Notes on the Brown, Douglas and Fillmore Theorem

Contents

1 Chapter 1: Compact and Fredholm Operators

1 Overview and Polar Decomposition

1.1 Overview

A normal operator on a finite dimensional inner product space can be diagonalised and the eigenvalues together with their multiplicities are a complete set of unitary invariants for the operator, while on a infinite dimensional Hilbert space the spectral theorem provides a model and a complete set of unitary invariants for such operators. Thus we view the theory of normal operators to be well understood. It is natural to study operators which may be thought of in some sense to be nearly normal. One hope is that it would be possible to provide canonical models and a complete set of invariants for such operators. Since an operator is normal if the commutator $[T, T^*] = TT^* - T^*T = 0$, one may say an operator is nearly normal if $[T, T^*]$ is small in some appropriate sense, for example, finite rank, trace class or compact. In these notes, we will take the last of these three measures of smallness for $[T, T^*]$ and make the following definition.

Definition 1.2. An operator T is *essentially normal* if $[T, T^*]$ is compact.

Our goal would be to classify the essentially normal operators with respect to some suitable notion of equivalence. Since we are considering compact operators to be small, the correct notion of equivalence would seem to be the following.

Definition 1.3. Two operators T_1 and T_2 are said to be *essentially equivalent* if there exists a unitary operator U and a compact operator K such that $UT_1U^* = T_2 + K$, in this case we write, $T_1 \sim T_2$.

(The *goal* of these notes is to describe, $\{$essentially normal operators$\}/\sim$.)

We will very closely follow the basic work of Brown, Douglas and Fillmore [1, 2].

Why should this problem be tractable at all? To answer this question, we have to look at some early history preceeding the work of Brown, Douglas and Fillmore [1, 2].

1.4 Brief History

In 1909, Weyl defined the essential spectrum of a self adjoint operator to be all points in its spectrum except the isolated eigen values of finite multiplicity.

He proved that if two self adjoint operators S and T are essentially equivalent then S and T have the same essential spectrum. Some twenty years later, von Neumann proved a striking converse, that is, if the essential spectrum of two self adjoint operators are equal then they are essentially equivalent. In response to a question of Halmos, Berg and Sikonia, independent of each other, showed in 1973 that the Weyl von Neumann theorem actually holds for normal operators.

What all this has to do with essentially normal operators? The point is that, if $\mathcal{C}(\mathcal{H})$ is the set of compact operators and $\pi : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})/\mathcal{C}(\mathcal{H})$ is the natural quotient map then an operator T is essentially normal if and only if the class $\pi(T)$ is normal in the Calkin algebra $\mathcal{L}(\mathcal{H})/\mathcal{C}(\mathcal{H})$. It is not very hard to see that the essential spectrum $\sigma_{\rm ess}(N)$, of a normal operator N is the same as the spectrum $\sigma(\pi(N))$ of the class $\pi(N)$ of the operator N in the Calkin algebra $\mathcal{L}(\mathcal{H})/\mathcal{C}(\mathcal{H})$. Let

$$\mathcal{N} + \mathcal{C} = \{N + K : N \text{ is normal and } K \text{ is comapct}).$$

For an operator T in $\mathcal{N} + \mathcal{C}$, we see that $\sigma_{\rm ess}(T) = \sigma_{\rm ess}(N+K) = \sigma(\pi(N+K)) = \sigma(\pi(N))$, so that the Weyl–von Neumann–Berg theorem actually extends to operators in the class $\mathcal{N} + \mathcal{C}$.

