THE RELATIONSHIP OF THE GAUSSIAN CURVATURE WITH THE CURVATURE OF A COWEN-DOUGLAS OPERATOR

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Abstract. It has been recently shown that if K is a sesqui-analytic scalar valued non-negative definite kernel on a domain Ω in $\mathbb{C}^m$, then the function $\big(K^2\partial_i\bar\partial_j \log K\big)_{i,j=1}^m$, is also a non-negative definite kernel on Ω. In this paper, we discuss two consequences of this result. The first one strengthens the curvature inequality for operators in the Cowen-Douglas class $B_1(\Omega)$ while the second one gives a relationship of the reproducing kernel of a submodule of certain Hilbert modules with the curvature of the associated quotient module.

1. Introduction

Let X be an arbitrary set and let $K : X \times X \to \mathcal{M}_n(\mathbb{C})$, $n \geq 1$, be a function. We say that K is a non-negative definite kernel (resp. positive definite kernel) if for any subset $\{x_1, \ldots , x_p\}$ of X, the $np\times np$ matrix $\Big(\!\!\Big(K(x_i, x_j) \Big)\!\!\Big)_{i,j=1}^p$ is non-negative definite (resp. positive definite). A Hilbert space $\mathcal{H}$ consisting of functions on X is said to be a reproducing kernel Hilbert space with reproducing kernel K if

(i) for each $x \in X$ and $\eta \in \mathbb{C}^n$, $K(\cdot, x)\eta \in \mathcal{H}$
(ii) for each $f \in \mathcal{H}$ and $x \in X$, $\langle f, K(\cdot, x)\eta\rangle_{\mathcal{H}} = \langle f(x), \eta\rangle_{\mathbb{C}^n}$.

The kernel K of a reproducing kernel Hilbert space $\mathcal{H}$ is non-negative definite. Conversely, corresponding to each non-negative definite kernel K there exists a unique reproducing kernel Hilbert space $(\mathcal{H}, K)$ whose reproducing kernel is K (see [2], [16]). For $K : X \times X \to \mathcal{M}_n(\mathbb{C})$, we write $K(x, y) \succeq 0$, $(x, y) \in X\times X$, whenever K is non-negative definite. Analogously, we write $K(x, y) \preceq 0$, $(x, y) \in X\times X$ if $-K$ is non-negative definite. For $K_1, K_2 : X \times X \to \mathcal{M}_n(\mathbb{C})$, we write $K_1(x, y) \succeq K_2(x, y)$ if $K_1(x, y) - K_2(x, y) \succeq 0$ and we write $K_1(x, y) \preceq K_2(x, y)$ if $K_1(x, y) - K_2(x, y) \preceq 0$, $(x, y) \in X \times X$. For any domain Ω in $\mathbb{C}^m$, $m \geq 1$, a function $K : \Omega \times \Omega \to \mathcal{M}_n(\mathbb{C})$ is said to be sesqui-analytic if it is holomorphic in first m-variables and anti-holomorphic in the second set of m-variables. In this paper, we will deal with non-negative definite kernels which are sesqui-analytic.

We now discuss an important class of operators introduced by Cowen and Douglas (see [4], [7]). Let $\boldsymbol{T} := (T_1, ..., T_m)$ be a m-tuple of commuting bounded linear operators on a separable Hilbert space $\mathcal{H}$. Let $D_{\boldsymbol{T}} : \mathcal{H} \to \mathcal{H} \oplus \cdots \oplus \mathcal{H}$ be the operator defined by $D_{\boldsymbol{T}}(x) = (T_1x, ..., T_mx),\ x \in \mathcal{H}$.

Definition 1.1 (Cowen-Douglas class operator). *Let $\Omega \subset \mathbb{C}^m$ be a bounded domain. A commuting m-tuple $\boldsymbol{T}$ on $\mathcal{H}$ is said to be in the Cowen-Douglas class $B_n(\Omega)$ if $\boldsymbol{T}$ satisfies the following requirements:*

(i) $\dim \ker D_{\boldsymbol{T}-w} = n,\ w \in \Omega$
(ii) $\operatorname{ran} D_{\boldsymbol{T}-w}$ is closed for all $w \in \Omega$
(iii) $\bigvee\big\{ \ker D_{\boldsymbol{T}-w} : w \in \Omega\big\} = \mathcal{H}$.

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If $\boldsymbol{T} \in B_n(\Omega)$, then for each $w \in \Omega$, there exist functions $\gamma_1, \ldots, \gamma_n$ holomorphic in a neighbourhood $\Omega_0 \subseteq \Omega$ containing w such that $\ker D_{\boldsymbol{T}-w'} = \bigvee\{\gamma_1(w'), \ldots, \gamma_n(w')\}$ for all $w' \in \Omega_0$ (cf. [5]). Consequently, every $\boldsymbol{T} \in B_n(\Omega)$ corresponds to a rank n holomorphic hermitian vector bundle $E_{\boldsymbol{T}}$ defined by

$$E_{\boldsymbol{T}} = \{(w, x) \in \Omega \times \mathcal{H} : x \in \ker D_{\boldsymbol{T}-w}\}$$

and $\pi(w, x) = w$, $(w, x) \in E_{\boldsymbol{T}}$. For a bounded domain Ω in $\mathbb{C}^m$, let $\Omega^* = \{z : \bar{z} \in \Omega\}$. It is known that if T is an operator in $B_n(\Omega^*)$, then for each $w \in \Omega$, T is unitarily equivalent to the adjoint of the multiplication tuple $M = (M_1, \ldots, M_m)$ on some reproducing kernel Hilbert space $(\mathcal{H}, K) \subseteq \mathrm{Hol}(\Omega_0, \mathbb{C}^n)$ for some open subset $\Omega_0 \subseteq \Omega$ containing w. If $T \in B_1(\Omega^*)$, the curvature matrix $\mathcal{K}_T(\bar{w})$ at a fixed but arbitrary point $\bar{w} \in \Omega^*$ is defined by

$$\mathcal{K}_T(\bar{w}) = -\Big(\!\!\Big(\partial_i \bar{\partial}_j \log \|\gamma(\bar{w})\|^2 \Big)\!\!\Big)_{i,j=1}^{m},$$

where γ is a holomorphic frame of E_T defined on some open subset $\Omega_0^* \subseteq \Omega^*$ containing $\bar{w}$. Here, ∂_i and $\bar{\partial}_j$ denote $\frac{\partial}{\partial w_i}$ and $\frac{\partial}{\partial \bar{w}_j}$, respectively. If T is realized as the adjoint of the multiplication tuple M on some reproducing kernel Hilbert space $(\mathcal{H}, K) \subseteq \mathrm{Hol}(\Omega_0)$, where $w \in \Omega_0$, the curvature $\mathcal{K}_T(\bar{w})$ is then equal to

$$-\Big(\!\!\Big(\partial_i \bar{\partial}_j \log K(w, w) \Big)\!\!\Big)_{i,j=1}^{m}.$$

Let $\Omega \subset \mathbb{C}$ be open and $\rho : \Omega \to \mathbb{R}_+$ be a C^2-smooth function. The Gaussian curvature of the metric ρ is given by the formula

$$\mathcal{G}_\rho(z) = -\frac{\big(\partial\bar{\partial} \log \rho\big)(z)}{\rho(z)^2}, z \in \Omega. \tag{1}$$

If $K : \Omega \times \Omega \to \mathbb{C}$ is a non-negative definite kernel with $K(z, z) > 0$, then the function $\frac{1}{K}$ defines a metric on Ω and its Gaussian curvature is given by the formula

$$\mathcal{G}_{K^{-1}}(z) = K(z, z)^2 \Big(\partial\bar{\partial} \log K \Big)(z, z),\ z \in \Omega.$$

Since $\mathcal{G}_{K^{-1}}(z)$ can also be written as $K(z, z)\partial\bar{\partial}K(z, z) - \partial K(z, z)\bar{\partial}K(z, z)$, it follows that $\mathcal{G}_{K^{-1}}(z)$ can be extended to a sesqui-analytic function $\mathcal{G}_{K^{-1}}(z, w)$ on $\Omega \times \Omega$. It is therefore natural to extend the definition of the Gaussian curvature to an open subset $\Omega \subset \mathbb{C}^m$. Thus, for any non-negative definite kernel K on Ω, we define

$$\mathcal{G}_{K^{-1}}(z, w) := \Big(\!\!\Big(K(z, w)\partial_i\bar{\partial}_j K(z, w) - \partial_i K(z, w)\bar{\partial}_j K(z, w) \Big)\!\!\Big)_{i,j=1}^{m},\ \ z, w \in \Omega, \tag{2}$$

where, with a slight abuse of notation, we let the symbols ∂_i and $\bar{\partial}_j$ also stand $\frac{\partial}{\partial z_i}$ and $\frac{\partial}{\partial \bar{w}_j}$, respectively.

Proposition 1.2. ([14, Proposition 2.3]) Let $\Omega \subset \mathbb{C}^m$ be a domain and $K : \Omega \times \Omega \to \mathbb{C}$ be a sesqui-analytic function. Let α, β be two positive real numbers. Suppose that K^α and K^β, defined on $\Omega \times \Omega$, are non-negative definite for some α, $\beta > 0$. Then the function $\mathbb{K}^{(\alpha,\beta)} : \Omega \times \Omega \to \mathcal{M}_m(\mathbb{C})$ defined by

$$\mathbb{K}^{(\alpha,\beta)}(z, w) := K^{\alpha+\beta}(z, w)\Big(\!\!\Big(\big(\partial_i\bar{\partial}_j \log K\big)(z, w) \Big)\!\!\Big)_{i,j=1}^{m},\ z, w \in \Omega,$$

is a non-negative definite kernel on $\Omega \times \Omega$ taking values in $\mathcal{M}_m(\mathbb{C})$.

