JOURNAL OF FUNCTIONAL ANALYSIS 88, 118–134 (1990)

Bounded Modules, Extremal Problems, and a Curvature Inequality

GADADHAR MISRA

Division of Theoretical Statistics and Mathematics, Indian Statistical Institute, 8th Mile, Mysore Road, RV College Post, Bangalore 560 059, India

AND

N. S. NARSIMHA SASTRY

Division of Theoretical Statistics and Mathematics, Indian Statistical Institute, 203 Barrackpore Trunk Road, Calcutta 700 035, India

Communicated by D. Sarason

Received October 2, 1987

In this paper we study certain finite dimensional Hilbert modules over the function algebra $\mathscr{A}(\Omega)$, $\Omega \subseteq \mathbb{C}^n$. These modules appear as localizations of a Cowen–Douglas operator. We show that these modules are always bounded, where the bound is related to the solution of an extremal problem. In particular, we obtain necessary sufficient conditions for such a module to be contractive. We apply the above results to produce an example of a contractive module over $\mathscr{A}(\mathbb{B}^2)$, which is not completely contractive.

1. INTRODUCTION

1.1. In this paper, we choose to work in the framework of Hilbert modules introduced by R. G. Douglas [9]. A *Hilbert module* $\mathscr{H}$ over a normed (not necessarily complete) complex algebra $\mathscr{A}$ consists of a complex Hilbert space $\mathscr{H}$ together with a continuous map $(a, f) \to a \cdot f$ from $\mathscr{A} \times \mathscr{H}$ to $\mathscr{H}$ satisfying the following conditions: For a, $b \in \mathscr{A}$, h, $h_i \in \mathscr{H}$, and α, $\beta \in \mathbb{C}$,

(i) $1 \cdot h = h$,

(ii) $(a \cdot b) \cdot h = a \cdot (b \cdot h)$,

(iii) $(a + b) \cdot h = a \cdot h + b \cdot h$, and

(iv) $a \cdot (\alpha h_1 + \beta h_2) = \alpha(a \cdot h_1) + \beta(a \cdot h_2)$.

0022-1236/90 $3.00

The Hilbert module is *bounded* if there exists a constant K such that

$$\|a \cdot h\|_{\mathscr{H}} \leqslant K \|a\|_{\mathscr{A}} \|h\|_{\mathscr{H}} \quad \text{for all} \quad a \in \mathscr{A} \quad \text{and} \quad h \in \mathscr{H},$$

and is *contractive* if $K \leqslant 1$.

For any region Ω in $\mathbb{C}^m$, let $\mathscr{A}(\Omega)$ denote the closure of the algebra $\mathscr{P}(\Omega)$ of the polynomial functions on Ω with respect to the supremum norm $\|\cdot\|_\infty$ on Ω. We recall that, for a given m-tuple $\mathbf{T} = (T_1, ..., T_m)$ of pairwise commuting operators on a Hilbert spce $\mathscr{H}$, the closure $\bar{\Omega}$ of Ω is a *K-spectral set* for $\mathbf{T}$ if

$$\|p(T)\| = \|p(T_1, .., T_m)\| \leqslant K \|p\|_\infty \quad \text{for all} \quad \mathscr{P} \in \mathbb{C}[Z_1, ..., Z_m],$$

and $\bar{\Omega}$ is a *spectral* set if $K \leqslant 1$.

1.2. The Hilbert $\mathscr{P}(\Omega)$-module structure on the Hilbert space $\mathscr{H}$ determines, and is completely determined by, a commuting m-tuple $\mathbf{T} = (T_1, ..., T_m)$ of continuous operators on $\mathscr{H}$ defined by $T_i(h) = z_i \cdot h$ for $h \in H$, $1 \leqslant i \leqslant m$. $\mathscr{H}$ is a bounded (respectively contractive) $\mathscr{P}(\Omega)$-module with bound K if and only if $\bar{\Omega}$ is a K-spectral (respectively spectral) set for $\mathbf{T}$; and, in this case, $\mathscr{H}$ can be made into an $\mathscr{A}(\Omega)$-module and is denoted by $\mathscr{H}_{\mathbf{T}}$. On the other hand, by an easy application of the uniform boundedness principle, if $\mathscr{H}$ is an $\mathscr{A}(\Omega)$-module and if $\mathbf{T}$ is the m-tuple of pairwise commuting operators on $\mathscr{H}$ corresponding to the action of $z_i \in \mathscr{P}(\Omega)$, $1 \leqslant i \leqslant m$, on $\mathscr{H}$, then $\bar{\Omega}$ is a K-spectral set for $\mathbf{T}$ for some K. Thus, the concept of $\mathscr{H}$ being an $\mathscr{A}(\Omega)$-module is equivalent to $\bar{\Omega}$ being a K-spectral set for some m-tuple of pairwise commuting operators on $\mathscr{H}$.

1.3. The central object of study in this paper is a Hilbert $\mathscr{A}(\Omega)$-module $\mathbb{C}_N^{n+1}$ described in 2.1, for a region Ω as in 3.1, which appears as the localization of a Cowen–Douglas operator (see [5]). We show that this module is bounded (Remark 3.3) by a quantity $M_\Omega(V, w)$ associated with Ω and $\mathbf{N}$ which can be realized as $\|L\| \, \|L^{-1}\|$ for some linear transformation L of $\mathbb{C}^{n+1}$ such that $\mathbb{C}^{n+1}_{LNL^{-1}}$ is a contractive $\mathscr{A}(\Omega)$-module (Theorem 3.5). We interpret $M_\Omega(V, w)$ as the solution of an extremal problem, a particular case of which is familiar (see Remark 4.4, [3, p, 772]). Our extremal problem itself, in turn, can be seen in a more general perspective (Remark 4.5). Using Theorems 3.4 and 4.1, we obtain a curvature inequality (Theorem 5.2) for an m-tuple of Hilbert space operators in the Cowen–Douglas class $\mathbf{P}_1(\Omega)$ and, in 5.4, we produce an example of a pair of joint weighted shift operators for which equality is attained. Finally, we apply the above results to solve in 6.1 a problem of Paulsen [21] and to show in 6.2 that, for an appropriate choice of $\mathbf{N}$, $\mathbb{C}^3_{\mathbf{N}}$ is a contractive, but not a completely contractive, module over $\mathscr{A}(B^2)$. The existence of such examples over various function algebras was conjectured in [1, p. 222]. However, the only known examples are due to Parrott [20] over the tri-disk algebra.

2. Preliminaries

2.1. For $\mathbf{v} \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}$, we define the $(n+1) \times (n+1)$-matrix

$$N(\mathbf{v}, \lambda) = \begin{bmatrix} \lambda & \mathbf{v} \\ 0 & \lambda I_n \end{bmatrix}.$$

For $\mathbf{v}^i = (v_1^i, ..., v_n^i) \in \mathbb{C}^n$, $1 \leqslant i \leqslant m$, and $w = (w_1, ..., w_m)$ in a region Ω in $\mathbb{C}^m$, we consider the m-tuple of pairwise commuting $(n+1) \times (n+1)$-matrices

$$\mathbf{N} = (N_1, ..., N_m) = (N(v^1, w_1), ..., N(v^m, w_m)).$$

The Hilbert $\mathscr{A}(\Omega)$-module structure $\mathbb{C}_{\mathbf{N}}^{n+1}$ on then Hilbert space $\mathbb{C}^{n+1}$ defined using the m-tuple $\mathbf{N}$ of operators (as in 1.2 when $\bar{\Omega}$ is a K-spectral set for $\mathbf{N}$ for some K) is a central object of our study here.

