RIGIDITY OF THE FLAG STRUCTURE FOR A CLASS OF COWEN-DOUGLAS OPERATORS

KUI JI, CHUNLAN JIANG, DINESH KUMAR KESHARI, AND GADADHAR MISRA

Abstract. The explicit description of irreducible homogeneous operators in the Cowen-Douglas class and the localization of Hilbert modules naturally leads to the definition of a smaller class possessing a flag structure. These operators are shown to be irreducible. It is also shown that the flag structure is rigid, that is, the unitary equivalence class of the operator and the flag structure determine each other. A complete set of unitary invariants, which are somewhat more tractable than those of an arbitrary operator in the Cowen-Douglas class, are obtained.

1. Introduction

In their very influential paper [1], Cowen and Douglas initiated the study of the following important class of operators.

Definition 1.1. For a connected open subset Ω of $\mathbb{C}$ and a positive integer n, let

$$B_n(\Omega) = \begin{array}{ll} \{ & T \in \mathcal{L}(\mathcal{H}) \mid \Omega \subset \sigma(T), \\ & \operatorname{ran}(T-w) = \mathcal{H} \text{ for } w \in \Omega, \\ & \bigvee_{w\in\Omega} \ker(T-w) = \mathcal{H}, \\ & \dim\ \ker(T-w) = n \text{ for } w \in \Omega \ \}, \end{array}$$

where $\mathcal{L}(\mathcal{H})$ is the algebra of all bounded linear operators on a complex separable Hilbert space $\mathcal{H}$ and $\sigma(T)$ is the spectrum of the operator T.

It is shown in [1, Proposition 1.12] that if T is in $B_n(\Omega)$, then it is possible to choose n eigenvectors in $\ker(T-w)$, which are holomorphic as functions of $w \in \Omega$. Thus $w \mapsto \ker(T-w)$ defines a rank n holomorphic Hermitian vector bundle E_T over Ω. It therefore follows that the holomorphic Hermitian vector bundle E_T is the sub-bundle of the trivial holomorphic Hermitian bundle $\Omega \times \mathcal{H}$ defined by

$$E_T = \{(w,x) \in \Omega \times \mathcal{H} : x \in \ker(T-w)\}$$

with the natural projection map $\pi : E_T \to \Omega$, $\pi(w,x) = w$ (cf. [1]). Here is one of the main results from [1].

Theorem 1.2. *The operators T and $\widetilde{T}$ in $B_n(\Omega)$ are unitarily equivalent if and only if the corresponding holomorphic Hermitian vector bundles E_T and $E_{\widetilde{T}}$ are equivalent.*

2010 *Mathematics Subject Classification.* 47B32, 47B35.

Key words and phrases. The Cowen-Douglas class, strongly irreducible operator, homogeneous operator, curvature, second fundamental form.

The work of K. Ji was supported by the Foundation for the Author of National Excellence Doctoral Dissertation of China (Grant No. 201116). The work of C. Jiang was supported by National Natural Science Foundation of China (Grant No. A010602). The work of D.K. Keshari was supported by a Research Associateship of the Indian Institute of Science. The work of G. Misra was supported by J C Bose National Fellowship and the UGC, SAP-IV.

They also find a set of complete invariant for this equivalence consisting of the curvature of E_T and its covariant derivatives. Unfortunately, these invariants are not easy to compute except when the rank of the bundle is 1. In this case, the curvature

$$\mathcal{K}(w)\, dw \wedge d\bar{w} = -\frac{\partial^2 \log \parallel \gamma(w) \parallel^2}{\partial w \partial \overline{w}} dw \wedge d\bar{w}$$

of the line bundle E_T, defined with respect to a non-zero holomorphic section γ of E_T, is a complete unitary invariant of the operator T. The definition of the curvature, in this case, is independent of the choice of the non-vanishing section γ: If γ_0 is another holomorphic (non-vanishing) section of E, then $\gamma_0 = \phi\gamma$ for some holomorphic function ϕ on an open subset Ω_0 of Ω, consequently the harmonicity of $\log|\phi|$ completes the verification. However, if the rank of the vector bundle is strictly greater than 1, then only the eigenvalues of the curvature are independent of the choice of the holomorphic frame. This limits the use of the curvature and its covariant derivative if the rank of the bundle is not 1. It is difficult to determine, in general, when an operator $T \in B_n(\Omega)$ is irreducible, again except in the case $n = 1$. In this case, the rank of the vector bundle is 1 and therefore it is irreducible and so is the operator T.

One may therefore ask: For what class of holomorphic Hermitian vector bundles, defined on a bounded open connected set $\Omega \subseteq \mathbb{C}$, of rank n, the curvature remains a complete invariant. Refining the proof of Lemma 3.2 of [1], one may infer that the curvature is a complete invariant for the class consisting of the n-fold direct sum of line bundles. Examples were given in [13, Example 2.1] to show that the class of the curvature alone does not determine the class of the vector bundle except in the case of a line bundle. The splitting of a holomorphic Hermitian vector bundle into a direct sum is determined by the vanishing of the second fundamental form (see [10, Proposition 6.14]). In this paper, we isolate those irreducible holomorphic Hermitian vector bundles, namely, the ones possessing a flag structure, for which the curvature together with the second fundamental form is a complete set of invariants. Among these, we study in detail the ones that correspond to irreducible operators in the Cowen-Douglas class $B_2(\Omega)$. All irreducible homogeneous operators in $B_2(\mathbb{D})$ are in this class. We obtain, using the methods developed in this paper, a description of all these operators. This classification was given earlier by D. Wilkins [14] using a sophisticated mix of Riemannian geometry and operator theory. We also investigate the case of $n > 2$, where together with the curvature and the second fundamental form, we find a set $\frac{n(n-1)}{2} + 1$ invariants, which are easy to compute. Finally, we show that these are a complete set of unitary invariants.

We discuss this new class of operators in $B_2(\Omega)$ separately and then provide the details for the case of $n > 2$. One important reason for separating out the case of $n = 2$ is that the proofs that appear in this case are often necessary to begin an inductive proof in the case of an arbitrary $n \in \mathbb{N}$.

In a forthcoming paper, we construct similarity invariants for the operators in this new class. A generalization to the case of commuting tuples of operators is apparent which we intend to consider in future work.

The results of this paper were announced in [7] and was the topic of a talk presented by the last author in the Workshop “Hilbert Modules and Complex Geometry” held during Apr 20 - 26, 2014 at Oberwolfach.

Acknowledgement. The research reported here was started during a post doctoral visit of Kui Ji to the Indian Institute of Science and was completed during a month long research visit of G. Misra to Hebei Normal university. We thank both of these Institutions for their admirable hospitality.

The authors thank the referee for several constructive suggestions. One of these suggestions simplified significantly the main step in the proof of the Rigidity theorem.

2. A new class of operators in $B_2(\Omega)$

2.1. Definitions. If T is an operator in $B_2(\Omega)$, then there exists a pair of operators T_0 and T_1 in $B_1(\Omega)$ and a bounded operator S such that $T = \begin{pmatrix} T_0 & S \\ 0 & T_1 \end{pmatrix}$. This is Theorem 1.49 of [8, page 48]. We show, the other way round, that two operators T_0 and T_1 from $B_1(\Omega)$ combine with the aid of an arbitrary bounded linear operator S to produce an operator in $B_2(\Omega)$.

Proposition 2.1. *Let T be a bounded linear operator of the form $\begin{pmatrix} T_0 & S \\ 0 & T_1 \end{pmatrix}$. Suppose that the two operators T_0, T_1 are in $B_1(\Omega)$. Then the operator T is in $B_2(\Omega)$.*

Proof. Suppose T_0 and T_1 are defined on the Hilbert spaces $\mathcal{H}_0$ and $\mathcal{H}_1$, respectively. Elementary considerations from index theory of Fredholm operators shows that the operator T is Fredholm and $\operatorname{ind}(T) = \operatorname{ind}(T_0) + \operatorname{ind}(T_1)$ (cf. [2, page 360]). Therefore, to complete the proof that T is in $B_2(\Omega)$, all we have to do is prove that the vectors in the kernel $\ker(T - w)$, $w \in \Omega$, span the Hilbert space $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$.

Let γ_0 and t_1 be non-vanishing holomorphic sections for the two line bundles E_0 and E_1 corresponding to the operators T_0 and T_1, respectively. For each $w \in \Omega$, the operator $T_0 - w$ is surjective. Therefore we can find a vector $\alpha(w)$ in $\mathcal{H}_0$ such that $(T_0 - w)\alpha(w) = -S(t_1(w))$, $w \in \Omega$. Setting $a(w) = \alpha(w) + t_1(w)$, we see that

$$(T - w)a(w) = 0 = (T - w)\gamma_0(w).$$

Thus $\{\gamma_0(w), a(w)\} \subseteq \ker(T - w)$ for w in Ω. If x is any vector orthogonal to $\ker(T - w)$, $w \in \Omega$, then in particular it is orthogonal to the vectors $\gamma_0(w)$ and $a(w)$, $w \in \Omega$, forcing it to be the zero vector. $\square$

We impose one additional condition on these operators, namely, $T_0 S = S T_1$ and assume that the operator S is non-zero. With this seemingly innocuous hypothesis, we show that *(i)* it is irreducible, *(ii)* and that any intertwining unitary operator between two of these operators must be diagonal and *(iii)* the curvature of E_{T_0} together with the second fundamental form of the inclusion $E_{T_0} \subseteq E_T$ form a complete set of unitary invariants for the operator T. It is therefore natural to isolate this class of operators.

Definition 2.2. We let $\mathcal{F}B_2(\Omega)$ denote the set of all bounded linear operators T of the form $T = \begin{pmatrix} T_0 & S \\ 0 & T_1 \end{pmatrix}$, where the two operators T_0, T_1 are assumed to be in the Cowen-Douglas class $B_1(\Omega)$ and the operator S is assumed to be a non-zero intertwiner between them, that is, $T_0 S = T_1 S$.

Specifically, if the operator T_i, $i = 0, 1$, is defined on the separable complex Hilbert space $\mathcal{H}_i$, then S is assumed to be a non-zero bounded linear operator from $\mathcal{H}_1$ to $\mathcal{H}_0$ such that $T_0 S = T_1 S$. The operator T is defined on the Hilbert space $\mathcal{H} := \mathcal{H}_0 \oplus \mathcal{H}_1$.

Each of the operators in $\mathcal{F}B_2(\Omega)$ is also in the Cowen-Douglas class $B_2(\Omega)$ by virtue of Proposition 2.1. Thus $\mathcal{F}B_2(\Omega) \subseteq B_2(\Omega)$.

Although, in the definition of the class $\mathcal{F}B_2(\Omega)$ given above, we have only assumed that S is non-zero, its range must be dense as is shown below.

Proposition 2.3. *Suppose T_0 and T_1 are two operators in $B_1(\Omega)$, and S is a bounded operator intertwining T_0 and T_1, that is, $T_0 S = S T_1$. Then S is non zero if and only if range of S is dense if and only if S^* is injective.*

Proof. Let γ be a holomorphic frame of E_{T_1}. Assume that S is a non zero operator. The intertwining relationship $T_0 S = S T_1$ implies that $S \circ \gamma$ is a section of E_{T_0}. Clearly, there exists an open set Ω_0 contained in Ω such that $S \circ \gamma$ is not zero on Ω_0, otherwise S has to be zero. Since $S(\gamma)$

is a holomorphic frame of E_{T_0} on Ω_0, it follows that the closure of the linear span of the vectors $\{S(\gamma(w)) : w \in \Omega_0\}$ must equal $\mathcal{H}_0$. Hence the range of the operator S is dense. $\square$

The following Proposition provides several equivalent characterizations of operators in the class $\mathcal{F}B_2(\Omega)$.

Proposition 2.4. *Suppose T is a bounded linear operator on a Hilbert space $\mathcal{H}$, which is in $B_2(\Omega)$. Then the following conditions are equivalent.*

(i) *There exist an orthogonal decomposition $\mathcal{H}_0 \oplus \mathcal{H}_1$ of $\mathcal{H}$ and operators $T_0 : \mathcal{H}_0 \to \mathcal{H}_0$, $T_1 : \mathcal{H}_1 \to \mathcal{H}_1$, and $S : \mathcal{H}_1 \to \mathcal{H}_0$ such that $T = \begin{pmatrix} T_0 & S \\ 0 & T_1 \end{pmatrix}$, where $T_0, T_1 \in B_1(\Omega)$ and $T_0 S = S T_1$, that is, $T \in \mathcal{F}B_2(\Omega)$.*

(ii) *There exists a holomorphic frame $\{\gamma_0, \gamma_1\}$ of E_T such that $\frac{\partial}{\partial w}\|\gamma_0(w)\|^2 = \langle \gamma_1(w), \gamma_0(w)\rangle$.*

(iii) *There exists a holomorphic frame $\{\gamma_0, \gamma_1\}$ of E_T such that $\gamma_0(w)$ and $\frac{\partial}{\partial w}\gamma_0(w) - \gamma_1(w)$ are orthogonal for all w in Ω.*

Proof. $(i) \implies (ii)$: Pick any two non-vanishing holomorphic sections t_0 and t_1 for the line bundles E_{T_0} and E_{T_1} respectively. Then

$$\begin{aligned} (T - w)t_1(w) &= (T_1 - w)t_1(w) + S(t_1(w)) \\ &= S(t_1(w). \end{aligned}$$

Since $T_0 S = S T_1$, it induces a bundle map from E_{T_1} to E_{T_0}, so $S(t_1(w)) = \psi(w)t_0(w)$ for some holomorphic function ψ defined on Ω. Thus $(T - w)t_1(w) = \psi(w)t_0(w)$. Setting $\gamma_0(w) := \psi(w)t_0(w)$ and $\gamma_1(w) := \frac{\partial}{\partial w}\gamma_0(w) - t_1(w)$, we see that $\{\gamma_0(w), \gamma_1(w)\} \subset \ker(T - w)$. Now assume that

$$\alpha_0\gamma_0(w) + \alpha_1\gamma_1(w) = 0 \tag{2.1}$$

for a pair of complex numbers α_0 and α_1. Then

$$\begin{aligned} 0 &= \langle \alpha_0\gamma_0(w) + \alpha_1\gamma_1(w), t_1(w)\rangle \\ &= \alpha_1\langle \gamma_1(w), t_1(w)\rangle \\ &= -\alpha_1\|t_1(w)\|^2. \end{aligned} \tag{2.2}$$

From equations (2.1) and (2.2), it follows that $\alpha_0 = \alpha_1 = 0$. Thus $\{\gamma_0, \gamma_1\}$ is a holomorphic frame of E_T. Since $\langle t_1(w), \gamma_0(w)\rangle = 0$, we see that

$$\frac{\partial}{\partial w}\|\gamma_0(w)\|^2 = \langle \gamma_1(w), \gamma_0(w)\rangle.$$

$(ii) \iff (iii)$: This equivalence is evident from the definition.

$(iii) \implies (i)$: Set $t_1(w) := \frac{\partial}{\partial w}\gamma_0(w) - \gamma_1(w)$. Let $\mathcal{H}_0$ and $\mathcal{H}_1$ be the closed linear span of $\{\gamma_0(w) : w \in \Omega\}$ and $\{t_1(w) : w \in \Omega\}$, respectively. Set $T_0 = T_{|\mathcal{H}_0}$, $T_1 = P_{\mathcal{H}_1}T_{|\mathcal{H}_1}$ and $S = P_{\mathcal{H}_0}T_{|\mathcal{H}_1}$.