We obtain the following corollary, saying that $\mathcal{G}_{K^{-1}}(z, w)$ is a non-negative definite kernel whenever K is non-negative definite, by setting $\alpha = 1 = \beta$.

Corollary 1.3. *Let Ω be a domain in $\mathbb{C}^m$. Suppose that $K : \Omega \times \Omega \to \mathbb{C}$ is a sesqui-analytic non-negative definite kernel. Then $\mathcal{G}_{K^{-1}}$ is also a non-negative definite kernel on Ω taking values in $\mathcal{M}_m(\mathbb{C})$.*

The introduction of the Gaussian curvature has many advantages and Corollary 1.3 serves as a handy tool for many proofs. This is already apparent from [3], many more examples are given in Section 2 of this paper. We have attempted to strengthen the curvature inequality in the hope of obtaining a criterion for contractivity of operators in $B_1(\mathbb{D})$. We haven't succeeded in doing this yet but several partial answers that we have obtained indicate that one of these inequalities may do the job. In Section 2, we establish a monotonicity property of the Gaussian curvature. We conclude Section 2 by showing that the partial derivatives from $(\mathcal{H}, K)$ to $(\mathcal{H}, \mathcal{G}_{K^{-1}})$ are bounded. In the third Section we discuss the decomposition of the tensor product of two Hilbert modules, say $\mathcal{M}_1 \subset \mathrm{Hol}(\Omega)$ and $\mathcal{M}_2 \subset \mathrm{Hol}(\Omega)$. For the basic notions of Hilbert modules, submodules and quotient modules we refer the reader to [13]. The tensor product $\mathcal{M}_1 \otimes \mathcal{M}_2$ consists of holomorphic functions on $\Omega \times \Omega$. We consider the nested set of submodules $\mathcal{M}_1 \otimes \mathcal{M}_2 \supset \mathcal{A}_0 \supset \mathcal{A}_1 \supset \cdots \supset \mathcal{A}_k \supset \cdots$, where $\mathcal{A}_k$ is the submodule of functions in $\mathcal{M}_1 \otimes \mathcal{M}_2$ vanishing on the diagonal subset Δ of $\Omega \times \Omega$ along with their derivatives to order k. Setting $\mathcal{S}_k := \mathcal{A}_{k-1} \ominus \mathcal{A}_k$, we have the direct sum decomposition

$$\mathcal{M}_1 \otimes \mathcal{M}_2 = \bigoplus_{k=1}^{\infty} \mathcal{S}_k,$$

which one may think of as the Clebsch-Gordon decomposition for Hilbert modules. We also have the short exact sequence of Hilbert modules: $0 \longrightarrow \mathcal{A}_0 \xrightarrow{\ i\ } \mathcal{M}_1 \otimes \mathcal{M}_2 \xrightarrow{\ \pi\ } \mathcal{S}_0 \longrightarrow 0$. It is important to be able to find invariants for $\mathcal{S}_0$ from the inclusion $\mathcal{A}_0 \subset \mathcal{M}_1 \otimes \mathcal{M}_2$. In Section 3, in a large class of examples, we find such an invariant, see Theorem 3.4 and the Remark following it. In this Section, we also show that the Gaussian curvature can be obtained as a limit that we believe is revealing, see Corollary 3.6 and the discussion preceding it.

2. Remarks on Curvature inequality

In this section, we will discuss the curvature inequality for a contractive operator $T : \mathcal{H} \to \mathcal{H}$ in the Cowen-Douglas class $B_1(\mathbb{D})$ taking into account Corollary 1.3. First, since the operator $T \in B_1(\mathbb{D})$, it follows that the map $\gamma_T : \Omega \to Gr(\mathcal{H}, 1)$, $\gamma_T(w) = \ker(T - w)$, $w \in \mathbb{D}$, is holomorphic. Here, $Gr(\mathcal{H}, 1)$ is the Grassmannian of $\mathcal{H}$ consisting of the 1 dimensional subspaces. Second, any operator T in $B_1(\mathbb{D})$ is unitarily equivalent to the adjoint M^* of the operator M of multiplication by the coordinate function z on some reproducing kernel Hilbert space $(\mathcal{H}, K) \subseteq \mathrm{Hol}(\mathbb{D})$. In particular, any contraction T in $B_1(\mathbb{D})$, modulo unitary equivalence, is of this form. Also, $M^* K(\cdot, w) = \overline{w} K(\cdot, w)$, therefore we can take the map $\gamma_T(\overline{w}) = \mathbb{C}[K(\cdot, w)]$ and with a slight abuse of notation, we shall write $\gamma_T(\overline{w}) = K(\cdot, w)$. It is then easy to verify that $(M^* - \overline{w}I)\overline{\partial} K(\cdot, w) = K(\cdot, w)$. Consequently, setting $\mathcal{N}(w)$ to be the 2 dimensional space $\{K(\cdot, w), \overline{\partial} K(\cdot, w)\}$, we have that $(M - wI)^*_{|\mathcal{N}(w)} = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. However if we represent $(M - wI)^*_{|\mathcal{N}(w)}$ with respect to the orthonormal basis $e_1(w), e_2(w)$ obtained by applying the Gram-Schmidt process to the pair of vectors $K(\cdot, w), \overline{\partial} K(\cdot, w)$, then we have the representation:

$$N_T(w) := (M - wI)^*_{|\mathcal{N}(w)} = \begin{pmatrix} 0 & (-\mathcal{K}_T(\overline{w}))^{-\frac{1}{2}} \\ 0 & 0 \end{pmatrix}, \ w \in \mathbb{D}.$$

The contractivity of the operator M, or equivalently, that of M^* implies that the local operators $N_T(w) + \overline{w}I$, $w \in \mathbb{D}$, must be contractive. Since a 2×2 matrix of the form $\begin{pmatrix} w & \lambda \\ 0 & w \end{pmatrix}$ is contractive if and only if $|\lambda| \leq 1 - |w|^2$, we obtain the curvature inequality of [15] reproduced in the form of a proposition below.

Proposition 2.1. *If T is contraction in $B_1(\mathbb{D})$, then the curvature of T is bounded above by the curvature of the backward shift operator S^*.*

Without loss of generality, we may assume that the operator T has been realized as the adjoint of the multiplication operator M on some Hilbert space of holomorphic functions $(\mathcal{H}, K)$. Note that $-\mathcal{K}_T(w) = \partial\bar\partial \log K(w,w)$ and the curvature $\mathcal{K}_{S^*}(w)$ of the backward shift operator S^* is $-\partial\bar\partial \log \mathbb{S}_{\mathbb{D}}(z,z)$, where $\mathbb{S}_{\mathbb{D}}(z,w) = \frac{1}{1-z\bar w}$ is the Szegö kernel of the unit disc. In other words, for a contractive operator M^* in $B_1(\mathbb{D})$, the curvature inequality takes the form (see [3]):

$$-\partial\bar\partial \log K(z,z) \le -\partial\bar\partial \log \mathbb{S}_{\mathbb{D}}(z,z) = -\tfrac{1}{(1-|z|^2)^2},\ z \in \mathbb{D}. \tag{3}$$

From the discussion preceding Proposition 2.1, it is clear that the curvature inequality of a contractive operator in $B_1(\mathbb{D})$ is nothing but the contractivity of its restriction to the 2 dimensional subspaces $\mathcal{N}(w)$, $w \in \mathbb{D}$. So, it is clear that the curvature inequality, in general, is not enough to ensure contractivity. We reproduce an example from [3] illustrating this phenomenon.

Let $K_0(z,w) = \frac{8+8z\bar w-(z\bar w)^2}{1-z\bar w}$, $z,w \in \mathbb{D}$. Note that $K_0(z,w)$ can be written in the form $8 + 16z\bar w + 15\frac{(z\bar w)^2}{1-z\bar w}$, therefore it defines a non-negative definite kernel on the unit disc. It is not hard to see that, in this case

$$\mathcal{K}_{M^*}(w) - \mathcal{K}_{S^*}(w) = -\frac{8(8-4|w|^2-|w|^4)}{(8+8|w|^2-|w|^4)^2} \leqslant 0,\ w \in \mathbb{D}.$$

Recall that for any reproducing kernel Hilbert space $(\mathcal{H}, K)$, the operator M^* on $(\mathcal{H}, K)$ is a contraction if and only if that the function $G(z,w) := (1-z\bar w)K(z,w)$ is non-negative definite on $\mathbb{D}\times\mathbb{D}$ (see [1, Corollary 2.37]). Since $(1-z\bar w)K_0(z,w) = 8 + 8z\bar w - (z\bar w)^2$ which is not a non-negative definite kernel on the unit disc, it follows that the operator M^* on $(\mathcal{H}, K_0)$ is not a contraction.

Since the curvature is a complete unitary invariant in the class $B_1(\mathbb{D})$, one attempts to strengthen the curvature inequality in the hope of finding a criterion for contractivity in terms of the curvature. One such possibility is discussed in the paper [3] replacing the point-wise inequality of (3) by requiring that $0 \preceq \partial\bar\partial \log K(z,w) - \partial\bar\partial \log \mathbb{S}_{\mathbb{D}}(z,w)$, that is,

$$\Big(\!\Big(\partial\bar\partial \log K(w_i,w_j) - \partial\bar\partial \log \mathbb{S}_{\mathbb{D}}(w_i,w_j)\Big)\!\Big)_{i,j=1}^n$$

is non-negative definite for all finite subsets $\{w_1,\dots,w_n\}$ of $\mathbb{D}$ and $n \in \mathbb{N}$. Here, we discuss a different strengthening of the curvature inequality (3).