2.2. Let $\mathscr{M}_n(\mathbb{C})$ denote the algebra of complex matrices of size n with respect to the operator norm

$$\|A\|^2 = \max \{|\lambda| : \lambda \text{ is an eigen value of } AA^*\}.$$

2.2.1. Lemma. *For $\mathbf{u}, \mathbf{v} \in \mathbb{C}^n$ and $a, \lambda, \mu \in \mathbb{C}$, we have*

(a) $N(\mathbf{u}, \lambda)\, N(\mathbf{v}, \mu) = N(\lambda \mathbf{v} + \mu \mathbf{u}, \lambda\mu)$,

(b) $N(\mathbf{u}, \lambda)^{-1} = N(-(1/\lambda^2)\, \mathbf{u}, 1/\lambda)$ *if* $\lambda \neq 0$,

(c) $\|N(\mathbf{u}, \lambda)\| = \|N(\|\mathbf{u}\|, \lambda)\|$, *and*

(d) $\|N(a, \lambda)\|^2 = \frac{1}{2}[|a|^2 + 2|\lambda|^2 + |a| \sqrt{|a^2| + 4|\lambda|^2}]$.

Proof. Part (c) follows because the characteristic polynomial in X of $N(u, \lambda)\; N(u, \lambda)^*$ is $(X - |\lambda|^2)^{n-2}$ times the characteristic polynomial of $N(\|u\|, \lambda)\; N(\|u\|, \lambda)^*$ in X. The rest is straightforward.

2.2.2. Lemma. *Let $\mathscr{A}$ be a complex algebra, $\theta : \mathscr{A} \to \mathbb{C}$ be a continuous algebra homomorphism, and $\varphi : \mathscr{A} \to \mathbb{C}^n$ be a continuous linear map such that $\varphi(ab) = \theta(a)\, \varphi(b) + \theta(b)\, \varphi(a)$. Then, the map $a \mapsto N(\varphi(a), \theta(a))$ is a continuous algebra homomorphism from $\mathscr{A}$ to $\mathscr{M}_{n+1}(\mathbb{C})$.*

Proof. To see the continuity of the map, we use (c) and (d) of the lemma above. The rest is straightforward.

For $w \in \mathbb{C}^m$, let $H(w)$ denote the algebra of the germs of complex valued analytic functions at w. Note that a sequence $\{f_n\}$ in $H(w)$ converges to f in $H(w)$ if and only if there exists a compact neighbourhood K of w on which a representative of f_n is defined for each n and the sequence of these representatives converges to a representative of f in K. We denote by $\mathscr{R}(w)$ the (incomplete!) algebra of the elements of $H(w)$ admitting a rational function defined in a neighbourhood of w as a representative.

2.2.3. PROPSITION. *Let V be an $n \times n$-matrix, $\mathbf{v}^k$ its kth-row, and let $\mathbf{N}$ be as in 2.1. Then, the map $f \mapsto N(\nabla f(w) \cdot V, f(w))$ is a continuous algebra homomorphism from $H(w)$ to $M_{n+1}(C)$ coinciding with the (evaluation) map $r \to r(N)$ on $\mathscr{R}(w)$.*

Proof. We first observe that if $\{f_n\}$ is a sequence of functions defined on a compact neighbourhood U of w converging to a function f, then, by the Weierstrass theorem [19, p. 159], $(\partial/\partial z_k) f_n$ converges to $(\partial/\partial z_k) f$ on U and so, $\|(\nabla f_n - \nabla f)(w)\|_2$ and $\|\nabla(f_n - f)(w) \cdot V\|_2$ tend to zero as n approaches ∞. Now, the first part of the proposition follows by Lemma 2.2.2. above applied to $H(w)$ by taking θ to be the map $f \mapsto f(w)$ and φ to be the map $f \mapsto \nabla f(w) \cdot V$. The second part of the proposition follows because the map is an (abstract) algebra homomorphism and $z_k \mapsto N(\mathbf{v}^k, w_k)$, $1 \leqslant k \leqslant m$, under this map.

2.2.4. DEFINITION. For any complex holomorphic function f defined in a neighbourhood of w, we define $f(\mathbf{N}) = f_V(\mathbf{N}) = N(\nabla f(w) \cdot V, f(w))$. If $\{p_n\}$ is a sequence of polynomial functions, all defined in a compact neighbourhood K of w, and converges to f on K, then $p_n(\mathbf{N}) \to f(\mathbf{N})$ in norm.

2.3. HYPOTHESIS ON Ω. Throughout this paper, Ω always denotes a bounded open neighbourhood of 0 in $\mathbb{C}^m$ which (a) is convex (i.e., $\lambda\Omega + (1-\lambda)\,\Omega \subseteq \Omega$ for $0 \leqslant \lambda \leqslant 1$); (b) is balanced (i.e., $\lambda\Omega \subseteq \Omega$ for all $\lambda \in \mathbb{C}$ with $|\lambda| \leqslant 1$); and (c) admits a group of biholomorphic automorphisms of Ω which is transitive on Ω.

We note that (a) implies that Ω is polynomially convex [12, p. 67] and so, by Oka's theorem [12, p. 84], $\mathscr{A}(\Omega)$ contains all functions holomorphic in a neighbourhood of Ω; (a) and (b) imply that Ω can be considered as the unit ball in $\mathbb{C}^m$ with respect to a suitable norm $\|\cdot\|_\Omega$ on $\mathbb{C}^m$ and that the Schwarz lemma [23, Theorem 8.12, p. 161] applies to Ω. The bounded symmetric domains [17, 1.6 and 4.6] satisfy our hypothesis.

Throughout, $(\mathbb{C}^m, \|\cdot\|_\Omega)$ denotes the complex linear space $\mathbb{C}^m$ equipped with the norm $\|\cdot\|_\Omega$; $\Omega^* \subseteq \mathbb{C}^m$ denotes the set defined by the property: $\{\varphi_z \in \mathrm{Hom}(\mathbb{C}^m, \mathbb{C})$ defined by $\varphi_z(x) = z \cdot x : z \in \Omega^*\}$ is the unit ball in $(\mathbb{C}^m, \|\cdot\|_\Omega)^*$; and, for a linear map f from $\mathbb{C}^n$ to $\mathbb{C}^m$ and domains $\Omega \subseteq \mathbb{C}^n$ and

$\Omega' \subseteq \mathbb{C}^m$ $\|f\|_{\Omega}^{\Omega'}$ (respectively $\|f\|_{\Omega^*}^{\Omega'}$) denotes the norm of f as a linear map from $(\mathbb{C}^n, \|\cdot\|_{\Omega})$ (respectively $(\mathbb{C}^n, \|\cdot\|_{\Omega})^*$) to $(\mathbb{C}^m, \|\cdot\|_{\Omega'})$.

3. Contractive and Bounded Modules

For $w \in \Omega$, and $V \in \mathcal{M}_m(\mathbb{C})$, define $\mathrm{Hol}_w(\Omega, \mathbb{D}) = \{f$: holomorphic in a neighbourhood of $\bar{\Omega}$, $f(w) = 0$ and $\|f\|_\infty \leqslant 1\}$; $\mathbf{D}\Omega(w) = \nabla f(w) : f \in \mathrm{Hol}_w(\Omega, \mathbb{D})\} \subseteq \mathbb{C}^m$; and $M_\Omega(V, w) = \sup\{\|z \cdot V\| : z \in \mathbf{D}\Omega(w)\}$.

