HOMOGENEOUS HERMITIAN HOLOMORPHIC VECTOR BUNDLES AND THE COWEN-DOUGLAS CLASS OVER BOUNDED SYMMETRIC DOMAINS

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Abstract. It is known that all the vector bundles of the title can be obtained by holomorphic induction from representations of a certain parabolic group on finite dimensional inner product spaces. The representations, and the induced bundles, have composition series with irreducible factors. We give a condition under which the bundle and the direct sum of its irreducible constituents are intertwined by an equivariant constant coefficient differential operator. We show that in the case of the unit ball in $\mathbb{C}^2$ this condition is always satisfied. As an application we show that all homogeneous pairs of Cowen-Douglas operators are similar to direct sums of certain basic pairs.

Résumé. Il est bien connu que les fibrés vectoriels homogènes holomorphes hermitiens peuvent être obtenus par induction holomorphe à partir des representations à dimension finie d'un certain groupe parabolique. Les représentations, ainsi que les fibrés induits, ont des séries de composition. On donne une condition qui assure que le fibré et la somme directe des termes de sa série de composition soient entrelacés par un opérateur différentiel invariant à coefficients constants. Cette condition est toujours satisfaite au cas de la boule unité de deux dimensions complexes. Comme application on montre que tous les couples d'opérateurs homogènes de la classe de Cowen-Douglas associés à la boule sont similaires à des sommes directes de certains couples fondamentaux.

1. Holomorphic vector bundles

Let $\mathfrak{g}$ be a simple non-compact Lie algebra with Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ such that $\mathfrak{k}$ is not semi-simple. Then $\mathfrak{k}$ is the direct sum of its center and of its semisimple part, $\mathfrak{k} = \mathfrak{z} + \mathfrak{k}_{\rm ss}$, and there is an element $\hat{z}$ which generates $\mathfrak{z}$ and $\mathrm{ad}(\hat{z})$ is a complex structure on $\mathfrak{p}$.

The complexification $\mathfrak{g}^{\mathbb{C}}$ is then the direct sum $\mathfrak{p}^+ + \mathfrak{k}^{\mathbb{C}} + \mathfrak{p}^-$ of the $i, 0, -i$ eigenspaces of $\mathrm{ad}(\hat{z})$ We let $G^{\mathbb{C}}$ denote the simply connected Lie group with Lie algebra $\mathfrak{g}^{\mathbb{C}}$ and we let $G, K^{\mathbb{C}}, K, P^{\pm}, \ldots$ be the analytic subgroups corresponding to $\mathfrak{g}^{\mathbb{C}}, \mathfrak{k}^{\mathbb{C}}, \mathfrak{k}, \mathfrak{p}^{\pm}, \ldots$. We denote by $\tilde{G}$ the universal covering group of the group G and by $\tilde{K}, \tilde{K}_{\rm ss}, \ldots$ its analytic subgroups corresponding to $\mathfrak{k}, \mathfrak{k}_{\rm ss}, \ldots$.

$K^{\mathbb{C}}P^-$ is a parabolic subgroup of $G^{\mathbb{C}}$. $P^+K^{\mathbb{C}}P^-$ is open dense in $G^{\mathbb{C}}$. The corresponding decomposition $g^+g^0g^-$ of any g in $P^+K^{\mathbb{C}}P^-$ is unique. The natural map $G/K \to G^{\mathbb{C}}/K^{\mathbb{C}}P^-$ is a holomorphic imbedding, its image is in the orbit of P^+. Applying now $\exp_{\mathfrak{p}^+}^{-1}$ we get the Harish-Chandra realization of G/K as a bounded symmetric domain $\mathcal{D} \subset \mathfrak{p}^+$. The action of $g \in G$ on $z \in \mathcal{D}$, written $g \cdot z$, is then defined by $\exp(g \cdot z) = (g \exp z)^+$. We will use the notation $k(g, z) = (g \exp z)^0$ and $\exp Y(g, z) = (g \exp z)^-$, so we have

$$g \exp z = (\exp(g \cdot z))k(g, z) \exp(Y(g, z)).$$

The $\tilde{G}$ - homogeneous Hermitian holomorphic vector bundles (`hHhvb`) over $\mathcal{D}$ are obtained by holomorphic induction from representations (ρ, V) of $\mathfrak{k}^{\mathbb{C}} + \mathfrak{p}^-$ on finite dimensional inner product spaces V such that $\rho(\mathfrak{k})$ is skew Hermitian. We write ρ^0, ρ^- for the restrictions of ρ to $\mathfrak{k}^{\mathbb{C}}$ and $\mathfrak{p}^-$,