Proposition 2.2. *Let $T \in B_1(\mathbb{D})$ be a contraction. Assume that T is unitarily equivalent to the operator M^* on $(\mathcal{H}, K)$ for some non-negative definite kernel K on the unit disc. Then the following inequality holds:*

$$K^2(z,w) \preceq\ \mathbb{S}_{\mathbb{D}}^{-2}(z,w)\mathcal{G}_{K^{-1}}(z,w),\ (z,w) \in \mathbb{D}\times\mathbb{D}, \tag{4}$$

that is, the matrix

$$\Big(\!\Big(\mathbb{S}_{\mathbb{D}}^{-2}(w_i,w_j)\mathcal{G}_{K^{-1}}(w_i,w_j) - K^2(w_i,w_j)\Big)\!\Big)_{i,j=1}^n$$

is non-negative definite for every subset $\{w_1,\dots,w_n\}$ of $\mathbb{D}$ and $n \in \mathbb{N}$.

Proof. Setting $G(z,w) = (1-z\bar w)K(z,w)$, we see that

$$\begin{aligned} &-G(z,w)^2\partial\bar\partial \log G(z,w) \\ &\quad= (1-z\bar w)^2K^2(z,w)\big(-\partial\bar\partial \log K(z,w) + \partial\bar\partial \log \mathbb{S}_{\mathbb{D}}(z,w)\big),\ z,w \in \mathbb{D}. \end{aligned}$$

Therefore, since $G(z,w)$ is non-negative definite on $\mathbb{D}\times\mathbb{D}$, applying Corollary 1.3 for $G(z,w)$, we obtain that

$$(1-z\bar w)^2K(z,w)^2\big(-\partial\bar\partial \log K(z,w) + \partial\bar\partial \log \mathbb{S}_{\mathbb{D}}(z,w)\big) \preceq 0.$$

Since $\mathbb{S}_{\mathbb{D}}(z,w)^{-2}\partial\bar\partial \log \mathbb{S}_{\mathbb{D}}(z,w) = 1$, the proof is complete. □

In particular, evaluating (4) at a fixed but arbitrary point, the inequality (3) is evident. However, for any contraction T in $B_1(\mathbb{D})$ (realized as M^* on $(\mathcal{H}, K)$), the inequality (4) gives a much stronger (curvature) inequality as shown in the computation given below. Conversely, whether it is strong enough to force contractivity of the operator M^* is not clear. For a different approach, see [18].

In order to show that the inequality (4) is stronger than the inequality (3), it suffices to prove the kernel K_0 does not satisfy (4). Setting $G_0(z,w) = (1-z\bar{w})K_0(z,w)$, we get $G_0(z,w) = 8+8z\bar{w}-(z\bar{w})^2$, $z, w \in \mathbb{D}$. Thus

$$\begin{aligned} G_0(z,w)^2\partial\bar{\partial}\log G_0(z,w) &= G_0(z,w)\partial\bar{\partial}G_0(z,w) - \partial G_0(z,w)\bar{\partial}G_0(z,w) \\ &= (8+8z\bar{w}-(z\bar{w})^2)(8-4z\bar{w}) - (8z-2z^2\bar{w})(8\bar{w}-2z\bar{w}^2) \\ &= 64 - 32z\bar{w} - 8(z\bar{w})^2, \end{aligned}$$

which is clearly not a non-negative definite kernel. Hence the operator M^* on $(\mathcal{H}, K_0)$ does not satisfy inequality (4).

Remark 2.3. *We now have the following remarks.*

(i) *Under the assumptions of Proposition 2.2, it follows from* [16, Theorem 5.1] *that the Hilbert space* $(\mathcal{H}, K^2)$ *is contained in the Hilbert space* $(\mathcal{H}, \mathbb{S}_{\mathbb{D}}^{-2}\mathcal{G}_{K^{-1}})$*, and the inclusion map from* $(\mathcal{H}, K^2)$ *to* $(\mathcal{H}, \mathbb{S}_{\mathbb{D}}^{-2}\mathcal{G}_{K^{-1}})$ *is contractive.*

(ii) *Recall that unitary equivalence class of the operator M acting on a reproducing kernel Hilbert space $(\mathcal{H}, K)$ is determined by the kernel K modulo pre- and post-multiplication by a non-vanishing holomorphic function and its conjugate, see* [7, Theorem 3.7] *and the remark following it. The Gaussian curvature $\mathcal{G}_{K^{-1}}$ of a non-negative definite kernel K clearly depends on the choice of the kernel K and therefore it is not a function of the unitary equivalence class of the operator M. However, we note that the validity of the inequality* (4) *depends only on the unitary equivalence class of the operator M.*

Let Ω be a finitely connected bounded planar domain and $\mathrm{Rat}(\Omega^*)$ be the ring of rational functions with poles off $\overline{\Omega^*}$. Let T be an operator in $B_1(\Omega^*)$ with $\sigma(T) = \overline{\Omega^*}$. Suppose that the homomorphism $q_T : \mathrm{Rat}(\Omega^*) \to B(\mathcal{H})$ given by

$$q_T(f) = f(T), \;\; f \in \mathrm{Rat}(\Omega^*),$$

is contractive, that is, $\|f(T)\| \leq \|f\|_{\Omega^*,\infty}$, $f \in \mathrm{Rat}(\Omega^*)$. As before, we think of T as the adjoint M^* of the multiplication operator M on some reproducing kernel Hilbert space $(\mathcal{H}, K) \subset \mathrm{Hol}(\Omega)$. Setting $G_f(z,w) = (1-f(z)\overline{f(w)})K(z,w)$ and using the contractivity of $f(M^*)$, $\|f\|_{\infty,\Omega} \leq 1$, we have that $G_f \succeq 0$. Applying Corollary 1.3, we conclude that

$$\begin{aligned} 0 \preceq & G_f(z,w)^2\partial\bar{\partial}\log G_f(z,w) \\ = & G_f(z,w)^2\Big(-\tfrac{f'(z)\overline{f'(w)}}{(1-f(z)\overline{f(w)})^2} + \partial\bar{\partial}\log K(z,w)\Big) \\ = & -K(z,w)^2 f'(z)\overline{f'(w)} + (1-f(z)\overline{f(w)})^2 K(z,w)^2\partial\bar{\partial}\log K(z,w) \end{aligned}$$

for any rational function f with poles off $\overline{\Omega}$ and $|f(z)| \leq 1$, $z \in \Omega$. Also, if f' is a non-vanishing function on Ω, then the pull-back of the metric induced by the Szegö kernel is the metric $f^*(\mathbb{S}_{\mathbb{D}})(z,z) = \frac{|f'(z)|}{1-|f(z)|^2}$, $z \in \Omega$. Thus, if f' is not zero on Ω, then the curvature inequality takes the form

$$K(z,w)^2 \preceq f^*(\mathbb{S}_{\mathbb{D}})(z,w)^{-2}\mathcal{G}_{K^{-1}}(z,w), \;\; z, w \in \Omega,$$

where $f^*(\mathbb{S}_{\mathbb{D}})(z,w)^2$ is the kernel $\frac{f'(z)\overline{f'(w)}}{(1-f(z)\overline{f(w)})^2}$. As in the case of the disc, in particular, evaluating this inequality at a fixed but arbitrary point $z \in \Omega$, we have

$$\partial\bar{\partial}\log K(z,z) \geq \sup\Big\{\tfrac{|f'(z)|^2}{(1-|f(z)|^2)^2} : f \in \mathrm{Rat}(\Omega), \|f\|_{\Omega,\infty} \leq 1\Big\} = \mathbb{S}_{\Omega}(z,z)^2,$$

where $\mathbb{S}_{\Omega}$ is the Szegö kernel of the domain Ω. This is the curvature inequality for contractive homomorphisms (see [15, Corollary 1.2']) and also [17]).

We now show that an analogue of Proposition 2.2 is also valid for spherical contractions in $B_1(\mathbb{B}^m)$, where $\mathbb{B}^m$ is the m-dimensional unit ball in $\mathbb{C}^m$. Recall that a commuting m-tuple $T = (T_1, \ldots, T_m)$ of operators on $\mathcal{H}$ is said to be a row contraction if $\sum_{i=1}^m T_iT_i^* \leq I$. Let $K : \mathbb{B}^m \times \mathbb{B}^m \to \mathbb{C}$ be a sesqui-analytic positive definite kernel. Assume that the commuting m-tuple $M = (M_1, \ldots, M_m)$ of multiplication by the coordinate functions on $(\mathcal{H}, K)$ is in $B_1(\mathbb{B}^m)$. We let $B_m(z,w) := \frac{1}{1-\langle z,w\rangle}$, $z, w \in \mathbb{B}^m$, be the reproducing kernel of the Drury-Arveson space. By [11, Corollary 2], M is a row contraction if and only if $B_m^{-1}(z,w)K(z,w)$ is non-negative definite on $\mathbb{B}^m$. Thus, if M on $(\mathcal{H}, K)$ is a row contraction in $B_1(\mathbb{B}^m)$, applying Corollary 1.3 for $B_m^{-1}(z,w)K(z,w)$ we obtain the following inequality:

$$K^2(z,w)B_m^{-2}(z,w)\Big(\!\Big(\partial_i\bar{\partial}_j \log B_m(z,w)\Big)\!\Big)_{i,j=1}^m \preceq B_m^{-2}(z,w)\mathcal{G}_{K^{-1}}(z,w). \tag{5}$$

As before, evaluating at a fixed but arbitrary point z in $\mathbb{B}^m$, we obtain [3, Corollary 2.3].

We now prove that the Gaussian curvature $\mathcal{G}_{K^{-1}}$ is monotone.