Theorem 1.5. *(Weyl–von Neumann–Berg theorem). Any two operators T_1 and T_2 in $\mathcal{N} + \mathcal{C}$ are essentially equivalent if and only if $\sigma_{\rm ess}(T_1) = \sigma_{\rm ess}(T_2)$. Moreover, if X is any compact subset of the complex plane $\mathbb{C}$ then there is a normal operator N such that $\sigma_{\rm ess}(N) = X$.*

This theorem shows that the essential spectrum of an operator $\mathcal{N} + \mathcal{C}$ is complete invariant for unitary equivalence modulo compact and the classification problem for such operators is complete. Are all essentially normal operators in $\mathcal{N} + \mathcal{C}$? To give an example of an essentially normal operator not in $\mathcal{N} + \mathcal{C}$, consider the Toeplitz operator T_z on the Hardy space $H^2(\mathbb{T})$. Note that $I - T_z T_z^* = P$ and $I - T_z^* T_z = 0$, where P is a rank one projection, therefore T_z is an essentially unitary operator. An operator T is called *Fredholm* if it has closed range and the dimension of its kernel and cokernel are finite. For a Fredholm operator T, define

$$\text{ind}(T) = \dim\ker(T) - \dim\ker(T^*).$$

It will be shown, that if T is Fredholm and K is compact then

$$\text{ind}(T + K) = \text{ind}(T).$$

If N is a normal operator which is also Fredholm then its index is zero. It is easy to see that the Toeplitz operator T_z is Fredholm, $\dim\ker(T_z) = 0$

and $\dim\ker(T_z^*) = 1$. If in addition T_z is also in $\mathcal{N}+\mathcal{C}$, then we would have

$$-1 = \operatorname{ind}(T_z) - \operatorname{ind}(N+K) = 0.$$

Secondly, note that the Multiplication operator M_z and the Toeplitz operator T_z both have the same essential spectrum, namely the unit circle $\mathbb{T}$. If these two operators were essentially equivalent then we would have

$$-1 = \operatorname{ind}(T_z) = \operatorname{ind}(T_z + K) = \operatorname{ind}(U^* M_z U) = \operatorname{ind}(M_z) = 0.$$

This shows that the essential spectrum is not the only invariant for our equivalence. The remarkable theorem of Brown, Douglas and Fillmore says that the essential spectrum together with a certain index data is complete set of invariants for essential equivalence.

We end this brief introduction, with a discussion of the Polar Decomposition Theorem. In these notes, we assume that all Hilbert spaces are separable.

Polar Decomposition

If λ is a complex number then $\lambda = |\lambda|e^{i\theta}$, for some θ; this is the polar decomposition of λ. For operators, is it possible to find an analogy? To answer this question, we may ask, what is the analogy of $|\lambda|$ and $e^{i\theta}$ among operators. A little thought would show that the analogy for $|\lambda|$ ought to be $(T^*T)^{1/2}$, the analogy for $e^{i\theta}$ would seem to be either an unitary or an isometry. However, none of these is correct for an operator on an infinite dimensional Hilbert space.

Definition 1.6. An Operator V on a Hilbert space $\mathcal{H}$ is a *partial isometry* if $\|Vf\| = \|f\|$ for f orthogonal to $\ker V$; if in addition the kernel of V is $\{0\}$ then V is an isometry. The initial space of V is the closed subspace orthogonal to $\ker V$.

It turns out that the correct analogy for $e^{i\theta}$ is a partial isometry.

Theorem 1.7. *If T is an operator on the Hilbert space $\mathcal{H}$ then there exists a positive operator P and a partial isometry V such that $T = VP$. Moreover, P and V are unique if* $\ker V = \ker P$.

Proof. If we set $P = (T^*T)^{1/2}$, then

$$\|Pf\|^2 = (Pf, Pf) = (P^2 f, f) = (T^*Tf, f) = \|Tf\|^2 \text{for } f \text{ in } \mathcal{H}$$