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respectively. The representation space V is the orthogonal direct sum of its subspaces V_λ ($\lambda \in \mathbb{R}$) on which $\rho^0(\hat{z}) = i\lambda$. It is easy to see that $\rho^-(Y)V_\lambda \subset V_{\lambda-1}$ for $Y \in \mathfrak{p}^-$. We also have

$$\rho^-([Z,Y]) = [\rho^0(Z), \rho^-(Y)],\ Z \in \mathfrak{k}^{\mathbb{C}},\ Y \in \mathfrak{p}^-. \tag{1}$$

We note that if representations ρ^0 and ρ^- of $\mathfrak{k}^{\mathbb{C}}$ and $\mathfrak{p}^-$, respectively, are given, then they will together give a representation of $\mathfrak{k}^{\mathbb{C}} + \mathfrak{p}^-$ if and only if equation (1) holds. We call (ρ, V) and the induced bundle, indecomposable if it is not the orthogonal sum of sub-representations, respectively, sub-bundles. We restrict ourselves to describing these.

Proposition 1.1. *Every indecomposable holomorphic homogeneous Hermitian vector bundle E can be written as a tensor product $L_{\lambda_0} \otimes E'$, where L_{λ_0} is the line bundle induced by a character χ_{λ_0} and E' is the lift to $\tilde{G}$ of a G - homogeneous holomorphic Hermitian vector bundle which is the restriction to G and $\mathcal{D}$ of a $G^{\mathbb{C}}$ - homogeneous vector bundle induced in the holomorphic category by a representation of $K^{\mathbb{C}}P^-$.*

The proof involves some structural properties of $G^{\mathbb{C}}$, which we omit in this short Announcement.

As shown in [1], $P^+ \times \tilde{K}^{\mathbb{C}} \times P^-$ can be given a structure of complex analytic local group such that (writing $\pi : \tilde{K}^{\mathbb{C}} \to K^{\mathbb{C}}$) $\mathrm{id} \times \pi \times \mathrm{id}$ is the universal local group covering of $P^+K^{\mathbb{C}}P^-$. We write $\tilde{G}_{\mathrm{loc}}$ for this local group and abbreviate $\mathrm{id} \times \pi \times \mathrm{id}$ to π. By [1], $\tilde{G}$, $\tilde{K}^{\mathbb{C}}P^-$, $P^+\tilde{K}^{\mathbb{C}}$ are closed subgroups of $\tilde{G}^{\mathbb{C}}_{\mathrm{loc}}$ and $\tilde{G} \exp \mathcal{D} \subset \tilde{G}^{\mathbb{C}}_{\mathrm{loc}}$. Defining $g \cdot z = \pi(g) \cdot z$ and $Y(g,z) = Y(\pi(g), z)$ we have the decomposition

$$g \exp z = (\exp g \cdot z)\tilde{k}(g,z) \exp Y(g,z),\ (g \in \tilde{G}, z \in \mathcal{D})$$

in $\tilde{G}_{\mathrm{loc}}$. We write $\tilde{b}(g,z) = \tilde{k}(g,z) \exp Y(g,z)$; then $\tilde{b}(g,z)$ satisfies the multiplier identity and $\tilde{b}(kp^-, 0) = kp^-$ for $kp^- \in \tilde{K}^{\mathbb{C}}P^-$.

Hence given a representation (ρ, V) of $\mathfrak{k}^{\mathbb{C}} + \mathfrak{p}^-$ as above, the holomorphically induced bundle has a canonical trivialization such that the sections are the elements of $\mathrm{Hol}(\mathcal{D}, V)$, and $\tilde{G}$ acts via the multiplier

$$\rho(\tilde{b}(g,z)) = \rho^0(\tilde{k}(g,z))\rho^-(\exp Y(g,z)).$$

If $f \in \mathrm{Hol}(\mathcal{D}, V)$, then we write Df for the derivative: $Df(z)X = (D_X f)(z)$ for $X \in \mathfrak{p}^+$. Thus $Df(z)$ is a $\mathbb{C}$ - linear map from $\mathfrak{p}^+$ to V.