Proposition 2.4. *Let $\Omega \subset \mathbb{C}^m$ be a domain. Suppose that K_1 and K_2 are two scalar valued positive definite kernels on Ω satisfying $K_1(z,w) \succeq K_2(z,w)$, $z, w \in \Omega$. Then*

$$\mathcal{G}_{K_1^{-1}}(z,w) \succeq \mathcal{G}_{K_2^{-1}}(z,w),\ (z,w) \in \Omega \times \Omega.$$

Proof. Set $K_3 = K_1 - K_2$. By hypothesis, K_3 is non-negative definite on Ω. For $1 \le i,j \le m$, a straightforward computation shows that

$$\begin{aligned} K_1^2\partial_i\bar{\partial}_j \log K_1 &= K_2^2\partial_i\bar{\partial}_j \log K_2 + K_3^2\partial_i\bar{\partial}_j \log K_3 \\ &\quad + K_2\partial_i\bar{\partial}_j K_3 + K_3\partial_i\bar{\partial}_j K_2 - \partial_i K_2 \bar{\partial}_j K_3 - \partial_i K_3 \bar{\partial}_j K_2. \end{aligned} \tag{6}$$

Now set $\gamma_i(w) = K_2(\cdot,w) \otimes \bar{\partial}_i K_3(\cdot,w) - \bar{\partial}_i K_2(\cdot,w) \otimes K_3(\cdot,w), 1 \le i \le m,\ w \in \Omega$. For $1 \le i,j \le m$ and $z, w \in \Omega$, then we have

$$\begin{aligned} &\langle \gamma_j(w), \gamma_i(z)\rangle \\ &\quad = \big(K_2\partial_i\bar{\partial}_j K_3\big)(z,w) + \big(K_3\partial_i\bar{\partial}_j K_2\big)(z,w) - \big(\partial_i K_2\bar{\partial}_j K_3\big)(z,w) - \big(\partial_i K_3\bar{\partial}_j K_2\big)(z,w). \end{aligned} \tag{7}$$

Combining (6) and (7), we obtain

$$\begin{aligned} &\big(\big(K_1^2\partial_i\bar{\partial}_j \log K_1\big)(z,w)\big)_{i,j=1}^m \\ &\quad = \big(\big(K_2^2\partial_i\bar{\partial}_j \log K_2\big)(z,w)\big)_{i,j=1}^m + \big(\big(K_3^2\partial_i\bar{\partial}_j \log K_3\big)(z,w)\big)_{i,j=1}^m + \big(\langle \gamma_j(w), \gamma_i(z)\rangle\big)_{i,j=1}^m. \end{aligned}$$

Note that $(z,w) \mapsto \big(\langle \gamma_j(w), \gamma_i(z)\rangle\big)_{i,j=1}^m$ is a non-negative definite kernel on Ω (see [14, Lemma 2.1]). The proof is now complete since sum of two non-negative definite kernels remains non-negative definite. □

As a consequence of Proposition 2.4, we obtain the following inequality for row contractions involving the Gaussian curvature.

Corollary 2.5. *Let $K : \mathbb{B}^m \times \mathbb{B}^m \to \mathbb{C}$ be a sesqui-analytic positive definite kernel. Assume that K is normalized at the origin, that is, $K(z,0) = 1$, $z \in \mathbb{B}^m$. Suppose that the commuting tuple M of multiplication by the coordinate functions is a row contraction on $(\mathcal{H}, K)$. Then*

$$\mathcal{G}_{K^{-1}}(z,w) \succeq \mathcal{G}_{B_m^{-1}}(z,w),\ (z,w) \in \mathbb{B}^m \times \mathbb{B}^m. \tag{8}$$

Proof. Since the tuple M on $(\mathcal{H}, K)$ is a row contraction, $\tilde{K}(z,w) := B_m^{-1}(z,w)K(z,w)$ defines a non-negative definite kernel on $\mathbb{B}^m$. The kernel $\tilde{K}$ is normalized at 0 since K is normalized at 0. Thus $1 = \tilde{K}(\cdot,0) \in (\mathcal{H}, \tilde{K})$ and

$$\|1\|^2_{(\mathcal{H},\tilde{K})} = \langle \tilde{K}(\cdot,0), \tilde{K}(\cdot,0)\rangle_{(\mathcal{H},\tilde{K})} = \tilde{K}(0,0) = 1.$$

Hence it follows from [16, Theorem 3.11] that $\tilde{K} \succeq 1$. Since the product of two non-negative definite kernels remain non-negative definite, multiplying both sides with B_m, we get $K \succeq B_m$. The proof is now complete by applying Proposition 2.4. □

Remark 2.6. *We point out that Corollary 2.5 can also be derived from (5). In particular, in case $m = 1$, Corollary 2.5 is a consequence of Proposition 2.2. But since the kernel K_0 satisfies the inequality $K_0(z,w) \succeq \mathbb{S}_{\mathbb{D}}(z,w)$, $(z,w) \in \mathbb{D}\times\mathbb{D}$, it follows from Theorem 2.7 that $\mathcal{G}_{K_0^{-1}}(z,w) \succeq \mathcal{G}_{\mathbb{S}_{\mathbb{D}}^{-1}}(z,w), (z,w) \in \mathbb{D}\times\mathbb{D}$. Therefore, the inequality (8) is weaker than the inequality (4) in case $m = 1$.*

After establishing a lower bound for the Gaussian curvature of a non-negative definite kernel, we show that the partial derivatives are bounded from $(\mathcal{H}, K)$ to $(\mathcal{H}, \mathcal{G}_{K^{-1}})$. We recall from [3, Lemma 3.1] that $\left(\!\!\left(\partial_i\bar{\partial}_j K(z,w) \right)\!\!\right)_{i,j=1}^m$ is a non-negative definite kernel whenever K is non-negative definite.

Theorem 2.7. *Let $\Omega \subset \mathbb{C}^m$ be a domain. Let $K : \Omega\times\Omega \to \mathbb{C}$ be a non-negative definite kernel. Suppose that the Hilbert space $(\mathcal{H}, K)$ contains the constant function 1. Then*

$$\left(\!\!\left(\partial_i\bar{\partial}_j K(z,w)\right)\!\!\right)_{i,j=1}^m \preceq c\, \mathcal{G}_{K^{-1}}(z,w), \quad z,w \in \Omega,$$

where $c = \|1\|^2_{(\mathcal{H},K)}$.

Proof. Set $c = \|1\|^2_{(\mathcal{H},K)}$. Choose an orthonormal basis $\{e_n(z)\}_{n\geq 0}$ of $(\mathcal{H}, K)$ with $e_0(z) = \frac{1}{\sqrt{c}}$. Then

$$K(z,w) - \frac{1}{c} = \sum_{i=1}^{\infty} e_i(z)\overline{e_i(w)},\ z,w \in \Omega.$$

Hence $K(z,w) - \frac{1}{c}$ is non-negative definite on $\Omega\times\Omega$, or equivalently $cK - 1$ is non-negative definite on $\Omega\times\Omega$. Therefore, by Corollary 1.2, it follows that $\left(\!\!\left((cK-1)^2\partial_i\bar{\partial}_j \log(cK-1) \right)\!\!\right)_{i,j=1}^m$ is non-negative definite on $\Omega\times\Omega$. Note that, for $z,w \in \Omega$, we have

$$\begin{aligned}
&\big((cK-1)^2\partial_i\bar{\partial}_j \log(cK-1)\big)(z,w)\\
&\qquad =(cK-1)(z,w)\big(\partial_i\bar{\partial}_j(cK-1)\big)(z,w) - \big(\partial_i(cK-1)\big)(z,w)\big(\bar{\partial}_j(cK-1)\big)(z,w)\\
&\qquad =c^2K(z,w)\partial_i\bar{\partial}_jK(z,w) - c\partial_i\bar{\partial}_jK(z,w) - c^2\partial_iK(z,w)\bar{\partial}_jK(z,w)\\
&\qquad =c^2K^2\partial_i\bar{\partial}_j\log K(z,w) - c\partial_i\bar{\partial}_jK(z,w).
\end{aligned}$$

Hence we conclude that

$$\left(\!\!\left(\partial_i\bar{\partial}_j K(z,w) \right)\!\!\right)_{i,j=1}^m \preceq c\, \mathcal{G}_{K^{-1}}(z,w),\ z,w \in \Omega.$$

□

Corollary 2.8. *Let $\Omega \subset \mathbb{C}^m$ be a domain. Let $K : \Omega\times\Omega \to \mathbb{C}$ be a non-negative definite kernel. Then the linear operator $\boldsymbol{\partial} : (\mathcal{H}, K) \to (\mathcal{H}, \mathcal{G}_{K^{-1}})$, where $\boldsymbol{\partial} f = (\partial_1 f, \ldots, \partial_m f)^{\mathrm{tr}}$, $f \in (\mathcal{H}, K)$, is bounded with $\|\boldsymbol{\partial}\| \leq \|1\|_{(\mathcal{H},K)}$. Moreover if K is normalized at the point $w_0 \in \Omega$, that is, $K(\cdot, w_0)$ is the constant function 1, then the linear operator $\boldsymbol{\partial} : (\mathcal{H}, K) \to (\mathcal{H}, \mathcal{G}_{K^{-1}})$ is contractive.*

