i$ admits a solution $(f_1, \dots, f_m)$ in $\mathcal{O}(\Omega)^m$ and if the module $\mathcal{M}$ is privileged, then the solution is in $\mathcal{M}^m$. This shows that $f \in \mathcal{M}^{(w_0)}$. Thus for Hilbert modules which are privileged, we have

$$\sharp\{minimal\ generators\ for\ S^{\mathcal{M}}_w\} = \dim\ \cap_{j=1}^m \ker(M_j - w_{0j})^*.$$

In accordance with the terminology of local spectral theory, see [21], we isolate the following observation.

Corollary 4.3. *Assume that the analytic module* $\mathcal{N} = \operatorname{coker}(A : L^2_a(\Omega)^p \longrightarrow L^2_a(\Omega)^q)$ *is Hausdorff, where* A *and* Ω *are as in the Theorem. Then* $\mathcal{N}$ *is a Hilbert analytic quasi-coherent module, and for every Stein open subset* U *of* $\mathbb{C}^m$*, the associated sheaf model is*

$$\widehat{\mathcal{N}}(U) = \mathcal{O}(U) \hat{\otimes}_{\mathcal{A}(\Omega)} \mathcal{N} =$$
$$\operatorname{coker}(A : \mathcal{H}(U)^p \longrightarrow \mathcal{H}(U)^q) =$$
$$\operatorname{coker}(z - w : \mathcal{O}(U) \hat{\otimes} \mathcal{N}^m \longrightarrow \mathcal{O}(U) \hat{\otimes} \mathcal{N}),$$

where $\mathcal{H}$ *denotes the sheaf model of the Bergman space.*

Remark 4.4. We recall that (see [21])

$$\mathcal{H}(U) = \{f \in \mathcal{O}(U \cap \Omega);\ \|f\|_{2,K} < \infty,\quad K \ \text{ compact in }\ U\}.$$

Since $\mathcal{H}|_\Omega = \mathcal{O}|_\Omega$ we infer that the restriction $\widehat{\mathcal{N}}|_\Omega$ is a coherent sheaf, with finite free resolution

$$0 \to \mathcal{O}|_\Omega^{n_p} \xrightarrow{d_p} \cdots \to \mathcal{O}|_\Omega^{n_1} \xrightarrow{d_1} \mathcal{O}|_\Omega^{n_0} \to \widehat{\mathcal{N}}|_\Omega \to 0.$$

5. Coincidence of sheaf models

Besides the expected relaxations of the main result above, for instance from convex to pseudoconvex domains, a natural problem to consider at this stage is the classification of the analytic Hilbert modules $\mathcal{N} = \operatorname{coker}(A : L^2_a(\Omega)^m \longrightarrow L^2_a(\Omega)^n)$ appearing in the Theorem 1.8 above. This question fits into the framework of quasi-free Hilbert modules introduced in [13]. That the resulting parameter space is wild, there is no doubt, as all Artinian modules M (over the polynomial algebra) supported by a fix point $w_0 \in \Omega$ enter into our discussion. Specifically, we can take

$$\mathcal{N} = \operatorname{coker}((\varphi_1, ..., \varphi_p) : L^2_a(\Omega)^p \longrightarrow L^2_a(\Omega)),$$

where $\varphi_1, ..., \varphi_p$ are polynomials with the only common zero $\{w_0\}$. Then in virtue of Theorem 2.1, the analytic module M is finite dimensional and privileged with respect to the Bergman space $L^2_a(\Omega)$. An algebraic reduction of the classification of all finite co-dimension analytic Hilbert modules of the Bergman space associated of a smooth, strictly convex domain can be found in [31, 32].

In order to better relate the Cowen-Douglas theory to the above framework, we consider together with the map

$$A : L^2_a(\Omega)^p \longrightarrow L^2_a(\Omega)^q$$

whose cokernel was supposed to be Hausdorff, the dual, anti-analytic map

$$A^* : L^2_a(\Omega)^q \longrightarrow L^2_a(\Omega)^p.$$

It is the linear system, in the terminology of Grothendieck [34] or [24], with its associated Hermitian structure induced from the embedding into Bergman space,

$$\ker A^*(z) \subset L^2_a(\Omega)^q, \quad z \in \Omega,$$

which was initially considered in operator theory, see [18].

Traditionally one works with the torsion-free module

$$\mathcal{M} = \operatorname{ran}(A : L^2_a(\Omega)^p \longrightarrow L^2_a(\Omega)^q),$$

rather than the cokernel $\mathcal{N}$ studied in the previous section. A short exact sequence relates the two modules:

$$0 \longrightarrow \mathcal{M} \longrightarrow L^2_a(\Omega)^q \longrightarrow \mathcal{N} \longrightarrow 0.$$

Proposition 5.1. *Assume, in the conditions of Theorem 1.8, that the range $\mathcal{M}$ of the module map A is closed. Then $\mathcal{M}$ is an analytic Hilbert quasi-coherent module, with associated sheaf model*

$$\widehat{\mathcal{M}}(U) = \operatorname{ran}(A : \mathcal{H}(U)^p \longrightarrow \mathcal{H}(U)^q),$$

for every Stein open subset U of $\mathbb{C}^m$.

In particular, for every point $w_0 \in \Omega$, there are finitely many elements $g_1, ..., g_d \in \mathcal{M} \subset L^2_a(\Omega)^q$, such that the stalk $\widehat{\mathcal{M}}_{w_0}$ coincides with the $\mathcal{O}_{w_0}$-module generated in $\mathcal{O}^q_{w_0}$ by $g_1, ..., g_d$.

Proof. The first assertion follows from the main result of the previous section and the yoga of quasi-coherent sheaves. In particular we obtain an exact complex of coherent analytic sheaves

$$0 \longrightarrow \widehat{\mathcal{M}}|_\Omega \longrightarrow \mathcal{O}^n_\Omega \longrightarrow \widehat{\mathcal{N}}|_\Omega \longrightarrow 0.$$

For the proof of the second assertion, recall that the quasi-coherence of $\mathcal{M}$ yields a finite presentation, derived from the associated Koszul complex,

$$\mathcal{O}^m_{w_0} \hat{\otimes} \mathcal{M} \stackrel{z-w}{\longrightarrow} \mathcal{O}_{w_0} \hat{\otimes} \mathcal{M} \longrightarrow \widehat{\mathcal{M}}_{w_0} \longrightarrow 0.$$

By evaluating the presentation at $w = w_0$, we obtain the exact complex

$$\mathcal{M}^m \stackrel{z-w_0}{\longrightarrow} \mathcal{M} \longrightarrow \widehat{\mathcal{M}}(w_0) \longrightarrow 0.$$

Above we denote by $w = (w_1, ..., w_m)$ the m-tuple of local coordinates in the ring $\mathcal{O}_{w_0}$, while $z = (z_1, ..., z_d)$ stands for the d-tuple of coordinate functions in the base space of the Hilbert module $L^2_a(\Omega)$.