3.1. *Remark.* $\mathcal{A}(\Omega)_1$, the unit bell of $\mathcal{A}(\Omega)$, is closed with respect to the uniform convergence on compact subsets of Ω and is a bounded family on Ω. In view of Montel's theorem [19, p. 159], $\mathcal{A}(\Omega)_1$ is a normal family. Since $\mathcal{A}(\Omega)_1$ is closed and normal, it must be compact. Since the map $f \mapsto \|\nabla f(w) \cdot V\|$ is continuous on $\mathcal{A}(\Omega)$, $M_\Omega(V, w)$ is attained in $\overline{\mathrm{Hol}_w(\Omega, \mathbb{D})}$.

3.2. Lemma. *Let* $\mathbf{N}$ *be as defined in* 2.1 *and* $\|f(\mathbf{N})\| \leqslant k$ *for all* $f \in \mathrm{Hol}_w(\Omega, \mathbb{D})$. *Then* $\|g(\mathbf{N})\| \leqslant \|g\|_\infty \cdot \max\{k, 1\}$ *for all* $g \in \mathcal{A}(\Omega)$.

Proof. We first note that, by 2.2.4, 2.2.3, and 2.2.1(d),

$$\|g(\mathbf{N})\| \leqslant l \Leftrightarrow \|\nabla g(w) \cdot V\| \leqslant l - l^{-1} |g(w)|^2. \tag{$*$}$$

Let $0 \neq g \in \mathcal{A}(\Omega)$. Clearly, we can assume that $\|g\|_\infty = 1$. Consider $f = \varphi_{g(w)} \circ g \in \mathrm{Hol}_w(\Omega, \mathbb{D})$, where $\varphi_{g(w)} \in \mathrm{Aut}(\mathbb{D})$ is defined by $\varphi_{g(w)}(z) = z - g(w))/(1 - g(w)z)$. The chain rule, our hypothesis, and $(*)$ imply that

$$\|\nabla f(w) \cdot V\| = \|\nabla g(w) \cdot V\| (1 - |g(w)|^2)^{-1} \leqslant k$$

and so, $\|\nabla g(w) \cdot V\| \leqslant k(1 - |g(w)|^2)$. Now, since $k(1 - |g(w)|^2)$ is at most $(1 - |g(w)|^2)$ if $k \leqslant 1$ and is at most $k - k^{-1} |g(w)|^{-2}$ if $k \geqslant 1$, the lemma follows from $(*)$.

3.3. *Remark.* By the Schwarz lemma [23, Theorem 8.1.2, p. 161] and Theorem 4.1(a), $\mathbf{D}\Omega(w) \subseteq \mathbb{C}^m$ and so $M_\Omega(V, w)$ is bounded. Therefore, by 2.2.1(d), $(*)$, and 3.2, $\{\|g(\mathbf{N})\| : g \in \mathcal{A}(\Omega)$ with $\|g\|_\infty = 1\}$ is bounded by $\max\{1, M_\Omega(V, w)\}$ and $\mathbb{C}_{\mathbf{N}}^{n+1}$ is a bounded $\mathcal{A}(\Omega)$-module with $\mathrm{Max}\{1, M_\Omega(V, w)\}$ as a bound.

From $(*)$, we have

3.4. Theorem. $\mathbb{C}_{\mathbf{N}}^{m+1}$ *is a contractive* $\mathcal{A}(\Omega)$*-module if and only if* $M_\Omega(V, w) \leqslant 1$.

3.5. THEOREM. $\mathbb{C}_{\mathbf{N}}^{m+1}$ *is a bounded* $\mathscr{A}(\Omega)$*-module with the bound* $K = \max\{1, M_\Omega(V, w)\}$. *Further, if* $M_\Omega(V, w) > 1$ *then there exists an invertible* $(m+1) \times (m+1)$*-matrix* L *such that* $\|L\|\,\|L^{-1}\| = M_\Omega(V, w)$ *and* $\mathbb{C}_{L\mathbf{N}L^{-1}}^{m+1}$ *is a contractive* $\mathscr{A}(\Omega)$*-module.*

Proof. Let $K_w = \sup\{\|f(\mathbf{N})\| : f \in \mathrm{Hol}_w(\Omega, \mathbb{D})\}$ and let K be the bound of $\mathbb{C}_{\mathbf{N}}^{m+1}$. Clearly, $K_w \leqslant K$ and by 2.2.1(d), $K_w = M_\Omega(V, w)$. We note that if L is an invertible operator such that $L\mathbf{N}L^{-1} = L(N_1, ..., N_m)\,L^{-1} = (LN_1L^{-1}, ..., LN_mL^{-1})$ determines a contractive module, then it is easy to see that $\mathbf{N}$ determines a bounded module with $\|L\| \cdot \|L^{-1}\|$ as a bound. Therefore K would then be at most $\|L\| \cdot \|L^{-1}\|$.

Choose L to be the diagonal matrix of size $(m+1)$ with 1 in the $(1, 1)$-position and $M_\Omega(V, w)$ in the (r, r)th-position for $2 \leqslant r \leqslant m+1$, then

$$L\mathbf{N}L^{-1} = \left(N\left(\frac{\mathbf{v}^1}{M_\Omega(V, w)}, w\right), ..., N\left(\frac{\mathbf{v}^m}{M_\Omega(V, w)}, w_m\right)\right).$$

Let $V_L = M_\Omega^{-1}(V, w) \cdot V$. Since $M_\Omega(V_L, w) = M_\Omega(V, w)/M_\Omega(V, w) = 1$, it follows that $C_{L\mathbf{N}L^{-1}}^{m+1}$ is a contractive $\mathscr{A}(\Omega)$-module 3.4. But $\|L\| \cdot \|L^{-1}\| = M_\Omega(V, w)$. Therefore, $M_\Omega(V, w) = K_w \leqslant K \leqslant \|L\| \cdot \|L^{-1}\| = M_\Omega(V, w)$.

4. AN EXTREMAL QUANTITY

The following alternative description of $\mathbf{D}\Omega(w)$ and $M_\Omega(V, w)$ is crucial in this article.

4.1. THEOREM. *Let* $w \in \Omega$ *and* θ_w *be a biholomorphic automorphism of* Ω *such that* $\theta_w(w) = 0$. *Then,*

(a) $\mathbf{D}\Omega(w) = \mathbf{D}\Omega(0) \cdot (D\theta_w(w))$,

(b) $\mathbf{D}\Omega(0) = \Omega^*$,

(c) $M_\Omega(V, w) = M_\Omega(D\theta_w(w) \cdot V, 0)$, *where the linear map* $D\theta_w$ $\mathbb{C}^m \to \mathbb{C}^m$ *is the usual Fréchet derivative, and*

(d) $M_\Omega(V, 0) = \|V\|_{\Omega^*} = \sup\{\|Vx\|_2 : x \in \Omega^*\}$.

Proof. Since the map $f \mapsto f \circ \theta_w$ defines a bijection from $\mathrm{Hol}_0(\Omega, \mathbb{D})$ to $\mathrm{Hol}_w(\Omega, \mathbb{D})$, (a) follows by chain rule.