Thus, if we define

$$\tilde{V} : \text{ran } P \to \mathcal{H} \text{ such that } \tilde{V}Pf = Tf,$$

then $\tilde{V}$ is well defined, in fact it is isometric and extends uniquely to an isometric mapping from clos [ran P] to $\mathcal{H}$. If we further define $V : \mathcal{H} \to \mathcal{H}$ by

$$Vf = \begin{cases} \tilde{V}f & \text{for } f \text{ in clos[ran } P] \\ 0 & \text{for} f \text{ in [ran } P]^{\perp} \end{cases}$$

then V is a partial isometry satisfying $T = VP$ and

$$\ker V = [\text{ran } P]^{\perp} = \ker P.$$

For the uniqueness, note first that if W is a partial isometry then for f in $\mathcal{H}$,

$$((I - W^*W)f, f) = (f, f) - (W^*Wf, f) = \|f\|^2 - \|Wf\|^2 \geq 0.$$

Thus, $(I - W^*W)^{1/2}$ is a well defined positive operator. Now, if $f \perp [\ker W]$ then $\|Wf\| = \|f\|$, and therefore, $((I - W^*W)f, f) = 0$. Since,

$$\|(I - W^*W)^{1/2}f\|^2 = ((I - W^*W)f, f) = 0,$$

we have, $(I - W^*W)^{1/2}f = 0$ or $W^*Wf = f$. Therefore, W^*W is the projection onto the initial space of W.

Now, if $T = WQ$, where W is a partial isometry, Q is positive and $\ker W = \ker Q$ then

$$P^2 = T^*T = QW^*WQ = Q^2,$$

since W^*W is projection onto

$$[\ker W]^{\perp} = [\ker Q]^{\perp} = \text{clos[ran } Q].$$

Thus, by the uniqueness of the square root, we have $P = Q$ and hence $WP = VP$. Therefore, $W = V$ on ran P. But

$$[\text{ran } P]^{\perp} = \ker P = \ker W = \ker V,$$

and hence $W = V$ on [ran $P]^{\perp}$. Therefore, $V = W$ and the proof is complete. □

Some times a polar decomposition in which the order of the factors are reversed is useful.

Corollary 1.8. *If T is an operator on the Hilbert space $\mathcal{H}$, then there exists a positive operator Q and a partial isometry W such that $T = QW$. Moreover, W and Q are unique if ran $W = [\ker Q]^{\perp}$.*

Proof. Apply the theorem to the operator T^*. Observe that $W = V^*$ and $Q = P$ so that $\ker P = \ker V$ if and only if ran $W[\ker Q]^{\perp}$, the uniqueness now follows from the theorem. □

2 Compact and Fredholm Operators

Compact Operators

We will show that an operator is compact if and only if it is the norm limit of a sequence of finite rank operators. Thus the compact operators are the natural generalisation of finite dimensional operators in a topological sense.

However, we first show that any closed subspace of the Hilbert space $\mathcal{H}$ in the range of a compact operator must be finite dimensional. It turns out, any operator whether compact or not, possessing this property can be approximated in norm by a sequence of finite rank operators. Thus, we obtain another characterisation of the compact operators, namely an operator is compact if and only if the only closed subspaces of the Hilbert space $\mathcal{H}$ in its range are finite dimensional.

Definition 2.1. An operator T is *finite rank* if the dimension of its range is finite and *compact* if the image of the unit ball under T is compact. Let $\mathcal{T}(\mathcal{H})$ and $\mathcal{C}(\mathcal{H})$ denote the set of all finite rank and compact operators respectively.

Most of the elementary properties of compact operators are collected together in the following.

Proposition 2.2. *If $\mathcal{H}$ is a Hilbert space then $\mathcal{T}(\mathcal{H})$ is a minimal two sided *-ideal in $\mathcal{L}(\mathcal{H})$.*

Proof. The two inclusions

$$\mathrm{ran}(S+T) \subseteq \mathrm{ran}(S) + \mathrm{ran}(T) \text{ and } \mathrm{ran}(ST) \subseteq \mathrm{ran}(S)$$

show that $\mathcal{T}(\mathcal{H})$ is a left ideal in $\mathcal{L}(\mathcal{H})$. The identity

$$\mathrm{ran}\, T^* = T^*[\ker T^*]^{\perp} = T^*(\mathrm{clos}[\mathrm{ran}\, T])$$

shows that T is in $\mathcal{T}(\mathcal{H})$ if and only if T^* is in $\mathcal{T}(\mathcal{H})$. Finally, if S is in $\mathcal{L}(\mathcal{H})$ and T is in $\mathcal{T}(\mathcal{H})$ then T^*S^* is in $\mathcal{T}(\mathcal{H})$ and hence $ST = (T^*S^*)^*$ is in $\mathcal{T}(\mathcal{H})$. Therefore, $\mathcal{T}(\mathcal{H})$ is a two sided *-ideal in $\mathcal{L}(\mathcal{H})$.