Lemma 1.2. *For any holomorphic representation τ of $\tilde{K}^{\mathbb{C}}$ and any $g \in \tilde{G}$, $z \in \mathcal{D}$, $X \in \mathfrak{p}^+$,*

$$D_X \tau\big(\tilde{k}(g,z)^{-1}\big) = -\tau\big([Y(g,z), X]\big)\tau\big(\tilde{k}(g,z)^{-1}\big).$$

Furthermore,

$$D_X Y(g,z) = \frac{1}{2}\big[Y(g,z), [Y(g,z), X]\big].$$

This is proved by refining the arguments of [4, p. 65]

2. The main results about vector bundles

If in the set up of Section 1 each subspace V_λ is irreducible under $\mathfrak{k}^{\mathbb{C}}$ we call the corresponding representations and the vector bundles *filiform*. We consider this case first.

We have seen that every indecomposable filiform representation is a direct sum of subspaces $V_{\lambda-j}$, which we denote by V_j, carrying an irreducible representation ρ_j^0 of $\mathfrak{k}^{\mathbb{C}}$ ($0 \le j \le m$), furthermore, we have non-zero $\mathfrak{k}^{\mathbb{C}}$-equivariant maps $\rho_j^- : \mathfrak{p}^- \to \mathrm{Hom}(V_{j-1}, V_j)$. The space of such maps is 1-dimensional: This is an equivalent restatement of the known fact that $\mathfrak{p}^- \otimes V_{j-1}$ as a representation of $\mathfrak{k}^{\mathbb{C}}$ is multiplicity free [2, Corollary 4.4]. We denote the orthogonal projection from $\mathfrak{p}^- \otimes V_{j-1}$ to V_j by P_j. We define for $Y \in \mathfrak{p}^-$, $v \in V_{j-1}$,

$$\tilde{\rho}_j(Y)v = P_j(Y \otimes v). \tag{2}$$

Then $\tilde\rho_j$ has the $\mathfrak{k}^{\mathbb{C}}$-equivariant property, and it follows that $\rho_j^- = y_j\tilde\rho_j$ with some $y_j \neq 0$. We write $y = (y_1, \dots, y_m)$ and denote by E^y the induced vector bundle. We observe here that the vector bundle E^y is uniquely determined by $\rho_0^0, P_1, \dots, P_m$ and y, but these data cannot be arbitrarily chosen: The $\tilde\rho_j$ $(1 \le j \le m)$ together must give a representation of the abelian Lie algebra $\mathfrak{p}^-$. In terms of P_j, this condition amounts to

$$P_{j+1}\big(Y' \otimes P_j(Y \otimes v)\big) = P_{j+1}\big(Y \otimes P_j(Y' \otimes v)\big) \tag{3}$$

for all $Y, Y' \in \mathfrak{p}^-$ and $v \in V_{j-1}$.

We denote by ι the identification of $(\mathfrak{p}^+)^*$ with $\mathfrak{p}^-$ under the Killing form, and for any vector space W, extend it to a map from $\mathrm{Hom}(\mathfrak{p}^+, W)$ to $\mathfrak{p}^- \otimes W$, that is, for $Y \in \mathfrak{p}^-, w \in W$,

$$\iota(B(\cdot, Y)w) = Y \otimes w.$$

In what follows n denotes the complex dimension of $\mathcal{D}$.

Lemma 2.1. *Given ρ_{j-1}^0, ρ_j^0, as above, there exists a constant c_j, independent of λ, such that for all $Y \in \mathfrak{p}^-$, we have*

$$P_j\iota\rho_{j-1}^0([Y,\cdot]) = (c_j - \tfrac{\lambda-j+1}{2n})\tilde\rho_j(Y).$$

This follows from $\mathfrak{k}^{\mathbb{C}}$-equivariance and a character computation. The following two lemmas can be proved by computations based on Lemmas 1.2 and 2.1.