Proof. To prove the first assertion of the corollary, note that the map $\boldsymbol{\partial}$ is unitary from $\ker\boldsymbol{\partial}^\perp$ to $(\mathcal{H}, (\partial_i\bar{\partial}_jK)_{i,j=1}^m)$, and therefore is contractive from $(\mathcal{H}, K)$ to $(\mathcal{H}, (\partial_i\bar{\partial}_jK)_{i,j=1}^m)$. To complete the proof, it is therefore enough to show that $(\mathcal{H}, (\partial_i\bar{\partial}_jK)_{i,j=1}^m)$ is contained in $(\mathcal{H}, \mathcal{G}_{K^{-1}})$ and the inclusion map is bounded by $\|1\|_{(\mathcal{H},K)}$. This follows from Theorem 2.7 using [16, Theorem 6.25]. For the second assertion, note that $\|1\|^2_{(\mathcal{H},K)} = \langle K(\cdot,w_0), K(\cdot,w_0)\rangle_{(\mathcal{H},K)} = K(w_0,w_0) = 1$ by hypothesis and use Theorem 2.7 to complete the proof. □

3. A limit Computation

Let Ω be a bounded domain in $\mathbb{C}^m$. Let $\boldsymbol{M}^* \in B_1(\Omega^*)$ be the adjoint of the m-tuple $\boldsymbol{M}$ of multiplication by the coordinate functions on a reproducing kernel Hilbert space $(\mathcal{H}, K)$ consisting of holomorphic functions on $\Omega \subset \mathbb{C}^m$. Let $\mathcal{A}(\Omega)$ be the function algebra of all those functions holomorphic in some open neighbourhood of the compact set $\overline{\Omega}$ equipped with the supremum norm on $\overline{\Omega}$. The map $\mathbf{m}_f : h \mapsto f \cdot h,\ f \in \mathcal{A}(\Omega),\ h \in (\mathcal{H}, K)$, where $(f \cdot h)(z) = f(z)h(z)$, defines a module multiplication for $(\mathcal{H}, K)$ over the algebra $\mathcal{A}(\Omega)$. We let $\mathcal{M} := (\mathcal{H}, K)$ denote this Hilbert module. Let $\mathcal{M}_0 \subseteq \mathcal{M}$ be a submodule. We now have a short exact sequence of Hilbert modules

$$0 \longrightarrow \mathcal{M}_0 \xrightarrow{\ i\ } \mathcal{M} \xrightarrow{\ \pi\ } \mathcal{Q} \longrightarrow 0$$

where i is the inclusion map and π is the quotient map. The problem of finding invariants for $\mathcal{Q}$ given the inclusion $\mathcal{M}_0 \subset \mathcal{M}$ has been studied in several papers (cf. [10, 12]). A variant of this problem occurs by replacing the inclusion map with some other module map, for instance, one might set $\mathcal{M}_0 = \varphi\mathcal{M}$ for some $\varphi \in \mathcal{A}(\Omega)$, see [13, p. 94] for the case of the Hardy module over the disc and the Beurling phenomenon. Here we are going to consider the case of submodules $\mathcal{M}_0$ consisting of the maximal set of functions in $\mathcal{M}$ vanishing on some fixed subset $\mathcal{Z}$ of Ω. A description of the specific examples we consider here follows.

Let K_1 and K_2 be two scalar valued non-negative definite kernels on Ω. Assume that both the kernels sesqui-analytic. It is well known that $(\mathcal{H}, K_1) \otimes (\mathcal{H}, K_2)$ is the reproducing kernel Hilbert space determined by the non-negative definite kernel $K_1 \otimes K_2$, where $K_1 \otimes K_2 : (\Omega \times \Omega) \times (\Omega \times \Omega) \to \mathbb{C}$ is given by

$$(K_1 \otimes K_2)(z, \zeta; w, \rho) = K_1(z, w)K_2(\zeta, \rho), \ \ z, \zeta, w, \rho \in \Omega.$$

We assume that the operator M_{z_i} of multiplication by the coordinate function z_i is bounded on $(\mathcal{H}, K_1)$ as well as on $(\mathcal{H}, K_2)$ for $i = 1, \ldots, m$. Then $(\mathcal{H}, K_1) \otimes (\mathcal{H}, K_2)$ may be realized as a Hilbert module over the polynomial ring $\mathbb{C}[z_1, \ldots, z_{2m}]$ with the module action defined by

$$\mathbf{m}_p(h) = ph,\ h \in (\mathcal{H}, K_1) \otimes (\mathcal{H}, K_2),\ p \in \mathbb{C}[z_1, \ldots, z_{2m}].$$

The Hilbert space $(\mathcal{H}, K_1) \otimes (\mathcal{H}, K_2)$ admits a natural direct sum decomposition as follows.

For a non-negative integer k, let $\mathcal{A}_k$ be the subspace of $(\mathcal{H}, K_1) \otimes (\mathcal{H}, K_2)$ defined by

$$\mathcal{A}_k := \big\{f \in (\mathcal{H}, K_1) \otimes (\mathcal{H}, K_2) : \big(\big(\tfrac{\partial}{\partial \zeta}\big)^{\boldsymbol{i}} f(z, \zeta)\big)_{|\Delta} = 0,\ |\boldsymbol{i}| \leq k\big\}, \tag{9}$$

where $\boldsymbol{i} = (i_1, \ldots, i_m) \in \mathbb{Z}_+^m$, $|\boldsymbol{i}| = i_1 + \cdots + i_m$, $\big(\frac{\partial}{\partial \zeta}\big)^{\boldsymbol{i}} = \frac{\partial^{|\boldsymbol{i}|}}{\partial \zeta_1^{i_1} \cdots \partial \zeta_m^{i_m}}$, and $\big(\big(\frac{\partial}{\partial \zeta}\big)^{\boldsymbol{i}} f(z, \zeta)\big)_{|\Delta}$ is the restriction of $\big(\frac{\partial}{\partial \zeta}\big)^{\boldsymbol{i}} f(z, \zeta)$ to the diagonal set $\Delta := \{(z, z) : z \in \Omega\}$. It is easily verified that each of the subspaces $\mathcal{A}_k$ is closed and invariant under multiplication by any polynomial in $\mathbb{C}[z_1, \ldots, z_{2m}]$ and therefore they are submodules of $(\mathcal{H}, K_1) \otimes (\mathcal{H}, K_2)$. Setting $\mathcal{S}_0 = \mathcal{A}_0^{\perp}$, $\mathcal{S}_k := \mathcal{A}_{k-1} \ominus \mathcal{A}_k$, $k = 1, 2, \ldots$, we obtain a direct sum decomposition of the Hilbert space $(\mathcal{H}, K_1) \otimes (\mathcal{H}, K_2)$ as follows

$$(\mathcal{H}, K_1) \otimes (\mathcal{H}, K_2) = \bigoplus_{k=0}^{\infty} \mathcal{S}_k.$$

Define a linear map $\mathcal{R}_1 : (\mathcal{H}, K^\alpha) \otimes (\mathcal{H}, K^\beta) \to \mathrm{Hol}(\Omega, \mathbb{C}^m)$ by setting

$$\mathcal{R}_1(f) = \frac{1}{\sqrt{\alpha\beta(\alpha + \beta)}} \begin{pmatrix} (\beta\partial_1 f - \alpha\partial_{m+1} f)_{|\Delta} \\ \vdots \\ (\beta\partial_m f - \alpha\partial_{2m} f)_{|\Delta} \end{pmatrix} \tag{10}$$

for $f \in (\mathcal{H}, K^\alpha) \otimes (\mathcal{H}, K^\beta)$. Let $\iota : \Omega \to \Omega \times \Omega$ be the map $\iota(z) = (z, z)$, $z \in \Omega$. Any Hilbert module $\mathcal{M}$ over the polynomial ring $\mathbb{C}[z_1, \ldots, z_m]$ may be thought of as a module $\iota_\star\mathcal{M}$ over the ring $\mathbb{C}[z_1, \ldots, z_{2m}]$ by re-defining the multiplication: $\mathbf{m}_p(h) = (p \circ \iota)h$, $h \in \mathcal{M}$ and $p \in \mathbb{C}[z_1, \ldots, z_{2m}]$. The module $\iota_\star\mathcal{M}$

over $\mathbb{C}[z_1, \ldots, z_{2m}]$ is defined to be the push-forward of the module $\mathcal{M}$ over $\mathbb{C}[z_1, \ldots, z_m]$ under the inclusion map ι.

Recall that for $\alpha, \beta > 0$ and a sesqui-analytic function $K : \Omega \times \Omega \to \mathbb{C}$, the function $\mathbb{K}^{(\alpha,\beta)} : \Omega \times \Omega \to \mathcal{M}_m(\mathbb{C})$ is defined by

$$\mathbb{K}^{(\alpha,\beta)}(z, w) := K^{\alpha+\beta}(z, w)\Big(\!\Big(\big(\partial_i \bar{\partial}_j \log K\big)(z, w) \Big)\!\Big)_{i,j=1}^{m}, \ z, w \in \Omega.$$

Theorem 3.1. ([14, Theorem 3.5.]) Suppose $K : \Omega \times \Omega \to \mathbb{C}$ is a sesqui-analytic function such that the functions K^α and K^β, defined on $\Omega \times \Omega$, are non-negative definite for some $\alpha, \beta > 0$. Then the followings hold:

(1) $\ker \mathcal{R}_1 = \mathcal{S}_1^\perp$ and $\mathcal{R}_1$ maps $\mathcal{S}_1$ isometrically onto $\big(\mathcal{H}, \mathbb{K}^{(\alpha,\beta)}\big)$.
(2) Suppose that the operator M_i of multiplication by the coordinate function z_i is bounded on both $(\mathcal{H}, K^\alpha)$ and $(\mathcal{H}, K^\beta)$ for $i = 1, 2, \ldots, m$. Then the Hilbert module $\mathcal{S}_1$ is isomorphic to the push-forward module $\iota_\star\big(\mathcal{H}, \mathbb{K}^{(\alpha,\beta)}\big)$ via the module map $\mathcal{R}_{1|\mathcal{S}_1}$.