To show that $\mathcal{T}(\mathcal{H})$ is minimal, assume that $\mathcal{J}$ is a non zero ideal in $\mathcal{L}(\mathcal{H})$. Thus there exists an operator $T \neq 0$ in $\mathcal{J}$ hence there is a non zero vector f and a unit vector g in $\mathcal{H}$ such that $Tf = g$. Let $T_{h,k}$ be the rank one operator defined by

$$T_{h,k}(\ell) = (\ell, h)k.$$

Note that,

$$T_{g,k}TT_{h,f}(\ell) = (\ell, h)k = T_{h,k}(\ell)$$

and therefore, $T_{h,k}$ is in $\mathcal{T}(\mathcal{H})$ for any pair of vectors h and k in $\mathcal{H}$. However,

$$\{T \in \mathcal{L}(\mathcal{H}) : T \text{ is rank one}\} = \{T_{h,k} : h \text{ and } k \text{ in } \mathcal{H}\}.$$

Thus, $\mathcal{T}(\mathcal{H})$ contains all the rank one operators and hence all finite rank operators. This completes the proof. $\square$

next, we obtain a very useful alternate characterisation of compact operators. The proof is elementary and left out.

Lemma 2.3. *If $\mathcal{H}$ is a Hilbert space and T is in $\mathcal{L}(\mathcal{H})$ then T is compact if and only if for every bounded sequence $\{f_n\}$ which converges to f weakly it is true that $\{Tf_n\}$ converges in norm to Tf.*

Lemma 2.4. *The closed unit ball $(\mathcal{H})_1$ in an infinite dimensional Hilbert space $\mathcal{H}$ is compact if and only if $\mathcal{H}$ is finite dimensional.*

Proof. If $\mathcal{H}$ is finite dimensional then it is isometrically isomorphic to $\mathbb{C}^n$ and hence its unit ball is compact. On the other hand if $\mathcal{H}$ is infinite dimensional there exists an infinite orthonormal sequence $\{e_n\}$ in the closed unit ball $(\mathcal{H})_1$. The fact that

$$\|e_n - e_m\| = \sqrt{2} \text{ for } n \neq m$$

shows that the sequence $\{e_n\}$ has no convergent subsequence. Thus, closed unit ball $(\mathcal{H})_1$ can not be compact. $\square$

Proposition 2.5. *If K is a compact operator on an infinite dimensional Hilbert space $\mathcal{H}$ and $\mathcal{M}$ is a closed subspace contained in the range of K then subspace $\mathcal{M}$ is finite dimensional.*

Proof. If $P_{\mathcal{M}}$ is the projection onto the subspace $\mathcal{M}$ then $P_{\mathcal{M}}T$ is also compact. If $A : \mathcal{H} \to \mathcal{M}$ is the operator defined by $Af = P_{\mathcal{M}}Tf$ then A is bounded and maps $\mathcal{H}$ onto $\mathcal{M}$. By the open mapping theorem A is an open map. Therefore,

$$A(\mathcal{H})_1 \supseteq B_\delta(0) \text{ for some } \delta > O.$$

Since the compact set $P_{\mathcal{M}}T(\mathcal{H})_1$ contains the closed ball $\overline{B_\delta(0)}$, it follows that $\mathcal{M}$ is finite dimensional by the preceeding corollary. $\square$