To prove (b), we need to show that $\{\varphi_z \in \mathrm{Hom}(\mathbb{C}^n, \mathbb{C})$ defined by $\varphi_z(x) = z \cdot x : z \in \mathbf{D}\Omega(D)\}$ is the unit ball of $(\mathbb{C}^m, \|\cdot\|_\Omega)^*$. If $f \in \mathrm{Hol}_0(\Omega, \mathbb{D})$, then $\nabla f(0) \in \mathbf{D}\Omega(0)$ and $|\nabla f(0)\,\lambda| \leqslant 1$ for all $\lambda \in \Omega$ by the Schwarz lemma [23, Theorem 8.1.2, p. 161]. Therefore, $\nabla f(0) \in \Omega^*$, on the other hand, if $z \in \Omega^*$, then $\varphi_z \in \mathrm{Hol}_0(\Omega, \mathbb{D})$ and $z = D\varphi_z(D) \in \mathbf{D}\Omega(0)$. Therefore, $\mathbf{D}\Omega(0) = \Omega^*$.

Finally, (c) and (d) follow from the definitions of $M_\Omega(V, w)$ and $\| V \|_{\Omega^*}$.

The following corollary has an obvious generalization.

4.2. COROLLARY. *$\mathbb{C}_{\mathbf{N}}^{n+1}$ is a contractive $\mathscr{A}(\mathbb{D}^m)$-module if $\mathbb{C}_{N_k}^{n+1}$ is a contractive $\mathscr{A}(\mathbb{D})$-module for each k, $1 \leqslant k \leqslant m$.*

Proof. By 3.4, 4.1(c), and 4.1(d), $\mathbb{C}_{\mathbf{N}}^{n+1}$ is a contractive $\mathscr{A}(\mathbb{D}^m)$-module if and only if $\| D\theta_w(w) \cdot V \| \leqslant 1$ for some $\theta_w \in \mathrm{Aut}(\mathbb{D}^m)$ such that $\theta(w) = 0$. We can take θ as $\theta_{w_1} \times \cdots \times \theta_{w_m}$, where $\theta_{w_i} \in \mathrm{Aut}(\mathbb{D})$ such that $\theta_{w_i}(w_i) = 0$. Then,

$$D\theta_w(w) = \mathrm{diag}\left[\frac{d}{dz_1}\,\theta_{w_1}(w_1), \ldots, \frac{d}{dz_m}\,\theta_{w_m}(w_m)\right].$$

Since $\mathbb{D}^m$ is the unit ball in $\mathbb{C}^m$ with respect to the l_∞-norm, the dual norm is the l_1-norm. Thus,

$$\begin{aligned}
&\| D\theta_w(w) \cdot V \|_{(\mathbb{D}^n)^*} \\
&\quad = \sup\{ \| z \cdot (D\theta_w(w) \cdot V) \|_2 : z = (z_1, \ldots, z_m) \text{ and } \| z \|_1 = 1 \} \\
&\quad \leqslant \sup\left\{ \sum_{k=1}^m |z_k| \left\| \frac{d}{dz_k}\,\theta_{w_k}(w_k) \cdot \mathbf{V}^k \right\|_2 : z \in \mathbb{C}^m, \| z \|_1 = 1 \right\} \\
&\quad = \max\left\{ \left\| \frac{d}{dz_k}\,\theta_{w_k}(w_k) \cdot \mathbf{v}^k \right\|_2 : 1 \leqslant k \leqslant m \right\}.
\end{aligned}$$

Since equality is attained for some $\zeta = e_k$, we have that

$$\| D\theta_w(w) \cdot V \|_{(\mathbb{D}^n)^*} = \max\left\{ \left\| \frac{d}{dz_k}\,\theta_{w_k}(w_k) \cdot \mathbf{v}^k \right\|_2 : 1 \leqslant k \leqslant m \right\}.$$

But $\mathbb{C}_{N_k}^{n+1}$ is contractive if and only if $\| (d/dz_k)\,\theta_{w_k}(w_k) \cdot \mathbf{V}^k \| \leqslant 1$. Therefore, the corollary follows.

4.3. *Remark.* If V has only one non-zero column, then our extremal problem, in view of Remark 4.4, turns out to be a familiar one [3, 15, Section 2], namely, to find, for $\Omega \subseteq \mathbb{C}^m$ and $\mathbf{v} \in \mathbb{C}^m$,

$$\sup\{ |\partial_{\mathbf{v}} f(w)| : f \text{ holomorphic on } \Omega, \| f \|_\infty \leqslant 1 \text{ and } f(w) = 0 \}.$$

4.4. *Remark.* In computing $M_\Omega(V, w)$ it makes no difference even if we allow all functions $f\colon \Omega \to \mathbb{D}$, $f(w) = 0$, and f holomorphic on Ω and not merely holomorphic in a neighbourhood of $\bar{\Omega}$, since the description of $\mathbf{D}\Omega(0)$ remains the same in both cases.

4.5. *Remark.* In view of Theorem 4.1, an alternate description of $M_\Omega(V, w)$ is as $\sup\{\|\mathbf{u} \cdot V\| : \mathbf{u} \in \Omega^* \cdot D\theta_w(w) \subseteq \mathbb{C}^m\}$. If $f\colon \Omega \to \mathbb{C}^n$ is any holomorphic function such that $Df(w)$ is V, then $\mathbf{u} \cdot V$ represents the covariant derivative of f in the direction $\mathbf{u}$ [13, p. 18]. Thus computing $M_\Omega(V, w)$ amounts to finding the maximum (in the sense of l_2-norm) of the covariant derivative of a function over the set of vectors $\Omega^* \cdot D\theta_w(w)$ in $\mathbb{C}^m$. In particular, when V has only one non-zero column, computing $M_\Omega(V, w)$ amounts to maximizing the directional derivative of some $f\colon \Omega \to \mathbb{C}$, holomorphic over the set of vectors $\Omega^* \cdot D\theta_w(w)$ in $\mathbb{C}^m$.

5. A Curvature Inequality

5.1. The following class $\mathbf{P}_1(\Omega)$ of m-tuples of operators on some Hilbert space $\mathscr{H}$ was introduced by Cowen and Douglas [6, p. 334] (see also [7]): An m-tuple $\mathbf{T} = (T_1, ..., T_m)$ is in $\mathbf{P}_1(\Omega)$ if

(1) $T_1, ..., T_m$ pairwise commute,

(2) $\dim \bigcap_{k=1}^m \operatorname{Ker}(T_k - w_k) = 1$ for all $w = (w_1, ..., w_m) \in \Omega$,

(3) the operator $T_w\colon \mathscr{H} \to \mathscr{H} \otimes \mathbb{C}^m$ defined by

$$T_w(h) = \bigoplus_{k=1}^{m} (T_k - w_k)\, h \qquad \text{for} \quad h \in \mathscr{H}$$

has closed range for all $w \in \Omega$, and

(4) $\operatorname{span}\{\bigcap_{k=1}^m \operatorname{Ker}(T_k - w_k) : w \in \Omega\} = \mathscr{H}$.

It was shown in [6] that each m-tuple $\mathbf{T}$ in $\mathbf{P}_1(\Omega)$ determines a non-zero holomorphic map $\gamma\colon \Omega \mapsto \mathscr{H}$ such that $\gamma(w) \in \bigcap_{k=1}^m \operatorname{Ker}(T_k - w_k)$ for all $w \in \Omega$ and such that the curvature matrix

$$\mathscr{K}_{\mathbf{T}}(w) = \left(\frac{\partial^2}{\partial w_i \partial \bar{w}_j} \log \|\gamma(w)\|^2\right)$$

is a complete unitary invariant for $\mathbf{T}$. It is therefore of interest to relate the properties of $\mathbf{T}$ to its curvature. In this section, we obtain an inequality for the curvature when $\mathscr{H}_{\mathbf{T}}$ is a contractive $\mathscr{A}(\Omega)$-module.