By coherence, $\dim \widehat{\mathcal{M}}(w_0) < \infty$, and it remains to choose the k-tuple of elements $g = (g_1, \ldots, g_d)$ as a basis of the ortho-complement of $\operatorname{ran}(z - w_0 : \mathcal{M}^m \longrightarrow \mathcal{M})$. Then the map

$$\mathcal{O}^m_{w_0} \hat{\otimes} (\mathcal{M} \oplus \mathbb{C}^m) \stackrel{z-w;g}{\longrightarrow} \mathcal{O}_{w_0} \hat{\otimes} \mathcal{M}$$

is onto. Consequently, the functions $g_1, ..., g_d$ generate $\widehat{\mathcal{M}}_{w_0}$ as a submodule of $\mathcal{O}^q_{w_0}$. As a matter of fact the same functions will generate $\widehat{\mathcal{M}}_w$ for all points w belonging to a neighborhood of w_0. $\square$

Corollary 5.2. *Under the assumptions of the Proposition, the restriction to Ω of the sheaf model $\widehat{\mathcal{M}} = \widehat{\operatorname{ran} A}$ coincides with the analytic subsheaf of $\mathcal{O}^q$ generated by all functions $f|_\Omega$, $f \in \mathcal{M}$.*

The dual picture emerges easily: let w_0 be a fixed point of Ω, under the assumptions of Theorem 1.8, the map $A_{w_0}(z) := (z_1 - w_{01}, \ldots, z_m - w_{0m}) : \mathcal{M}^m \longrightarrow \mathcal{M}$ has finite dimensional cokernel. Choose a basis $v_1, ..., v_\ell$ of $\ker A_{w_0}(z)^*$ and denote by P_w the orthogonal projection onto $\ker A_w(z)^*$. Then for w belonging to a small enough open neighborhood V of w_0, the elements $P_w(v_1), ..., P_w(v_\ell)$ generate $\ker A_w(z)^*$ as a vector space, but they need not remain linearly independent on V. Never the less, starting with a module $\mathcal{M}$ in $\mathfrak{B}_1(\Omega)$, in the next section, we will provide a construction of a holomorphic Hermitian vector bundle $E_{\mathcal{M}}$ on V.

6. Classification of Hilbert modules and curvature invariants

Let $\mathcal{M}$ be a Hilbert module in $\mathfrak{B}_1(\Omega)$ and $w_0 \in \Omega$ be fixed. The vectors $K^{(i)}_w \in \mathcal{M}$, $1 \leq i \leq d$, produced in part (ii) of the decomposition theorem 1.5 are independent in some small neighborhood, say Ω_0 of w_0. However, while the choice of these vectors is not canonical, in general, we provide below a recipe for finding the vectors $K^{(i)}_w$, $1 \leq i \leq d$, satisfying

$$K(\cdot, w) = g^0_1(w)K^{(1)}_w + \cdots + g^0_n(w)K^{(d)}_w,\ w \in \Omega_0$$

following [7]. We note that $\mathfrak{m}_w\mathcal{M}$ is a closed submodule of $\mathcal{M}$. We assume that we have equality in (1.3) for the module $\mathcal{M}$ at the point $w_0 \in \Omega$, that is, $\operatorname{span}_{\mathbb{C}}\{K^{(i)}_{w_0} : 1 \leq i \leq d\} = \dim \cap_{j=1}^m \ker(M_j - w_{0j})^*$.

Let $D_{(\mathbf{M}-w)^*} = V_{\mathbf{M}}(w)|D_{(\mathbf{M}-w)^*}|$ be the polar decomposition of $D_{(\mathbf{M}-w)^*}$, where $|D_{(\mathbf{M}-w)^*}|$ is the positive square root of the operator $\big(D_{(\mathbf{M}-w)^*}\big)^* D_{(\mathbf{M}-w)^*}$ and $V_{\mathbf{M}}(w)$ is the partial isometry mapping $\big(\ker D_{(\mathbf{M}-w)^*}\big)^\perp$ isometrically onto $\operatorname{ran} D_{(\mathbf{M}-w)^*}$. Let $Q_{\mathbf{M}}(w)$ be the positive operator:

$$Q_{\mathbf{M}}(w)\big|_{\ker D_{(\mathbf{M}-w)^*}} = 0 \text{ and } Q_{\mathbf{M}}(w)\big|_{(\ker D_{(\mathbf{M}-w)^*})^\perp} = \big(|D_{(\mathbf{M}-w)^*}|\big|_{(\ker D_{(\mathbf{M}-w)^*})^\perp}\big)^{-1}.$$

Let $R_{\mathbf{M}}(w) : \mathcal{M} \oplus \cdots \oplus \mathcal{M} \to \mathcal{M}$ be the operator $R_{\mathbf{M}}(w) = Q_{\mathbf{M}}(w)V_{\mathbf{M}}(w)^*$. The two equations, involving the operator $D_{(\mathbf{M}-w)^*}$, stated below are analogous to the semi-Fredholmness property of a single operator (cf. [5, Proposition 1.11]):

$$R_{\mathbf{M}}(w)D_{(\mathbf{M}-w)^*} = I - P_{\ker D_{(\mathbf{M}-w)^*}} \tag{6.1}$$

$$D_{(\mathbf{M}-w)^*}R_{\mathbf{M}}(w) = P_{\operatorname{ran} D_{(\mathbf{M}-w)^*}}, \tag{6.2}$$

where $P_{\ker D_{(\mathbf{M}-w)^*}}, P_{\operatorname{ran} D_{(\mathbf{M}-w)^*}}$ are orthogonal projection onto $\ker D_{(\mathbf{M}-w)^*}$ and $\operatorname{ran} D_{(\mathbf{M}-w)^*}$ respectively. Consider the operator

$$P(\bar{w}, \bar{w}_0) = I - \{I - R_{\mathbf{M}}(w_0)D_{\bar{w}-\bar{w}_0}\}^{-1}R_{\mathbf{M}}(w_0)D_{(\mathbf{M}-w)^*},\ w \in B(w_0; \| R(w_0) \|^{-1}),$$

where $B(w_0; \| R(w_0) \|^{-1})$ is the ball of radius $\| R(w_0) \|^{-1}$ around w_0. Using the equations (6.1) and (6.2) given above, we write

$$P(\bar{w}, \bar{w}_0) = \{I - R_{\mathbf{M}}(w_0)D_{\bar{w}-\bar{w}_0}\}^{-1}P_{\ker D_{(\mathbf{M}-w)^*}}, \tag{6.3}$$

where $D_{\bar{w}-\bar{w}_0}f = ((\bar{w}_1 - \bar{w}_{01})f_1, \ldots, (\bar{w}_m - \bar{w}_{0m})f_m)$. From definition of $P(\bar{w}, \bar{w}_0)$ it follows that $P(\bar{w}, \bar{w}_0)P_{\ker D_{(\mathbf{M}-w)^*}} = P_{\ker D_{(\mathbf{M}-w)^*}}$ which implies $\ker D_{(\mathbf{M}-w)^*} \subset \operatorname{ran} P(\bar{w}, \bar{w}_0)$ for $w \in \Delta(w_0; \varepsilon)$.

Consequently $K(\cdot,w) \in \operatorname{ran} P(\bar w, \bar w_0)$ and therefore

$$K(\cdot,w) = \sum_{i=1}^{d} \overline{a_i(w)} P(\bar w, \bar w_0) K_{w_0}^{(i)},$$

for some complex valued functions $a_1, \ldots, a_d$ on $\Delta(w_0;\varepsilon)$. We will show that the functions a_i, $1 \le i \le d$, are holomorphic and their germs form a minimal set of generators for $S_{w_0}^{\mathcal{M}}$. Now

$$R_{\mathbf{M}}(w_0) D_{\bar w - \bar w_0} K(\cdot,w) = R_{\mathbf{M}}(w_0) D_{(\mathbf{M}-w_0)^*} K(\cdot,w) = (I - P_{\ker D_{(\mathbf{M}-w_0)^*}}) K(\cdot,w).$$

Hence we have,

$$\{I - R_{\mathbf{M}}(w_0) D_{\bar w - \bar w_0}\} K(\cdot,w) = P_{\ker D_{(\mathbf{M}-w_0)^*}} K(\cdot,w).$$

Since $K(\cdot,w) \in \operatorname{ran} P(\bar w, \bar w_0)$, we also have

$$P(\bar w, \bar w_0)^{-1} K(\cdot,w) = P_{\ker D_{(\mathbf{M}-w_0)^*}} K(\cdot,w).$$