We see that the closed linear span of the vectors $\{\gamma_0(w), t_1(w) : w \in \Omega\}$ is $\mathcal{H}$: Suppose x in $\mathcal{H}$ is orthogonal to this set of vectors. Then clearly, $x \perp \gamma_0(w)$ and $x \perp t_1(w)$ for all w in Ω. Or, equivalently $x \perp \gamma_0(w)$ and $x \perp \gamma_1(w)$ for all w in Ω. Therefore x must be the 0 vector. Next, we show that the two operators T_0 and T_1 are in $B_1(\Omega)$.

Clearly, $(T_1 - w)$ is onto. Thus $\operatorname{index}(T_1 - w) = \dim\ \ker(T_1 - w)$ and $2 = \operatorname{index}(T - w) = \operatorname{index}(T_0 - w) + \operatorname{index}(T_1 - w)$. It follows that $\dim\ \ker(T_1 - w) = 1$ or 2.

Suppose $\dim\ker(T_1 - w) = 2$ and $\{s_1(w), s_2(w)\}$ be a holomorphic choice of linearly independent vectors in $\ker(T_1 - w)$. Then we can find holomorphic functions ϕ_1, ϕ_2 defined on Ω such that

$S(s_1(w)) = \phi_1(w)\gamma_0(w)$ and $S(s_2(w)) = \phi_2(w)\gamma_0(w)$. Setting

$$\begin{aligned}\tilde\gamma_0(w) &:= \gamma_0(w),\\ \tilde\gamma_1(w) &:= \frac{\partial}{\partial w}(\phi_1(w)\gamma_0(w)) - s_1(w) \text{ and}\\ \tilde\gamma_2(w) &:= \frac{\partial}{\partial w}(\phi_2(w)\gamma_0(w)) - s_2(w),\end{aligned}$$

we see that $(T-w)(\tilde\gamma_i(w)) = 0$ for $0 \le i \le 2$. If $\sum_{i=0}^2 \alpha_i\tilde\gamma_i(w) = 0$, $\alpha_i \in \mathbb{C}$, then

$$\alpha_0\gamma_0(w) + \tfrac{\partial}{\partial w}\big((\alpha_1\phi_1(w) + \alpha_2\phi_2(w))\gamma_0(w)\big) + \alpha_1 s_1(w) + \alpha_2 s_2(w) = 0.$$

It follows that $\alpha_1 s_1(w) + \alpha_2 s_2(w) = 0$ since $\mathcal{H}_0$ is orthogonal to $\mathcal{H}_1$. Hence $\alpha_1 = \alpha_2 = 0$ implying $\alpha_0 = 0$. Thus we have $\dim\ \ker(T-w) \ge 3$. This contradiction proves that $\dim\ker(T_0 - w) = 1$ and hence T_1 is in $B_1(\Omega)$.

To show that T_0 is in $B_1(\Omega)$, pick any $x \in \mathcal{H}_0$, and note that there exist $z \in \mathcal{H}$ such that $(T-w)z = x$ since $T-w$ is onto. Let $z_{\mathcal{H}_1}$ and $z_{\mathcal{H}_0}$ be the projections of z to the subspaces $\mathcal{H}_0$ and $\mathcal{H}_1$, respectively. We have $[(T_0 - w)z_{\mathcal{H}_0} + S(z_{\mathcal{H}_1})] + (T_1 - w)z_{\mathcal{H}_1} = x$. Therefore $(T_1 - w)z_{\mathcal{H}_1} = 0$ and $(T_0 - w)z_{\mathcal{H}_0} + S(z_{\mathcal{H}_1}) = x$. Since dim ker $(T_1 - w) = 1$, so $z_{\mathcal{H}_1} = c_1 t_1(w)$, it follows that

$$\begin{aligned} x &= (T_0 - w)z_{\mathcal{H}_0} + S(z_{\mathcal{H}_1})\\ &= (T_0 - w)z_{\mathcal{H}_0} + S(c_1 t_1(w))\\ &= (T_0 - w)z_{\mathcal{H}_0} + c_1\gamma_0(w)\\ &= (T_0 - w)z_{\mathcal{H}_0} + (T_0 - w)(c_1\tfrac{\partial}{\partial w}\gamma_0(w))\\ &= ((T_0 - w)(z_{\mathcal{H}_0} + c_1\tfrac{\partial}{\partial w}\gamma_0(w)).\end{aligned}$$

Thus $T_0 - w$ is onto. We have $2 = \dim\ \ker\,(T-w) = \dim\ \ker\,(T_0 - w) + \dim\ \ker\,(T_1 - w)$. Hence $\dim\ \ker\,(T_0 - w) = 1$ and we see that T_0 is in $B_1(\Omega)$.

Finally, since $S(t_1(w)) = \gamma_0(w)$, it follows that $T_0 S = S T_1$. □

2.2. Models for operators in $\mathcal{F}B_2(\Omega)$. An operator $T \in \mathcal{F}B_2(\Omega)$ is also in $B_2(\Omega)$, therefore as is well-known (cf. [1, 3]), it can be realized as the adjoint of a multiplication operator on some reproducing kernel Hilbert space of holomorphic $\mathbb{C}^2$-valued functions. These functions are defined on $\Omega^* := \{w : \bar w \in \Omega\}$. An explicit description for operators in $\mathcal{F}B_2(\Omega)$ follows.

Let E_T be the holomorphic Hermitian vector bundle over Ω corresponding to the operator T. Since T is in $\mathcal{F}B_2(\Omega)$, we may find a holomorphic frame $\gamma = \{\gamma_0, \gamma_1\}$ such that $\gamma_0(w)$ and $\frac{\partial}{\partial w}\gamma_0(w) - \gamma_1(w)$ are orthogonal for all w in Ω. Define $\Gamma : \mathcal{H} \to \mathcal{O}(\Omega^*, \mathbb{C}^2)$ as follows:

$$\Gamma(x)(z) = \big(\langle x, \gamma_0(\bar z)\rangle, \langle x, \gamma_1(\bar z)\rangle\big)^{\mathrm{tr}} \qquad z \in \Omega^*,\ x \in \mathcal{H},$$

where $\mathcal{O}(\Omega^*, \mathbb{C}^2)$ is the space of holomorphic functions defined on Ω^* which take values in $\mathbb{C}^2$. Here $(\,\cdot\,,\,\cdot\,)^{\mathrm{tr}}$ denotes the transpose of the vector $(\,\cdot\,,\,\cdot\,)$.

The map Γ is injective and therefore transplanting the inner product from $\mathcal{H}$ on the range of Γ, we make it unitary from $\mathcal{H}$ onto $\mathcal{H}_\Gamma := \operatorname{ran}\Gamma$. Define K_Γ to be the function on $\Omega^* \times \Omega^*$ taking values in the 2×2 matrices $\mathcal{M}_2(\mathbb{C})$:

$$\begin{aligned} K_\Gamma(z,w) &= \big(\!\big(\langle\gamma_j(\bar w), \gamma_i(\bar z)\rangle\big)\!\big)_{i,j=0}^1\\ &= \begin{pmatrix} \langle\gamma_0(\bar w), \gamma_0(\bar z)\rangle & \frac{\partial}{\partial\bar w}\langle\gamma_0(\bar w), \gamma_0(\bar z)\rangle\\ \frac{\partial}{\partial z}\langle\gamma_0(\bar w), \gamma_0(\bar z)\rangle & \frac{\partial^2}{\partial z\partial\bar w}\langle\gamma_0(\bar w), \gamma_0(\bar z)\rangle + \langle t_1(\bar w), t_1(\bar z)\rangle\end{pmatrix}\\ &= \begin{pmatrix} K_0(z,w) & \frac{\partial}{\partial\bar w}K_0(z,w)\\ \frac{\partial}{\partial z}K_0(z,w) & \frac{\partial^2}{\partial z\partial\bar w}K_0(z,w)\end{pmatrix} + \begin{pmatrix} 0 & 0\\ 0 & K_1(z,w)\end{pmatrix}, \end{aligned} \tag{2.3}$$

where $t_1(\bar{w}) = \frac{\partial}{\partial \bar{w}}\gamma_0(\bar{w}) - \gamma_1(\bar{w})$, $K_0(z,w) = \langle \gamma_0(\bar{w}), \gamma_0(\bar{z})\rangle$ and $K_1(z,w) = \langle t_1(\bar{w}), t_1(\bar{z})\rangle$ for z, w in Ω^*. Set $(K_\Gamma)_w(\cdot) = K_\Gamma(\cdot, w)$. It is then easily verified that K_Γ has the following properties:

(1) The reproducing property: $\langle \Gamma(x)(\cdot), (K_\Gamma)_w(\cdot)\eta\rangle_{\operatorname{ran}\Gamma} = \langle \Gamma(x)(w), \eta\rangle_{\mathbb{C}^2}$, $x \in \mathcal{H}$, $\eta \in \mathbb{C}^2$, $w \in \Omega^*$.
(2) The unitary operator Γ intertwines the operators T defined on $\mathcal{H}$ and M^* defined on $\mathcal{H}_\Gamma$, namely, $\Gamma T^* = M_z \Gamma$.
(3) Each w in Ω is an eigenvalue with eigenvector $(K_\Gamma)_{\bar{w}}(\cdot)\eta$, $\eta \in \mathbb{C}^2$, for the operator $M^* = \Gamma T \Gamma^*$.

2.3. Rigidity. Once we represent an operator T from $\mathcal{F}B_2(\Omega)$ in this form, the possibilities for the change of frame are limited. The admissible ones are described in the following lemma.

Lemma 2.5. *Let T be an operator in $\mathcal{F}B_2(\Omega)$. Suppose $\{\gamma_0, \gamma_1\}$, $\{\tilde{\gamma}_0, \tilde{\gamma}_1\}$ are two frames of the vector bundle E_T such that $\gamma_0(w) \perp (\frac{\partial}{\partial w}\gamma_0(w) - \gamma_1(w))$ and $\tilde{\gamma}_0(w) \perp (\frac{\partial}{\partial w}\tilde{\gamma}_0(w) - \tilde{\gamma}_1(w))$ for all $w \in \Omega$. If $\phi = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22}\end{pmatrix}$ is any change of frame between $\{\gamma_0, \gamma_1\}$ and $\{\tilde{\gamma}_0, \tilde{\gamma}_1\}$, that is,*

$$\{\tilde{\gamma}_0, \tilde{\gamma}_1\} = \{\gamma_0, \gamma_1\} \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22}\end{pmatrix},$$

then $\phi_{21} = 0$, $\phi_{11} = \phi_{22}$ and $\phi_{12} = \phi'_{11}$.

Proof. Define the unitary map Γ, as above, using the holomorphic frame $\gamma = \{\gamma_0, \gamma_1\}$. The operator T is then unitarily equivalent to the adjoint of the multiplication operator on the Hilbert space $\mathcal{H}_\Gamma$ possessing a reproducing kernel K_Γ of the form (2.3). Let e_1 and e_2 be the standard unit vectors in $\mathbb{C}^2$. Clearly, $(K_\Gamma)_w(\cdot)e_1$ and $(K_\Gamma)_w(\cdot)e_2$ are two linearly independent eigenvectors of M^* with eigenvalue $\bar{w}$.

Similarly, corresponding to the holomorphic frame $\tilde{\gamma} = \{\tilde{\gamma}_0, \tilde{\gamma}_1\}$, the operator T is unitarily equivalent to the adjoint of multiplication operator on the Hilbert space $\mathcal{H}_{\tilde{\Gamma}}$. The reproducing kernel $K_{\tilde{\Gamma}}$ is again of the form (2.3) except that K_0 and K_1 must be replaced by $\tilde{K}_0$ and $\tilde{K}_1$, respectively.

For $i = 0, 1$, set $s_i(w) := (K_\Gamma)(w)e_i$, and $\tilde{s}_i(w) := (K_{\tilde{\Gamma}})(w)e_i$. Let $\phi(w) := \begin{pmatrix} \phi_{00}(w) & \phi_{01}(w) \\ \phi_{10}(w) & \phi_{11}(w)\end{pmatrix}$ be the holomorphic function, taking values in 2×2 matrices, such that

$$(\tilde{s}_0(w), \tilde{s}_1(w)) = (s_0(w), s_1(w))\phi(w).$$

This implies that

$$\tilde{s}_0(w) = \phi_{00}(w)s_0(w) + \phi_{10}(w)s_1(w) \tag{2.4}$$

and

$$\tilde{s}_1(w) = \phi_{01}(w)s_0(w) + \phi_{11}(w)s_1(w). \tag{2.5}$$

From Equation (2.4), equating the first and the second coordinates separately, we have

$$(\tilde{K}_0)_w(\cdot) = \phi_{00}(w)(K_0)_w(\cdot) + \phi_{10}(w)\frac{\partial}{\partial \bar{w}}(K_0)_w(\cdot) \tag{2.6}$$

and

$$\frac{\partial}{\partial z}(\tilde{K}_0)_w(\cdot) = \phi_{00}(w)\frac{\partial}{\partial z}(K_0)_w(\cdot) + \phi_{10}(w)\frac{\partial^2}{\partial z \partial \bar{w}}(K_0)_w(\cdot) + \phi_{10}(w)(K_1)_w(\cdot). \tag{2.7}$$

From these two equations, we get

$$\phi_{00}(w)\frac{\partial}{\partial z}(K_0)_w(\cdot) + \phi_{10}(w)\frac{\partial^2}{\partial z \partial \bar{w}}(K_0)_w(\cdot) =$$
$$\phi_{00}(w)\frac{\partial}{\partial z}(K_0)_w(\cdot) + \phi_{10}(w)\frac{\partial^2}{\partial z \partial \bar{w}}(K_0)_w(\cdot) + \phi_{10}(w)(K_1)_w(\cdot),$$

which implies that $\phi_{10} = 0$. Finally, from Equation (2.5), we have

$$\frac{\partial}{\partial \bar{w}}(\tilde{K}_0)_w(\cdot) = \phi_{01}(w)(K_0)_w(\cdot) + \phi_{11}(w)\frac{\partial}{\partial \bar{w}}(K_0)_w(\cdot) \tag{2.8}$$

The Equations (2.5) and (2.8) together give

$$\phi_{01} = \phi_{00}' \text{ and } \phi_{00} = \phi_{11}$$

completing the proof. □

A very important consequence of this Lemma is that the decomposition of the operators in the class $\mathcal{F}B_2(\Omega)$ is unique in the sense described in the following proposition.

Proposition 2.6. *Let $T, \tilde{T} \in \mathcal{F}B_2(\Omega)$ be two operators of of the form $\begin{pmatrix} T_0 & S \\ 0 & T_1 \end{pmatrix}$ and $\begin{pmatrix} \tilde{T}_0 & \tilde{S} \\ 0 & \tilde{T}_1 \end{pmatrix}$ with respect to the decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ and $\tilde{\mathcal{H}} = \tilde{\mathcal{H}}_0 \oplus \tilde{\mathcal{H}}_1$, respectively. Let $U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$: $\mathcal{H}_0 \oplus \mathcal{H}_1 \to \tilde{\mathcal{H}}_0 \oplus \tilde{\mathcal{H}}_1$ be an unitary operator such that*

$$\begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} T_0 & S \\ 0 & T_1 \end{pmatrix} = \begin{pmatrix} \tilde{T}_0 & \tilde{S} \\ 0 & \tilde{T}_1 \end{pmatrix} \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix},$$

then $U_{12} = U_{21} = 0$.