For $\mathbf{T} \in \mathbf{P}_1(\Omega)$, we define $\mathscr{H}(\mathbf{T})$ to be the subspace $\bigcap_{j,k=1}^m \operatorname{Ker} T_j T_K$ of $\mathscr{H}$ and the localization $\mathbf{N}(w)$ of $\mathbf{T}$ at w as the m-tuple $\mathbf{N} = (N_1, ..., N_m)$, where

$$N_k = w_k I_{m+1} + (T_K - w_k)|_{\mathscr{H}((T_1 - w_1), ..., (T_m - w_m))}$$

with respect to the basis $\{\gamma(\omega), (\partial\gamma/\partial w_1)(w), ..., (\partial\gamma/\partial w_m)(w)\}$ for $\mathscr{H}((T_1 - w_1), ..., (T_m - w_m))$; the matrix for N_k is $N(\mathbf{v}^k, w_k)$ defined in 2.1. Note that the first k-entries in $\mathbf{v}^k$ are zero.

The curvature and the localization of $\mathbf{T}$ at w are related by

$$\mathscr{K}_{\mathbf{T}}(w) = (\operatorname{tr}((N_K - w_K I) \cdot (\overline{N_j - w_j I})^t))^{-1} = (V\bar{V}^t)^{-1},$$

where the kth-row of V is the vector $\mathbf{v}^k$ [6, pp. 336–337]: Note that the matrix V corresponding to the localization $N(w)$ depends on w.

5.2. THEOREM. *If $\mathscr{H}_{\mathbf{T}}$ is a contractive $\mathscr{A}(\Omega)$-module and θ_w is a biholomorphic automorphism of Ω such that $\theta_w(w) = 0$, then*

$$\| D\theta_w(w)\, \mathscr{K}_{\mathbf{T}}(w)^{-1}\, \overline{D\theta_w(w)}^t \|_{\Omega^*}^{\Omega} \leqslant 1.$$

Proof. Since $\mathscr{H}_{\mathbf{T}}$ is a contractive $\mathscr{A}(\Omega)$-module, we see that $\mathbb{C}^{m+1}_{\mathbf{N}(w)}$ is also a contractive $\mathscr{A}(\Omega)$-module. Thus, by 3.4, 4.1(c), and 4.1(d), $\| D\theta_w(w) \cdot V \|_{\Omega^*} \leqslant 1$, where $D\theta_w(w) \cdot V \colon (\mathbb{C}^n, \|\cdot\|_{\Omega})^* \to (\mathbb{C}^n, \|\cdot\|_2)$ is given by $(D\theta_w(w) \cdot V)(z) = z \cdot (D\theta_w(w)\, V)$. But the matrix for the localization operator in this section was computed with respect to the right action. Thus, we write $D\theta_w(w)\, V(z) = (D\theta_w(w)\, V)^t z^t$ and note that in this notation, our hypothesis implies that $\|(D\theta_w(w)\, V)^t\|_{\Omega^*} \leqslant 1$. Therefore,

$$\begin{aligned} &\| D\theta_w(w)\, \mathscr{K}_{\mathbf{T}}(w)^{-1}\, \overline{D\theta_w(w)}^t \|_{\Omega^*}^{\Omega} \\ &\quad = \|((D\theta_w(w)\, V)^t)^t\, (\overline{D\theta_w(w)\, V})^t \|_{\Omega^*}^{\Omega} \\ &\quad \leqslant \|(D\theta_w(w)\, V)^t\|_{\Omega^*}\, \|(D\theta_w(w)\, v)^t\|_{\Omega^*} \leqslant 1. \end{aligned}$$

5.3. *Remark.* Any m-tuple of operators in $\mathbf{P}_1(\Omega)$ for which equality occurs in the foregoing inequality is called an *extremal operator.* In particular, if Ω is either the poly disk $\mathbb{D}^m$ or the unit ball $\mathbb{B}^m$ in $\mathbb{C}^m$ and the m-tuple $\mathbf{T}$ in $\mathbf{P}_1(\Omega)$ is such that $\mathscr{K}_{\mathbf{T}}(w) = \overline{D\theta_w(w)}^t$, $D\theta_w(w)$ for some $\theta_w \in \operatorname{Aut}(\Omega)^-$ with $\theta_w(w) = 0$, then $\mathbf{T}$ is extremal. While $(M_{z_1}^*, ..., M_{z_m}^*)$ on $H^2(\Omega)$ is an extremal m-tuple in this sense for $\mathbb{D}^m$, it fails to be for $\mathbb{B}^m$. However, we construct below an m-tuple of joint weighted shifts in m-variables for the ball which is extremal. The extremal operator for the unit disk is unique (cf. [18]) while it does not seem to be so for either $\mathbb{D}^m$ or $\mathbb{B}^m$ when $m \geqslant 1$.

5.4. *An Extremal Operator.* (a) For a pair $\mathbf{T} = (T_1, T_2)$ of operators in $\mathbf{P}_1(\mathbb{B}^2)$, the localizations are of the form N_1 and N_2 with $v_1^2 = 0$. If $\mathscr{H}_{\mathbf{T}}$ is a contractive $A(\mathbb{B}^2)$-module, then

$$\| D\theta_w(w)\, \mathscr{K}_{\mathbf{T}}(w)^{-1}\, \overline{D\theta_w(w)}^t \|_2 \leqslant 1$$

for any $\theta_w \in \operatorname{Aut}(B_2)$ with $\theta_w(w) = 0$.

After introducing some concepts from the theory of joint weighted shift operators following [14, p. 208], we produce a pair $\mathbf{T}$ of operators in $\mathbf{P}_1(\mathbb{B}_2)$ such that $\mathscr{K}_{\mathbf{T}}(w) = \overline{D\varphi_w(w)}^t D\varphi_w(w)$ for $\varphi_w \in \mathrm{Aut}(B^2)$ defined by $\varphi_w(z) = (w - p_w z - s_w Q_w z)/(1 - \langle z, w\rangle)$, $w \in B^2$, where $P_w z = \langle z, w\rangle / \langle w, w\rangle$, $w \neq 0$, $Q_w = I - P_w$, and $s_w^2 = (1 - |w|^2)$, so that $\mathbf{T}$ is an extremal operator. For the general properties of these automorphisms we refer to [23 Theorem 2.2.2, p. 26].

(b) Let $I = (i_1, ..., i_m)$ be a multi-index of non-negative integers. Let ε_j be the multi-index having i_k equal to 1 or 0 according as k is equal to j or not. Let $I \pm \varepsilon_k$ denote the multi-index $(i_1, ..., i_k \pm 1, ..., i_m)$. Let $\{e_I\}$ be an orthonormal basis for a complex Hilbert space $\mathscr{H}$ and let $\{w_{I,j} : j = 1, ..., m\}$ be a bounded sequence of complex numbers such that

$$w_{I,k} w_{I+\varepsilon_k, l} = w_{I,l} w_{I+\varepsilon_l, k}.$$

A *system of m-variable weighted shift operators with weights* $\{w_I\}$ is a family of m operators $T_1, ..., T_m$ such that

$$T_j e_I = w_{I,j} e_{I+e_j}, \qquad 1 \leqslant j \leqslant m,$$

for each multi-index I.