Theorem 2.6. *If $\mathcal{H}$ is an infinite dimensional Hilbert space then the norm closure of $\mathcal{T}(\mathcal{H})$ is contained in $\mathcal{C}(\mathcal{H})$. If the range of an operator T on the Hilbert space $\mathcal{H}$ does not contain any closed infinite dimensional subspace of $\mathcal{H}$ then T is in the norm closure of $\mathcal{T}(\mathcal{H})$. In particular, the norm closure of $\mathcal{T}(\mathcal{H})$ is $\mathcal{C}(\mathcal{H})$.*

Proof. First it is obvious that $\mathcal{T}(\mathcal{H})$ is contained in $\mathcal{C}(\mathcal{H})$. Secondly, to prove that $\mathcal{C}(\mathcal{H})$ is closed assume that $\{K_n\}$ is a sequence of compact operators which converges in norm to K. If $\{f_n\}$ is bounded sequence converging weakly to f and

$$M = \max\{1, \|f_n\| : n \in \mathbb{N}\}$$

then choose N such that $\|K-K_N\| < \epsilon/3M$. Since K_N is a compact operator, there exists an n_0 such that

$$\|K_N f_n - K_N f\| < \epsilon/3 \text{ for } n > n_0.$$

Thus, we have

$$\begin{aligned} \|Kf_n - Kf\| &\leq \|(K - K_N)f_n\| + \|K_N f_n - K_N f\| + \|(K_N - K)f\| \\ &\leq \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon \text{ for } n > n_0, \end{aligned}$$

and hence K is a compact operator. Therefore, the closure of $\mathcal{T}(\mathcal{H})$ is $\mathcal{C}(\mathcal{H})$.

Let T be any operator on the Hilbert space $\mathcal{H}$ such that the range of T does not contain any closed infinite dimensional subspace of $\mathcal{H}$ and let $T = PV$ be the polar decomposition for T. Consider the extended functional calculus for the operator P, defined for functions in $L^\infty(\nu)$ for some positive regular Borel measure ν. Let $\chi_\epsilon = \chi_{(\epsilon, \|P\|]}$ be the characterstic function of the interval $(\epsilon, \|P\|]$ and note that χ_ϵ is in $L^\infty(\nu)$. Thus,

$$\mathbb{E}_\epsilon = \chi_\epsilon(P) \text{ is a projection on } \mathcal{H}.$$

If we define ψ_ϵ on $(0, \|P\|)$ by

$$\psi_\epsilon = \begin{cases} 1/x & \text{for } \epsilon < x \leq \|P\| \\ 0 & \text{otherwise} \end{cases}$$

then $Q_\epsilon = \psi_\epsilon(P)$ satisfies

$$Q_\epsilon P = PQ_\epsilon = \mathbb{E}_\epsilon$$

Thus, we have

$$\operatorname{ran}(\mathbb{E}_\epsilon) = \operatorname{ran}\, P(Q_\epsilon) \subseteq \operatorname{ran}\, P = \operatorname{ran}\, T$$

and therefore the range of the projection $\mathbb{E}_\epsilon$ is finite dimensional by assumption. Hence $P_\epsilon = P(\mathbb{E}_\epsilon)$ is in $\mathcal{T}(H)$ and $P_\epsilon V$ is also in $\mathcal{T}(\mathcal{H})$. Finally,

$$\begin{aligned} \|K - P_\epsilon V\| &= \|PV - P_\epsilon V\| \leq \|P - P_\epsilon\| - \|P - P\mathbb{E}_\epsilon\| \\ &= \sup_{0 \leq x \leq \|P\|} \|x - x\chi_\varepsilon(x)\| \end{aligned}$$

Therefore, T is in the norm closure of $\mathcal{T}(\mathcal{H})$. The comment about the compact operators follows from the preceeding Proposition. □

For emphasis, we separately record the following corollary which is already contained in Proposition 2.3 and second half of the theorem.