We consider the example of the Hardy space. Let $K_1(z, w) = K_2(z, w) = \frac{1}{1 - z\bar{w}}$, $z, w \in \mathbb{D}$, be the Szegö kernel of the unit disc $\mathbb{D}$. In this case $(\mathcal{H}, K_1) \otimes (\mathcal{H}, K_2)$ is the Hardy space on the bidisc $\mathbb{D}^2$, and it is often denoted by $H^2(\mathbb{D}^2)$. Now, we can compute the kernel functions for $\mathcal{S}_0$ and $\mathcal{A}_0$ in this example as follows, see [9]. The vectors $\Big\{\frac{e_k}{\sqrt{k+1}}\Big\}_{k \geq 0}$ form an orthonormal basis of $\mathcal{S}_0$, where e_k is given by

$$e_k(z_1, z_2) = \sum_{j=0}^{k} z_1^j z_2^{k-j}, z_1, z_2 \in \mathbb{D}.$$

Therefore the reproducing kernel $K_{\mathcal{S}_0}$ of $\mathcal{S}_0$ is given by

$$K_{\mathcal{S}_0}(\boldsymbol{z}, \boldsymbol{w}) = \sum_{k \geq 0} \frac{e_k(\boldsymbol{z})\overline{e_k(\boldsymbol{w})}}{k+1} \ \boldsymbol{z} = (z_1, z_2), \boldsymbol{w} = (w_1, w_2) \in \mathbb{D}^2.$$

A closed expression for $K_{\mathcal{S}_0}$ is easily obtained:

$$K_{\mathcal{S}_0}(\boldsymbol{z}, \boldsymbol{z}) = \frac{1}{|z_1 - z_2|^2} \log \frac{|1 - z_1\bar{z}_2|^2}{(1 - |z_1|^2)(1 - |z_2|^2)}, \ \boldsymbol{z} = (z_1, z_2) \in \mathbb{D}^2.$$

Therefore it follows that

$$\begin{aligned} &K_{\mathcal{A}_0}(\boldsymbol{z}, \boldsymbol{z}) \\ &= \frac{1}{(1 - |z_1|^2)(1 - |z_2|^2)} - K_{\mathcal{S}_0}(\boldsymbol{z}, \boldsymbol{z}) \\ &= \frac{1}{(1 - |z_1|^2)(1 - |z_2|^2)} - \frac{1}{|z_1 - z_2|^2} \log \frac{|1 - z_1\bar{z}_2|^2}{(1 - |z_1|^2)(1 - |z_2|^2)} \\ &= \frac{1}{(1 - |z_1|^2)(1 - |z_2|^2)} - \frac{1}{|z_1 - z_2|^2}\Big(\frac{|z_1 - z_2|^2}{(1 - |z_1|^2)(1 - |z_2|^2)} - \frac{1}{2}\frac{|z_1 - z_2|^4}{(1 - |z_1|^2)^2(1 - |z_2|^2)^2} + \cdots\Big). \end{aligned}$$

Thus, we see that

$$\lim_{z_2 \to z_1} \frac{K_{\mathcal{A}_0}(\boldsymbol{z}, \boldsymbol{z})}{|z_1 - z_2|^2} = \frac{1}{2}\frac{1}{(1 - |z_1|^2)^4}, (z_1, z_2) \in \mathbb{D}^2, \tag{11}$$

and hence, this limit coincides (after taking the pushforward under the map ι) with the metric of the module $\mathcal{S}_1$, up to a scalar multiple, see Theorem 3.1(2).

Consider the short exact sequence $0 \to \mathcal{A}_0 \to H^2(\mathbb{D}^2) \to \mathcal{S}_0 \to 0$. It is known that the quotient module $\mathcal{S}_0$ is the pushforward of the Bergman module on the disc. Also, by Theorem 3.1(2), the module $\mathcal{S}_1$ is the pushforward of $(\mathcal{H}, \frac{1}{(1-z\bar{w})^4})$. It then follows that

$$\mathcal{K}_{\mathcal{S}_1}(\boldsymbol{z}) = \mathcal{K}_{\mathcal{S}_0}(\boldsymbol{z}) - \frac{2}{(1-|z|^2)^2},\ \boldsymbol{z} = (z, z), z \in \mathbb{D}.$$

Thus the restriction of $\mathcal{K}_{\mathcal{A}_0}$ to the zero set Δ might serve as an invariant for the inclusion $\mathcal{M}_0 \subset \mathcal{M}$. This possibility is explored below in a class of examples.

Let $\Omega \subset \mathbb{C}^m$ be a bounded domain and $K : \Omega \times \Omega \to \mathbb{C}$ be a sesqui-analytic function such that the functions K^α and K^β are non-negative definite on $\Omega \times \Omega$ for some $\alpha, \beta > 0$. For a non-negative integer p, let $K_{\mathcal{A}_p}$ be the reproducing kernel of $\mathcal{A}_p$, where $\mathcal{A}_p$ is defined in (9).

To prove the main result of this section, we need the following two lemmas. One way to prove both of the lemmas is to make the change of variables

$$u_1(z,\zeta) = \tfrac{1}{2}(z_1 - \zeta_1), \ldots, u_m(z,\zeta) = \tfrac{1}{2}(z_m - \zeta_m);\ v_1(z,\zeta) = \tfrac{1}{2}(z_1 + \zeta_1), \ldots, v_m(z,\zeta) = \tfrac{1}{2}(z_m + \zeta_m).$$

We give the details for the proof of the first lemma. The proof for the second one follows by similar arguments.

Lemma 3.2. *Let $\Omega \subset \mathbb{C}^m$ be a domain and let Δ be the diagonal set $\{(z, z) : z \in \Omega\}$. Suppose that $f : \Omega \times \Omega \to \mathbb{C}$ is a holomorphic function satisfying $f_{|\Delta} = 0$. Then for each $z_0 \in \Omega$, there exists a neighbourhood $\Omega_0 \subset \Omega$ (independent of f) of z_0 and holomorphic functions $f_1, f_2, \ldots, f_m$ on $\Omega_0 \times \Omega_0$ such that*

$$f(z,\zeta) = \sum_{i=1}^{m} (z_i - \zeta_i) f_i(z,\zeta),\ z = (z_1, \ldots, z_m), \zeta = (\zeta_1, \ldots, \zeta_m) \in \Omega_0.$$

Proof. Note that the image of the diagonal set $\Delta \subseteq \Omega \times \Omega$ under the map $\varphi : \Omega \times \Omega \to \mathbb{C}^{2m}$, where

$$\varphi(z,\zeta) := (u_1(z,\zeta), \ldots, u_m(z,\zeta), v_1(z,\zeta), \ldots, v_m(z,\zeta)),$$

is the set $\{(0, w) : w \in \Omega\}$. Therefore we may choose a neighbourhood of $(0, z_0)$ which is a polydisc contained in $\hat{\Omega} := \varphi(\Omega \times \Omega)$. Suppose f is a holomorphic function on $\Omega \times \Omega$ vanishing on the set Δ. Setting $g := f \circ \varphi^{-1}$ on $\hat{\Omega}$, we see that g is a holomorphic function on $\hat{\Omega}$ vanishing on the set $\{(0, w) : w \in \Omega\}$. Therefore g has a power series representation around $(0, z_0)$ of the form $\sum_{\boldsymbol{i},\boldsymbol{j} \in \mathbb{Z}_+^m} a_{\boldsymbol{i}\boldsymbol{j}} u^{\boldsymbol{i}} (v - z_0)^{\boldsymbol{j}}$, where $\sum_{\boldsymbol{j} \in \mathbb{Z}_+^m} a_{0\boldsymbol{j}} (v - z_0)^{\boldsymbol{j}} = 0$ on the chosen polydisc. Hence $a_{0\boldsymbol{j}} = 0$ for all $\boldsymbol{j} \in \mathbb{Z}_+^m$, and the power series of g is of the form $\sum_{\ell=1}^m u_\ell g_\ell(u, v)$, where

$$g_\ell(u,v) = \sum_{\boldsymbol{i}\boldsymbol{j}} a_{\boldsymbol{i}\boldsymbol{j}} u^{\boldsymbol{i} - e_\ell} (v - z_0)^{\boldsymbol{j}},\ 1 \le \ell \le m.$$

Here the sum is over all multi-indices $\boldsymbol{i} = (i_1, \ldots, i_m)$ satisfying $i_1 = 0, \ldots, i_{\ell-1} = 0, i_\ell \ge 1$ while $\boldsymbol{j}$ remains arbitrary. Pulling this expression back to $\Omega \times \Omega$ under the bi-holomorphic map φ, we obtain the expansion of f in a neighbourhood of (z_0, z_0) as prescribed in the Lemma 3.2. □

Lemma 3.3. *Suppose that $f : \Omega \times \Omega \to \mathbb{C}$ is a holomorphic function satisfying $f_{|\Delta} = 0$ and $\big((\frac{\partial}{\partial \zeta_j}) f(z,\zeta)\big)_{|\Delta} = 0$, $j = 1, \ldots, m$. Then for each $z_0 \in \Omega$, there exists a neighbourhood $\Omega_0 \subset \Omega$ (independent of f) of z_0 and holomorphic functions f_{ij}, $1 \le i \le j \le m$, on $\Omega_0 \times \Omega_0$ such that*

$$f(z,\zeta) = \sum_{1 \le i \le j \le m} (z_i - \zeta_i)(z_j - \zeta_j) f_{ij}(z,\zeta),\ z, \zeta \in \Omega_0.$$