Let $v_1, \ldots, v_d$ be the orthonormal basis for $\ker D_{(\mathbf{M}-w_0)^*}$. Let $g_1, \ldots, g_d$ denotes the minimal set of generators for the stalk at $\mathcal{S}_{w_0}^{\mathcal{M}}$. Then there exist a neighborhood U, small enough such that $v_j = \sum_{i=1}^{d} g_i f_i^j$, $1 \le j \le d$, and for some holomorphic functions $f_i^j, 1 \le i,j \le d$, on U. We then have

$$\begin{aligned}
P(\bar w, \bar w_0)^{-1} K(\cdot,w) &= P_{\ker D_{(\mathbf{M}-w_0)^*}} K(\cdot,w) = \sum_{j=1}^{d} \langle K(\cdot,w), v_j\rangle v_j \\
&= \sum_{j=1}^{d} \langle K(\cdot,w), \sum_{i=1}^{d} g_i f_i^j\rangle v_j = \sum_{i=1}^{d}\sum_{j=1}^{d} \overline{g_i(w) f_i^j(w)} v_j \\
&= \sum_{i=1}^{d} \overline{g_i(w)} \{\sum_{j=1}^{d} \overline{f_i^j(w)} v_j\}.
\end{aligned}$$

So $K(z,w) = \sum_{i=1}^{d} \overline{g_i(w)} \{\sum_{j=1}^{d} \overline{f_i^j(w)} P(\bar w, \bar w_0) v_j(z)\}$. Let

$$\tilde K_w^{(i)} = \sum_{j=1}^{d} \overline{f_i^j(w)} P(\bar w, \bar w_0) v_j.$$

Since the vectors $K_{w_0}^{(i)}$, $1 \le i \le d$ are uniquely determined as long as $g_1, \ldots, g_d$ are fixed and $P(\bar w_0, \bar w_0) = P_{\ker D_{(\mathbf{M}-w_0)^*}}$, it follows that $K_{w_0}^{(i)} = \tilde K_{w_0}^{(i)} = \sum_{j=1}^{d} \overline{f_i^j(w_0)} v_j$, $1 \le i \le d$. Therefore, the $d \times d$ matrix $(\overline{f_i^j(w_0)})_{i,j=1}^d$ has a non-zero determinant. As *Det* $(\overline{f_i^j(w)})_{i,j=1}^d$ is an anti-holomorphic function, there exist a neighbourhood of w_0, say $\Delta(w_0;\varepsilon), \varepsilon > 0$, such that *Det* $(\overline{f_i^j(w)})_{i,j=1}^d \neq 0$ for all $w \in \Delta(w_0;\varepsilon)$. The set of vectors $\{P(\bar w, \bar w_0) v_j\}_{j=1}^n$ is linearly independent since $P(\bar w, \bar w_0)$ is injective on $\ker D_{(\mathbf{M}-w_0)^*}$. Let $(\alpha_{ij})_{i,j=1}^d = \{(\overline{f_i^j(w_0)})_{i,j=1}^d\}^{-1}$, in consequence, $v_j = \sum_{l=1}^{d} \alpha_{jl} K_{w_0}^{(l)}$. We then have

$$\begin{aligned}
K(\cdot,w) &= \sum_{i=1}^{d} \overline{g_i(w)} \{\sum_{j=1}^{d} \overline{f_i^j(w)} P(\bar w, \bar w_0) (\sum_{l=1}^{d} \alpha_{jl} K_{w_0}^{(l)})\} \\
&= \sum_{l=1}^{d} \{\sum_{i,j=1}^{d} \overline{g_i(w)}\,\overline{f_i^j(w)} \alpha_{jl}\} P(\bar w, \bar w_0) K_{w_0}^{(l)}).
\end{aligned}$$

Since the matrices $(\overline{f_i^j(w)})_{i,j=1}^d$ and $(\alpha_{ij})_{i,j=1}^d$ are invertible, the functions

$$a_l(z) = \sum_{i,j=1}^{d} g_i(z) f_i^j(z)\alpha_{jl},\ 1 \le l \le d,$$

form a minimal set of generators for the stalk $S_{w_0}^{\mathcal{M}}$ and hence we have the canonical decomposition,

$$K(\cdot, w) = \sum_{i=1}^{d} \overline{a_i(w)} P(\bar{w}, \bar{w}_0) K_{w_0}^{(i)}.$$

Let $\mathcal{P}_w = \operatorname{ran} P(\bar{w}, \bar{w}_0) P_{\ker D_{(\mathbf{M}-w_0)^*}}$ for $w \in B(w_0; \| R_{\mathbf{M}}(w_0) \|^{-1})$. Since $P(\bar{w}, \bar{w}_0)$ restricted to the $\ker D_{(\mathbf{M}-w_0)^*}$ is one-one, $\dim \mathcal{P}_w$ is constant for $w \in B(w_0; \| R_{\mathbf{M}}(w_0) \|^{-1})$. Thus to prove Lemma 1.9, we will show that $\mathcal{P}_w = \ker \mathbb{P}_0 D_{(\mathbf{M}-w)^*}$.

Proof of Lemma 1.9. From [7, pp. 453], it follows that $\mathbb{P}_0 D_{(\mathbf{M}-w)^*} P(\bar{w}, \bar{w}_0) = 0$. So, $\mathcal{P}_w \subseteq \ker \mathbb{P}_0 D_{(\mathbf{M}-w)^*}$. Using (6.1) and (6.2), we can write

$$\begin{aligned}
\mathbb{P}_0 D_{(\mathbf{M}-w)^*} &= D_{(\mathbf{M}-w_0)^*} R_{\mathbf{M}}(w_0)\{D_{(\mathbf{M}-w_0)^*} - D_{(\bar{w}-\bar{w}_0)}\} \\
&= D_{(\mathbf{M}-w_0)^*}\{I - P_{\ker D_{(\mathbf{M}-w_0)^*}} - R_{\mathbf{M}}(w_0) D_{(\bar{w}-\bar{w}_0)}\} \\
&= D_{(\mathbf{M}-w_0)^*}\{I - R_{\mathbf{M}}(w_0) D_{(\bar{w}-\bar{w}_0)}\}.
\end{aligned}$$

Since $\{I - R_{\mathbf{M}}(w_0) D_{(\bar{w}-\bar{w}_0)}\}$ is invertible for $w \in B(w_0; \| R_{\mathbf{M}}(w_0) \|^{-1})$, we have

$$\dim \mathcal{P}_w = \dim D_{(\mathbf{M}-w_0)^*} \ge \dim \ker \mathbb{P}_0 D_{(\mathbf{M}-w)^*}.$$

This completes the proof. $\square$

From the construction of the operator $P(\bar{w}, \bar{w}_0)$, it follows that, the association $w \to \mathcal{P}_w$ forms a Hermitian holomorphic vector bundle of rank m over $\Omega_0^* = \{\bar{z} : z \in \Omega_0\}$ where $\Omega_0 = B(w_0; \| R_{\mathbf{M}}(w_0) \|^{-1})$. Let $\mathcal{P}$ denote this Hermitian holomorphic vector bundle.

Proof of Theorem1.10. Since $\mathcal{M}$ and $\tilde{\mathcal{M}}$ are equivalent Hilbert modules, there exist a unitary $U : \mathcal{M} \to \tilde{\mathcal{M}}$ intertwining the adjoint of the module multiplication, that is, $UM_j{}^* = \tilde{M}_j^* U$, $1 \le j \le m$. Here $\tilde{M}_j$ denotes the multiplication by co-ordinate function $z_j, 1 \le j \le m$ on $\tilde{\mathcal{M}}$. It is enough to show that $UP(\bar{w}, \bar{w}_0) = \tilde{P}(\bar{w}, \bar{w}_0)U$ for $w \in B(w_0; \| R_{\mathbf{M}}(w_0) \|^{-1})$.