Corollary 2.7. *If $\mathcal{H}$ is an infinite dimensional Hilbert space and T is an operator on $\mathcal{H}$ then T is compact if and only if range T does not contain any closed infinite dimensional subspaces.*

Corollary 2.8. *If $\mathcal{H}$ is an infinite dimensional Hilbert space then $\mathcal{C}(\mathcal{H})$ is the only proper closed two sided *-ideal in $\mathcal{L}(\mathcal{H})$.*

Proof. Since $\mathcal{C}(\mathcal{H})$ is the norm closure of $\mathcal{T}(\mathcal{H})$ and $\mathcal{T}(\mathcal{H})$ is a minimal two sided *-ideal, it follows, that $\mathcal{C}(\mathcal{H})$ is itself is a minimal two sided *-ideal. Next, we prove that it is the only such ideal. If an operator T is not compact then by the previous corollary, the range of T contains a closed infinite dimensional subspace $\mathcal{M}$. The operator $P_{\mathcal{M}}T$ maps $\mathcal{H}$ onto the subspace $\mathcal{M}$ and by the open mapping theorem, we find an operator S such that $TS = P_{\mathcal{M}}$ and hence any two sided ideal containing T must also contain I. This completes the proof. □

2.9 Fredholm Operators

We prove the basic spectral properties of compact operators after obtaining some elementary results for Fredholm operators.

Definition 2.10. If $\mathcal{H}$ is a Hilbert space then the quotient algebra $\mathcal{U}(\mathcal{H}) = \mathcal{L}(\mathcal{H})/\mathcal{C}(\mathcal{H})$ is a Banach algebra called the *Calkin algebra*.

In fact, if $\mathcal{I}$ is a closed ideal in any C^*-algebra $\mathcal{U}$ then the quotient $\mathcal{U}/\mathcal{I}$ is also a C^*-algebra. In particular the Calkin algebra is a C^*-algebra. The natural homomorphism from $\mathcal{L}(\mathcal{H})$ onto $\mathcal{U}(\mathcal{H})$ is denoted by π.

The following definition of Fredholm operators is equivalent to the classical one via Atkinsons Theorem, which will be proved below.

Definition 2.11. If $\mathcal{H}$ is a Hilbert space then T in $\mathcal{L}(\mathcal{H})$ is *Fredholm operator* if $\pi(T)$ is invertible in the Calkin algebra $\mathcal{U}(\mathcal{H})$. The collection of Fredholm operators on $\mathcal{H}$ is denoted by $\mathcal{F}(\mathcal{H})$.

Some elementary properties of Fredholm operators is immediate from the definition, they are collected together in the following.

Proposition 2.12. *If $\mathcal{H}$ is a Hilbert space then $\mathcal{F}(\mathcal{H})$ is an open subset of $\mathcal{L}(\mathcal{H})$, which is self adjoint, closed under multiplication and invariant under compact pertubations.*

Proof. Since $\pi : \mathcal{L}(\mathcal{H}) \to \mathcal{U}(\mathcal{H})$ is continuous and $\mathcal{F}(\mathcal{H})$ is the inverse image of the group of invertible elements in $\mathcal{U}(\mathcal{H})$, it follows that $\mathcal{F}(\mathcal{H})$ is open. Again the fact that π is multiplicative implies $\mathcal{F}(\mathcal{H})$ is closed under multiplication. The fact that $\mathcal{F}(\mathcal{H})$ is invariant under compact pertubations is all but obvious. Lastly, if T is in $\mathcal{F}(\mathcal{H})$ then there exists an operator S and compact operators K_1 and K_2 such that

$$ST = I + K_1 \text{ and } TS = I + K_2.$$

Taking adjoints, we see that $\pi(T^*)$ is invertible in the Calkin algebra $\mathcal{U}(\mathcal{H})$ and $\mathcal{F}(\mathcal{H})$ is self adjoint.