Theorem 3.4. *Let $\Omega \subset \mathbb{C}^m$ be a bounded domain and $K : \Omega \times \Omega \to \mathbb{C}$ be a sesqui-analytic function such that the functions K^α and K^β are non-negative definite on $\Omega \times \Omega$ for some $\alpha, \beta > 0$. For z in Ω and $1 \le i, j \le m$, we have*

$$\lim_{\substack{\zeta_i \to z_i \\ \zeta_j \to z_j}} \left(\frac{K_{\mathcal{A}_0}(z, \zeta; z, \zeta)}{(z_i - \zeta_i)(\bar{z}_j - \bar{\zeta}_j)} \bigg|_{\zeta_l = z_l, l \ne i,j} \right) = \tfrac{\alpha\beta}{(\alpha+\beta)} K(z,z)^{\alpha+\beta} \partial_i \bar{\partial}_j \log K(z,z),$$

where $K_{\mathcal{A}_0}$ *is the reproducing kernel of the subspace* $\mathcal{A}_0$*, and* $\frac{K_{\mathcal{A}_0}(z,\zeta;z,\zeta)}{(z_i-\zeta_i)(\bar{z}_j-\bar{\zeta}_j)}\Big|_{\zeta_l=z_l,l\neq i,j}$ *is the restriction of the function* $\frac{K_{\mathcal{A}_0}(z,\zeta;z,\zeta)}{(z_i-\zeta_i)(\bar{z}_j-\bar{\zeta}_j)}$ *to the set* $\big\{(z,\zeta)\in\Omega\times\Omega : z_l=\zeta_l, l=1,\dots,m, l\neq i,j\big\}$.

Proof. Let $K_{\mathcal{A}_0\ominus\mathcal{A}_1}(z,\zeta;w,\nu)$ be the reproducing kernels of $\mathcal{A}_0\ominus\mathcal{A}_1$. Fix a point z_0 in Ω. Choose a neighbourhood Ω_0 of z_0 in Ω such that the conclusions of Lemma 3.2 and Lemma 3.3 are valid. Now we restrict the kernels K^{α} and K^{β} to $\Omega_0\times\Omega_0$.

Let f be an arbitrary function in $\mathcal{A}_1$. Then, by definition, f satisfies the hypothesis of Lemma 3.3, and therefore, it follows that

$$\lim_{\zeta_i\to z_i}\left(\frac{f(z,\zeta)}{(z_i-\zeta_i)}\Big|_{z_l=\zeta_l,l\neq i}\right)=0,\ \ i=1,\dots,m. \tag{12}$$

Let $\{h_n\}_{n\in\mathbb{Z}_+}$ be an orthonormal basis of $\mathcal{A}_1$. Since the series $\sum_{n=0}^{\infty}h_n(z,\zeta)\overline{h_n(z,\zeta)}$ converges uniformly to $K_{\mathcal{A}_1}(z,\zeta;z,\zeta)$ on the compact subsets of $\Omega_0\times\Omega_0$, using (12) we see that

$$\begin{aligned}\lim_{\substack{\zeta_i\to z_i\\ \zeta_j\to z_j}}\left(\frac{K_{\mathcal{A}_1}(z,\zeta;z,\zeta)}{(z_i-\zeta_i)(\bar{z}_j-\bar{\zeta}_j)}\Big|_{\zeta_l=z_l,l\neq i,j}\right)&=\sum_{n=0}^{\infty}\lim_{\zeta_i\to z_i}\left(\frac{h_n(z,\zeta)}{(z_i-\zeta_i)}\Big|_{z_l=\zeta_l,l\neq i}\right)\overline{\lim_{\zeta_j\to z_j}\left(\frac{h_n(z,\zeta)}{(z_j-\zeta_j)}\Big|_{z_l=\zeta_l,l\neq j}\right)}\\&=0.\end{aligned}$$

Since $K_{\mathcal{A}_0}=K_{\mathcal{A}_0\ominus\mathcal{A}_1}+K_{\mathcal{A}_1}$, the above equality leads to

$$\lim_{\substack{\zeta_i\to z_i\\ \zeta_j\to z_j}}\left(\frac{K_{\mathcal{A}_0}(z,\zeta;z,\zeta)}{(z_i-\zeta_i)(\bar{z}_j-\bar{\zeta}_j)}\Big|_{\zeta_l=z_l,l\neq i,j}\right)=\lim_{\substack{\zeta_i\to z_i\\ \zeta_j\to z_j}}\left(\frac{K_{\mathcal{A}_0\ominus\mathcal{A}_1}(z,\zeta;z,\zeta)}{(z_i-\zeta_i)(\bar{z}_j-\bar{\zeta}_j)}\Big|_{\zeta_l=z_l,l\neq i,j}\right).$$

Now let $\{e_n\}_{n\in\mathbb{Z}_+}$ be an orthonormal basis of $\mathcal{A}_0\ominus\mathcal{A}_1$. Since each $e_n\in\mathcal{A}_0$, by Lemma 3.2, there exist holomorphic functions $e_{n,i}$, $1\le i\le m$, on $\Omega_0\times\Omega_0$ such that

$$e_n(z,\zeta)=\sum_{i=1}^{m}(z_i-\zeta_i)e_{n,i}(z,\zeta),\ z,\zeta\in\Omega_0.$$

Thus, for $1\le i\le m$, we have

$$\lim_{\zeta_i\to z_i}\left(\frac{e_n(z,\zeta)}{(z_i-\zeta_i)}\Big|_{z_l=\zeta_l,l\neq i}\right)=e_{n,i}(z,z),\ z\in\Omega_0. \tag{13}$$

Since the series $\sum_{n=0}^{\infty}e_n(z,\zeta)\overline{e_n(z,\zeta)}$ converges to $K_{\mathcal{A}_0\ominus\mathcal{A}_1}$ uniformly on compact subsets of $\Omega_0\times\Omega_0$, using (13), we see that

$$\lim_{\substack{\zeta_i\to z_i\\ \zeta_j\to z_j}}\left(\frac{K_{\mathcal{A}_0\ominus\mathcal{A}_1}(z,\zeta;z,\zeta)}{(z_i-\zeta_i)(\bar{z}_j-\bar{\zeta}_j)}\Big|_{\zeta_l=z_l,l\neq i,j}\right)=\sum_{n=0}^{\infty}e_{n,i}(z,z)\overline{e_{n,j}(z,z)},\ z\in\Omega_0. \tag{14}$$

Recall that by Theorem 3.1, the map $\mathcal{R}_1:\mathcal{A}_0\ominus\mathcal{A}_1\to(\mathcal{H},\mathbb{K}^{(\alpha,\beta)})$ given by

$$\mathcal{R}_1 f=\frac{1}{\sqrt{\alpha\beta(\alpha+\beta)}}\begin{pmatrix}(\beta\partial_1 f-\alpha\partial_{m+1}f)_{|\Delta}\\ \vdots\\ (\beta\partial_m f-\alpha\partial_{2m}f)_{|\Delta}\end{pmatrix},\ f\in\mathcal{A}_0\ominus\mathcal{A}_1$$

is unitary. Hence $\{\mathcal{R}_1(e_n)\}_n$ is an orthonormal basis for $(\mathcal{H},\mathbb{K}^{(\alpha,\beta)})$ and consequently,

$$\sum_{n=0}^{\infty}\mathcal{R}_1(e_n)(z)\mathcal{R}_1(e_n)(w)^*=\mathbb{K}^{(\alpha,\beta)}(z,w),\ z,w\in\Omega_0. \tag{15}$$

A direct computation shows that

$$\big((\beta\partial_i - \alpha\partial_{m+i})e_n(z,\zeta)\big)_{|\Delta} = (\alpha+\beta)e_{n,i}(z,\zeta)_{|\Delta},\ 1 \le i \le m,\ n \ge 0.$$

Therefore $\mathcal{R}_1(e_n)(z) = \sqrt{\frac{\alpha+\beta}{\alpha\beta}}\begin{pmatrix} e_{n,1}(z,z) \\ \vdots \\ e_{n,m}(z,z)\end{pmatrix}$. Thus, using (15), we obtain

$$\sum_{n=0}^{\infty}\begin{pmatrix} e_{n,1}(z,z) \\ \vdots \\ e_{n,m}(z,z)\end{pmatrix}\begin{pmatrix} e_{n,1}(z,z) \\ \vdots \\ e_{n,m}(z,z)\end{pmatrix}^* = \tfrac{\alpha\beta}{(\alpha+\beta)}\mathbb{K}^{(\alpha,\beta)}(z,z),\quad z \in \Omega_0.$$

Now the proof is complete using (14). □

Remark 3.5. *Let $\mathcal{M}$ be a Hilbert modules over the polynomial ring $\mathbb{C}[\boldsymbol{z}]$ and $\mathcal{M}_0 \subseteq \mathcal{M}$ be a submodule. Let $\mathcal{Q}$ be the quotient module, i.e.,*

$$0 \longrightarrow \mathcal{M}_0 \longrightarrow \mathcal{M} \longrightarrow \mathcal{Q} \longrightarrow 0 \tag{16}$$

is a short exact sequence of Hilbert modules. Finding an invariant for the equivalence class of $\mathcal{Q}$ from that of the pair $\mathcal{M}_0, \mathcal{M}$ is one of the main problems of Sz.-Nagy - Foias model theory. An invariant is an object associated to the pair $\mathcal{M}_0, \mathcal{M}$ that depends only on the quotient $\mathcal{M}/\mathcal{M}_0$. Such an invariant is also said to be an invariant of the short exact sequence (16)*. Indeed, taking $\mathcal{M} = H^2(\mathbb{D})$ and $\mathcal{S}_i$, $i = 1, 2$, to be any two submodules of $H^2(\mathbb{D})$, then we know that while $\mathcal{S}_1$ is always equivalent to $\mathcal{S}_2$, the quotient modules $\mathcal{Q}_i := H^2(\mathbb{D})/\mathcal{S}_i$ are not necessarily equivalent, see the discussion after the Theorem on page 65 of* [6]*. Thus, finding invariants for quotient modules is an interesting problem.*