Lemma 2.2. *For all $1 \le j \le m-1$, and holomorphic $F : \mathcal{D} \to V_j$,*

$$\begin{aligned} &P_{j+1}\iota D^{(z)}\big(\rho_j^0(\tilde k(g,z)^{-1})F(gz)\big) \\ &\quad = \; -(c_{j+1} - \tfrac{\lambda-j}{2n})\tilde\rho_{j+1}\big(Y(g,z)\big)\big(\rho_j^0(\tilde k(g,z)^{-1})F(gz)\big) + \rho_{j+1}^0(\tilde k(g,z)^{-1})\big((P_{j+1}\iota DF)(gz)\big), \end{aligned}$$

where $D^{(z)}$ denotes the differentiation with respect to z.

Lemma 2.3. *For all $1 \le j \le m-1$, with the constants c_j of Lemma 2.1,*

$$P_{j+1}\iota D^{(z)}\tilde\rho_j(Y(g,z)) = \frac{1}{2}(c_j - c_{j+1} - \tfrac{1}{2n})\tilde\rho_{j+1}(Y(g,z))\tilde\rho_j(Y(g,z)).$$

Now let E^y be an indecomposable filiform hHhvb as described above. Writing $0 = (0, \dots, 0)$, E^0 makes sense, it is the direct sum of irreducible vector bundles in the composition series of E^y.

If $f \in \mathrm{Hol}(\mathcal{D}, V)$, we write f_j for the component of f in V_j, that is, the projection of f onto V_j.

Theorem 2.4. *Assume that in E^y, the constants c_j of Lemma 2.1 are of the form*

$$c_j = u + (j-1)v,\ 1 \le j \le m, \tag{\#}$$

with some constants u, v and λ is regular in the sense that

$$c_{jk} = \tfrac{1}{(j-k)!}\prod_{i=1}^{j-k}\big\{u - \tfrac{\lambda}{2n} + \tfrac{2k+i-1}{2}\big(v + \tfrac{1}{2n}\big)\big\}^{-1}$$

is meaningful for $0 \le k \le j \le m$. Then the operator $\Gamma : \mathrm{Hol}(\mathcal{D}, V) \to \mathrm{Hol}(\mathcal{D}, V)$ given by

$$(\Gamma f_j)_\ell = \begin{cases} c_{\ell j}\, y_\ell \cdots y_{j+1}(P_\ell\iota D)\cdots(P_{j+1}\iota D)f_j & \text{if } \ell > j, \\ f_j & \text{if } \ell = j, \\ 0 & \text{if } \ell < j \end{cases}$$

intertwines the actions of $\tilde G$ on the trivialized sections of E^0 and E^y.

The proof is by induction based on the preceding lemmas.

We note that condition (#) is vacuous when $m = 1$ or 2. Otherwise, it can be shown that (#) is also necessary for the theorem to hold.

Next we pass from the filiform case to the general case. Now (ρ_0, V_0) is a direct sum of representations $(\rho_j^{0^\alpha}, V_j^\alpha)$ with inequivalent irreducible representations α of $\mathfrak{k}_{\rm ss}^{\mathbb{C}}$, and $\rho_j^{0^\alpha} = \chi_{\lambda-j}(I_{m_{j\alpha}} \otimes \alpha)$. For pairs of (α, β) that are admissible in the sense that$\beta \subset \mathrm{Ad}_{\mathfrak{p}^-} \otimes \alpha$ we write $P_{\alpha\beta}$ for the corresponding projection and define maps $\tilde\rho_{\alpha\beta}$ for $Y \in \mathfrak{p}^-$. Then

$$\rho_j^-(Y) = \oplus_{\alpha,\beta} y_j^{\alpha\beta} \otimes \tilde\rho_{\alpha\beta}(Y)$$

with $y_j^{\alpha\beta} \in \mathrm{Hom}(\mathbb{C}^{m_\alpha}, \mathbb{C}^{m_\beta})$ such that $y_{j+1}^{\beta\gamma} y_j^{\alpha\beta} = 0$ unless $(\alpha\beta)$ and $(\beta\gamma)$ are admissible and the analogue of (3) holds. We let $\mathbb{E}^y$ denote the bundle holomorphically induced by ρ, and let $\mathbb{E}^0$ be the (direct sum) bundle gotten by changing all the $y^{\alpha\beta}$ to 0. The general version of Γ is now going to be (for $j < \ell$)

$$(\Gamma f_j)_\ell = \oplus_{\alpha_j,\dots,\alpha_\ell} c_{\ell j}^{\alpha_j,\dots,\alpha_\ell} \big(y_\ell^{\alpha_{\ell-1}\alpha_\ell} \cdots y_{j+1}^{\alpha_j \alpha_{j+1}}\big) \otimes \big(P_{\alpha_{\ell-1}\alpha_\ell} \iota D\big) \cdots \big(P_{\alpha_j j+1} \iota D\big) f_j^\alpha.$$