Proof. Let $\{\gamma_0, \gamma_1\}$ and $\{\tilde{\gamma_0}, \tilde{\gamma_1}\}$ be holomorphic frames of E_T and $E_{\tilde{T}}$ respectively with the property that $\gamma_0 \perp (\frac{\partial}{\partial w}\gamma_0 - \gamma_1)$ and $\tilde{\gamma}_0 \perp (\frac{\partial}{\partial w}\tilde{\gamma}_0 - \tilde{\gamma}_1)$. Set $t_1 := (\frac{\partial}{\partial w}\gamma_0 - \gamma_1)$ and $\tilde{t}_1 := (\frac{\partial}{\partial w}\tilde{\gamma}_0 - \tilde{\gamma}_1)$. Since U intertwines T and $\tilde{T}$, it follows that $\{U\gamma_0, U\gamma_1\}$ is a second holomorphic frame of $E_{\tilde{T}}$ with the property $U\gamma_0 \perp (\frac{\partial}{\partial w}(U\gamma_0) - U\gamma_1) = U(t_1)$. By Lemma 2.5, we have that

$$U(\gamma_0) = \phi\tilde{\gamma}_0 \tag{2.9}$$

and

$$U(\gamma_1) = \phi'\tilde{\gamma}_0 + \phi\tilde{\gamma}_1. \tag{2.10}$$

From equations (2.9) and (2.10), we get

$$U(t_1) = \phi\,\tilde{t}_1. \tag{2.11}$$

From equations (2.9) and (2.11), it follows that U maps $\mathcal{H}_0$ to $\mathcal{H}_0$ and $\mathcal{H}_1$ to $\mathcal{H}_1$. Thus U is a block diagonal from $\mathcal{H}_0 \oplus \mathcal{H}_1$ onto $\tilde{\mathcal{H}}_0 \oplus \tilde{\mathcal{H}}_1$ □

Remark 2.7. In summary, we note that a holomorphic change of frame for the vector bundle E_T, preserving the orthogonality relation between γ_0 and $\frac{\partial}{\partial w}\gamma_0(w) - \gamma_1(w)$, must be of the form $\begin{pmatrix} \varphi & \varphi' \\ 0 & \varphi \end{pmatrix}$. Thus such a change of frame for the vector bundle E_T induces change of frame $\begin{pmatrix} \varphi & 0 \\ 0 & \varphi \end{pmatrix}$ for the vector bundle $E_{\begin{pmatrix} T_0 & 0 \\ 0 & T_1 \end{pmatrix}}$ and vice-versa.

Corollary 2.8. *For $i = 0, 1$, let T_i be any two operators in $B_1(\Omega)$. Let S and $\tilde{S}$ be bounded linear operators such that $T_0 S = ST_1$ and $T_0\tilde{S} = \tilde{S}T_1$. If $T = \begin{pmatrix} T_0 & S \\ 0 & T_1 \end{pmatrix}$ and $\tilde{T} = \begin{pmatrix} T_0 & \tilde{S} \\ 0 & T_1 \end{pmatrix}$, then T is unitarily equivalent to $\tilde{T}$ if and only if $\tilde{S} = e^{i\theta}S$ for some real number θ.*

Proof. Suppose that $UT = \tilde{T}U$ for some unitary operator U. We have just shown that such an operator U must be diagonal, say $U = \begin{pmatrix} U_{11} & 0 \\ 0 & U_{22} \end{pmatrix}$. Hence we have

$$U_{11}T_0 = T_0U_{11},\ U_{22}T_1 = T_1U_{22},\ U_{11}S = \tilde{S}U_{22}. \tag{2.12}$$

Since U_{11} is unitary, the first of the equations (2.12) implies that

$$U_{11} \in \{T_0, T_0^*\}' := \{W \in \mathcal{L}(\mathcal{H}_0) : WT_0 = T_0W \text{ and } WT_0^* = T_0^*W\}.$$

Since T_0 is an irreducible operator, we conclude that $U_{11} = e^{i\theta_1} I_{\mathcal{H}_0}$ for some $\theta_1 \in \mathbb{R}$. Similarly, $U_{22} = e^{i\theta_2} I_{\mathcal{H}_1}$ for some $\theta_2 \in \mathbb{R}$. Hence the third equation in (2.12) implies that $\tilde{S} = e^{i(\theta_1-\theta_2)} S$.

Conversely suppose that $\tilde{S} = e^{i\theta} S$ for some real number θ. Then evidently the operator $U := \begin{pmatrix} \exp\left(i\frac{\theta}{2}\right) I_{\mathcal{H}_0} & 0 \\ 0 & \exp\left(-i\frac{\theta}{2}\right) I_{\mathcal{H}_1} \end{pmatrix}$ is unitary on $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ and $UT = \tilde{T}U$. $\square$

Corollary 2.9. *For $i = 0, 1$, let T_i be two operators in $B_1(\Omega)$. Let S be a non-zero bounded linear operators such that $T_0 S = S T_1$. If $T_\mu = \begin{pmatrix} T_0 & \mu S \\ 0 & T_1 \end{pmatrix}$ and $T_{\tilde{\mu}} = \begin{pmatrix} T_0 & \tilde{\mu} S \\ 0 & T_1 \end{pmatrix}$, μ, $\tilde{\mu} > 0$, then T_μ is unitarily equivalent to $T_{\tilde{\mu}}$ if and only if $\mu = \tilde{\mu}$.*

2.4. A complete set of unitary invariants. The following theorem lists a complete set of unitary invariants for operators in $\mathcal{F}B_2(\Omega)$.

Theorem 2.10. *Suppose that $T = \begin{pmatrix} T_0 & S \\ 0 & T_1 \end{pmatrix}$ and $\tilde{T} = \begin{pmatrix} \tilde{T}_0 & \tilde{S} \\ 0 & \tilde{T}_1 \end{pmatrix}$ are any two operators in $\mathcal{F}B_2(\Omega)$. Then the operators T and $\tilde{T}$ are unitarily equivalent if and only if $\mathcal{K}_{T_1} = \mathcal{K}_{\tilde{T}_1}$ (or, $\mathcal{K}_{T_0} = \mathcal{K}_{\tilde{T}_0}$) and $\frac{\|S(t_1)\|^2}{\|t_1\|^2} = \frac{\|\tilde{S}(\tilde{t}_1)\|^2}{\|\tilde{t}_1\|^2}$, where t_1 and $\tilde{t}_1$ are non-vanishing holomorphic sections for the vector bundles E_{T_1} and $E_{\tilde{T}_1}$, respectively.*

Proof. On a small small open subset of Ω, we can assume that $S(t_1)$ and $\tilde{S}(\tilde{t}_1)$ are holomorphic frames of the bundle E_{T_0} and $E_{\tilde{T}_0}$, respectively. First suppose that $\bar{\partial}\partial \log \|S(t_1)\|^2 = \bar{\partial}\partial \log \|\tilde{S}(\tilde{t}_1)\|^2$ and $\frac{\|S(t_1)\|^2}{\|t_1\|^2} = \frac{\|\tilde{S}(\tilde{t}_1)\|^2}{\|\tilde{t}_1\|^2}$. Then we claim that T and $\tilde{T}$ are unitarily equivalent. The equality of the curvatures, namely, $\bar{\partial}\partial \log \|S(t_1)\|^2 = \bar{\partial}\partial \log \|\tilde{S}(\tilde{t}_1)\|^2$ implies that $\|S(t_1)\|^2 = |\phi|^2 \|\tilde{S}(\tilde{t}_1)\|^2$ for some non-vanishing holomorphic function ϕ on Ω. It may be that we have to shrink, without loss of generality, to a smaller open set Ω_0. The second of our assumptions gives $\|t_1\|^2 = |\phi|^2 \|\tilde{t}_1\|^2$. Let $\gamma_0(w) := S(t_1(w))$ and $\tilde{\gamma}_0(w) := \tilde{S}(\tilde{t}_1(w))$; $\gamma_1(w) := \frac{\partial}{\partial w}\gamma_0(w) - t_1(w)$ and $\tilde{\gamma}_1(w) := \frac{\partial}{\partial w}\tilde{\gamma}_0(w) - \tilde{t}_1(w)$. It follows that $\{\gamma_0, \gamma_1\}$ and $\{\tilde{\gamma}_0, \tilde{\gamma}_1\}$ are holomorphic frames of E_T and $E_{\tilde{T}}$, respectively. Define the map $\Phi : E_T \to E_{\tilde{T}}$ as follows:

(1) $\Phi(\gamma_0(w)) = \phi(w)\tilde{\gamma}_0(w)$,
(2) $\Phi(\gamma_1(w)) = \phi'(w)\tilde{\gamma}_0(w) + \phi(w)\tilde{\gamma}_1(w)$.

Clearly, Φ is holomorphic. Note that

$$\begin{aligned} \langle \Phi(\gamma_0(w)), \Phi(\gamma_1(w)) \rangle &= \langle \phi(w)\tilde{\gamma}_0(w), \phi'(w)\tilde{\gamma}_0(w) + \phi(w)\tilde{\gamma}_1(w) \rangle \\ &= \langle \phi(w)\tilde{\gamma}_0(w), \phi'(w)\tilde{\gamma}_0(w) + \phi(w)(\tfrac{\partial}{\partial w}\tilde{\gamma}_0(w) - \tilde{t}_1(w)) \rangle \\ &= \langle \phi(w)\tilde{\gamma}_0(w), \tfrac{\partial}{\partial w}(\phi(w)\tilde{\gamma}_0(w)) - \phi(w)\tilde{t}_1(w) \rangle \\ &= \tfrac{\partial}{\partial \bar{w}} \|\phi(w)\tilde{\gamma}_0(w)\|^2 \\ &= \tfrac{\partial}{\partial \bar{w}} \|\gamma_0(w)\|^2 \end{aligned}$$

and

$$\begin{aligned} \langle \gamma_0(w), \gamma_1(w) \rangle &= \langle \gamma_0(w), \tfrac{\partial}{\partial w}\gamma_0(w) - t_1(w) \rangle \\ &= \tfrac{\partial}{\partial \bar{w}} \|\gamma_0(w)\|^2. \end{aligned}$$

Hence we have $\langle \Phi(\gamma_0(w)), \Phi(\gamma_1(w)) \rangle = \langle \gamma_0(w), \gamma_1(w) \rangle$. Similarly, $\|\Phi(\gamma_0(w))\| = \|\gamma_0(w)\|$ and $\|\Phi(\gamma_1)\| = \|\gamma_1\|$. Thus E_T and $E_{\tilde{T}}$ are equivalent as holomorphic Hermitian vector bundles. Hence T and $\tilde{T}$ are unitarily equivalent by Theorem 1.2 of Cowen and Douglas.

Conversely, suppose T and $\tilde{T}$ are unitarily equivalent. Let $U : \mathcal{H} \to \tilde{\mathcal{H}}$ be the unitary map such that $UT = \tilde{T}U$. By proposition 2.6, U takes the form $\begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix}$ for some pair of unitary operators

U_1 and U_2. Hence we have $U_1(S(t_1)) = \phi_1(\tilde{S}(\tilde{t}_1))$ and $U_2 t_1 = \phi_2 \tilde{t}_1$. The intertwining relation $U_1 S = \tilde{S} U_2$ implies that $\phi_1 = \phi_2$. Thus $\mathcal{K}_{T_0} = \mathcal{K}_{\tilde{T}_0}$ and

$$\frac{\|S(t_1)\|^2}{\|t_1\|^2} = \frac{\|U_1(S(t_1))\|^2}{\|U_2(t_1)\|^2} = \frac{\|\phi_1\tilde{S}(\tilde{t}_1)\|^2}{\|\phi_2\tilde{t}_1\|^2} = \frac{\|\tilde{S}(\tilde{t}_1)\|^2}{\|\tilde{t}_1\|^2}.$$

This verification completes the proof. □

2.5. The second fundamental form. We relate the invariants of Theorem 2.10 to the second fundamental form of the inclusion $E_0 \subseteq E$. The computation of the second fundamental form is given below following [6, page. 2244]. Here E_0, is the line bundle corresponding to the operator T_0 and E is the vector bundle of rank 2 corresponding to the operator T in $\mathcal{F}B_2(\Omega)$. Let $\{\gamma_0, \gamma_1\}$ be a holomorphic frame for E such that γ_0 and $t_1 := \partial\gamma_0 - \gamma_1$ are orthogonal. One obtains an orthonormal frame, say, $\{e_0, e_1\}$, from the holomorphic frame $\{\gamma_0, \gamma_1\}$ by the usual Gram-Schmidt process – Set $h = \langle\gamma_0, \gamma_0\rangle$, and observe that

$$e_1 = h^{-1/2}\gamma_0,\ e_2 = \frac{\gamma_1 - \frac{\gamma_0\langle\gamma_1,\gamma_0\rangle}{\|\gamma_0\|^2}}{(\|\gamma_1\|^2 - \frac{|\langle\gamma_1,\gamma_0\rangle|^2}{\|\gamma_0\|^2})^{1/2}}$$

are orthogonal. The canonical hermitian connection D for the vector bundle E_T is given, in terms of e_1 and e_2 by the formula:

$$\begin{aligned} D\,e_1 &= D^{1,0}e_1 + D^{0,1}e_1 \\ &= \alpha_{11}e_1 + \alpha_{21}e_2 + \bar{\partial}e_1 \\ &= (\alpha_{11} - \bar{\partial}(\log h))e_1 + \alpha_{21}e_2 \\ &= \theta_{11}e_1 + \theta_{21}e_2, \end{aligned}$$

where α_{11}, α_{21} are $(1,0)$ forms to be determined. Similarly, we have

$$\begin{aligned} D\,e_2 &= D^{1,0}e_2 + D^{0,1}e_2 \\ &= \alpha_{12}e_1 + \alpha_{22}e_2 + \bar{\partial}e_2 \\ &= \left(\alpha_{12} - h^{1/2}\frac{\bar{\partial}(h^{-1}\langle\gamma_2,\gamma_1\rangle)}{(\|\gamma_2\|^2 - \frac{|\langle\gamma_2,\gamma_1\rangle|^2}{\|\gamma_1\|^2})^{1/2}}\right)e_1 + \left(\alpha_{22} - \frac{1}{2}\frac{\bar{\partial}(\|\gamma_2\|^2 - \frac{\langle\gamma_2,\gamma_1\rangle}{\|\gamma_1\|^2})}{(\|\gamma_2\|^2 - \frac{\langle\gamma_2,\gamma_1\rangle}{\|\gamma_1\|^2})}\right)e_2 \\ &= \theta_{12}e_1 + \theta_{22}e_2, \end{aligned}$$

where α_{12}, α_{22} are $(1,0)$ forms to be determined. Since we are working with an orthonormal frame, the compatibility of the connection with the Hermitian metric gives

$$\begin{aligned} \langle D\,e_i, e_j\rangle + \langle e_i, D\,e_j\rangle &= \theta_{ji} + \bar{\theta}_{ij} \\ &= 0 \quad \text{for} \quad 1 \le i,j \le 2. \end{aligned}$$

For $1 \le i,j \le 2$, equating $(1,0)$ and $(0,1)$ forms separately to zero in the equation $\theta_{ij} + \bar{\theta}_{ji} = 0$, we obtain $\alpha_{11} = \partial(\log h)$, $\alpha_{12} = 0$, $\alpha_{21} = h^{1/2}\frac{\bar{\partial}(h^{-1}\langle\gamma_1,\gamma_0\rangle)}{(\|\gamma_1\|^2 - \frac{|\langle\gamma_1,\gamma_0\rangle|^2}{\|\gamma_0\|^2})^{1/2}}$ and $\alpha_{22} = \frac{1}{2}\frac{\bar{\partial}(\|\gamma_1\|^2 - \frac{\langle\gamma_1,\gamma_0\rangle}{\|\gamma_0\|^2})}{(\|\gamma_1\|^2 - \frac{\langle\gamma_1,\gamma_0\rangle}{\|\gamma_0\|^2})}$. Hence the second fundamental form for the inclusion $E_0 \subset E$ is given by the formula:

$$\theta_{12} = -h^{1/2}\frac{\bar{\partial}(h^{-1}\langle\gamma_1,\gamma_0\rangle)}{\left(\|\gamma_1\|^2 - \frac{|\langle\gamma_1,\gamma_0\rangle|^2}{\|\gamma_0\|^2}\right)^{1/2}} = -\frac{\frac{\partial^2}{\partial z\partial\bar{z}}\log h\,d\bar{z}}{\left(\frac{\|_1\|^2}{\|\gamma_0\|^2} + \frac{\partial^2}{\partial z\partial\bar{z}}\log h\right)^{1/2}},$$

where $t_1 = \gamma_0' - \gamma_1$. If $T = \begin{pmatrix} T_0 & S \\ 0 & T_1 \end{pmatrix}$ is an operator in $\mathcal{F}B_2(\Omega)$ and t_1 is a non-vanishing holomorphic section of the vector bundle E_1 corresponding to the operator T_1, then we may assume, without loss of generality, that $S(t_1)$ is a holomorphic frame of E_0. The second fundamental form θ_{12} of the inclusion $E_0 \subseteq E$, in this case, is therefore equal to

$$-\frac{\frac{\partial^2}{\partial z \partial \bar{z}} \log \|S(t_1)\|^2 d\bar{z}}{\left(\frac{\|t_1\|^2}{\|S(t_1)\|^2} + \frac{\partial^2}{\partial z \partial \bar{z}} \log \|S(t_1)\|^2\right)^{1/2}}.$$

It follows from Theorem 2.10 that the second fundamental form of the inclusion $E_0 \subseteq E$ and the curvature of E_1 form a complete set of invariants for the operator T. We restate Theorem 2.10 using the second fundamental form θ_{12}.