As in the one variable case, a commuting system of weighted shift operators are m multiplication operators on a suitable Hilbert space of formal power series defined as follows.

Let $\{\beta_I : I \geqslant 0\}$ be a set of strictly positive numbers such that $\beta_0 = 1$, and let

$$H^2(\beta) = \left\{ f = \sum_{I \geqslant 0} f_I z^I : \|f\|^2 = \sum |f_I|^2 \beta_I^2 < +\infty \right\}.$$

Clearly, $H^2(\beta)$ is a Hilbert space with respect to the inner product

$$\langle f, g\rangle = \sum f_I \bar{g}_I \beta_I^2.$$

For $1 \leqslant j \leqslant \leqslant m$, let T_{z_j} denote the multiplication operator on $H^2(\beta)$ defined by

$$T_{z_j} \cdot f = z_j f = \sum_{I \geqslant 0} f_I z^{I+\varepsilon_j}.$$

Then, $\mathbf{T} = (T_1, ..., T_m)$ is a commuting m-tuple of weighted shift operators with weights

$$w_{I,j} = \beta_{I+\varepsilon_j} \beta_I^{-1}.$$

(c) If $\beta(I)^{-2}$ denotes the coefficient of $w^I \bar{w}^I$ in the multinomial expansion of $(1 - \|w\|^2)^{-1}$, then the Kernel function for $H^2(\beta)$ is given by

$$K_\beta(z, w) = (1 - \langle z, w \rangle)^{-1}$$

(see [14, p. 219]) and the corresponding m-tuple $\mathbf{T}_\beta$ of weighted shift operators is determined by the weights

$$w_{I,j} = \beta_{I+\varepsilon_j} \beta_I^{-1} = (i_j + 1)^{1/2} (|I| + 1)^{-1/2}.$$

Now, since

$$\inf \{ (w_{I-\varepsilon_1,1})^2 + (w_{I-\varepsilon_2,2})^2 \}^{1/2} = 1,$$

$\mathbf{T}_\beta \in \mathbf{P}_1(\mathbb{B}^2)$ when $m = 2$ by a theorem of Curto and Salinas [8, Theorem 4.9(h), p. 128]. Further

$$\begin{aligned}
\mathscr{K}_{\mathbf{T}}\beta(w) &= \left(\frac{\partial^2}{\partial w_i \partial \bar{w}_j} \log(1 - \langle w, w \rangle)^{-1} \right) \\
&= \frac{1}{(1 - \|w\|^2)} \begin{bmatrix} 1 - |w_2|^2 & -\bar{w}_1 w_2 \\ -w_1 \bar{w}_2 & 1 - |w_1|^2 \end{bmatrix} \\
&= \overline{D\varphi_w(w)}^t \, D\varphi_w(w),
\end{aligned}$$

where φ_w is as in **a**. Therefore $\mathbf{T}_\beta$ is a pair of extremal operators in $\mathbf{P}_1(\mathbb{B}^2)$.

(d) *Remark.* It was observed by Lubin [16, Proposition 4, p. 842] that $\mathbf{T}_\beta$ is a subnormal pair which does not possess a pair of commuting normal extensions. However, we observe that localization of $\mathbf{T}_\beta$ at 0 is $\mathbf{N}$ with $\mathbf{v}^1 = (1, 0)$ and $\mathbf{v}^2 = (0, 1)$ (as in 2.1). We show in 6.2.2 that this pair of local operators does not determine a completely contractive module. Thus $\mathbf{T}_\beta$ cannot determine a completely contractive module either. Equivalently, $\mathbf{T}_\beta$ does not possess a normal dilation in the sense of [2, p. 279].

6. Applications

6.1. *Solution to a Question of Paulsen*

Let $\mathfrak{D}_1$ and $\mathfrak{D}_2$ be domains in $\mathbb{C}$ and let $\mathfrak{D} = \mathfrak{D}_1 \times \mathfrak{D}_2$ and $\partial_d(\mathfrak{D}) = \partial\mathfrak{D}_1 \times \partial\mathfrak{D}_2$. Let $\mathscr{C}(\mathfrak{D})$ dote the algebra of continuous complex-valued functions on $\mathfrak{D}$ with the supremum norm, let $\mathscr{R}(\mathfrak{D})$ denote the subalgebra of $\mathscr{C}(\mathfrak{D})$ consisting of rational functions with poles $\mathfrak{D}$, and let $\mathscr{R}_d(\mathfrak{D})$ denote the subalgebra of $\mathscr{C}(\partial_d \mathfrak{D})$ generated by $\mathscr{R}(\mathfrak{D}_i)$, $i = 1, 2$. The algebra $\mathscr{R}_d(\mathfrak{D})$ is contained in $\mathscr{C}(\mathfrak{D})$ and is algebraicaly isomorphic to $\mathscr{R}(\mathfrak{D}_1) \otimes \mathscr{R}(\mathfrak{D}_2)$, but is not necessarily dense in $\mathscr{R}(\mathfrak{D})$; see [22, p. 170].

Question [21, p. 29]. Let $T_1, T_2 \in \mathcal{L}(\mathcal{H})$ be such that $T_1 T_2 = T_2 T_1$ and such that the spectrum $\rho(T_i)$ of T_i is contained in $\mathfrak{D}_i$, $i=1, 2$. Let $\mathcal{H}_{T_1}$, $\mathcal{H}_{T_2}$, and $\mathcal{H}_{(T_1, T_2)}$ denote $\mathcal{H}$, considered as a Hilbert module over $\mathcal{R}(\mathfrak{D}_1)$, $\mathcal{R}(\mathfrak{D}_2)$, and $\mathcal{R}_d(\mathfrak{D})$, respectively. Assume that $\mathcal{H}_{T_i}$ is a contractive Hilbert $\mathcal{R}(\mathfrak{D}_i)$-module, $i=1, 2$. Then, is $\mathcal{H}_{(T_1, T_2)}$ necessarily a contractive $\mathcal{R}_d(\mathfrak{D})$-module?

For $k=1, 2$ and w_k in $\mathfrak{D}_k$, let

$$R_k = \sup\{|f'(w_k)| \mid f\colon \mathfrak{D} \to \mathbb{D} \text{ is holomorphic and } f(w_k)=0\}$$

and $F_k\colon \mathfrak{D}_k \to \mathbb{D}$ be the Ahlfors function [11, Theorem 1.6, p. 114]; i.e., F is holomorphic $F(w_k)=0$ and $F'(w_k)=R_k$. Note that $\mathbf{D}\mathfrak{D}_k(w_k) = \mathbb{D}_{R_k}$ and for s, t positive $se^{i\theta_1}F_1 + te^{i\theta_2}F_2$ maps $\mathfrak{D}$ to $\mathbb{D}$ whenever $s+t \leqslant 1$, which shows that

$$\mathbf{D}\mathfrak{D}(w_1, w_2) \supseteq \{(\alpha_1, \alpha_2)\colon |\alpha_1|/R_1 + |\alpha_2|/R_2 \leqslant 1\}.$$

The fact that these two sets are in fact equal is the consequence of a theorem of Royden on the Caratheodory metric of product domains, which we recall.