Let $| D_{\mathbf{M}^*} |= \{\sum_{j=1}^m M_j M_j{}^*\}^{\frac{1}{2}}$, that is, the positive square root of $(D_{\mathbf{M}^*})^* D_{\mathbf{M}^*}$. We have

$$\sum_{j=1}^{m} M_j M_j{}^* = U^*(\sum_{j=1}^{m} \tilde{M}_j \tilde{M}_j^*)U = (U^* \mid D_{\tilde{\mathbf{M}}^*} \mid U)^2.$$

Clearly, $| D_{\mathbf{M}^*} |= U^* \mid D_{\tilde{\mathbf{M}}^*} \mid U$. Similar calculation gives $| D_{(\mathbf{M}-w_0)^*} |= U^* \mid D_{(\tilde{\mathbf{M}}-w_0)^*} \mid U$. Let $P_i : \mathcal{M} \oplus \mathcal{M} \cdots \oplus \mathcal{M}($ m times$) \longrightarrow \mathcal{M}$ be the orthogonal projection on the i-th component. In this notation, we have $P_j D_{\mathbf{M}^*} = M_j{}^*, 1 \le j \le m$. Then,

$$\begin{aligned}
\tilde{P}_j D_{(\tilde{\mathbf{M}}-w_0)^*} &= UP_j D_{(\mathbf{M}-w_0)^*} U^* = UP_j V_{\mathbf{M}}(w_0) U^* U \mid D_{(\mathbf{M}-w_0)^*} \mid U^* \\
&= UP_j V_{\mathbf{M}}(w_0) U^* \mid D_{(\tilde{\mathbf{M}}-w_0)^*} \mid .
\end{aligned}$$

But $\tilde{P}_j D_{(\tilde{\mathbf{M}}-w_0)^*} = \tilde{P}_j V_{\tilde{\mathbf{M}}}(w_0) \mid D_{(\tilde{M}-w_0)^*} \mid$. The uniqueness of the polar decomposition implies that $\tilde{P}_j V_{\tilde{\mathbf{M}}}(w_0) = UP_j V_{\mathbf{M}}(w_0) U^*$, $1 \le j \le m$. It follows that $Q_{\tilde{M}}(w_0) = UQ_M(w_0)U^*$.

Note that $P_j{}^* : \mathcal{M} \longrightarrow \mathcal{M} \oplus \cdots \oplus \mathcal{M}$ is given by $P_j{}^* h = (0, \ldots, h, \ldots, 0)$, $h \in \mathcal{M}, 1 \le j \le m$. So we have, $V_{\tilde{\mathbf{M}}}(w_0)^* \tilde{P}_j^* = UV_{\mathbf{M}}(w_0)^* P_j{}^* U^*$, $1 \le j \le m$. Let $\tilde{D}_{\bar{w}} : \mathcal{M} \longrightarrow \mathcal{M} \oplus \cdots \oplus \mathcal{M}$ be the operator: $\tilde{D}_{\bar{w}}f = (\bar{w}_1 f, \ldots, \bar{w}_m f)$, $f \in \tilde{\mathcal{M}}$. Clearly, $\tilde{D}_{\bar{w}} = UD_{\bar{w}}U^*$, that is, $U^*\tilde{P}_j\tilde{D}_{\bar{w}} = P_jD_{\bar{w}}U^*$, $1 \leq j \leq m$. Finally,

$$\begin{aligned}
&R_{\tilde{\mathbf{M}}}(w_0)\tilde{D}_{\bar{w}-\bar{w}_0}\\
&\quad= Q_{\tilde{\mathbf{M}}}(w_0)V_{\tilde{\mathbf{M}}}(w_0)^*\tilde{D}_{\bar{w}-\bar{w}_0} \;=\; Q_{\tilde{\mathbf{M}}}(w_0)V_{\tilde{\mathbf{M}}}(w_0)^*(\tilde{P}_1\tilde{D}_{\bar{w}-\bar{w}_0}, \ldots, \tilde{P}_m\tilde{D}_{\bar{w}-\bar{w}_0})\\
&\quad= Q_{\tilde{\mathbf{M}}}(w_0)V_{\tilde{\mathbf{M}}}(w_0)^*(\sum_{j=1}^m \tilde{P}_j^*\tilde{P}_j\tilde{D}_{\bar{w}-\bar{w}_0})\\
&\quad= Q_{\tilde{\mathbf{M}}}(w_0)UV_{\mathbf{M}}(w_0)^*(\sum_{j=1}^m P_j{}^*U^*\tilde{P}_j\tilde{D}_{\bar{w}-\bar{w}_0})\\
&\quad= UQ_{\mathbf{M}}(w_0)V_{\mathbf{M}}(w_0)^*(\sum_{j=1}^m P_j{}^*P_jD_{\bar{w}-\bar{w}_0}U^*) \;=\; UQ_{\mathbf{M}}(w_0)V_{\mathbf{M}}(w_0)^*D_{\bar{w}-\bar{w}_0}U^*\\
&\quad= UR_{\mathbf{M}}(w_0)D_{\bar{w}-\bar{w}_0}U^*.
\end{aligned}$$

Hence $\{R_{\tilde{\mathbf{M}}}(w_0)\tilde{D}_{\bar{w}-\bar{w}_0}\}^k = U\{R_{\mathbf{M}}(w_0)D_{\bar{w}-\bar{w}_0}\}^kU^*$ for all $k \in \mathbb{N}$. From (6.3), $P(\bar{w}, \bar{w}_0) = \sum_{k=0}^\infty \{R_{\mathbf{M}}(w_0)D_{\bar{w}-\bar{w}_0}\}^kP_{\ker D_{(\mathbf{M}-w_0)^*}}$. Also as U maps $\ker D_{(\mathbf{M}-w)^*}$ onto $\ker D_{(\tilde{M}-w)^*}$ for each w, we have in particular, $UP_{\ker D_{(\mathbf{M}-w_0)^*}} = P_{\ker D_{(\tilde{M}-w_0)^*}}U$. Therefore,

$$\begin{aligned}
&UP(\bar{w}, \bar{w}_0)\\
&\quad= \sum_{k=0}^\infty U\{R_{\mathbf{M}}(w_0)D_{\bar{w}-\bar{w}_0}\}^kP_{\ker D_{(\mathbf{M}-w_0)^*}} \;=\; \sum_{k=0}^\infty \{R_{\tilde{\mathbf{M}}}(w_0)\tilde{D}_{\bar{w}-\bar{w}_0}\}^kUP_{\ker D_{(\mathbf{M}-w_0)^*}}\\
&\quad= \sum_{k=0}^\infty \{R_{\tilde{\mathbf{M}}}(w_0)\tilde{D}_{\bar{w}-\bar{w}_0}\}^kP_{\ker D_{(\tilde{M}-w_0)^*}}U \;=\; \tilde{P}(\bar{w}, \bar{w}_0)U,
\end{aligned}$$

for $w \in B(w_0; \| R_{\mathbf{M}}(w_0) \|^{-1})$. □

Remark 6.1. For any commuting m-tuple $D_{\mathbf{T}} = (T_1, \ldots, T_m)$ of operator on $\mathcal{H}$, the construction given above, of the Hermitian holomorphic vector bundle, provides a unitary invariant, assuming only that $\mathrm{ran}D_{\mathbf{T}-w}$ is closed for w in $\Omega \subseteq \mathbb{C}^m$. Consequently, the class of this Hermitian holomorphic vector bundle is an invariant for any semi-Fredholm Hilbert module over $\mathbb{C}[\underline{z}]$.