Theorem 2.11. *Suppose that* $T = \begin{pmatrix} T_0 & S \\ 0 & T_1 \end{pmatrix}$ *and* $\tilde{T} = \begin{pmatrix} \tilde{T}_0 & \tilde{S} \\ 0 & \tilde{T}_1 \end{pmatrix}$ *are any two operators in* $\mathcal{F}B_2(\Omega)$. *Then the operators* T *and* $\tilde{T}$ *are unitarily equivalent if and only if* $\mathcal{K}_{T_1} = \mathcal{K}_{\tilde{T}_1}$ *(or* $\mathcal{K}_{T_0} = \mathcal{K}_{\tilde{T}_0}$*) and* $\theta_{12} = \tilde{\theta}_{12}$.

2.6. Application to homogeneous operators. We use the machinery developed here to list the unitary equivalence classes of homogeneous operators in $B_n(\mathbb{D})$, $n = 2$. For $n = 1$ this was done in [12] and in [14] for $n = 2$. The classification of homogeneous operators in $B_n(\mathbb{D})$ was given in [11] for an arbitrary n. The proofs of [14] and [11] use tools from Differential geometry and the representation theory of Lie groups respectively. While the description below is very close to the spirit of [12].

Definition 2.12. An operator T is said to be homogeneous if $\varphi(T)$ is unitarily equivalent to T for all φ in Möb which are analytic on the spectrum of T.

Proposition 2.13 ([12])**.** *An operator* T *in* $\mathcal{B}_1(\mathbb{D})$ *is homogeneous if and only if*

$$\mathcal{K}_T(w) = -\lambda(1 - |w|^2)^{-2}$$

for some positive real number λ.

Remark 2.14. From the Proposition 2.13, it follows that T is unitarily equivalent to the adjoint of the multiplication operator $M^{(\lambda)}$ acting on the reproducing kernel Hilbert space $(\mathcal{H}^{(\lambda)}, K^{(\lambda)})$, where the reproducing kernel $K^{(\lambda)}$ is of the form $\frac{1}{(1-z\bar{w})^{\lambda}}$, $z, w \in \mathbb{D}$.

Proposition 2.15. *Let* T *be an operator in* $\mathcal{F}B_2(\mathbb{D})$ *and let* t_1 *be a non-vanishing holomorphic section of the bundle* E_1 *corresponding to the operator* T_1. *For any* φ *in Möb, set* $t_{1,\varphi} = t_1 o \varphi^{-1}$. *The operator* T *is homogeneous if and only if* T_0, T_1 *are homogeneous and* $\frac{\|S(t_{1,\varphi})\|^2}{\|t_{1,\varphi}\|^2} = |(\varphi^{-1})'|^2 \frac{\|S(t_1)\|^2}{\|t_1\|^2}$ *for all* φ *in Möb.*

Proof. Using the intertwining property in the class $\mathcal{F}B_2(\mathbb{D})$, we see that

$$\varphi(T) = \begin{pmatrix} \varphi(T_0) & S\varphi'(T_1) \\ 0 & \varphi(T_1) \end{pmatrix}.$$

Suppose that T is homogeneous, that is, T is unitarily equivalent to $\varphi(T)$ for φ in Möb. From Theorem 2.10, it follows that T_0 is unitarily equivalent to $\varphi(T_0)$, T_1 is unitarily equivalent to $\varphi(T_1)$ and

$$\frac{\|S\varphi'(T_1)(t_{1,\varphi}(w))\|^2}{\|t_{1,\varphi}(w)\|^2} = \frac{\|S(t_1(w))\|^2}{\|t_1(w)\|^2}. \tag{2.13}$$

Now, we have

$$\begin{aligned} \frac{\|S\,\varphi'(T_1)(t_{1,\varphi}(w))\|^2}{\|t_{1,\varphi}(w)\|^2} &= \frac{\|S\varphi'(\varphi^{-1}(w))(t_{1,\varphi}(w))\|^2}{\|t_{1,\varphi}(w)\|^2} \\ &= \frac{|\varphi'(\varphi^{-1}(w))|^2 \|S(t_{1,\varphi}(w))\|^2}{\|t_{1,\varphi}(w)\|^2} \\ &= \frac{|(\varphi^{-1})'(w)|^{-2} \|S(t_{1,\varphi}(w))\|^2}{\|t_{1,\varphi}(w)\|^2}. \end{aligned} \tag{2.14}$$

From equations (2.13) and (2.14), it follows that

$$\frac{\|S(t_{1,\varphi}(w))\|^2}{\|t_{1,\varphi}(w)\|^2} = |(\varphi^{-1})'(w)|^2 \frac{\|S(t_1(w))\|^2}{\|t_1(w)\|^2}. \tag{2.15}$$

Conversely suppose that T_0, T_1 are homogeneous operators and

$$\frac{\|S(t_{1,\varphi}(w))\|^2}{\|t_{1,\varphi}(w)\|^2} = |(\varphi^{-1})'(w)|^2 \frac{\|S(t_1(w))\|^2}{\|t_1(w)\|^2}$$

for all φ in Möb. From equations (2.14), (2.15) and Theorem 2.10, it follows that T is a homogeneous operator. $\square$

Corollary 2.16. *An operator T in $\mathcal{F}B_2(\mathbb{D})$ is a homogeneous if and only if*

(i) *T_0 and T_1 are homogeneous operators;*
(ii) *$\mathcal{K}_{T_1}(w) = \mathcal{K}_{T_0}(w) + \mathcal{K}_{B^*}(w)$, $w \in \mathbb{D}$, where B is the forward Bergman shift;*
(iii) *$S(t_1(w)) = \alpha\gamma_0(w)$ for some positive real number α and $\|t_1(w)\|^2 = \frac{1}{(1-|w|^2)^{\lambda+2}}$, $\|\gamma_0(w)\|^2 = \frac{1}{(1-|w|^2)^{\lambda}}$.*

Proof. Suppose T is a homogeneous operator. Then Proposition 2.15 shows that T_0 and T_1 are homogeneous operators. We may therefore find non-vanishing holomorphic sections γ_0 and t_1 of E_0 and E_1, respectively, such that $\|\gamma_0(w)\|^2 = (1-|w|^2)^{-\lambda}$ and $\|t_1(w)\|^2 = (1-|w|^2)^{-\mu}$ for some positive real λ and μ. For φ in Möb, set $\gamma_{0,\varphi} = \gamma_0 \circ \varphi^{-1}$ and $t_{1,\varphi} = t_1 \circ \varphi^{-1}$. Clearly, $\|\gamma_{0,\varphi}(w)\|^2 = |(\varphi^{-1})'(w)|^{-\lambda}\|\gamma_0(w)\|^2$ and $\|t_{1,\varphi}(w)\|^2 = |(\varphi^{-1})'(w)|^{-\mu}\|t_1(w)\|^2$. Let $S(t_1(w)) = \psi(w)\gamma_0(w)$ for some holomorphic function ψ on $\mathbb{D}$. We have $S(t_{1,\varphi}(w)) = S(t_1(\varphi^{-1}(w))) = \psi(\varphi^{-1}(w))\gamma_0(\varphi^{-1}(w)) = \psi(\varphi^{-1}(w))\gamma_{0,\varphi}(w)$ and

$$\frac{\|S(t_{1,\varphi}(w))\|^2}{\|t_{1,\varphi}(w)\|^2} = |(\varphi^{-1})'(w)|^2 \frac{\|S(t_1(w))\|^2}{\|t_1(w)\|^2}. \tag{2.16}$$

Combining these we see that

$$\begin{aligned} \frac{\|S(t_{1,\varphi}(w))\|^2}{\|t_{1,\varphi}(w)\|^2} &= |\psi(\varphi^{-1}(w))|^2 \frac{\|(\gamma_{0,\varphi}(w))\|^2}{\|t_{1,\varphi}(w)\|^2} \\ &= |\psi(\varphi^{-1}(w))|^2 |(\varphi^{-1})'(w)|^{\mu-\lambda} \frac{\|(\gamma_0(w))\|^2}{\|t_1(w)\|^2}. \end{aligned} \tag{2.17}$$

From the equations (2.16) and (2.17), we get

$$|\psi(w)|^2 |(\varphi^{-1})'(w)|^{\lambda+2-\mu} = |\psi(\varphi^{-1}w)|^2 \tag{2.18}$$

Pick $\varphi = \varphi_u$, where $\varphi_u(w) = \frac{w-u}{1-\bar{u}w}$ and put $w = 0$ in the equation (2.18). Then

$$|\psi(0)|^2 (1-|u|^2)^{\lambda+2-\mu} = |\psi(u)|^2. \tag{2.19}$$

If $\psi(0) = 0$ then equation (2.19) implies that $\psi(u) = 0$ for all $u \in \mathbb{D}$, which makes $S = 0$ leading to a contradiction. Thus $\psi(0) \neq 0$. Differentiating of both sides the equation (2.19), we see that

$$(\lambda + 2 - \mu)\frac{\partial^2}{\partial u \partial \bar{u}} \log(1-|u|^2) = 0.$$

Hence we conclude that $\mu = \lambda + 2$. Putting $\mu = \lambda + 2$ in the equation (2.19) we find that ψ must be a constant function. Hence there is a constant α such that $S(t_1(w)) = \alpha\gamma_0(w)$ for all $w \in \Omega$. Finally,

$$\begin{aligned} \mathcal{K}_{T_1}(w) &= \bar{\partial}\partial \log \|t_1(w)\|^2 \\ &= \bar{\partial}\partial \log(1-|w|^2)^{-\mu} \\ &= \bar{\partial}\partial \log(1-|w|^2)^{-\lambda-2} \\ &= \bar{\partial}\partial \log(1-|w|^2)^{-\lambda} + \bar{\partial}\partial \log(1-|w|^2)^{-2} \\ &= \bar{\partial}\partial \log \|\gamma_0(w)\|^2 + \bar{\partial}\partial \log(1-|w|^2)^{-2} \\ &= \mathcal{K}_{T_0}(w) + \mathcal{K}_{B^*}(w). \end{aligned}$$

Conversely, suppose that conditions $(i), (ii)$ and (iii) are met. We need to show that T is a homogeneous operator. Condition (ii) is equivalent to $\mu = \lambda + 2$. By Proposition 2.15, it is sufficient to show that

$$\frac{\|S(t_{1,\varphi}(w))\|^2}{\|t_{1,\varphi}(w)\|^2} = |(\varphi^{-1})'(w)|^2 \frac{\|S(t_1(w))\|^2}{\|t_1(w)\|^2}.$$

However, we have

$$\begin{aligned} \frac{\|S(t_{1,\varphi}(w))\|^2}{\|t_{1,\varphi}(w)\|^2} &= |\alpha|^2 \frac{\|(\gamma_{0,\varphi}(w))\|^2}{\|t_{1,\varphi}(w)\|^2} \\ &= |\alpha|^2 |(\varphi^{-1})'(w)|^{\mu-\lambda} \frac{\|(\gamma_0(w))\|^2}{\|t_1(w)\|^2} \\ &= |\alpha|^2 |(\varphi^{-1})'(w)|^2 \frac{\|(\gamma_0(w))\|^2}{\|t_1(w)\|^2} \\ &= |(\varphi^{-1})'(w)|^2 \frac{\|S(t_1(w))\|^2}{\|t_1(w)\|^2}. \end{aligned}$$

□

2.7. Irreducibility and strong irreducibility in $\mathcal{F}B_2(\Omega)$. In this subsection, we show that an operator T in $\mathcal{F}B_2(\Omega)$ is irreducible. Furthermore, if the intertwining operator S is invertible, then T is strongly irreducible. (Recall that an operator T is said to be strongly irreducible if the commutant $\{T\}'$ of the operator T contains no idempotent operator.) We also provide a more direct proof of proposition 2.6, which easily generalizes to the case of an arbitrary n.

Definition 2.17. Let T_1 and T_2 be any two bounded linear operators on the Hilbert space $\mathcal{H}$. Define $\sigma_{T_1,T_2} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ to be the operator

$$\sigma_{T_1,T_2}(X) = T_1X - XT_2,\ X \in \mathcal{L}(\mathcal{H}).$$

Let $\sigma_T : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ be the operator $\sigma_{T,T}$.

An operator T defined on a Hilbert space $\mathcal{H}$ is said to be quasi-nilpotent if $\lim_{n\to\infty} \|T^n\|^{1/n} = 0$.

Lemma 2.18. *Suppose T is in $B_1(\Omega)$ and X is a quasi-nilpotent operator such that $TX = XT$. Then $X = 0$.*

Proof. Let γ be a non-vanishing holomorphic section for E_T. Since $TX = XT$, we see that $X(\gamma)$ is also a holomorphic section of E_T. Hence $X(\gamma(w)) = \phi(w)\gamma(w)$ for some holomorphic function ϕ defined on Ω. Clearly, $X^n(\gamma(w)) = \phi(w)^n\gamma(w)$. Now, we have

$$\begin{aligned} |\phi(w)|^n \|\gamma(w)\| &= \|\phi(w)^n\gamma(w)\| \\ &= \|X^n(\gamma(w))\| \\ &\leq \|X^n\|\|\gamma(w)\| \end{aligned}$$

Thus, for $n \in \mathbb{N}$ and $w \in \Omega$, we have $|\phi(w)| \leq \|X^n\|^{1/n}$ implying $\phi(w) = 0$, $w \in \Omega$. Hence $X = 0$. □

The following theorem from [9] is the key to an alternative proof of the proposition 2.6 and its generalization in the following section.