For $j \geq \ell$, it is unchanged.

Theorem 2.5. *Suppose that (#) holds for all indecomposable filiform subbundles of $\mathbb{E}^y$. Then there exist constants $c_{\ell j}^{\alpha_j,\dots,\alpha_\ell}$ such that Γ intertwines the actions of $\tilde G$ on the trivialized sections of $\mathbb{E}^0$ and $\mathbb{E}^y$.*

We note that if $\mathcal{D}$ is the disc in one variable then (#) always holds with $u = v = 0$. So, Theorem 2.5 contains Theorem 3.1 of [3]. We proceed to discuss cases where (#) is satisfied.

The Cartan product of two irreducible representations of $\mathfrak{k}^{\mathbb{C}}$ is the irreducible component of the tensor product whose highest weight is the sum of the highest weights of the original representations.

Lemma 2.6. *Let ρ_0^0 be any irreducible representation of $\mathfrak{k}^{\mathbb{C}}$ and define ρ_j^0 $(1 \leq j \leq m)$ inductively as the Cartan product of $\mathrm{Ad}_{\mathfrak{p}^-}$ and ρ_{j-1}^0. Then with $\tilde\rho_j$ as in (2) and $\rho_j^- = y_j \tilde\rho_j$ $(y_j \neq 0)$ we obtain a filiform representation ρ of $\mathfrak{k}^{\mathbb{C}} + \mathfrak{p}^-$. In this case (#) is automatically satisfied.*

Both the statements in this Lemma are proved by using weight theory.

Finally, we specialize to the case where $\mathcal{D}$ is the unit ball in $\mathbb{C}^2$. Then $G = SU(2,1)$, $\mathfrak{k}_{\rm ss}^{\mathbb{C}} = \mathfrak{sl}(2,\mathbb{C})$. It is well known that the irreducible representations of $\mathfrak{sl}(2,\mathbb{C})$ are just the symmetric tensor powers of the natural representation, which is equivalent to $\mathrm{Ad}'_{\mathfrak{p}^-}$, the restriction of $\mathrm{Ad}_{\mathfrak{p}^-}$ to $\mathfrak{k}_{\rm ss}^{\mathbb{C}}$. Consequently, a complete description of indecomposable representations of $\mathfrak{k}^{\mathbb{C}} + \mathfrak{p}^-$ is possible in this case. In the following, an exponent in brackets, $[k]$, denotes the k-th symmetric tensor power.

Proposition 2.7. *For the complex ball in $\mathbb{C}^2$, there are only two types of indecomposable filiform representations of $\mathfrak{k}^{\mathbb{C}} + \mathfrak{p}^-$. For the first type, $\rho_j^0 = \chi_{\lambda-j} \otimes (\mathrm{Ad}'_{\mathfrak{p}^-})^{[k+j]}$, $0 \leq j \leq m$, with some $\lambda \in \mathbb{R}$ and $k, m \in \mathbb{N}$. For the second type, $\rho_j^0 = \chi_{\lambda-j} \otimes (\mathrm{Ad}'_{\mathfrak{p}^-})^{[k-j]}$, $0 \leq j \leq m$, with $\lambda \in \mathbb{R}$, and $m \leq k$. For both types the condition (#) is satisfied.*

Sketch of the proof. By Clebsch-Gordan, for $j \geq 1$,

$$\mathrm{Ad}'_{\mathfrak{p}^-} \otimes (\mathrm{Ad}'_{\mathfrak{p}^-})^{[j]} = (\mathrm{Ad}'_{\mathfrak{p}^-})^{[j+1]} \oplus (\mathrm{Ad}'_{\mathfrak{p}^-})^{[j-1]},$$

So for each P_j there are two possibilities, "up" or "down". One can show that unless all are up or all are down, the condition in equation (3) will fail.