If $X \subseteq \mathbb{C}^m$ is open and bounded then the infinitesimal form of the Caratheodory metric is given by

$$F_X(w, \mathbf{v}) = \sup\left\{\left|\sum_{j=1}^{m} \frac{\partial f}{\partial z_j}(w) \cdot v_k\right| \,\middle|\, f\colon X \to \mathbb{D} \text{ is holomorphic and } f(w)=0\right\}$$

Royden has proved the following theorem about the product of two complex spaces (cf. [15, Theorem 2.3, p. 362]).

THEOREM (Royden). *If X and Y are any two complex spaces then*

$$F_{X\times Y}((\mathbf{z}, w), (\mathbf{u}, \mathbf{v})) = \max\{F_X(\mathbf{z}, \mathbf{u}), F_Y(w, \mathbf{v}\}$$

for $\mathbf{z}$ in X and w in Y.

From this theorem it follows that

$$\mathbf{D}\mathfrak{D}(w_1, w_2) \subseteq \{(\alpha_1, \alpha_2)\colon |\alpha_1|/R_1 + |\alpha_2|/R_2 \leqslant 1\}.$$

However, $\mathbb{C}^{n+1}_{N_k}$ is a contractive module over $\mathcal{R}(\mathfrak{D}_k)$ if and only if [18, Corollary 1.1, p. 309] $\|\mathbf{v}_k\| \leqslant R_k^{-1}$. Thus as in Corollary 4.2 we deduce that $\mathbb{C}^{n+1}_{\mathbf{N}}$ is a contractive module over $\mathcal{R}(\mathfrak{D})$ if and only if $\mathbb{C}^{n+1}_{N_k}$ is a contractive module over $\mathcal{R}(\mathfrak{D}_k)$.

6.2. *Example of a Contractive Module Which is not Completely Contractive*

6.2.1. Following Arveson [2, p. 278], we introduce the notion of a completely contractive module: For any complex function algebra $\mathscr{A}$ and integer $k \geqslant 1$, let $\mathscr{M}_k(\mathscr{A}) \simeq \mathscr{A} \otimes_{\mathbb{C}} \mathscr{M}_k(\mathbb{C})$ denote the algebra of all $k \times k$-matrices with entries from $\mathscr{A}$,

Here, for $F = (f_{ij}) \in M_k(\mathscr{A})$, the norm $\|F\|$ of F is defined by

$$\|F\| = \sup\{\|(f_{ij}(z)\| : z \in \Omega\},$$

where Ω is the maximal ideal space for A. We note that, for the algebra $A = A(\Omega)$, the maximal ideal space can be identified with Ω [12, Theorem 1.2, p. 67] and thus,

$$\| F\| = \sup\{\|(F_{ij}(z)\| : z \in \Omega\}.$$

DEFINITION. If H is a bounded Hilbert A-module, then, clearly, $H \otimes_C C^k$ is a bounded $M_k(A)$-module. For each k, let n_k denote the smallest bound for $H \otimes C^k$. The Hilbert A-module H is *completely bounded* if $n_\infty = \lim_{n \to \infty} n_k < +\infty$ and is *completely contractive* if $n_\infty \leqslant 1$.

The importance of completely contractive modules lies in, among other things, a dilation theorem due to Arveson [2, Corollary, p. 279]. We refer the reader to the forthcoming book of Paulsen [22] for more details; related material can also be found in [1, 21].

6.2.2. Let $P = (p_{ij}) \in M_k(P(\Omega))$ and let, for $w \in \Omega$, $\mathscr{D}P(w)$ denote the $k \times mk$-matrix

$$\left(\frac{\partial}{\partial z_1} P(w), ..., \frac{\partial}{\partial z_m} P(w)\right), \qquad \text{where} \quad \frac{\partial}{\partial z_i} P(w) = \left(\frac{\partial}{\partial z_i} P_{ij}(w)\right).$$

Thus, if $\mathbf{N} = (N_1, ..., N_m)$ is as in 2.1 and V is the matrix whose columns are $\mathbf{v}^1, ..., \mathbf{v}^m \in \mathbb{C}^m$, then

$$P(\mathbf{N}) = (p_{ij}(\mathbf{N})) = (\mathbf{N}(\nabla p_{ij}(w) \cdot V, p_{ij}(w)))$$

and, after a suitable rearrangement of the rows and columns, this can be written in the form

$$\begin{bmatrix} P(w) & \mathscr{D}P(w) \cdot (V \otimes I_k) \\ 0 & I_m \otimes P(w) \end{bmatrix}.$$

Here, as usual, for a matrix $A=(a_{ij})$ of size a and B of size b, $A\otimes B$ denotes the matrix $(a_{ij}B)$ of size ab.

If $\mathbb{C}_{\mathbf{N}}^{m+1}$ is to be a completely contractive module, since by 2.2.1(d),

$$\|P(\mathbf{N})\| = \|\mathscr{D}P(w)\cdot V\otimes I_k\| \qquad \text{when} \quad P(w)=0, \tag{1}$$

we must at least have

$$\sup\{\|\mathscr{D}P(w)\cdot V\otimes I_k\| : P\in\mathscr{M}_k(\mathscr{P}(\Omega)),\ \|P\|\leqslant 1 \qquad \text{and} \quad P(w)=0\}\leqslant 1.$$

However,

$$\begin{aligned}\|\mathscr{D}P(w)\cdot V\otimes I_k\| = \sup\{&|\langle(\mathscr{D}P(w)\cdot V\otimes I_k)\,x, y\rangle| :\\ &x\in\mathbb{C}^{mk},\ y\in\mathbb{C}^k,\ \|x\|\leqslant 1=\|y\|\leqslant 1\}.\end{aligned}$$

We set, for $x=(x^1, ..., x^m)\in\mathbb{C}^{mk}$ with $x^i=(x^i_1, ..., x^i_m)\in\mathbb{C}^k$, and $y\in\mathbb{C}^k$,

$$DP_{x,y}(w)=(q_{ij}(w)), \qquad q_{ij}(w)=\frac{\partial}{\partial z_i}\langle P(w)\,x^j, y\rangle.$$

Then,

$$\begin{aligned}\langle(\mathscr{D}P(w)\cdot V\otimes I_k)\,x, y\rangle &= \langle\nabla P_1(w), \bar{v}^1\rangle + \cdots + \langle\nabla P_m(w), \bar{v}^m\rangle\\ &= \operatorname{tr}(DP_{x,y}(w)\cdot V^t),\end{aligned}$$

so that

$$\begin{aligned}\|\mathscr{D}P(w)\cdot V\otimes I_k\| = \operatorname{Sup}\{&|\operatorname{tr}(DP_{x,y}(w)\cdot V^t)| : x\in\mathbb{C}^{mk},\ y\in\mathbb{C}^k\\ &\text{with} \quad \|x\|\leqslant 1=\|y\|\leqslant 1\}.\end{aligned} \tag{2}$$

We illustrate that (1) and (2) can be used to generate examples of contractive modules which are not completely contractive. We choose Ω to be the unit ball $\mathbb{B}^2$ in $\mathbb{C}^2$.