Theorem 2.19. *Let P, T be two bounded linear operators. If $P \in ran\,\sigma_T \cap ker\,\sigma_T$, then P is a quasi-nilpotent.*

A second Proof of Proposition 2.6

Proof. Suppose T is unitarily equivalent to $\tilde{T}$ via the unitary U, namely, $UT = TU$. Then

$$U_{21}S + U_{22}T_1 = \tilde{T}_1U_{22} \tag{2.20}$$

$$U_{21}T_0 = \tilde{T}_1U_{21}. \tag{2.21}$$

Equivalently, we also have $TU^* = U^*\tilde{T}$, which gives an additional relationship:

$$T_1U_{12}^* \;=\; U_{12}^*\tilde{T}_0. \tag{2.22}$$

Using these equations, we compute

$$\begin{aligned}
U_{21}SU_{12}^*\tilde{S} &= (\tilde{T}_1U_{22} - U_{22}T_1)U_{12}^*\tilde{S}\\
&= \tilde{T}_1U_{22}U_{12}^*\tilde{S} - U_{22}T_1U_{12}^*\tilde{S}\\
&= \tilde{T}_1U_{22}U_{12}^*\tilde{S} - U_{22}U_{12}^*\tilde{T}_0\tilde{S}\\
&= \tilde{T}_1U_{22}U_{12}^*\tilde{S} - U_{22}U_{12}^*\tilde{S}\tilde{T}_1\\
&= \sigma_{\tilde{T}_1}(U_{22}U_{12}^*\tilde{S}),
\end{aligned}$$

and

$$\begin{aligned}
(U_{21}SU_{12}^*\tilde{S})\tilde{T}_1 &= U_{12}SU_{12}^*\tilde{T}_0\tilde{S}\\
&= U_{21}ST_1U_{12}^*\tilde{S}\\
&= U_{21}T_0SU_{12}^*\tilde{S}\\
&= \tilde{T}_1(U_{12}\tilde{S}U_{12}^*\tilde{S}).
\end{aligned}$$

Thus $U_{21}SU_{12}^*\tilde{S} \in \operatorname{ran}\sigma_{\tilde{T}_1} \cap \ker\sigma_{\tilde{T}_1}$. From Lemma 2.18 and Theorem 2.19, it follows that

$$U_{21}SU_{12}^*\tilde{S} = 0.$$

Since $\tilde{S}$ has dense range, we have $U_{21}SU_{12}^* = 0$. Let us consider the two possibilities for U_{12}^*, namely, either $U_{12}^* = 0$ or $U_{12}^* \neq 0$. If $U_{12}^* \neq 0$, then from equation (2.22), U_{12}^* must have dense range. Since S also has dense range, we have $U_{21} = 0$. To complete the proof, we consider two cases.

Case 1: Suppose $U_{21} = 0$. In this case, we have to prove that $U_{12} = 0$. From $U^*U = I$, we get $U_{11}^*U_{11} = I$ and $U_{12}^*U_{11} = 0$. From $UT = \tilde{T}U$, we get $U_{11}T_0 = \tilde{T}_0U_{11}$, so U_{11} has dense rang. Since U_{11} is an isometry and has dense range, it follows that U_{11} is onto. Hence U_{11} is unitary. Since U_{11} is unitary and $U_{12}^*U_{11} = 0$, it follows that $U_{12} = 0$.

Case 2: Suppose $U_{12} = 0$. In this case, we have to prove that $U_{21} = 0$. We have $U_{11}U_{11}^* = I$ and $U_{21}U_{11}^* = 0$. The intertwining relation $TU^* = U^*\tilde{T}$ gives $T_0U_{11}^* = U_{11}^*\tilde{T}_0$. So U_{11}^* has dense range. Since U_{11}^* is an isometry and it has dense range, we must conclude that U_{11}^* is onto. Hence U_{11} is unitary and we have $U_{21}U_{11}^* = 0$ forcing U_{21} to be the 0 operator. ∎

Proposition 2.20. *Any operator T in $\mathcal{F}B_2(\Omega)$ is irreducible. Also, if $T = \begin{pmatrix} T_0 & I \\ 0 & T_0 \end{pmatrix}$, then it is strongly irreducible.*

Proof. Let $P = (P_{ij})_{2\times 2}$ be a projection in the commutant $\{T\}'$ of the operator T, that is,

$$\begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}\begin{pmatrix} T_0 & S \\ 0 & T_1 \end{pmatrix} = \begin{pmatrix} T_0 & S \\ 0 & T_1 \end{pmatrix}\begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}.$$

This equality implies that $P_{11}T_0 = T_0P_{11} + SP_{21}$, $P_{11}S + P_{12}T_1 = T_0P_{12} + SP_{22}$, $P_{21}T_0 = T_1P_{21}$ and $P_{21}S + P_{22}T_1 = T_1P_{22}$. Now

$$(P_{21}S)T_1 \;=\; P_{21}(ST_1) \;=\; P_{21}(T_0S) \;=\; (P_{21}T_0)S \;=\; T_1(P_{21}S).$$

Thus $P_{21}S \in \ker\sigma_{T_1}$. Also note that

$$P_{21}S \;=\; T_1P_{22} - P_{22}T_1 \;=\; \sigma_{T_1}(P_{22}).$$

Hence $P_{21}S \in \operatorname{ran}\sigma_{T_1} \cap \ker\sigma_{T_1}$. Thus from Lemma 2.18 and Theorem 2.19, it follows that $P_{21}S = 0$. The operator P_{21} must be 0 since S has dense range.

To prove the first statement, we may assume that the operator P is self adjoint and conclude P_{12} is 0 as well. Since both the operators T_0 and T_1 are irreducible and the projection P is diagonal, it follows that T must be irreducible.

For the proof of the second statement, note that if P is an idempotent of the form $\begin{pmatrix} P_{11} & P_{12} \\ 0 & P_{22} \end{pmatrix}$, both P_{11} and P_{22} must be idempotents. By our hypothesis, P_{11} and P_{22} must also commute with T_0, which is strongly irreducible, hence $P_{11} = 0$ or I and $P_{22} = 0$ or I. By using Theorem 2.19, we see that if $P = \begin{pmatrix} I & P_{12} \\ 0 & 0 \end{pmatrix}$ or $P = \begin{pmatrix} 0 & P_{12} \\ 0 & I \end{pmatrix}$, then P does not commute with $\begin{pmatrix} T_0 & I \\ 0 & T_0 \end{pmatrix}$. Thus $P = \begin{pmatrix} I & P_{12} \\ 0 & I \end{pmatrix}$ or $P = \begin{pmatrix} 0 & P_{12} \\ 0 & 0 \end{pmatrix}$. Now, using the equation $P^2 = P$, we conclude that P_{12} must be zero. Thus $P = I$ or $P = 0$.

□

We now give a sufficient condition for an operator T in $\mathcal{F}B_2(\Omega)$ to be strongly irreducible.

Proposition 2.21. *Let $T = \begin{pmatrix} T_0 & S \\ 0 & T_1 \end{pmatrix}$ be an operator in $\mathcal{F}B_2(\Omega)$. If the operator S is invertible, then the operator T is strongly irreducible.*

Proof. By our hypothesis, the operator $X = \begin{pmatrix} I & 0 \\ 0 & S \end{pmatrix}$ is invertible. Now

$$\begin{aligned} XTX^{-1} &= \begin{pmatrix} I & 0 \\ 0 & S \end{pmatrix} \begin{pmatrix} T_0 & S \\ 0 & T_1 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & S \end{pmatrix}^{-1} \\ &= \begin{pmatrix} T_0 & I \\ 0 & ST_1S^{-1} \end{pmatrix} \\ &= \begin{pmatrix} T_0 & I \\ 0 & T_0 \end{pmatrix}. \end{aligned}$$

Thus T is similar to a strongly irreducible operator and consequently it is strongly irreducible. □

We conclude this section with a characterization of strong irreducibility in $\mathcal{F}B_2(\Omega)$.

Proposition 2.22. *An operator $T = \begin{pmatrix} T_0 & S \\ 0 & T_1 \end{pmatrix}$ in $\mathcal{F}B_2(\Omega)$ is strongly irreducible if and only if $S \notin$ ran σ_{T_0,T_1}.*

Proof. Let P be an idempotent in the commutant $\{T\}'$ of the operator T. The proof of the Proposition 2.20 shows that P must be upper triangular: $\begin{pmatrix} P_{11} & P_{12} \\ 0 & P_{22} \end{pmatrix}$. The commutation relation $PT = TP$ gives us $P_{11}T_0 = T_0P_{11}$, $P_{22}T_1 = T_1P_{22}$ and

$$P_{11}S - SP_{22} = T_0P_{12} - P_{12}T_1. \tag{2.23}$$

Since $P_{i+1i+1} \in \{T_i\}'$ for $0 \le i \le 1$, it follows that P_{ii} can be either I or 0. If either $P_{11} = I$ and $P_{22} = 0$ or $P_{11} = 0$ and $P_{22} = I$, then S is in ran σ_{T_0,T_1} contradicting our assumption. Thus P is of the form $\begin{pmatrix} I & P_{12} \\ 0 & I \end{pmatrix}$ or $\begin{pmatrix} 0 & P_{12} \\ 0 & 0 \end{pmatrix}$. Since P is an idempotent operator, we must have $P_{12} = 0$. Hence T is strongly irreducible.

Assume that the operator S is in ran σ_{T_0,T_1}. In this case, we show that T cannot be strongly irreducible completing the proof. Since $S \in$ ran σ_{T_0,T_1}, we can find an operator P_{12} such that

$$\begin{aligned} S &= \sigma_{T_0,T_1}(P_{12}) \\ &= T_0P_{12} - P_{12}T_1. \end{aligned} \tag{2.24}$$

The operator $P = \begin{pmatrix} I & P_{12} \\ 0 & 0 \end{pmatrix}$ is an idempotent operator. We have

$$\begin{pmatrix} I & P_{12} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} T_0 & S \\ 0 & T_1 \end{pmatrix} = \begin{pmatrix} T_0 & S + P_{12}T_1 \\ 0 & 0 \end{pmatrix} \tag{2.25}$$

and

$$\begin{pmatrix} T_0 & S \\ 0 & T_1 \end{pmatrix} \begin{pmatrix} I & P_{12} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} T_0 & T_0P_{12} \\ 0 & 0 \end{pmatrix}. \tag{2.26}$$

From these equations, we have $PT = TP$ proving that the operator T is not strongly irreducible. $\square$

3. Rigidity of the flag structure

We begin by describing, what one may think of as, the natural generalization of the class $\mathcal{F}B_2(\Omega)$ to operators in $B_n(\Omega)$ for an arbitrary $n \in \mathbb{N}$.

Definition 3.1. We let $\mathcal{F}B_n(\Omega)$ be the set of all bounded linear operators T defined on some complex separable Hilbert space $\mathcal{H} = \mathcal{H}_0 \oplus \cdots \oplus \mathcal{H}_{n-1}$, which are of the form

$$T = \begin{pmatrix} T_0 & S_{0,1} & S_{0,2} & \cdots & S_{0,n-1} \\ 0 & T_1 & S_{1,2} & \cdots & S_{1,n-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & T_{n-2} & S_{n-2,n-1} \\ 0 & \cdots & \cdots & 0 & T_{n-1} \end{pmatrix},$$

where the operator $T_i : \mathcal{H}_i \to \mathcal{H}_i$, defined on the complex separable Hilbert space $\mathcal{H}_i$, $0 \le i \le n-1$, is assumed to be in $B_1(\Omega)$ and $S_{i,i+1} : \mathcal{H}_{i+1} \to \mathcal{H}_i$, is assumed to be a non-zero intertwining operator, namely, $T_iS_{i,i+1} = S_{i,i+1}T_{i+1}$, $0 \le i \le n-2$.

Even without mandating the intertwining condition, the set of operators described above belong to the Cowen-Douglas class $B_n(\Omega)$. An inductive proof presents no difficulty starting with the base case of $n = 2$, which was proved in the previous section. Therefore, in particular, $\mathcal{F}B_n(\Omega) \subseteq B_n(\Omega)$. We begin with a preparatory Lemma for proving the rigidity theorem.

Lemma 3.2. *An invertible X that intertwines two operators in $\mathcal{F}B_n(\Omega)$. Let $Y = X^{-1}$. If $X = (\!(X_{i,j})\!)_{n\times n}$, $Y = (\!(Y_{i,j})\!)_{n\times n}$ are the block decompositions of the two operators X and Y, respectively, then $X_{n-1,j} = 0$, $0 \le j \le n-2$, and $Y_{n-1,j} = 0$, $0 \le j \le n-2$.*

Proof. Consider the three possibilities:

1. $X_{n-1,j} = 0$, $0 \le j \le n-2$, but $Y_{n-1,j} \ne 0$ for some $0 \le j \le n-2$.
2. $Y_{n-1,j} = 0$, $0 \le j \le n-2$, $X_{n-1,j} \ne 0$ for some $0 \le j \le n-2$.
3. $X_{n-1,j} \ne 0$ for some $0 \le j \le n-2$ and $Y_{n-1,k} \ne 0$ for some $0 \le k \le n-2$.

In each of these cases, we arrive at a contradiction proving the Lemma.

Case 1: Choose l to be the smallest index such that $Y_{n-1,l} \ne 0$, that is, $Y_{n-1,i} = 0$ for $0 \le i \le l-1$ but $Y_{n-1,l} \ne 0$. For this index l, the intertwining relation $TY = Y\tilde{T}$ implies $T_{n-1}Y_{n-1,l} = Y_{n-1,l}\tilde{T}_l$. Since $Y_{n-1,l} \ne 0$, it follows from Proposition 2.3 that $Y_{n-1,l}$ has dense range. From $XY = I$, we get $X_{n-1,n-1}Y_{n-1,l} = 0$ and $X_{n-1,n-1}Y_{n-1,n-1} = I$. Since $Y_{n-1,l}$ has dense range and $X_{n-1,n-1}Y_{n-1,l} = 0$, we conclude that $X_{n-1,n-1} = 0$. This contradicts the identity: $X_{n-1,n-1}Y_{n-1,n-1} = I$.

Case 2: The contradiction in this case is arrived at exactly in the same manner as in the first case after interchanging the roles of X and Y.