7. Examples

7.1. The (λ, μ) examples. Let $\mathcal{M}$ and $\tilde{\mathcal{M}}$ be two Hilbert modules in $\mathrm{B}_1(\Omega)$ and $\mathcal{I}$, $\mathcal{J}$ be two ideals in $\mathbb{C}[\underline{z}]$. Let $\mathcal{M}_{\mathcal{I}} := [\mathcal{I}] \subseteq \mathcal{M}$ (resp. $\tilde{\mathcal{M}}_{\mathcal{J}} := [\mathcal{J}] \subset \tilde{\mathcal{M}}$) denote the closure of $\mathcal{I}$ in $\mathcal{M}$ (resp. closure of $\mathcal{J}$ in $\tilde{\mathcal{M}}$). Also we let $\dim V(\mathcal{I}), \dim V(\mathcal{J}) \leq m-2$. It is then not hard to see that $\mathcal{M}_{\mathcal{I}}$ and $\tilde{\mathcal{M}}_{\mathcal{J}}$ are equivalent if and only if $\mathcal{I} = \mathcal{J}$ following the argument in the proof [2, Theorem 2.10] and using the characteristic space theory of [4, Chapter 2]. Assume $\mathcal{M}$ and $\tilde{\mathcal{M}}$ are minimal extensions of the two modules $\mathcal{M}_{\mathcal{I}}$ and $\tilde{\mathcal{M}}_{\mathcal{I}}$ respectively and that $\mathcal{M}_{\mathcal{I}}$ is equivalent to $\tilde{\mathcal{M}}_{\mathcal{I}}$. We ask if these assumptions force the extensions $\mathcal{M}$ and $\tilde{\mathcal{M}}$ to be equivalent. The answer for a class of examples is given below.

For $\lambda, \mu > 0$, let $H^{(\lambda,\mu)}(\mathbb{D}^2)$ be the reproducing kernel Hilbert space on the bi-disc determined by the positive definite kernel

$$K^{(\lambda,\mu)}(z, w) = \frac{1}{(1 - z_1\bar{w}_1)^\lambda(1 - z_2\bar{w}_2)^\mu},\; z, w \in \mathbb{D}^2.$$

As is well-known, $H^{(\lambda,\mu)}(\mathbb{D}^2)$ is in $\mathrm{B}_1(\mathbb{D}^2)$. Let I be the maximal ideal in $\mathcal{C}_2$ of polynomials vanishing at $(0, 0)$. Let $H_0^{(\lambda,\mu)}(\mathbb{D}^2) := [I]$. For any other pair of positive numbers λ', μ', we let

$H_0^{(\lambda',\mu')}(\mathbb{D}^2)$ denote the closure of I in the reproducing kernel Hilbert space $H^{(\lambda',\mu')}(\mathbb{D}^2)$. Let $K^{(\lambda',\mu')}$ denote the corresponding reproducing kernel. The modules $H^{(\lambda,\mu)}(\mathbb{D}^2)$ and $H^{(\lambda',\mu')}(\mathbb{D}^2)$ are in $\mathrm{B}_1(\mathbb{D}^2 \setminus \{(0,0)\})$ but not in $\mathrm{B}_1(\mathbb{D}^2)$. So, there is no easy computation to determine when they are equivalent. We compute the curvature, at $(0,0)$, of the holomorphic Hermitian bundle $\mathcal{P}$ and $\tilde{\mathcal{P}}$ of rank 2 corresponding to the modules $H_0^{(\lambda,\mu)}(\mathbb{D}^2)$ and $H_0^{(\lambda',\mu')}(\mathbb{D}^2)$ respectively. The calculation of the curvature show that if these modules are equivalent then $\lambda = \lambda'$ and $\mu = \mu'$, that is, the extensions $H^{(\lambda,\mu)}(\mathbb{D}^2)$ and $H^{(\lambda',\mu')}(\mathbb{D}^2)$ are then equivalent.

Since $H_0^{(\lambda,\mu)}(\mathbb{D}^2) := \{f \in H^{(\lambda,\mu)}(\mathbb{D}^2) : f(0,0) = 0\}$, the corresponding reproducing kernel $K_0^{(\lambda,\mu)}$ is given by the formula

$$K_0^{(\lambda,\mu)}(z,w) = \frac{1}{(1-z_1\bar{w}_1)^\lambda(1-z_2\bar{w}_2)^\mu} - 1,\ z,w \in \mathbb{D}^2.$$

The set $\{z_1^m z_2^n : m,n \geq 0, (m,n) \neq (0,0)\}$ forms an orthogonal basis for $H_0^{(\lambda,\mu)}(\mathbb{D}^2)$. Also $\langle z_1^l z_2^k, M_1^* z_1^{m+1}\rangle = \langle z_1^{l+1} z_2^k, z_1^{m+1}\rangle = 0$, unless $l = m, k = 0$ and $m > 0$. In consequence,

$$\langle z_1^m, M_1^* z_1^{m+1}\rangle = \langle z_1^{m+1}, z_1^{m+1}\rangle = \frac{1}{(-1)^{m+1}\binom{-\lambda}{m+1}} = \frac{(-1)^m\binom{-\lambda}{m}}{(-1)^{m+1}\binom{-\lambda}{m+1}}\langle z_1^m, z_1^m\rangle.$$

Then

$$\langle z_1^l z_2^k, M_1^* z_1^{m+1} - \frac{m+1}{\lambda+m} z_1^m\rangle = 0 \text{ for all } l,k \geq 0, (l,k) \neq (0,0),$$

where $\binom{-\lambda}{m} = (-1)^m \frac{\lambda(\lambda+1)\ldots(\lambda+m-1)}{m!}$. Now, $\langle z_1^l z_2^k, M_1^* z_1\rangle = \langle z_1^{l+1} z_2^k, z_1\rangle = 0$, $l,k \geq 0$ and $(l,k) \neq (0,0)$. Therefore, we have

$$M_1^* z_1^{m+1} = \begin{cases} \frac{m+1}{\lambda+m}\, z_1^m & m > 0 \\ 0 & m = 0. \end{cases}$$

Similarly,

$$M_2^* z_2^{n+1} = \begin{cases} \frac{n+1}{\lambda+n}\, z_1^n & n > 0 \\ 0 & n = 0. \end{cases}$$

We easily verify that $\langle z_1^l z_2^k, M_2^* z_1^{m+1}\rangle = \langle z_1^l z_2^{k+1}, z_1^{m+1}\rangle = 0$. Hence $M_2^* z_1^{m+1} = 0 = M_1^* z_2^{n+1}$ for $m,n \geq 0$. Finally, calculations similar to the one given above, show that

$$M_1^* z_1^{m+1} z_2^{n+1} = \frac{m+1}{\lambda+m} z_1^m z_2^{n+1} \text{ and } M_2^* z_1^{m+1} z_2^{n+1} = \frac{n+1}{\mu+n} z_1^{m+1} z_2^n, m.n \geq 0$$

Therefore we have

$$(M_1M_1^* + M_2M_2^*) : \begin{cases} z_1^{m+1} \longmapsto \frac{m+1}{\lambda+m} z_1^{m+1}, & \text{for } m > 0; \\ z_2^{n+1} \longmapsto \frac{n+1}{\mu+n} z_2^{n+1}, & \text{for } n > 0; \\ z_1^{m+1} z_2^{n+1} \longmapsto (\frac{m+1}{\lambda+m} + \frac{n+1}{\mu+n}) z_1^{m+1} z_2^{n+1}, & \text{for } m,n \geq 0; \\ z_1, z_2 \longmapsto 0. & \end{cases}$$

Also, since $D_{\mathbf{M}^*} f = (M_1^* f, M_2^* f)$, we have

$$D_{\mathbf{M}^*} : \begin{cases} z_1^{m+1} \longmapsto (\frac{m+1}{\lambda+m} z_1^m, 0), & \text{for } m > 0; \\ z_2^{n+1} \longmapsto (0, \frac{n+1}{\mu+n} z_2^n), & \text{for } n > 0; \\ z_1^{m+1} z_2^{n+1} \longmapsto (\frac{m+1}{\lambda+m} z_1^m z_2^{n+1}, \frac{n+1}{\mu+n} z_1^{m+1} z_2^n), & \text{for } m,n \geq 0; \\ z_1, z_2 \longmapsto (0,0). & \end{cases}$$