Let $H(z) = \big(\!\!\big(\langle s_i(z), s_j(z)\rangle\big)\!\!\big)_{i,j=1}^n$, $z \in \Omega$, be the Hermitian metric of a holomorphic (trivial) vector bundle E defined on Ω relative to the holomorphic frame $\{s_1, \ldots, s_n\}$. The curvature $\mathbf{K}_H$ of the vector bundle E is the $(1,1)$ *form*

$$\sum_{i,j=1}^{n}\overline{\partial}_j\big(H^{-1}\partial_i H\big)d\bar{z}_j \wedge dz_i.$$

The trace of the curvature $\mathbf{K}_H$ is obtained by replacing each of the coefficients $\overline{\partial}_j\big(H^{-1}\partial_i H\big)$ by their trace. Recall that the determinant bundle $\det E$ is a line bundle determined by the holomorphic frame $s_1 \wedge \cdots \wedge s_n$ and the Hermitian metric: $H(z) := \det H(z)$. The trace of the curvature of the vector bundle E and the curvature of the determinant bundle $\det E$ are equal, i.e., $\mathrm{trace}(\mathbf{K}_H) = \mathbf{K}_{\det H}$, see [8, Equation (4.6)].

Now, from Theorem 3.4, we see that the Hermitian structure for the Hilbert module $\mathcal{S}_1$ is $\mathbb{K}^{(\alpha,\beta)}$. Also, we have the following equality:

$$\mathrm{trace}\big(\mathbf{K}_{\mathbb{K}^{(\alpha,\beta)}}\big) = m\mathbf{K}_{K^{\alpha+\beta}} + \mathbf{K}_{\det(\mathcal{K}_K)}.$$

Thus, in these examples, we see that $\mathrm{trace}\big(\mathbf{K}_{\mathbb{K}^{(\alpha,\beta)}}\big)$ is a function of $\alpha+\beta$. Consequently, if $\alpha+\beta = \alpha'+\beta'$, then we have

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{A}_0 \arrow[r, "i"] & (\mathcal{H}, K^{\alpha}) \otimes (\mathcal{H}, K^{\beta}) \arrow[r, "\pi"] & (\mathcal{H}, K^{\alpha+\beta}) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{A}_0 \arrow[r, "i"] & (\mathcal{H}, K^{\alpha'}) \otimes (\mathcal{H}, K^{\beta'}) \arrow[r, "\pi"] & (\mathcal{H}, K^{\alpha'+\beta'}) \arrow[r] & 0,
\end{tikzcd}$$

and $\mathrm{trace}\big(\mathbf{K}_{\mathbb{K}^{(\alpha,\beta)}}\big) = \mathrm{trace}\big(\mathbf{K}_{\mathbb{K}^{(\alpha',\beta')}}\big)$. Replacing the equality of the quotient modules $(\mathcal{H}, K^{\alpha+\beta})$ and $(\mathcal{H}, K^{\alpha'+\beta'})$ by an isomorphism does not change the conclusion. In general, replace K^{α} by K_1; K^{β} by K_2, and $K^{\alpha'}$ by K_1'; $K^{\beta'}$ by K_2' and assume that $K_1'(w,w)K_2'(w,w) = \varphi(w)K_1(w,w)K_2(w,w)\overline{\varphi(w)}$ for *some non-vanishing holomorphic function defined on an open subset* $U \subset \Omega$. *This means that the quotient modules* $\mathcal{S}_0$ *and* $\mathcal{S}_0'$ *are equivalent. A straightforward computation then shows that* $\operatorname{trace}\big(\mathbf{K}_{K_{12}}\big) = \operatorname{trace}\big(\mathbf{K}_{K_{12}'}\big)$, *where* $K_{12} = \mathcal{G}_{(K_1K_2)^{-1}}$ *and similarly,* $K_{12}' = \mathcal{G}_{(K_1'K_2')^{-1}}$. *Hence* $\operatorname{trace}\big(\mathbf{K}_{K_{12}}\big)$ *is an invariant of the short exact sequences of the form*

$$0 \longrightarrow \mathcal{A}_0 \xrightarrow{\ i\ } (\mathcal{H}, K_1) \otimes (\mathcal{H}, K_2) \xrightarrow{\ \pi\ } (\mathcal{H}, K_1K_2) \longrightarrow 0.$$

We expect this to be the case in much greater generality.

The following corollary is immediate by choosing $\alpha = 1 = \beta$ in Theorem 3.4. It also gives an alternative for computing the Gaussian curvature defined in (1) whenever the metric is of the form $K(z,z)^{-1}$ for some positive definite kernel K defined on $\Omega \times \Omega$, where $\Omega \subset \mathbb{C}$ is a bounded domain. Indeed, the assumption that $\boldsymbol{T}$ is in $B_1(\Omega)$ is not necessary to arrive at the formula in the corollary below.

Corollary 3.6. *Let* $\boldsymbol{T}$ *be a commuting* m*-tuple in the Cowen-Douglas class* $B_1(\Omega)$ *realized as the adjoint of the* m*-tuple* M *of multiplication operators by coordinate functions on a reproducing kernel Hilbert space* $(\mathcal{H}, K) \subseteq \operatorname{Hol}(\Omega_0)$, *for some open subset* Ω_0 *of* Ω. *Then the curvature* $\mathcal{K}_{\boldsymbol{T}}(z) = \Big(\mathcal{K}_{\boldsymbol{T}}(z)_{i,j}\Big)_{i,j=1}^{m}$ *is given by the formula*

$$\mathcal{K}_{\boldsymbol{T}}(z)_{i,j} = \frac{2}{K(z,z)^2} \lim_{\substack{\zeta_i \to z_i \\ \zeta_j \to z_j}} \left(\frac{K_{\mathcal{A}_0}(z,\zeta; z,\zeta)}{(z_i - \zeta_i)(\bar{z}_j - \bar{\zeta}_j)} \bigg|_{\zeta_l = z_l, l \neq i,j} \right), \; z \in \Omega, \; 1 \le i,j \le m.$$

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References

[1] J. Agler and J. E. McCarthy, Pick interpolation and Hilbert function spaces, Graduate Studies in Mathematics, vol. 44, American Mathematical Society, Providence, RI, 2002.

[2] N. Aronszajn, *Theory of reproducing kernels*, Trans. Amer. Math. Soc. **68** (1950), 337–404.

[3] S. Biswas, D. K. Keshari, and G. Misra, *Infinitely divisible metrics and curvature inequalities for operators in the Cowen-Douglas class*, J. Lond. Math. Soc. (2) **88** (2013), 941–956.

[4] M. J. Cowen and R. G. Douglas, *Complex geometry and operator theory*, Acta Math. **141** (1978), no. 3-4, 187–261.

[5] ———, *Operators possessing an open set of eigenvalues*, Functions, series, operators, Vol. I, II (Budapest, 1980), Colloq. Math. Soc. János Bolyai, vol. 35, North-Holland, Amsterdam, (1983), 323–341.

[6] ——— *On moduli of invariant subspaces*, In: Apostol, C., Douglas, R.G., Sz.-Nagy, B., Voiculescu, D., Arsene, G. (eds) Invariant Subspaces and Other Topics. Operator Theory: Advances and Applications, vol 6. Birkhäuser, Basel.

[7] R. E. Curto and N. Salinas, *Generalized bergman kernels and the cowen-douglas theory*, American Journal of Mathematics **106** (1984), 447–488.

[8] J.-P. Demailly, *Complex Analytic and Differential Geometry*, Universite de Grenoble, 2007. http://www-fourier.ujf-grenoble.fr/~demailly/manuscripts/agbook.pdf

[9] R. G. Douglas and G. Misra, *Some calculations for Hilbert modules*, J. Orissa Math. Soc., **12-15** (1993-96), 75–85.

[10] ———, *On quotient modules*, In: Recent advances in operator theory and related topics (Szeged, 1999), 203–209, Oper. Theory Adv. Appl., 127, Birkhäuser, Basel, 2001.

[11] R. G. Douglas, G. Misra and J. Sarkar, *Contractive Hilbert modules and their dilations*, Israel J. Math. **187** (2012), 141–165

[12] R. G. Douglas, G. Misra and C. Varughese, *On quotient modules - the case of arbitrary multiplicity*, J. Funct. Anal. 174 (2000), 364–398.

[13] R. G. Douglas and V. I. Paulsen, *Hilbert modules over function algebras*, Longman Sc & Tech, 1989.

[14] S. Ghara and G. Misra, *Decomposition of the tensor product of two Hilbert modules*, Operator theory, operator algebras and their interactions with geometry and topology – Ronald G. Douglas memorial volume, Birkhäuser/Springer, Cham (2020), 221–265

[15] ———, *Curvature inequalities and extremal properties of bundle shifts*, J. Operator Theory **11** (1984), 305–317.

[16] V. I. Paulsen and M. Raghupathi, *An introduction to the theory of reproducing kernel Hilbert spaces*, Cambridge Studies in Advanced Mathematics **152**, Cambridge University Press, Cambridge, (2016).

[17] Md. R. Reza, *Curvature inequalities for operators in the Cowen-Douglas class of a planar domain*, Indiana Univ. Math. J. **67** (2018), 1255–1279.
[18] K. Wang and G. Zhang, *Curvature inequalities for operators of the Cowen-Douglas class*, Israel J. Math. **222** (2017), 279–296.

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