Case 3: Pick j, l to be the smallest index such that $X_{n-1,j} \neq 0$ and $Y_{n-1,l} \neq 0$. We have that $XT = \tilde{T}X$. Consequently,

$$X_{n-1,j}T_j = \tilde{T}_{n-1}X_{n-1,j}, \; X_{n-1,j}S_{j,j+1} + X_{n-1,j+1}T_{j+1} = \tilde{T}_{n-1}X_{n-1,j+1}. \tag{3.1}$$

Since $T_kS_{k,k+1} = S_{k,k+1}T_{k+1}$ for $k = 0, 1, 2, \cdots n-1$, multiplying the second equation in (3.1) by $S_{j+1,j+2} \cdots S_{n-2,n-1}$, and replacing $T_{j+1}S_{j+1,j+2} \cdots S_{n-2,n-1}$ with $S_{j+1,j+2} \cdots S_{n-2,n-1}T_{n-1}$, we have

$$\begin{aligned} X_{n-1,j}S_{j,j+1} \cdots S_{n-2,n-1} + X_{n-1,j+1}S_{j+1,j+2} \cdots S_{n-2,n-1}T_{n-1} \\ = \tilde{T}_{n-1}X_{n-1,j+1}S_{j+1,j+2} \cdots S_{n-2,n-1}. \end{aligned} \tag{3.2}$$

We also have $TY = Y\tilde{T}$, which gives us

$$T_{n-1}Y_{n-1,l} = Y_{n-1,l}\tilde{T}_l. \tag{3.3}$$

Now, multiply both sides of the equation (3.2) by $Y_{n-1,l}$, using the commutation $T_{n-1}Y_{n-1,l} = Y_{n-1,l}\tilde{T}_l$, then again multiplying both sides of the resulting equation by $\widetilde{S}_{l,l+1} \cdots \widetilde{S}_{n-2,n-1}$ and finally using the commutation relations $\tilde{T}_k\widetilde{S}_{k,k+1} = \widetilde{S}_{k,k+1}\tilde{T}_{k+1}, \; 0 \leq k \leq n-1$, we have

$$\begin{aligned} X_{n-1,j}S_{j,j+1} \cdots S_{n-2,n-1}Y_{n-1,l}\tilde{S}_{l,l+1} \cdots \tilde{S}_{n-2,n-1} \\ + X_{n-1,j+1}S_{j,j+1} \cdots S_{n-2,n-1}Y_{n-1,l}\tilde{S}_{l,l+1} \cdots \tilde{S}_{n-2,n-1}\tilde{T}_{n-1} \\ = \tilde{T}_{n-1}X_{n-1,j+1}S_{j+1,j+2} \cdots S_{n-2,n-1}Y_{n-1,l}\tilde{S}_{l,l+1} \cdots \tilde{S}_{n-2,n-1}. \end{aligned} \tag{3.4}$$

Therefore, we see that

$$X_{n-1,j}S_{j,j+1}S_{j+1,j+2} \cdots S_{n-2,n-1}Y_{n-1,l}\tilde{S}_{l,l+1} \cdots \tilde{S}_{n-2,n-1}$$

is in the range of the operator $\sigma_{\tilde{T}_{n-1}}$. Indeed it is also in the kernel of $\sigma_{\tilde{T}_{n-1}}$, as is evident from the following string of equalities:

$$\begin{aligned} & X_{n-1,j}S_{j,j+1}S_{j+1,j+2} \cdots S_{n-2,n-1}Y_{n-1,l}\tilde{S}_{l,l+1} \cdots \tilde{S}_{n-2,n-1}\tilde{T}_{n-1} \\ &= X_{n-1,j}S_{j,j+1}S_{j+1,j+2} \cdots S_{n-2,n-1}Y_{n-1,l}\tilde{T}_l\tilde{S}_{l,l+1} \cdots \tilde{S}_{n-2,n-1} \\ &= X_{n-1,j}S_{j,j+1}S_{j+1,j+2} \cdots S_{n-2,n-1}T_{n-1}Y_{n-1,l}\tilde{S}_{l,l+1} \cdots \tilde{S}_{n-2,n-1} \\ &= X_{n-1,j}T_jS_{j,j+1}S_{j+1,j+2} \cdots S_{n-2,n-1}Y_{n-1,l}\tilde{S}_{l,l+1} \cdots \tilde{S}_{n-2,n-1} \\ &= \tilde{T}_{n-1}X_{n-1,j}S_{j,j+1}S_{j+1,j+2} \cdots S_{n-2,n-1}Y_{n-1,l}\tilde{S}_{l,l+1} \cdots \tilde{S}_{n-2,n-1}. \end{aligned}$$

Thus

$$X_{n-1,j}S_{j,j+1}S_{j+1,j+2} \cdots S_{n-2,n-1}Y_{n-1,l}\tilde{S}_{l,l+1} \cdots \tilde{S}_{n-2,n-1} \in \ker \sigma_{\tilde{T}_{n-1}} \cap \operatorname{ran} \sigma_{\tilde{T}_{n-1}}.$$

Consequently, using Lemma 2.18 and Theorem 2.19, we conclude that

$$X_{n-1,j}S_{j,j+1}S_{j+1,j+2} \cdots S_{n-2,n-1}Y_{n-1,l}\tilde{S}_{l,l+1} \cdots \tilde{S}_{n-2,n-1} = 0.$$

By hypothesis, all the operators $S_{k,k+1}, \tilde{S}_{k,k+1}, k = 0, 1, \cdots n-2$ have dense range. Since $Y_{n-1,l} \neq 0$, then equation (3.3) and Proposition 2.3 ensure that $Y_{n-1,l}$ has dense range. Hence $X_{n-1,j} = 0$. This contradicts the assumption $X_{n-1,j} \neq 0$. □

The following proposition is the first step in the proof of the rigidity theorem.

Proposition 3.3. *If X is an invertible operator intertwining two operators T and $\tilde{T}$ from $\mathcal{F}B_n(\Omega)$, then X and X^{-1} are upper triangular.*

Proof. The proof is by induction on n. The validity of the case $n = 2$, is immediate from Lemma 3.2. Let us write the two operators $T, \tilde{T}$ in the form of 2×2 block matrix:

$$T = \begin{pmatrix} T_{n-1\times n-1} & T_{n-1\times 1} \\ 0 & T_{n-1,n-1} \end{pmatrix}, \tilde{T} = \begin{pmatrix} \tilde{T}_{n-1\times n-1} & \tilde{T}_{n-1\times 1} \\ 0 & \tilde{T}_{n-1,n-1} \end{pmatrix}.$$

Using Lemma 3.2, the operators X, Y can be written in the form of 2×2 block matrix:

$$X = \begin{pmatrix} X_{n-1\times n-1} & X_{n-1\times 1} \\ 0 & X_{n-1,n-1} \end{pmatrix}, Y = \begin{pmatrix} Y_{n-1\times n-1} & Y_{n-1\times 1} \\ 0 & Y_{n-1,n-1} \end{pmatrix}$$

without loss of generality. Here $X_{n-1\times n-1}$ and $Y_{n-1\times n-1}$ are the operators $\left(\!\left(X_{i,j}\right)\!\right)_{i,j=0}^{n-2}$ and $\left(\!\left(Y_{i,j}\right)\!\right)_{i,j=0}^{n-2}$ respectively and

$$T_{n-1\times n-1} = \begin{pmatrix} T_0 & S_{0,1} & S_{0,2} & \cdots & S_{0,n-2} \\ 0 & T_1 & S_{1,2} & \cdots & S_{1,n-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & T_{n-3} & S_{n-3,n-2} \\ 0 & \cdots & \cdots & 0 & T_{n-2} \end{pmatrix}, \tilde{T}_{n-1\times n-1} = \begin{pmatrix} \tilde{T}_0 & \tilde{S}_{0,1} & \tilde{S}_{0,2} & \cdots & \tilde{S}_{0,n-2} \\ 0 & \tilde{T}_1 & \tilde{S}_{1,2} & \cdots & \tilde{S}_{1,n-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \tilde{T}_{n-3} & \tilde{S}_{n-3,n-2} \\ 0 & \cdots & \cdots & 0 & \tilde{T}_{n-2} \end{pmatrix}.$$

From the relations $XT = \tilde{T}X$, $TY = Y\tilde{T}$ and $XY = YX = I$, we get $X_{n-1\times n-1}T_{n-1\times n-1} = \tilde{T}_{n-1\times n-1}X_{n-1\times n-1}$, $T_{n-1\times n-1}Y_{n-1\times n-1} = Y_{n-1\times n-1}\tilde{T}_{n-1\times n-1}$ and

$$X_{n-1\times n-1}Y_{n-1\times n-1} = Y_{n-1\times n-1}X_{n-1\times n-1} = I.$$

Now, to complete the proof by induction, we assume that any invertible operator X and its inverse X^{-1} intertwining two operators $T, \tilde{T}$ in $\mathcal{F}B_k(\Omega)$ are upper triangular for all $k < n$. Thus the induction hypothesis guarantees that $X_{n-1\times n-1}$ and $Y_{n-1\times n-1}$ must be upper triangular completing the proof. □

Employing these techniques, we show that any operator X, not necessarily invertible, in the commutant of $T \in \mathcal{F}B_n(\Omega)$, must be upper triangular.

Proposition 3.4. *Suppose T is in $\mathcal{F}B_n(\Omega)$ and X is a bounded linear operator in the commutant of T. Then X is upper triangular.*

Proof. The proof is is by induction n. To begin the induction, for $n = 2$, following the method of the proof in Proposition 2.20. we see that an operator commute with an operator in $\mathcal{F}B_2(\Omega)$ must be upper triangular. Now, assume that any operator commute with an operator in $\mathcal{F}B_k(\Omega)$ is upper triangular for all $k < n$.

Step 1: We claim that $X_{n-1,i} = 0$ for $0 \le i \le n-2$. Suppose on contrary this is not true. Then let l, $0 \le l \le n-2$, be the smallest index such that $X_{n-1,l} \neq 0$. For this index l, the commuting relation $XT = TX$ implies that

$$X_{n-1,l}T_l = T_{n-1}X_{n-1,l} \text{ and } \sum_{k=0}^{l} X_{n-1,k}S_{k,l+1} + X_{n-1,i+1}T_{l+1} = T_{n-1}X_{n-1,l+1}. \tag{3.5}$$

From equation (3.5), we have

$$X_{n-1,l}S_{l,l+1}S_{1,2}\dots S_{n-2,n-1} \in \ker \sigma_{T_{n-1}},$$
$$X_{n-1,l}S_{l,l+1}S_{1,2}\dots S_{n-2,n-1} = \sigma_{T_{n-1}}(X_{n-1,l+1}S_{l+1,l+2}, \dots S_{n-2,n-1}).$$

Therefore $X_{n-1,l}S_{l,l+1}S_{l+1,l+2}\dots S_{n-2,n-1}$ is in $\operatorname{ran}\sigma_{T_{n-1}} \cap \ker \sigma_{T_{n-1}}$. Combining Proposition 2.3 with Lemma 2.18 and Theorem 2.19, we conclude that $X_{n-1,l} = 0$. This contradicts the assumption $X_{n-1,l} \neq 0$.

Step 2: Write

$$X = \begin{pmatrix} X_{n-1\times n-1} & X_{n-1\times 1} \\ 0 & X_{n-1,n-1} \end{pmatrix}$$

and

$$T = \begin{pmatrix} T_{n-1\times n-1} & T_{n-1\times 1} \\ 0 & T_{n-1,n-1} \end{pmatrix},$$

where meaning of $X_{n-1\times n-1}$ and $T_{n-1\times n-1}$ are same as in Proposition 3.3. It follows from the commuting relation $XT = TX$ that

$$X_{n-1\times n-1}T_{n-1\times n-1} = T_{n-1\times n-1}X_{n-1\times n-1}.$$

Now, the induction hypothesis guarantees that $X_{n-1\times n-1}$ must be upper triangular completing the proof. □

3.1. Rigidity. Finally, we prove a rigidity theorem for the operators in $\mathcal{F}B_n(\Omega)$. In other words, we show that any intertwining unitary between two operators in the class $\mathcal{F}B_n(\Omega)$ must be diagonal. We refer to this phenomenon as "rigidity."

Theorem 3.5 (Rigidity)**.** *Any two operators T and $\tilde{T}$ in $\mathcal{F}B_n(\Omega)$ are unitarily equivalent if and only if there exists unitary operators U_i, $0 \le i \le n-1$, such that $U_iT_i = \tilde{T}_iU_i$ and $U_iS_{i,j} = \tilde{S}_{i,j}U_j$, $i < j$.*

Proof. Clearly, it is enough to prove the necessary part of this statement. Let U be an unitary operator such that $UT = \tilde{T}U$. By Proposition 3.3, both U and $U^* = U^{-1}$ must be upper triangular, that is,

(a) $U = \big(\!\big(U_{ij}\big)\!\big)_{i,j=1}^n$, $U_{ij} = 0$ whenever $i > j$;
(b) $U^* = \big(\!\big(U_{ji}^*\big)\!\big)_{i,j=1}^n$, $U_{j,i}^* = 0$ whenever $i > j$.

It follows that the operator U must be diagonal. □

We use the rigidity theorem just proved to extract a complete set of unitary invariants for operators in the class $\mathcal{F}B_n(\Omega)$.

Theorem 3.6. *Suppose T is an operator in $\mathcal{F}B_n(\Omega)$ and t_{n-1} is a non-vanishing holomorphic section of $E_{T_{n-1}}$. Then*

(i) *the curvature $\mathcal{K}_{T_{n-1}}$,*
(ii) $\frac{\|t_{i-1}\|}{\|t_i\|}$, *where $t_{i-1} = S_{i-1,i}(t_i)$, $1 \le i \le n-1$*
(iii) $\frac{\langle S_{i,j}(t_j), t_i\rangle}{\|t_i\|^2}$, *for $0 \le i < j \le n-2$ with $j - i \ge 2$*

are a complete set of unitary invariants for the operator T.

Proof. Suppose T, $\tilde{T}$ are in $\mathcal{F}B_n(\Omega)$ and that there is an unitary U such that $UT = \tilde{T}U$. Such an intertwining unitary must be diagonal, that is, $U = U_0 \oplus \cdots \oplus U_{n-1}$, for some choice of n unitary operators $U_0, \dots, U_{n-1}$.

Since $U_iT_i = \tilde{T}_iU_i, 0 \le i \le n-1$, and $U_iS_{i,i+1} = \tilde{S}_{i,i+1}U_{i+1}, 0 \le i \le n-2$, we have

$$U_i(t_i(w)) = \phi(w)\tilde{t}_i(w),\ 0 \le i \le n-1, \tag{3.6}$$

where ϕ is some non zero holomorphic function. Thus

$$\mathcal{K}_{T_{n-1}} = \mathcal{K}_{\tilde{T}_{n-1}} \quad \text{and} \quad \frac{\|t_{i-1}\|}{\|\tilde{t}_{i-1}\|} = \frac{\|t_i\|}{\|\tilde{t}_i\|},\ 1 \le i \le n-1.$$

For $0 \le i < j \le n-2$ with $j - i \ge 2$ and $w \in \Omega$, we have

$$\begin{aligned}
\frac{\langle S_{i,j}(t_j(w)), t_i(w)\rangle}{\|t_i(w)\|^2} &= \frac{\langle U_i(S_{i,j}(t_j(w))), U_i(t_i(w))\rangle}{\|U_i(t_i(w))\|^2} \\
&= \frac{\langle \tilde{S}_{i,j}(U_j(t_j(w))), U_i(t_i(w))\rangle}{\|U_i(t_i(w))\|^2} \\
&= \frac{\langle \tilde{S}_{i,j}(\phi(w)\tilde{t}_j(w)), \phi(w)\tilde{t}_i(w)\rangle}{\|\phi(w)\tilde{t}_i(w)\|^2} \\
&= \frac{\langle \tilde{S}_{i,j}(\tilde{t}_j(w)), \tilde{t}_i(w)\rangle}{\|\tilde{t}_i(w)\|^2}.
\end{aligned}$$

Conversely assume that T and $\tilde{T}$ are operators in $\mathcal{F}B_n(\Omega)$ for which these invariants are the same. Equality of the two curvature $\mathcal{K}_{T_{n-1}} = \mathcal{K}_{\tilde{T}_{n-1}}$ together with the equality of the second fundamental forms $\frac{\|t_{i-1}\|}{\|\tilde{t}_{i-1}\|} = \frac{\|t_i\|}{\|\tilde{t}_i\|}$, $1 \le i \le n-1$ implies that there exist a non-zero holomorphic function ϕ defined on Ω (if necessary, one may choose a domain $\Omega_0 \subseteq \Omega$ such that ϕ is non zero on Ω_0) such that

$$\|t_i(w)\| = |\phi(w)|\, \|\tilde{t}_i(w)\|, \quad 0 \le i \le n-1.$$

For $0 \le i \le n-1$, define $U_i : \mathcal{H}_i \to \widetilde{\mathcal{H}}_i$ by the formula

$$U_i(t_i(w)) = \phi(w)\tilde{t}_i(w), \ w \in \Omega.$$

and extend to the linear span of these vectors. For $0 \le i \le n-1$,

$$\begin{aligned}
\|U_i(t_i(w))\| &= \|\phi(w)\tilde{t}_i(w)\| \\
&= |\phi(w)|\|\tilde{t}_i(w)\| \\
&= \|t_i(w)\|.
\end{aligned}$$