It is easy to calculate $V_{\mathbf{M}}(0)$ and $Q_{\mathbf{M}}(0)$ and show that

$$V_{\mathbf{M}}(0):\begin{cases} z_1^{m+1}\longmapsto \sqrt{\frac{m+1}{\lambda+m}}(z_1^m,0), & \text{for } m>0;\\ z_2^{n+1}\longmapsto \sqrt{\frac{n+1}{\mu+n}}(0,z_2^n), & \text{for } n>0;\\ z_1^{m+1}z_2^{n+1}\longmapsto \frac{1}{\sqrt{\frac{m+1}{\lambda+m}+\frac{n+1}{\mu+n}}}(\frac{m+1}{\lambda+m}z_1^m z_2^{n+1},\frac{n+1}{\mu+n}z_1^{m+1}z_2^n), & \text{for } m,n\geq 0;\\ z_1,z_2\longmapsto (0,0), \end{cases}$$

while

$$Q_{\mathbf{M}}(0):\begin{cases} z_1^{m+1}\longmapsto \frac{1}{\sqrt{\frac{m+1}{\lambda+m}}}z_1^{m+1}, & \text{for } m>0;\\ z_2^{n+1}\longmapsto \frac{1}{\sqrt{\frac{n+1}{\mu+n}}}z_2^{n+1}, & \text{for } n>0;\\ z_1^{m+1}z_2^{n+1}\longmapsto \frac{1}{\sqrt{\frac{m+1}{\lambda+m}+\frac{n+1}{\mu+n}}}z_1^{m+1}z_2^{n+1}, & \text{for } m,n\geq 0;\\ z_1,z_2\longmapsto 0. \end{cases}$$

Now for $w\in\Delta(0,\varepsilon)^*$,

$$P(\bar w,0)=(I-R_{\mathbf{M}}(0)D_{\bar w})^{-1}P_{\ker D_{M^*}}=\sum_{n=0}^{\infty}(R_{\mathbf{M}}(0)D_{\bar w})^nP_{\ker D_{M^*}},$$

where $R_{\mathbf{M}}(0)=Q_{\mathbf{M}}(0)V_{\mathbf{M}}(0)^*$. The vectors z_1 and z_2 forms a basis for $\ker D_{\mathbf{M}^*}$ and therefore define a holomorphic frame: $\big(P(\bar w,0)z_1,P(\bar w,0)z_2\big)$. Recall that $P(\bar w,0)z_1=\sum_{n=0}^{\infty}(R_{\mathbf{M}}(0)D_{\bar w})^nz_1$ and $P(\bar w,0)z_2=\sum_{n=0}^{\infty}(R_{\mathbf{M}}(0)D_{\bar w})^nz_2$. To describe these explicitly, we calculate $(R_{\mathbf{M}}(0)D_{\bar w})z_1$ and $(R_{\mathbf{M}}(0)D_{\bar w})z_2$:

$$\begin{aligned}(R_{\mathbf{M}}(0)D_{\bar w})z_1 &= R_{\mathbf{M}}(0)(\bar w_1,z_1,\bar w_2z_2)\\ &= \bar w_1R_{\mathbf{M}}(0)(z_1,0)+\bar w_2R_{\mathbf{M}}(0)(0,z_2)\\ &= \bar w_1Q_{\mathbf{M}}(0)V_{\mathbf{M}}(0)^*(z_1,0)+\bar w_2Q_{\mathbf{M}}(0)V_{\mathbf{M}}(0)^*(0,z_2).\end{aligned}$$

We see that

$$V_{\mathbf{M}}(0)^*(z_1,0)=\sum_{l,k\geq 0,\,(l,k)\neq(0,0)}\langle V_{\mathbf{M}}(0)^*(z_1,0),\frac{z_1^lz_2^k}{\|z_1^lz_2^k\|}\rangle\frac{z_1^lz_2^k}{\|z_1^lz_2^k\|}.$$

Therefore,

$$\langle V_{\mathbf{M}}(0)^*(z_1,0),z_1^lz_2^k\rangle=\langle(z_1,0),V_{\mathbf{M}}(0)(z_1^lz_2^k)\rangle,\ l,k\geq 0,(l,k)\neq(0,0).$$

From the explicit form of $V_{\mathbf{M}}(0)$, it is clear that the inner product given above is 0 unless $l=2$, $k=0$. For $l=2$, $k=0$, we have

$$\langle(z_1,0),V_{\mathbf{M}}(0)z_1^2\rangle=\sqrt{\frac{2}{\lambda+1}}\|z_1\|^2=\sqrt{\frac{2}{\lambda+1}}\frac{1}{\lambda}.$$

Hence

$$V_{\mathbf{M}}(0)^*(z_1,0)=\sqrt{\frac{2}{\lambda+1}}\frac{1}{\lambda}\frac{z_1^2}{\|z_1^2\|^2}=\sqrt{\frac{2}{\lambda+1}}\frac{1}{\lambda}\frac{\lambda(\lambda+1)}{2}z_1^2=\sqrt{\frac{\lambda+1}{2}}z_1^2.$$

Again, to calculate $V_{\mathbf{M}}(0)^*(0,z_1)$, we note that $\langle V_{\mathbf{M}}(0)^*(0,z_1), z_1^l z_2^k\rangle$ is 0 unless $l=1$, $m=1$. For $l=1$, $m=1$, we have

$$\begin{aligned}\langle V_{\mathbf{M}}(0)^*(0,z_1), z_1z_2\rangle &= \langle (0,z_1), V_{\mathbf{M}}(0)z_1z_2\rangle\\ &= \langle \frac{1}{\sqrt{\frac{1}{\lambda}+\frac{1}{\mu}}}(\frac{1}{\lambda}z_2, \frac{1}{\mu}z_1),(0,z_1)\rangle\\ &= \frac{1}{\sqrt{\frac{1}{\lambda}+\frac{1}{\mu}}}\frac{1}{\mu}\parallel z_1\parallel^2 = \frac{1}{\sqrt{\frac{1}{\lambda}+\frac{1}{\mu}}}\frac{1}{\lambda\mu}.\end{aligned}$$

Thus

$$V_{\mathbf{M}}(0)^*(0,z_1) = \langle V_{\mathbf{M}}(0)^*(0,z_1), z_1z_2\rangle\frac{z_1z_2}{\parallel z_1z_2\parallel^2} = \frac{1}{\sqrt{\frac{1}{\lambda}+\frac{1}{\mu}}}z_1z_2.$$

Since

$$\begin{aligned}Q_{\mathbf{M}}(0)z_1^2 &= \sqrt{\frac{\lambda+1}{2}}z_1^2,\\ Q_{\mathbf{M}}(0)z_1z_2 &= \frac{1}{\sqrt{\frac{1}{\lambda}+\frac{1}{\mu}}}z_1z_2,\\ Q_{\mathbf{M}}(0)z_2^2 &= \sqrt{\frac{\mu+1}{2}}z_2^2,\end{aligned}$$

it follows that

$$R_{\mathbf{M}}(0)D_{\bar{w}}z_1 = \bar{w}_1\frac{\lambda+1}{2}z_1^2 + \bar{w}_2\frac{\lambda\mu}{\lambda+\mu}z_1z_2.$$

Similarly, we obtain the formula

$$R_{\mathbf{M}}(0)D_{\bar{w}}z_2 = \bar{w}_1\frac{\lambda\mu}{\lambda+\mu}z_1z_2 + \bar{w}_2\frac{\mu+1}{2}z_2^2.$$

We claim that

$$\langle (R_{\mathbf{M}}(0)D_{\bar{w}})^m z_i, (R_{\mathbf{M}}(0)D_{\bar{w}})^n z_j\rangle = 0 \text{ for all } m\neq n \text{ and } i,j=1,2. \tag{7.1}$$

This makes the calculation of

$$h(w,w) = \big(\!\big(\langle P(\bar{w},0)z_i, P(\bar{w},0)z_j\rangle\big)\!\big)_{1\le i,j\le 2},\ w\in U\subset\mathbb{D}^2,$$

which is the Hermitian metric for the vector bundle $\mathcal{P}$, on some small open set $U\subseteq\mathbb{D}^2$ around $(0,0)$, corresponding to the module $H_0^{(\lambda,\mu)}(\mathbb{D}^2)$, somewhat easier.