First, we note that, if

$$P(z)=\begin{bmatrix} az_1 & bz_2\\ cz_1 & dz_2\end{bmatrix},$$

then

$$DP_{x,y}(0) = \begin{bmatrix} x_1^1(ay_1 + cy_2) & x_1^2(ay_1 + cy_2) \\ x_2^1(by_1 + dy_2) & x_2^2(by_1 + dy_2) \end{bmatrix}$$
$$= \begin{bmatrix} \langle (a, c), (\bar{y}_1, \bar{y}_2) \rangle & 0 \\ 0 & \langle (b, d), (\bar{y}_1, \bar{y}_2) \rangle \end{bmatrix} \cdot \begin{bmatrix} x_1^1 & x_1^2 \\ x_2^1 & x_2^2 \end{bmatrix},$$

and

$$\| P \| = \sup \left\{ \left\| \begin{bmatrix} az_1 & bz_2 \\ cz_1 & dz_2 \end{bmatrix} \right\| : (z_1, z_2) \in \mathbb{B}^2 \right\}$$
$$= \left\| \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right\|_{\mathbb{B}^2 \circ \mathbb{B}^2},$$

where

$$\mathbb{B}^2 \circ \mathbb{B}^2 = \{ (z_1 w_1, z_2 w_2) : (z_1, z_2) \text{ and } (w_1, w_2) \text{ are in } \mathbb{B}^2 \}.$$

By elementary computations, one can check that

$$\mathbb{B}^2 \circ \mathbb{B}^2 = \{ (\lambda_1, \lambda_2) : |\lambda_1| + |\lambda_2| \leqslant 1 \}.$$

Therefore (or by direct checking), if we choose $a = b = c = d = 1/\sqrt{2}$, then $\| P \| \leqslant 1$. If we further choose $y = (y_1, y_2) = (1/\sqrt{2}, 1/\sqrt{2})$, then

$$DP_{x,y}(0) = \begin{bmatrix} x_1^1 & x_1^2 \\ x_2^1 & x_2^2 \end{bmatrix}.$$

Also, notice that if $T = [\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}] \in \mathcal{M}_2(\mathbb{C})$ with $\| T \| \leqslant 1/\sqrt{2}$, then $(p, q)\| \leqslant 1/\sqrt{2}$, $\|(r, s)\| \leqslant 1/\sqrt{2}$ so that $\|(p, q, r, s)\| \leqslant 1$. If we choose

$$x = (p, r, q, s) \qquad \text{then} \quad DP_{x,y}(0) = T.$$

Therefore,

$$\{ DP_{x,y}(0) : p \in \mathcal{M}_2(\mathcal{P}(\Omega)),\ \| P \| \leqslant 1,\ P(w) = 0,\ x \in \mathbb{C}^4,\ y \in \mathbb{C}^2$$
$$\text{with} \quad \| x \| = 1 = \| y \| \}$$
$$\supseteq \{ T \in \mathcal{M}_2(\mathbb{C}) : \| T \| \leqslant 1/\sqrt{2} \}. \tag{3}$$

For any $V \in \mathcal{M}_2(\mathbb{C})$, the norm of the map $L_V : \mathcal{M}_2(\mathbb{C}) \to \mathbb{C}$ defined by $L_V(P) = \operatorname{tr}(P \cdot V^t)$ is given by

$$\| L_V \| = \sup \{ |\operatorname{tr} P \cdot V^t| : \| P \| \leqslant 1 \} = \operatorname{tr} |V^t|.$$

The last equality follows by applying [4, Theorem 4.3, p. 20]. This remark, together with (1), (2), and (3), implies that

$$n_2 \geqslant \sup\{|\operatorname{tr}(C \cdot V^t)| : \|C\| \leqslant 1/\sqrt{2}\} = 1/\sqrt{2}\operatorname{tr}|V^t|. \tag{4}$$

Thus, we obtain the following

EXAMPLE. In the definition of $\mathbb{C}_N^3$ in 2.1, take Ω to be the unit ball in $\mathbb{C}^2$, w to be the origin of Ω, and $\mathbf{v}^1, \mathbf{v}^2 \in \mathbb{C}^2$ be such that if $V = \binom{\mathbf{v}_1}{\mathbf{v}_2}$, then $\|V\| \leqslant 1$ and $\operatorname{tr}|V^t| \geqslant \sqrt{2}$. Then, $\mathbb{C}_N^3$ is a contractive, but not a completely contractive, $\mathscr{A}(\Omega)$-module.

The contractive part follows from Theorem 4.1(d) and that $\mathbb{C}_N^3$ is not completely contractive follows from (4) above.

In particular, $V = I_2$ makes $\mathbb{C}_N^3$ into a contractive module which is not completely contractive.

Remark. Evidently, results on completely contractive modules are not in the final form. A more detailed study is under progress.

REFERENCES

1. W. B. ARVESON, Subalgebras of C^*-algebras, *Acta. Math.* **123** (1969), 141–224.
2. W. B. ARVESON, Subalgebras of C^*-algebras, *Acta. Math.* **128** (1972), 271–308.
3. J. BURBEA, The Catheodory metric and its majorant metrices, *Canad. J. Math.* **29** (1977), 771–780.
4. J. B. CONWAY, "Subnormal Operators," Pitman, New York, 1981.
5. M. J. COWEN AND R. G. DOUGLAS, Complex geometry and operator theory, *Acta. Math.* **141** (1978), 187–261.
6. M. J. COWEN AND R. G. DOUGLAS, Operators possessing an open set of eigen values, *in* "Proceedings, Fejer–Riesz Conference, Budapest, 1980."
7. R. E. CURTO AND N. SALINAS, Generalised Bergman kernels and the Cowen–Douglas theory, *Amer. J. Math.* **106** (1984), 447–448.
8. R. E. CURTO AND N. SALINAS, Spectral properties of Subnormal m-tuples, *Amer. J. Math.* **107** (1985), 113–138.
9. R. G. DOUGLAS, Hilbert modules for function algebras, *Oper. Theory: Adv. Appl.* **17** (1986), 125–139.
10. R. G. DOUGLAS, Hilbert modules for function algebras, Szechuan lectures, Fall 1986.
11. S. D. FISHER, "Function Theory on Planar Domains," Wiley, New York, 1983.
12. T. W. GAMELIN, "Uniform Algebras," Prentice–Hall, Englewood Cliffs, NJ, 1969.
13. N. J. HICKS, Notes on Diffeential Geometry," Van Nostrand, Princeton, NJ, 1965.
14. N. P. JEWEL AND A. R. LUBIN, Commuting weighted shifts and analytic function theory in several variables, *J. Operator Theory* **1** (1979), 207–223.
15. S. KOBAYASHI, Intrinsic Distances, Measures and Geometric Function Theory, *Bull. A.M.S.* **82** (1976), 357.
16. A. R. LUBIN, Weighted shifts and products of sub-normal operator, *Indiana Univ. Math. J.* **26** (1977), 839–845.

17. O. LOOS, Bounded symmetric domains and Jordan pairs, Lecture notes, Univ. of California, Irvine, 1977.
18. G. MISRA, Extremal properties of bundle shifts and curvature inequalities, *J. Operator Theory* **11** (1984), 305–317.
19. R. NARASIMHAN, "Complex Analysis in One Variable," Birkhäuser, Basel, 1985.
20. S. K. PARROTT, Unitary dilation for commuting contractions, *Pacific. J. Math.* **34** (1970), 481–490.
21. V. I. PAULSEN, Toword a theory of K-spectral sets, preprint.
22. V. I. PAULSEN, "Completely Positive Maps, Completely Bounded Maps, Similarities and Dilations," Research Notes in Mathematics, No. 146, Pitman, New York.
23. W. RUDIN, "Function Theory in the Unit Ball of C^n," Springer-Verlag, New York/Berlin, 1980.