Thus U_i extend to an isometry from $\mathcal{H}_i$ to $\widetilde{\mathcal{H}}_i$. Since U_i is isometric and $U_iT_i = \tilde{T}_iU_i$, it follows, using Proposition 2.3, that each U_i is unitary. It is easy to see that $U_iS_{i,i+1} = \tilde{S}_{i,i+1}U_{i+1}$ for $0 \le i \le n-2$ also. For $0 \le i < j \le n-2$ with $j - i \ge 2$ and $w \in \Omega$,

$$\begin{aligned}
\langle U_i(S_{i,j}(t_j(w))), U_i(t_i(w))\rangle &= \langle S_{i,j}(t_j(w)), t_i(w)\rangle \\
&= \tfrac{\|t_i(w)\|^2}{\|\tilde{t}_i(w)\|^2}\langle \tilde{S}_{i,j}(\tilde{t}_j(w)), \tilde{t}_i(w)\rangle \\
&= |\phi(w)|^2\langle \tilde{S}_{i,j}(\tilde{t}_j(w)), \tilde{t}_i(w)\rangle \\
&= \langle \phi(w)\tilde{S}_{i,j}(\tilde{t}_j(w)), \phi(w)\tilde{t}_i(w)\rangle \\
&= \langle \tilde{S}_{i,j}(\phi(w)\tilde{t}_j(w)), \phi(w)\tilde{t}_i(w)\rangle \\
&= \langle \tilde{S}_{i,j}(U_j(t_j(w))), U_i(t_i(w))\rangle.
\end{aligned}$$

Polarizing the real analytic functions $\langle U_i(S_{i,j}(t_j(w))), U_i(t_i(w))\rangle$ and $\langle \tilde{S}_{i,j}(U_j(t_j(w))), U_i(t_i(w))\rangle$ to functions which are holomorphic in the first and anti-holomorphic in the second variable, we obtain the equality:

$$\langle U_i(S_{i,j}(t_j(z))), U_i(t_i(w))\rangle = \langle \tilde{S}_{i,j}(U_j(t_j(z))), U_i(t_i(w))\rangle, \ z, w \in \Omega.$$

Hence for w in Ω and $0 \le i < j \le n-2$ with $j - i \ge 2$, we have

$$U_i(S_{i,j}(t_j(w))) = \tilde{S}_{i,j}(U_j(t_j(w)))$$

which implies that

$$U_iS_{i,j} = \tilde{S}_{i,j}U_j.$$

Now, setting $U = U_0 \oplus \cdots \oplus U_{n-1}$, we see that U is unitary and $UT = \tilde{T}U$ completing the proof. $\square$

Proposition 3.7. *If an operator T is in $\mathcal{F}B_n(\Omega)$, then it is irreducible.*

Proof. Let P be a projection in the commutant $\{T\}'$ of the operator T. The operator P must therefore be upper triangular by Proposition 3.4. It is also a Hermitian idempotent and therefore must be diagonal with projections $P_{ii}, 0 \leq i \leq n-1$, on the diagonal. We are assuming that $PT = TP$, which gives

$$P_{ii}S_{i,i+1} = S_{i,i+1}P_{i+1i+1},\ 0 \leq i \leq n-2.$$

None of the operators $S_{i,i+1},\ 0 \leq i \leq n-2$, are zero by hypothesis. It follows that $P_{ii} = 0$, if and only if $P_{i+1\,i+1} = 0$. Thus, for any projections $P_{ii} \in \{T_i\}'$, we have only two possibilities:

$$P_{00} = P_{11} = P_{22} = \cdots = P_{n-1n-1} = I,\ \text{or}\ P_{00} = P_{11} = P_{22} = \cdots = P_{n-1n-1} = 0.$$

Hence T is irreducible. □

4. An application to Module tensor products

The localization of a module at a point of the spectrum is obtained by tensoring with the one dimensional module of evaluation at that point. The localization technique has played a prominent role in the structure theory of modules. More recently, they have found their way into the study of Hilbert modules (cf. [4]). An initial attempt was made in [5] to see if higher order localizations would be of some use in obtaining invariants for quotient Hilbert modules. Here we give an explicit description of the module tensor products over the polynomial ring in one variable.

There are several different ways in which one may define the action of the polynomial ring on $\mathbb{C}^k$. The following lemma singles out the possibilities for the module action which evaluates a function at w along with a finite number of its derivatives, say $k-1$, at w. Let f be a polynomial in one variable. Set

$$\mathfrak{J}_{\boldsymbol{\mu}}(f)(z) = \begin{pmatrix} \mu_{1,1}f(z) & 0 & \cdots & 0 \\ \mu_{2,1}\frac{\partial}{\partial z}f(z) & \mu_{2,2}f(z) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{k,1}\frac{\partial^{k-1}}{\partial z^{k-1}}f(z) & \mu_{k-1,1}\frac{\partial^{k-2}}{\partial z^{k-2}}f(z) & \cdots & \mu_{k,k}f(z) \end{pmatrix}$$

where $\boldsymbol{\mu} = \left(\!\left(\mu_{i,j}\right)\!\right)$ is a lower triangular matrix of complex numbers with $\mu_{i,i} = 1,\ 1 \leq i \leq k$.

Lemma 4.1. *The following are equivalent.*

(1) $\mathfrak{J}_{\boldsymbol{\mu}}(fg) = \mathfrak{J}_{\boldsymbol{\mu}}(f)\mathfrak{J}_{\boldsymbol{\mu}}(g)$
(2) $(p+1-j-l)\mu_{p+1-j,l} = \mu_{p+1-j,l+1}\,\mu_{l+1,l},\ 1 \leq l \leq p-2,\ 1 \leq j < p-l+1$
(3) $\mu_{p,l}\,\mu_{l,i} = \binom{p-i}{l-i}\mu_{p,i},\ 1 \leq p,l,i \leq k,\ i \leq l \leq p$

Proof. All the implications of the Lemma are easy to verify except for one, which we verify here. For $1 \leq i,j \leq k$ and $i \leq j$, note that

$$\begin{aligned}
(\mathfrak{J}_{\boldsymbol{\mu}}(f)(z)\mathfrak{J}_{\boldsymbol{\mu}}(g)(z))_{i,j} &= \sum_{l=0}^{i-j}\mu_{i,j+l}\,\mu_{j+l,j}\big(\tfrac{\partial^{i-j-l}}{\partial z^{i-j-l}}f(z)\big)\big(\tfrac{\partial^l}{\partial z^l}g(z)\big) \\
&= \sum_{l=0}^{i-j}\tbinom{i-j}{i-j-l}\mu_{i,j}\big(\tfrac{\partial^{i-j-l}}{\partial z^{i-j-l}}f(z)\big)\big(\tfrac{\partial^l}{\partial z^l}g(z)\big) \\
&= \mu_{i,j}\sum_{l=0}^{i-j}\tbinom{i-j}{i-j-l}\big(\tfrac{\partial^{i-j-l}}{\partial z^{i-j-l}}f(z)\big)\big(\tfrac{\partial^l}{\partial z^l}g(z)\big) \\
&= \mu_{i,j}\tfrac{\partial^{i-j}}{\partial z^{i-j}}(fg)(z) \\
&= (\mathfrak{J}_{\boldsymbol{\mu}}(fg)(z))_{i,j}.
\end{aligned}$$

For $i > j$,

$$(\mathfrak{J}_{\boldsymbol{\mu}}(f)(z)\mathfrak{J}_{\boldsymbol{\mu}}(g)(z))_{i,j} = (\mathfrak{J}_{\boldsymbol{\mu}}(fg)(z))_{i,j} = 0.$$

Hence we have

$$\mathfrak{J}_{\boldsymbol{\mu}}(fg) = \mathfrak{J}_{\boldsymbol{\mu}}(f)\mathfrak{J}_{\boldsymbol{\mu}}(g).$$

□

For $\mathbf{x}$ in $\mathbb{C}^k$, and f in the polynomial ring $P[z]$, define the module action as follows:

$$f \cdot \mathbf{x} = \mathfrak{J}_{\boldsymbol{\mu}}(f)(w)\mathbf{x}.$$

Suppose $T_0 : \mathcal{M} \to \mathcal{M}$ is an operator in $B_1(\Omega)$. Assume that the operator T has been realized as the adjoint of a multiplication operator acting on a Hilbert space of functions possessing a reproducing kernel K. Then the polynomial ring acts on the Hilbert space $\mathcal{M}$ naturally by pointwise multiplication making it a module. We construct a module of k - jets by setting

$$J\mathcal{M} = \Big\{ \sum_{l=0}^{k-1} \tfrac{\partial^i}{\partial z^i} h \otimes \epsilon_{i+1} : h \in \mathcal{M} \Big\},$$

where ϵ_{i+1}, $0 \leq i \leq k-1$, are the standard basis vectors in $\mathbb{C}^k$. There is a natural module action on $J\mathcal{M}$, namely,

$$\Big(f, \sum_{l=0}^{k-1} \tfrac{\partial^i}{\partial z^i} h\Big) \mapsto \mathfrak{J}(f)\Big(\sum_{l=0}^{k-1} \tfrac{\partial^i}{\partial z^i} h \otimes \epsilon_{i+1}\Big),\ f \in P[z],\ h \in \mathcal{M},$$

where

$$\mathfrak{J}(f)_{i,j} = \begin{cases} \binom{i-1}{j-1}\partial^{i-j} f & \text{if } i \geq j, \\ 0 & \text{otherwise.} \end{cases}$$

The module tensor product $J\mathcal{M} \otimes_{\mathcal{A}(\Omega)} \mathbb{C}^k_w$ is easily identified with the quotient module $\mathcal{N}^\perp$, where $\mathcal{N} \subseteq \mathcal{M}$ is the sub-module spanned by the vectors

$$\Big\{ \sum_{l=1}^{k} (J_f \cdot \mathbf{h}_l \otimes \epsilon_l - \mathbf{h}_l \otimes (\mathfrak{J}_{\boldsymbol{\mu}}(f))(w) \cdot \epsilon_l) : \mathbf{h}_l \in J\mathcal{M}, \epsilon_l \in \mathbb{C}^k, f \in P[z] \Big\}.$$

Following the proof of the lemma 4.2 in [5, Lemma 4.1], we can prove:

Lemma 4.2. *The module tensor product $J\mathcal{M} \otimes_{P[z]} \mathbb{C}^k_w$ is spanned by the vector $e_p(w)$ in $J\mathcal{M} \otimes_{\mathcal{A}(\Omega)} \mathbb{C}^k_w$, where*

$$e_p(w) = \sum_{l=1}^{p} b_{p,l} JK(\cdot, w)\epsilon_{p-l+1} \otimes \epsilon_l,\ 1 \leq p \leq k$$

and for a fixed p,

$$b_{p,l} = \frac{\mu_{p-j+1,l}}{\binom{p-l}{j-1}} b_{p,p-j+1},\ l + j < p + 1.$$

The set of vectors $\{e_p(w) : w \in \Omega^*, 1 \leq p \leq k\}$ define a natural holomorphic frame for a vector bundle, say $J_{\mathrm{loc}}(\mathcal{E})$. This vector bundle also inherits a Hermitian structure from that of $J\mathcal{M} \otimes_{\mathcal{A}(\Omega)} \mathbb{C}^k_w$, which furthermore defines a positive definite kernel on $\Omega \times \Omega$:

$$\begin{aligned} J_{\mathrm{loc}} K(z, w) &= \big(\!\!\big(\langle e_p(w), e_q(z) \rangle\big)\!\!\big) \\ &= \sum_{l=1}^{k} D(l) J_{k-l+1} K(z, w) D(l), \end{aligned}$$

where $J_r K(z, w) = \begin{pmatrix} 0_{k-r \times k-r} & 0_{k-r \times r} \\ 0_{r \times k-r} & \tilde{J}_r K(z, w) \end{pmatrix}$ and $D(l)$ is diagonal. Moreover, $D(l)_{m,m} = b_{m+l-1,l}$ and

$$\tilde{J}_r K(z,w) = \begin{pmatrix} K(z,w) & \frac{\partial}{\partial \bar{w}} K(z,w) & \cdots & \frac{\partial^{r-1}}{\partial \bar{w}^{r-1}} K(z,w) \\ \frac{\partial}{\partial z} K(z,w) & \frac{\partial^2}{\partial z \partial \bar{w}} K(z,w) & \cdots & \frac{\partial^{r}}{\partial z \partial \bar{w}^{r-1}} K(z,w) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^{r-1}}{\partial z^{r-1}} K(z,w) & \frac{\partial^{r}}{\partial z^{r-1} \partial \bar{w}} K(z,w) & \cdots & \frac{\partial^{2r-2}}{\partial z^{r-1} \partial \bar{w}^{r-1}} K(z,w) \end{pmatrix}.$$

The two Hilbert spaces $\mathcal{M}$ and $\mathcal{M} \otimes \mathbb{C}^k$ may be identified via the map J_{k-l+1}, which is given by the formula

$$J_{k-l+1}(h) = \sum_{p=0}^{k-l} b_{p+l-1,l} \tfrac{\partial^p}{\partial z^p} h \otimes \epsilon_{p+l}.$$

Since J_{k-l+1} is injective, we may choose an inner product on $J_{p-l+1}\mathcal{M}$ making it unitary.

Proposition 4.3. [5, Proposition 4.2] *The Hilbert module $J_{loc}(\mathcal{M})$ admits a direct sum decomposition of the form $\oplus_{l=1}^k J_{k-l+1}\mathcal{M}$, and the corresponding reproducing kernel is the sum*

$$\sum_{l=1}^k D(l) J_{k-l+1} K(z,w) D(l).$$

Let γ_0 be a non-vanishing holomorphic section for the line bundle E corresponding to the operator T_0. Put $b_{1,1}t_0(w) = \gamma_0(w)$ and for $1 \le l \le k-1$, let

(1) $t_l(w) := \sum_{i=0}^{k-l-1} \bar{b}_{l+1+i,l+1} \frac{\partial^i}{\partial z^i} K(\cdot, w) \otimes \epsilon_{l+1+i}$,
(2) $\gamma_l(w) = \sum_{i=1}^{l+1} b_{l+1,i} \frac{\partial^{l+1-i}}{\partial \bar{w}^{l+1-i}} t_{i-1}(w)$.

Now, $\{\gamma_0, \gamma_1, \cdots, \gamma_{k-1}\}$ are eigenvectors of the operator $M_z^* - \bar{w}$ acting on the Hilbert space $\mathcal{M}_{\rm loc}$.

Since $(M_z^* - \bar{w})\gamma_1(w) = 0$, it follows that $(M_z^* - \bar{w})t_1(w) = -\frac{b_{2,1}}{b_{2,2}} t_0(w)$, which is equivalent to $(M_z^* - \bar{w})t_1(w) = -\mu_{2,1}t_0(w)$.