We will prove the claim by showing that $(R_{\mathbf{M}}(0)D_{\bar{w}})^n z_i$ consists of terms of degree $n+1$. For this, it is enough to calculate $V_{\mathbf{M}}(0)^*(z_1^l z_2^k, 0)$ and $V_{\mathbf{M}}(0)^*(0, z_1^l z_2^k)$ for different l, $k\ge 0$ such that $(l,k)\neq(0,0)$. Calculations similar to that of $V_{\mathbf{M}}(0)^*$ show that

$$\begin{aligned}&V_{\mathbf{M}}(0)^*(z_1^m,0) = \sqrt{\frac{\lambda+m}{m+1}}z_1^{m+1}, V_{\mathbf{M}}(0)^*(0,z_2^n) = \sqrt{\frac{\mu+n}{n+1}}z_2^{n+1} \text{ and,}\\ &V_{\mathbf{M}}(0)^*(z_1^m z_2^{n+1},0) = V_{\mathbf{M}}(0)^*(0,z_1^{m+1}z_2^n) = \frac{1}{\sqrt{\frac{m+1}{\mu+n}+\frac{n+1}{\mu+n}}}z_1^{m+1}z_2^{n+1}.\end{aligned}$$

Recall that $(R_{\mathbf{M}}(0)D_{\bar{w}})z_i$ is of degree 2. From the equations given above, inductively, we see that $(R_{\mathbf{M}}(0)D_{\bar{w}})^n z_i$ is of degree $n+1$. Since monomials are orthogonal in $H^{(\lambda,\mu)}(\mathbb{D}^2)$, the proof of claim

(7.1) is complete. We then have

$$P(\bar{w},0)z_1 = z_1 + \bar{w}_1\frac{\lambda+1}{2}z_1^2 + \bar{w}_2\frac{\lambda\mu}{\lambda+\mu}z_1z_2 + \sum_{n=2}^{\infty}(R_{\mathbf{M}}(0)D_{\bar{w}})^n z_1 \text{ and}$$

$$P(\bar{w},0)z_2 = z_2 + \bar{w}_1\frac{\lambda\mu}{\lambda+\mu}z_1z_2 + \bar{w}_2\frac{\mu+1}{2}z_2^2 + \sum_{n=2}^{\infty}(R_{\mathbf{M}}(0)D_{\bar{w}})^n z_2.$$

Putting all of this together, we see that

$$h(w,w) = \begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix} + \sum a_{IJ}w^I\bar{w}^J,$$

where the sum is over all multi-indices I, J satisfying $|I|, |J| > 0$ and $w^I = w_1^{i_1}w_2^{i_2}$, $\bar{w}^J = \bar{w}_1^{j_1}\bar{w}_2^{j_2}$. The metric h is (almost) normalized at $(0,0)$, that is, $h(w,0) = \left(\begin{smallmatrix}\lambda & 0\\ 0 & \mu\end{smallmatrix}\right)$. The metric h_0 obtained by conjugating the metric h by the invertible (constant) linear transformation $\left(\begin{smallmatrix}\sqrt{\lambda} & 0\\ 0 & \sqrt{\mu}\end{smallmatrix}\right)$ induces an equivalence of holomorphic Hermitian bundles. The vector bundle $\mathcal{P}$ equipped with the Hermitian metric h_0 has the additional property that the metric is normalized: $h_0(w,0) = I$. The coefficient of $dw_i \wedge d\bar{w}_j$, $i,j = 1,2$, in the curvature of the holomorphic Hermitian bundle $\mathcal{P}$ at $(0,0)$ is then the Taylor coefficient of $w_i\,\bar{w}_j$ in the expansion of h_0 around $(0,0)$ (cf. [38, Lemma 2.3]).

Thus the normalized metric $h_0(w,w)$, which is real analytic, is of the form

$$\begin{aligned} h_0(w,w) &= \begin{pmatrix} \lambda\langle P(\bar{w},0)z_1, P(\bar{w},0)z_1\rangle & \sqrt{\lambda\mu}\langle P(\bar{w},0)z_1, P(\bar{w},0)z_2\rangle \\ \sqrt{\lambda\mu}\langle P(\bar{w},0)z_2, P(\bar{w},0)z_1\rangle & \mu\langle P(\bar{w},0)z_2, P(\bar{w},0)z_2\rangle \end{pmatrix} \\ &= I + \begin{pmatrix} \frac{\lambda+1}{2}|w_1|^2 + \frac{\lambda^2\mu}{(\lambda+\mu)^2}|w_2|^2 & \frac{1}{\sqrt{\lambda\mu}}\left(\frac{\lambda\mu}{\lambda+\mu}\right)^2 w_1\bar{w}_2 \\ \frac{1}{\sqrt{\lambda\mu}}\left(\frac{\lambda\mu}{\lambda+\mu}\right)^2 w_2\bar{w}_1 & \frac{\lambda\mu^2}{(\lambda+\mu)^2}|w_1|^2 + \frac{\mu+1}{2}|w_2|^2 \end{pmatrix} + O(|w|^3), \end{aligned}$$

where $O(|w|^3)_{i,j}$ is of degree $\geq$ 3. Explicitly, it is of the form

$$\sum_{n=2}^{\infty}\langle (R_{\mathbf{M}}(0)D_{\bar{w}})^n z_i, (R_{\mathbf{M}}(0)D_{\bar{w}})^n z_j\rangle.$$

The curvature at $(0,0)$, as pointed out earlier, is given by $\bar{\partial}\partial h_0(0,0)$. Consequently, if $H_0^{(\lambda,\mu)}(\mathbb{D}^2)$ and $H_0^{(\lambda',\mu')}(\mathbb{D}^2)$ are equivalent, then the corresponding holomorphic Hermitian vector bundles $\mathcal{P}$ and $\tilde{\mathcal{P}}$ of rank 2 must be equivalent. Hence their curvatures, in particular, at $(0,0)$, must be unitarily equivalent. The curvature for $\mathcal{P}$ at $(0,0)$ is given by the 2×2 matrices

$$\begin{pmatrix} \frac{\lambda+1}{2} & 0 \\ 0 & \frac{\lambda\mu^2}{(\lambda+\mu)^2}\end{pmatrix}, \begin{pmatrix} 0 & \frac{1}{\sqrt{\lambda\mu}}\left(\frac{\lambda\mu}{\lambda+\mu}\right)^2 \\ 0 & 0\end{pmatrix}, \begin{pmatrix} 0 & 0 \\ \frac{1}{\sqrt{\lambda\mu}}\left(\frac{\lambda\mu}{\lambda+\mu}\right)^2 & 0\end{pmatrix}, \begin{pmatrix} \frac{\lambda^2\mu}{(\lambda+\mu)^2} & 0 \\ 0 & \frac{\mu+1}{2}\end{pmatrix}.$$