When all P_j are "up" (the first type), we are in the situation of Lemma 2.6, so (#) holds. The second type is the contragredient of a representation of the first type: it follows that (#) holds again. □

Theorem 2.8. *In the case of the unit ball in $\mathbb{C}^2$ the conclusion of Theorem 2.5 holds for all indecomposable hHhvb-s with regular λ.*

This is because Proposition 2.7 shows that the condition of Theorem 2.5 is satisfied.

3. Hilbert spaces of sections

With notations preserved, for general $\mathcal{D}$, we consider first the case where ρ is irreducible. Then automatically ρ^0 is also irreducible and $\rho^- = 0$. We write $\rho^0 = \chi_\lambda \otimes \sigma$, where σ is an irreducible representation of $\mathfrak{k}_{\rm ss}$. For every σ, there is an (explicitly known) set of λ-s such that the sections of the corresponding holomorphically induced vector bundle have a $\tilde{G}$-invariant inner product. This is Harish-Chandra's holomorphic discrete series and its analytic continuation. In the canonical trivialization it gives Hilbert spaces $\mathcal{H}_{\rho^0} = \mathcal{H}_{\sigma,\lambda}$ which are known to have reproducing kernels $K_{\sigma,\lambda}(z,w)$. If we set

$$\tilde{\mathcal{K}}(z,w) = \tilde{k}(\exp -\bar{w}, z),$$

(where $\bar{w}$ denotes conjugation with respect to $\mathfrak{g}$) we have, slightly extending [4, Chap II, §5]

$$K_{\sigma,\lambda}(z,w) = (\chi_\lambda \otimes \sigma)(\tilde{\mathcal{K}}(z,w)).$$

In particular, it is known that the inner product is *regular* in the sense that all K - types (i.e polynomials) have non-zero norm in $\mathcal{H}_{\sigma,\lambda}$ if and only if $\lambda < \lambda_\sigma$ for a certain known constant λ_σ.

In the following theorem we consider a bundle $\mathbb{E}^y$ as in Section 2. The corresponding $\mathbb{E}^0$ is then a direct sum of irreducible bundles as above. Its sections have a $\tilde{G}$-invariant inner product if and only if this is true for each summand. In this case, we have a Hilbert space $\mathcal{H}^0 = \oplus \mathcal{H}_{\rho_j^0}$.

Theorem 3.1. *Suppose (#) holds for all filiform subbundles of $\mathbb{E}^y$. Then the sections of $\mathbb{E}^y$ have a $\tilde{G}$-invariant regular inner-product if and only if the same is true for $\mathbb{E}^0$. In this case, the map Γ is a unitary isomorphism of $\mathcal{H}^0$ onto the Hilbert space $\mathcal{H}^y$ of sections of $\mathbb{E}^y$. The space $\mathcal{H}^y$ (as well as $\mathcal{H}^0$) has a reproducing kernel.*

For the proof one observes that Γ has an inverse of the same form (only the constants c_{jk} change). Γ being a holomorphic differential operator, the image of $\mathcal{H}^0$ is also a Hilbert space of holomorphic functions with a reproducing kernel. One can verify that this is the sought after $\mathcal{H}^y$.

Theorem 3.2. *Suppose $\mathcal{D}$ is the unit ball in $\mathbb{C}^n$. Let σ_0, σ_1 be the irreducible representations of $\mathfrak{k}_{\rm ss}^{\mathbb{C}}$ such that $\sigma_1 \subset \mathrm{Ad}_{\mathfrak{p}^-} \otimes \sigma_0$ and let P be the corresponding projection. Then if $\lambda < \lambda_{\sigma_0}$, we have $\lambda - 1 < \lambda_{\sigma_1}$ and $P\iota D$ is a bounded linear transformation from $\mathcal{H}_{\sigma_0,\lambda}$ to $\mathcal{H}_{\sigma_1,\lambda-1}$.*

By the theory of reproducing kernels, for this it is enough to prove that ($D^{(z)}$ and $D^{(w)}$ denote the differentiation with respect to the variable z and w respectively)

$$CK_{\sigma_1,\lambda-1}(z,w) - (P\iota D^{(z)})K_{\sigma_0,\lambda}(z,w)(P\iota D^{(w)})^*$$

is positive definite for some $C > 0$. (In general, we say that a kernel K taking values in $\mathrm{Hom}(V,V)$ is positive definite if, for any $z_1, \dots, z_n$ in $\mathcal{D}$ and $v_1, \dots, v_n$ in V,

$$\sum_{j,k=1}^{n} \langle K(z_j, z_k) v_k, v_j \rangle \geq 0$$

holds.)