Suppose $(M_z^* - \bar{w})t_l(w) = -\mu_{l+1,l}t_{l-1}(w)$ for $1 \le l \le r$. Again, since $(M_z^* - \bar{w})\gamma_{r+1}(w) = 0$, it follows that

$$\begin{aligned} &(M_z^* - \bar{w})t_{r+1}(w) \\ &\quad = \tfrac{1}{b_{r+2,r+2}} \big\{ (-(r+1) b_{r+2,1} \bar{\partial}^r t_0(w)) \\ &\qquad - \sum_{i=2}^{r+1} b_{r+2,i} (-\mu_{i,i-1} \bar{\partial}^{r+2-i} t_{i-2}(w) + (r+2-i) \bar{\partial}^{r+1-i} t_{i-1}(w)) \big\} \\ &\quad = \tfrac{1}{b_{r+2,r+2}} \big\{ \sum_{i=1}^{r} (-(r+2-i) b_{r+2,i} + b_{r+2,i+1}\mu_{i+1,i}) \bar{\partial}^{r+1-i} t_{i-1}(w) - b_{r+2,r+1} t_r(w) \big\} \\ &\quad = \tfrac{b_{r+2,r+1}}{b_{r+2,r+2}} t_r(w) \\ &\quad = \mu_{r+2,r+1} t_r(w) \end{aligned}$$

Let $\Gamma := J_k \oplus J_{k-1} \oplus \ldots \oplus J_1$, be the unitary from $\widetilde{\mathcal{M}} := \mathcal{M}_0 \oplus \cdots \mathcal{M}_{k-1}$ to $\mathcal{M}_{\rm loc}$, where each of the summands $\mathcal{M}_0, \ldots, \mathcal{M}_{k-1}$ is equal to $\mathcal{M}$. Let $K_l(\cdot, w) := J_{k-l}^* t_l(w) = K(\cdot, w)$, $0 \le l \le k-1$. Now, we describe the operator $T := \Gamma^* M^* \Gamma$, where M is the multiplication operator on $\mathcal{M}_{\rm loc}$. For $1 \le l \le k-1$, set $T_l := P_{\mathcal{M}_l} T_{|\mathcal{M}_l}$ and note that

$$\begin{aligned} T(K_l(\cdot, w)) &= (\Gamma^* M^* \Gamma) K_l(\cdot, w) \\ &= \Gamma^* M_z^* t_l(w) \\ &= \Gamma^* (\bar{w} t_l(w) + \mu_{l+1,l} t_{l-1}(w)) \\ &= \bar{w} K_l(\cdot, w) + \mu_{l+1,l} K_{l-1}(\cdot, w). \end{aligned}$$

Now,

$$\begin{aligned} T_l(K_l(\cdot,w)) &= P_{\mathcal{M}_l}T_{|\mathcal{M}_l}(K_l(\cdot,w)) \\ &= P_{\mathcal{M}_l}T(K_l(\cdot,w)) \\ &= P_{\mathcal{M}_l}(\bar{w}K_l(\cdot,w)+\mu_{l+1,l}K_{l-1}(\cdot,w)) \\ &= \bar{w}K_l(\cdot,w). \end{aligned}$$

Let $S_{l-1,l}:\mathcal{M}_l\to\mathcal{M}_{l-1}$ be the bounded linear operator defined by the rule $S_{l-1,l}(K_l(\cdot,w)):=\mu_{l+1,l}K_{l-1}(\cdot,w)$, $1\le l\le k-1$. Since $\mathcal{M}_l=\mathcal{M}_{l-1}=\mathcal{M}$, it follows that $S_{l-1,l}=\mu_{l+1,l}I$. Hence the operator T has the form:

$$T=\begin{pmatrix} T_0 & \mu_{2,1}I & 0 & \cdots & 0 & 0 \\ 0 & T_0 & \mu_{3,2}I & \cdots & 0 & 0 \\ 0 & 0 & T_0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \mu_{k-1,k-2}I & 0 \\ 0 & 0 & 0 & \cdots & T_0 & \mu_{k,k-1}I \\ 0 & 0 & 0 & \cdots & 0 & T_0 \end{pmatrix}.$$

Thus T is in $\mathcal{F}B_k(\Omega)$ and defines, up to unitary equivalence via the unitary Γ, the module action in $\mathcal{M}_{\rm loc}$. In consequence, setting $\mathbb{C}^k_w[\boldsymbol{\mu}]$ to be the Hilbert module with the module action induced by $\mathcal{J}_{\boldsymbol{\mu}}(f)(w)$, we have the following theorem as a direct application of Theorem 3.6.

Theorem 4.4. *The Hilbert modules corresponding to the localizations $J\mathcal{M}\otimes_{P[z]}\mathbb{C}^k_w[\boldsymbol{\mu}_i]$, $i=1,2$, are in $\mathcal{F}B_k(\Omega)$ and they are isomorphic if and only if $\boldsymbol{\mu}_1=\boldsymbol{\mu}_2$.*

Appendix: Frames

As in Remark 2.7, we attempt to relate the frame of the holomorphic vector bundle E_T, T in $\mathcal{F}B_n(\Omega)$, to that of the direct sum of the line bundles $E_{T_0}\oplus\cdots\oplus T_{n-1}$. Let $\boldsymbol{t}=\{t_0,t_1,\ldots,t_{n-1}\}$ be a set of non-vanishing holomorphic sections for the line bundles $E_{T_0},\ldots,E_{T_{n-1}}$, respectively. Suppose that a suitable linear combination of these non-vanishing sections $t_i,\ i=0,\ldots,n-1$, and their derivatives produces a holomorphic frame $\boldsymbol{\gamma}:=\{\gamma_0,\ldots,\gamma_{n-1}\}$ for the vector bundle E_T, that is,

$$\gamma_i=t_0^{(i)}+\mu_{1,i}t_1^{(i-1)}+\cdots+\mu_{i-1,i}t_{i-1}^{(1)}+t_i$$

for some choice of non-zero constants $\mu_{1,i},\ldots,\mu_{i-1,i},\ 0\le i\le k-1$. The existence of such an orthogonal frame is not guaranteed except when $n=2$.. Assuming that it exists, the relationship between these vector bundles can be very mysterious as shown below. This justifies, to some extent, the choice of the smaller class of operators in the next section. If $\tilde{\boldsymbol{t}}$ is another set of non-vanishing sections for the line bundles $E_{T_1},\ldots,E_{T_{n-1}}$, then the linear combination of these with exactly the same constants μ_{ij} is a second holomorphic frame, say $\tilde{\boldsymbol{\gamma}}$ of the vector bundle E_T. Let Φ_k be a change of frame between the two sets of non-vanishing orthogonal frames $\boldsymbol{t}$ and $\tilde{\boldsymbol{t}}$, and Ψ_k be a change of frame between $\boldsymbol{\gamma}$ and $\tilde{\boldsymbol{\gamma}}$. We now describe the relationship between Φ_k and Ψ_k explicitly:

(1) $\Phi_k(i,j):=\phi_{i,j}=\psi_{i,j}:=\Psi_k(i,j)=0$, $i>j$, that is, Φ_k and Ψ_k are upper-triangular.

(2) For $0\le i\le k-1$, we have $\phi_{i,i}=\psi_{i,i}=\phi_{0,0}$, and for $i<k-1$, we have

$$\psi_{i,k-1}=C^i_{k-1}\phi_{0,0}^{(k-1-i)}+\cdots+C^i_{k-1-j}\mu_{j,k-1}\phi_{0,j}^{(k-1-j-i)}+\cdots+\mu_{k-1-i,k-1}\phi_{0,k-1-i},$$

where C^n_r stands for the binomial coefficient $\binom{n}{r}$.

(3) In particular, for $1\le i\le k-1$, if we choose $\phi_{0,i}$, then $\psi_{i,k-1}=C^i_{k-1}\phi_{0,0}^{(k-1-i)}$. In this case, we have

(a)

$$\Psi_k = \begin{pmatrix} \psi & \psi^{(1)} & \psi^{(2)} & \cdots & \psi^{(k-2)} & \psi^{(k-1)} \\ & \psi & 2\psi^{(1)} & \cdots & C^1_{k-2}\psi^{(k-3)} & C^1_{k-1}\psi^{(k-2)} \\ & & \psi & \ddots & \vdots & \vdots \\ & & & \ddots & \ddots & \vdots \\ & & & & \psi & C^{k-2}_{k-1}\psi^{(1)} \\ & & & & & \psi \end{pmatrix};$$

(b) and there are $\frac{(k-2)(k-1)}{2}$ equations in $\frac{(k-1)k}{2}$ variables, namely, $\mu_{i\,j},\ 1 \le i < j,\ j \le k-1$. Thus these coefficients are determined as soon we make an arbitrary choice of the coefficients $\mu_{1,k-1}, \dots, \mu_{k-2,k-1}$.

We prove the statements (1) and (2) by induction on k. These statements are valid for $k = 2$ as was noted in Remark 2.7. To prove their validity for an arbitrary $k \in \mathbb{N}$, assume them to be valid for $k-1$. Let Φ_k^i and Ψ_k^i denote the ith row of Φ and Ψ, respectively. Suppose that $(\widetilde{t}_0, \widetilde{t}_1, \cdots, \widetilde{t}_k) = (t_0, t_1, \cdots, t_k)\Phi_k$ and $(\widetilde{\gamma}_0, \widetilde{\gamma}_1, \cdots, \widetilde{\gamma}_k) = (\gamma_0, \gamma_1, \cdots, \gamma_k)\Psi_k$. Then we have

$$\widetilde{t}_j = (t_0, t_1, \cdots, t_{k-1})\Phi^j_{k-1} + t_k\psi_{k,j}, j < k.$$

For any $i < k$, we have

$$\begin{aligned} \widetilde{\gamma}_i &= (\gamma_0, \gamma_1, \cdots, \gamma_{k-1})\Psi^i_{k-1} + \gamma_k\psi_{k,i} \\ &= (\gamma_0, \gamma_1, \cdots, \gamma_{k-1})\Psi^i_{k-1} + (t_0^{(k)} + \mu_{1,k}t_1^{(k-1)} + \cdots + \mu_{i,k}t_i^{(k-i)} + \cdots + t_k)\psi_{k,i} \end{aligned}$$

and

$$\widetilde{\gamma}_i = \widetilde{t}_0^{(i)} + \mu_{1,i}\widetilde{t}_1^{(i-1)} + \cdots + \mu_{i-1,i}\widetilde{t}_{i-1}^{(1)} + \widetilde{t}_i,\ i < k.$$

From these equations, it follows that

$$\begin{aligned} &(\gamma_0, \gamma_1, \cdots, \gamma_{k-1})\Psi^i_{k-1} + (t_0^{(k)} + \mu_{1,k}t_1^{(k-1)} + \cdots + \mu_{i,k}t_i^{(k-i)} + \cdots + t_k)\psi_{k,i} \\ &\quad = \widetilde{t}_0^{(i)} + \mu_{1,i}\widetilde{t}_1^{(i-1)} + \cdots + \mu_{i-1,i}\widetilde{t}_{i-1}^{(1)} + \widetilde{t}_i. \end{aligned}$$

We Note that $\mu_{i,k}\psi_{k,i}t_i^{(k-i)}$ appears only once in this equation to conclude $\psi_{k,i} = 0,\ i < k$. Comparing the coefficients of t_i on both sides of the equation, we also conclude that $\psi_{k,i} = \phi_{k,i}, i < k$ completing the induction step for the first statement of our claim.

Our assumption that $(\widetilde{t}_0, \widetilde{t}_1, \cdots, \widetilde{t}_k) = (t_0, t_1, \cdots, t_k)\Phi_k$ and $(\widetilde{\gamma}_0, \widetilde{\gamma}_1, \cdots, \widetilde{\gamma}_k) = (\gamma_0, \gamma_1, \cdots, \gamma_k)\Psi_k$ gives

$$\sum_{i=0}^{k}(t_0^i + \mu_{1,i}t_1^{(i-1)} + \cdots + \mu_{i-1,i}t_{i-1}^{(1)} + t_i)\psi_{i,k} = \sum_{i=0}^{k}\mu_{i,k}(t_0\phi_{0,i} + \cdots + t_i\phi_{0,0})^{(k-i)}, i < k.$$

A comparison of the coefficients of $t_0^{(i)}$ leads to

$$\psi_{i,k} = C^i_k\phi_{0,0}^{(k-i)} + \cdots + C^i_{k-j}\mu_{j,k}\phi_{0,j}^{(k-j-i)} + \cdots + \mu_{k-i,k}\phi_{0,k-i}, i < k$$

completing the proof of the second statement. For the third statement, from the equations

$$\begin{aligned} &\sum_{i=0}^{k-1}(t_0^i + \mu_{1,i}t_1^{(i-1)} + \cdots + \mu_{i-1,i}t_{i-1}^{(1)} + t_i)\psi_{i,k-1} \\ &\qquad = \sum_{i=0}^{k-1}\mu_{i,k-1}(t_0\phi_{0,i} + \cdots + t_i\phi_{0,0})^{(k-1-i)}, i < k-1, \end{aligned}$$

setting $\phi_{0,i} = 0$, and comparing the coefficients of t_i, $i > 0$, we have that $\phi_{i,k-1} = c_{i,k-1}\phi_{0,0}^{(k-1-i)}$ for some $c_{i,k-1} \in \mathbb{C}$. Putting this back in the equation given above, we obtain $\frac{(k-2)(k-1)}{2}$ equations involving $\frac{(k-1)k}{2}$ coefficients. This completes the proof of the third statement.

References

[1] M. J. Cowen and R. G. Douglas, *Complex geometry and operator theory*, Acta Math. **141** (1978), 187 - 261.

[2] J. B. Conway, *A Course in functional analysis* Graduate Text in Mathematics, 96. Springer-Verling, New York, 1985.

[3] R. E. Curto and N. Salinas, *Generalized Bergman kernels and the Cowen-Douglas theory*, Amer. J. Math., 106 (1984), 447 - 488.

[4] R. G. Douglas and V. I. Paulsen, *Hilbert modules over function algebra*, Longman Research Notes, 217, 1989.

[5] R. G. Douglas, G. Misra, C. Varughese , *On quotient modules – the case of arbitrary multiplicity*, J. Funct. Anal. 174 (2000), 364 - 398.

[6] R. G. Douglas, G. Misra, *Equivalence of quotient Hilbert modules II*, Trans. Amer. Math. Soc. 360 (2008), 2229 - 2264.

[7] K. Ji, C. Jiang, D. K. Keshari, G. Misra, *Flag structure for operators in the Cowen-Douglas class*, Comptes Rendus Mathmatique, 352 (2014), 511 - 514.

[8] C. Jiang, Z. Wang, *Strongly irreducible operators on Hilbert space* Pitman Research Notes in Mathematics Series, 389. Longman, Harlow, 1998.

[9] D. C. Kleinecke, *On operator commutator*, Proc. Amer. Math. Soc., 8 (1957),535 - 536.

[10] S. Kobayashi, *Differential geometry of complex vector bundles*, Publications of the Mathematical Society of Japan, 15. Kanô Memorial Lectures, 5. Princeton University Press, Princeton, NJ; Iwanami Shoten, Tokyo, 1987.

[11] A. Koranyi and G. Misra, *A classification of homogeneous operators in the Cowen-Douglas class*, Adv. Math., 226 (2011) 5338 - 5360.

[12] G. Misra, *Curvature and the backward shift operators*, Proc. Amer. Math. Soc., 91 (1984), 105 - 107.

[13] G. Misra and S. Shyam Roy, *The curvature invariant for a class of Homogeneous operators*, Proc. London Math.Soc., 99 (2009), 557 - 584.

[14] D. R. Wilkins, *Homogeneous vector bundles and Cowen - Douglas operators*, International J. of Math,. 4 (1993), 503 – 520.

(Ji) Department of Mathematics, Hebei Normal University, Shijiazhuang, Heibei 050016, China

E-mail address: jikui@hebtu.edu.cn

(Jiang) Department of Mathematics, Hebei Normal University, Shijiazhuang, Heibei 050016, China

E-mail address: cljiang@hebtu.edu.cn

(Keshari) Department of Mathematics, Indian Institute of Science, Bangalore 560012

Current address: Indian Statistical Institute, R V College Post, Bangalore 560059

E-mail address: kesharideepak@gmail.com

(Misra) Department of Mathematics, Indian Institute of Science, Bangalore 560012, India

E-mail address: gm@math.iisc.ernet.in