The curvature for $\tilde{\mathcal{P}}$ has a similar form with λ' and μ' in place of λ and μ respectively. All of them are to be simultaneously equivalent by some unitary map. The only unitary that intertwines the 2×2 matrices

$$\begin{pmatrix} 0 & \frac{1}{\sqrt{\lambda\mu}}\left(\frac{\lambda\mu}{\lambda+\mu}\right)^2 \\ 0 & 0\end{pmatrix} \text{ and } \begin{pmatrix} 0 & \frac{1}{\sqrt{\lambda'\mu'}}\left(\frac{\lambda'\mu'}{\lambda'+\mu'}\right)^2 \\ 0 & 0\end{pmatrix}$$

is aI with $|a| = 1$. Since this fixes the unitary intertwiner, we see that the 2×2 matrices

$$\begin{pmatrix} \frac{\lambda+1}{2} & 0 \\ 0 & \frac{\lambda\mu^2}{(\lambda+\mu)^2}\end{pmatrix} \text{ and } \begin{pmatrix} \frac{\lambda'+1}{2} & 0 \\ 0 & \frac{\lambda'\mu'^2}{(\lambda'+\mu')^2}\end{pmatrix}$$

must be equal. Hence we have $\frac{\lambda+1}{2} = \frac{\lambda+1}{2}$, that is $\lambda = \lambda'$. Consequently, $\frac{\lambda\mu^2}{(\lambda+\mu)^2} = \frac{\lambda'\mu'^2}{(\lambda'+\mu')^2}$ gives $\frac{\mu^2}{(\lambda+\mu)^2} = \frac{\mu'^2}{(\lambda+\mu')^2}$ and then

$$\mu^2(\lambda^2 + 2\lambda\mu' + \mu'^2) = \mu'^2(\lambda^2 + 2\lambda\mu + \mu^2), \text{ that is, } (\mu - \mu')\{\lambda^2(\mu + \mu') + 2\lambda\mu\mu'\} = 0.$$

We then have $\mu = \mu'$. Therefore, $H_0^{(\lambda,\mu)}(\mathbb{D}^2)$ and $H_0^{(\lambda',\mu')}(\mathbb{D}^2)$ are equivalent if and only if $\lambda = \lambda'$ and $\mu = \mu'$.

7.2. The (n, k) examples. For a fixed natural number j, let I_j be the polynomial ideal generated by the set $\{z_1^n, z_1^{k_j} z_2^{n-k_j}\}$, $k_j \neq 0$. Let $\mathcal{M}_j$ be the closure of I_j in the Hardy space $H^2(\mathbb{D}^2)$. We claim that $\mathcal{M}_1$ and $\mathcal{M}_2$ are inequivalent as Hilbert module unless $k_1 = k_2$. From Lemma 1.3, it follows that both the modules $\mathcal{M}_1$ and $\mathcal{M}_2$ are in $\mathrm{B}_1(\mathbb{D}^2 \setminus X)$, where $X := \{(0, z) : |z| < 1\} \cup \{(z, 0) : |z| < 1\}$ is the zero set of the ideal I_j, $j = 1, 2$. However, there is a holomorphic Hermitian line bundle corresponding to these modules on the projectivization of $\mathbb{D}^2 \setminus X$ at $(0, 0)$ (cf. [17, pp. 264]). Following the proof of [17, Theorem 5.1], we see that if these modules are assumed to be equivalent, then the corresponding line bundles they determine must also be equivalent. This leads to contradiction unless $k_1 \neq k_2$.

Suppose $L : \mathcal{M}_1 \to \mathcal{M}_2$ is given to be a unitary module map. Let K_j, $j = 1, 2$, be the corresponding reproducing kernel. By our assumption, the localizations of the modules, $M_j(w)$ at the point $w \in \mathbb{D}^2 \setminus X$ are one dimensional and spanned by the corresponding reproducing kernel K_j, $j = 1, 2$. Since L intertwines module actions, it follows that $M_f^* LK_1(\cdot, w) = \overline{f(w)} LK_1(\cdot, w)$. Hence,

$$LK_1(\cdot, w) = \overline{g(w)} K_2(\cdot, w), \text{ for } w \notin X. \tag{7.2}$$

We conclude that g must be holomorphic on $\mathbb{D}^2 \setminus X$ since both $LK_1(\cdot, w)$ and $K_2(\cdot, w)$ are anti-holomorphic in w. For $j = 1, 2$, let E_j be the holomorphic line bundle on $\mathbb{P}^1$ whose section on the affine chart $U = \{w_1 \neq 0\}$ is given by

$$\begin{aligned} s_j(\theta) &= \lim_{w\to 0, \frac{\bar{w}_2}{\bar{w}_1}=\theta} \frac{K_j(z, w)}{\bar{w}_1^n} = \frac{z_1^n \bar{w}_1^n + z_1^{k_j} z_2^{n-k_j} \bar{w}_1^{k_j} \bar{w}_2^{n-k_j} + \text{higher order terms}}{\bar{w}_1^n} \\ &= z_1^n + \theta^{n-k_j} z_1^{k_j} z_2^{n-k_j}. \end{aligned}$$

Using the ideas from the proof of [17, Theorem 5.1], one shows that $|g(w)|$ has a finite limit at each point of the variety X. By the Riemann removable singularity theorem, it follows that g extends to a holomorphic function on all of $\mathbb{D}^2$. Then from (7.2), and the expression of $s_j(\theta)$, by a limiting argument, we find that $Ls_1(\theta) = g(\theta)s_2(\theta)$. The unitarity of the map L implies that

$$\|Ls_1(\theta)\|^2 = |g(\theta)|^2 \|s_2(\theta)\|^2$$

and consequently the bundles E_j determined by $\mathcal{M}_j$, $j = 1, 2$, on $\mathbb{P}^1$ are equivalent. We now calculate the curvature to determine when these line bundles are equivalent. Since the monomials are orthonormal, we note that the square norm of the section is given by

$$\| s_1(\theta) \|^2 = 1 + |\theta|^{2(n-k_j)}.$$

Consequently the curvature (actually coefficient of the $(1,1)$ form $d\theta \wedge d\bar{\theta}$) of the line bundle on the affine chart U is given by

$$\begin{aligned}
\mathcal{K}_j(\theta) &= -\partial_\theta \partial_{\bar{\theta}} \log \| s_1(\theta) \|^2 = -\partial_\theta \partial_{\bar{\theta}} \log(1+|\theta|^{2(n-k_j)}) \\
&= -\partial_\theta \frac{(n-k_j)\theta^{(n-k_j)}\bar{\theta}^{(n-k_j-1)}}{1+|\theta|^{2(n-k_j)}} \\
&= -\frac{(n-k_j)^2|\theta|^{2(n-k_j-1)}\{1+|\theta|^{2(n-k_j)}\} - (n-k_j)^2|\theta|^{2(n-k_j)}|\theta|^{2(n-k_j-1)}}{\{1+|\theta|^{2(n-k_j)}\}^2} \\
&= -\frac{(n-k_j)^2|\theta|^{2(n-k_j-1)}}{\{1+|\theta|^{2(n-k_j)}\}^2}.
\end{aligned}$$

So if the bundles are equivalent on $\mathbb{P}^1$, then $\mathcal{K}_1(\theta) = \mathcal{K}_2(\theta)$ for $\theta \in U$, and we obtain

$$\begin{aligned}
&(n-k_1)^2\{|\theta|^{2(n-k_1-1)} + 2|\theta|^{2(n-k_2)}|\theta|^{2(n-k_1-1)} + |\theta|^{4(n-k_2)}|\theta|^{2(n-k_1-1)}\} \\
&- \quad (n-k_2)^2\{|\theta|^{2(n-k_2-1)} + 2|\theta|^{2(n-k_1)}|\theta|^{2(n-k_2-1)} + |\theta|^{4(n-k_1)}|\theta|^{2(n-k_2-1)}\} = 0.
\end{aligned}$$

Since the equation given above must be satisfied by all θ corresponding to the affine chart U, it must be an identity. In particular, the coefficient of $|\theta|^{2\{(n-k_1)+(n-k_2)-1\}}$ must be 0 implying $(n-k_1)^2 = (n-k_2)^2$, that is, $k_1 = k_2$. Hence $\mathcal{M}_1$ and $\mathcal{M}_2$ are always inequivalent unless they are equal.

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