Ron Douglas, a personal account

Gadadhar Misra

I still can't believe that Ron is no more. I had first heard of him at Sambalpur university, in the year 1977, from one of his former students, Dr. Swadhin Pattanayk who taught a topics course to our class from the book "Banach Algebra Techniques in Operator Theory" written by R. G. Douglas. This book was unlike any other book on Functional Analysis that we had seen at that time. In a way, it set the tone for several such books which appeared later. In spite of the advanced nature of the topics covered in this book, some of us found it very instructive and my mind was made up to work with Professor Douglas if I should ever get a chance.

I first met Professor Douglas on September 13, 1979. He was the Director of Graduate Studies at the Mathematics Department of SUNY, Stony Brook. I had just arrived to begin my work for a PhD. I remember telling him at the very first meeting, in a somewhat awkward manner, that I have come to work with him. He smiled and said, you don't have to worry about all that right now. I didn't quite understand what he meant at that time. Finally, when I was ready to begin research, he said, "I will have very little time for you, maybe, you should consider working with someone else." Afraid that he may not take me as a student, I nervously assured him that I will not take much of his time. When he accepted me as a student, I was thrilled and filled with joy. Because of his several administrative duties, I did not get the opportunity to talk to him very much when I was in graduate school. None the less, he did point me in the direction of finding inequalities for the curvature invariant of a Fredholm operator possessing an open set Ω of eigenvalues and of index -1 there. Abstracting these properties, he along with M. J. Cowen, had introduced the important class of operators $B_n(\Omega)$ earlier, in the very substantial paper [1]. The operators in this class are now known as the Cowen-Douglas operators and they have been studied continuously and vigorously to this date without showing any sign of slowing down.

As if to make up for the lack of our interaction when I was a graduate student, he invited me to join him in carrying forward his programme on Hilbert Modules described in the book [2]. After I returned to India in 1985, I made several trips to Stony Brook during the summers starting from the summer of 1986 to work on this topic. It was a matter of great pride for me when he was able to visit India for the first time during the winter of 1996 just before moving to Texas A&M as the Provost. Indeed, if I recall correctly, he came to Bangalore from Stony Brook but returned to College Station. This trip was followed by several others to India. During each one of these trips, he would lecture extensively and talk to a number of young students. The two of us had organized a special session on multi-variate operator theory in the IndoAMS annual meeting. This was held at the Indian Institute of Science, Bangalore, in December of 2003. In a postscript to the hugely influential paper [3] containing the main ingredients for what is called the Arveson-Douglas conjecture, he says: This research was begun during a visit to India supported in part by the DST-NSF S&T Cooperation Programme. During conferences in Chennai and Bangalore, the author had the opportunity to speak with W.B. Arveson about his results. The author would like to acknowledge that these conversations prompted this work.

He had an uncanny knack for asking what might appear to be a simple question, which, more often than not, has resulted in deep and new insights. Let me give a couple of examples.

The class introduced in [1] consists of operators acting on some Hilbert space $\mathcal{H}$ possessing an open set $\Omega \subseteq \mathbb{C}$ of eigenvalues of (constant) multiplicity 1 and characterized by the existence of a holomorphic map $\gamma : \Omega \to \mathcal{H}$ such that $\gamma(w)$ is an eigenvector with eigenvalue $w \in \Omega$. One of the main features of the operator T in this class is that the curvature

$$\mathcal{K}_T(w) := -\partial\bar{\partial} \log \|\gamma_T(w)\|^2$$

of the holomorphic Hermitian line bundle E_T determined by the holomorphic map γ_T equipped with the Hermitian structure $\|\gamma_T(w)\|^2$ is a complete unitary invariant for the operator T. It is easy to see that if T is a contraction in the Cowen-Douglas class of the unit disc $\mathbb{D}$, then $\mathcal{K}_T(w) \leq \mathcal{K}_{S^*}(w)$, where S^* is the backward unilateral shift acting on ℓ^2. Choosing a holomorphic frame γ_{S^*}, say $\gamma_{S^*}(w) = (1, w, w^2, \ldots)$, it follows that $\|\gamma_{S^*}(w)\|^2 = (1 - |w|^2)^{-1}$ and that $\mathcal{K}_{S^*}(w) = -(1 - |w|^2)^{-2}$, $w \in \mathbb{D}$. Thus the the operator S^* is an extremal operator in the class of all contractive Cowen-Douglas operators. A very simple question that Ron asked many years ago was if the curvature $\mathcal{K}_T$ of a contraction T achieves equality in this inequality even at just one point, then does it follow that T must be unitarily equivalent to S^*? Once the question is raised it is easy to see that the answer is "no", in general. However, if T is homogeneous, namely, $U_\varphi^* T U_\varphi = \varphi(T)$ for each bi-holomorphic automorphism φ of the unit disc and some unitary U_φ, then the answer is "yes". Of course, it is then natural to ask what are all the homogeneous operators. Finding these has been a very rewarding experience. In the process, one discovers many interesting relationships between complex geometry, representation theory of Lie groups and operator theory. Also, it is natural to replace the unit disc by a more general domain either in $\mathbb{C}$, or even in $\mathbb{C}^n$ and make up similar questions. Finding answers to these is not entirely trivial and almost always involve adapting techniques from other related areas and having to find new ones.

Another topic, among many others, that was very close to his heart is the study of Hilbert modules, which he introduced in the mid-eighties based on his lectures at the Szechuan University. A Hilbert module is simply a Hilbert space together with an action of a function algebra. In all the familiar examples, one has a natural action, given by point-wise multiplication, of the ring of polynomials or the rational functions. What he called a Hilbert module, required this action to be continuous in both variables. One of the first observations he made was that all the submodules of the Hardy module are isomorphic to the Hardy module, that is, there exists an intertwining unitary module map between them. This immediately gives an alternative proof of Beurling's theorem describing all the submodules of the Hardy module. The point of this new proof was to ask what happens in the multi-variate case. For instance, if one considers the Hardy module over the poly-disc algebra, the situation is much more complicated. This new approach, however, explains why attempts to prove a theorem like that of Beurling had failed in the case of more than one variable. Also, it becomes apparent that finding the moduli space for the isomorphism classes of submodules of a Hilbert module is a very interesting problem. What Ron had observed is that the moduli space is a singleton for the Hardy module in one variable. It is surprising that while the study of isomorphism classes of quotient modules can be seen to be the familiar problem of the Sz.-Nagy - Foias model theory, a similar question involving submodules had to wait for the notion of Hilbert modules to be introduced.

The last time I met Ron was at the Workshop on "Analytic Hilbert Modules" held at the Yau Mathematical Sciences Center in May 2017. I was there along with several PhD students and young researchers from India. All of them were clearly very happy to have the opportunity to talk to Ron in person. I hardly imagined that it would be our last meeting. Now, all that is left is the memory of those few days that we spent together.

All of us here in India and others around the world, who have benefited greatly from his lectures, books and clear expositions will miss him very much in the years to come. I, for one, having had the chance to work with him and having come to depend on him over the past several years will take a very long time to find my feet again.

References

[1] M. J. Cowen and R. G. Douglas, *Complex geometry and operator theory*, Acta Math., 141 (1978), 187 - 261.

[2] R. G. Douglas and V. I. Paulsen, *Hilbert modules over function algebras*, Pitman Research Notes in Mathematics Series, 217, 1989.

[3] R. G. Douglas, *Essentially reductive Hilbert modules*, Journal of Operator Theory, 55 (2006), 117-133.

Email address: gm@iisc.ac.in

Department of Mathematics, Indian Institute of Science, Bangalore 56 0 012