Remark 3.3. When $\mathcal{D}$ is the unit ball in $\mathbb{C}^2$, the conditions of Theorem 3.1 are satisfied for any indecomposable $\mathbb{E}^y$. Furthermore, the spaces $\mathcal{H}^0$ and $\mathcal{H}^y$ are equal as sets. This follows from Theorem 3.2 and the closed graph theorem.

4. Homogeneous Cowen-Douglas pairs

For any bounded domain $\mathcal{D} \subseteq \mathbb{C}^n$, the n-tuple $\boldsymbol{T} = (T_1, \dots, T_n)$ of bounded linear operators on a Hilbert space $\mathcal{H}$ is said to be homogeneous (relative to the holomorphic automorphism group $\mathrm{Aut}(\mathcal{D})$) if the joint (Taylor) spectrum of $\boldsymbol{T}$ is in $\overline{\mathcal{D}}$ and for every g in $\mathrm{Aut}(\mathcal{D})$, the n-tuple $g(\boldsymbol{T}) = g(T_1, \dots, T_n)$ is unitarily equivalent to $\boldsymbol{T}$.

Another important class of n-tuples of commuting operators associated with the domain $\mathcal{D} \subseteq \mathbb{C}^n$ is the extended Cowen-Douglas class $\mathrm{B}'_k(\mathcal{D})$. Its elements are n-tuples of bounded operators that can be realized as adjoints of the multiplications by the coordinate functions on some Hilbert space of holomorphic $\mathbb{C}^k$- valued functions on $\mathcal{D}$ possessing a reproducing kernel K and containing all $\mathbb{C}^k$-valued polynomials as a dense set. (The strict Cowen-Douglas class $\mathrm{B}_k(\mathcal{D})$ as originally defined consists of the n-tuples of bounded operators $(T_1, \dots, T_n)$ that can be realized like this and in addition satisfy the condition that $\oplus(T_j - z_j)$ mapping the Hilbert space into the n-fold direct sum with itself has closed range.)

We wish to investigate, for bounded symmetric domains $\mathcal{D}$, the homogeneous n-tuples in $\mathrm{B}'_k(\mathcal{D})$. For the case of the unit disc, there is a complete description and classification of these in [3]. (In that case, it turns out that the homogeneous operators in $\mathrm{B}'_k(\mathbb{D})$ are the same as in $\mathrm{B}_k(\mathbb{D})$.)

Theorem 4.1. *For any bounded symmetric $\mathcal{D}$, an n-tuple $\boldsymbol{T}$ in $\mathrm{B}'_k(\mathcal{D})$ is homogeneous if and only if the corresponding holomorphic Hermitian vector bundle is homogeneous under $\tilde{G}$.*

The proof (not entirely trivial) is the same as in [3, Theorem 2.1].

For a bounded symmetric $\mathcal{D}$, we call a n-tuple $\boldsymbol{T}$ in $\mathrm{B}'_k(\mathcal{D})$ and its corresponding bundle E basic if E is induced by an irreducible ρ. From the results of Section 3, when $\mathcal{D}$ is the unit ball in $\mathbb{C}^n$, E is basic if and only if it is induced by some $\chi_\lambda \otimes \sigma$ with $\lambda < \sigma_\lambda$.

Theorem 4.2. *If $\mathcal{D}$ is the unit ball in $\mathbb{C}^2$, all homogenous pairs in $\mathrm{B}'_k(\mathcal{D})$ are similar to direct sums of basic homogeneous pairs.*

The proof is based on Remark 3.3. The similarity arises as the identity map between $\mathcal{H}^0$ to $\mathcal{H}^y$, which clearly intertwines the operators M_j on the respective Hilbert spaces.

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