While the vector sum of two closed subspaces $\mathcal{M}$ and $\mathcal{M}_0$ of a Hilbert space $\mathcal{H}$ is not closed in general, it is true true that the sum is closed if one of the subspaces say $\mathcal{M}_0$ is finite dimensional. To prove this, let E be the projection onto the subspace $\mathcal{M}^\perp$ and note that $E(\mathcal{M}_0)$ is finite dimensional, therefore, closed and $\mathcal{M} + \mathcal{M}_0 = E^{-1}(E(\mathcal{M}_0))$. □

Theorem 2.13. *(Atkinson). If $\mathcal{H}$ is a Hilbert space then T in $\mathcal{L}(\mathcal{H})$ is a Fredholm operator if and only if the range of T is closed,* $\dim\ker T$ *is finite and* $\dim\ker T^*$ *is finite.*

Proof. If T is a Fredholm operator, then there exists an operator S in $\mathcal{L}(\mathcal{H})$ and compact operator K such that $ST = I + K$. If f is a vector in the kernel of $I + K$ implies that $Kf = -f$, and hence f is in the range of K. Thus,

$$\ker T \subseteq \ker ST = \ker I + K \subseteq \text{ran } K$$

and therefore, $\dim\ker T$ is finite. Similarly, $\dim\ker T^*$ is finite. Moreover, there exists a finite rank operator F such that $\|K - F\| < \frac{1}{2}$. Hence for f in $\ker F$, we have

$$\begin{aligned} \|S\|\|Tf\| &\geq \|STf\| = \|f + Kf\| = \|f + Ff + Kf - Ff\| \\ &\geq \|f\| - \|Kf - Ff\| \geq \|f\|/2. \end{aligned}$$

Therefore, T is bounded below on $\ker F$, which implies that $T(\ker F)$ is a closed subspace of $\mathcal{H}$. To show range of T is closed, observe that $(\ker F)^\perp$ is finite dimensional and

$$\text{ran } T = T(\ker F) + T[(\ker F)^\perp].$$

Conversely, assume that range of T is closed and both kernel and cokernel of T are finite. The operator

$$T_0 : (\ker T)^\perp \to \text{ran } T \text{ defined by } T_0 f = Tf$$

is one to one and onto and hence invertible by the open mapping theorem. If we define the operator S on $\mathcal{H}$ by

$$Sf = \begin{cases} T_0^{-1}f & f \in \text{ran}\, T \\ 0 & f \in \text{ran}\, T \end{cases}$$

then S is a bounded,

$$ST = I - P_1 \text{ and } TS = I - P_2,$$

where P_1 is the projection onto $\ker T$ and P_2 is the projection onto $(\text{ran}\, T)^\perp = \ker T^*$. Therefore, $\pi(S)$ is the inverse for $\pi(T)$ in the Calkin algebra $\mathcal{U}(\mathcal{H})$ and the proof is complete. □

3 Index and Basic Spectral Properties

If $\mathcal{H}$ is a Hilbert space and $T : \mathcal{H} \to \mathcal{H}$ is a Fredholm operator then the two numbers

$$\alpha_T = \dim \ker T \text{ and } \beta_T = \dim \ker T^*$$

would seem to contain important information concerning the operator T. It turns out, their difference

$$\text{ind}\, T = \alpha_T - \beta_T$$

is of even greater importance.

If $T : V \to W$ is any finite dimensional operator then

$$\dim V - \alpha_T = \dim W - \beta_T = \text{rank}\; T.$$

Thus, for any such operator T maping one finite dimensional space to another,

$$\text{ind}\, T = \dim V - \dim W.$$

Let L and L' be any two invertible operators on the Hilbert space $\mathcal{H}$. The operator $T : \mathcal{H} \to \mathcal{H}$ is Fredholm if any only if LTL' is Fredholm. Moreover,

$$\alpha_{LTL'} = \alpha_T, \beta_{LTL'} = \beta_T \quad \text{and} \quad \text{ind}\, LTL' = \text{ind}\, T.$$

Index of any invertible operator is zero. Finally, if $T = T_1 \oplus T_2$ then $\ker T = \ker T_1 \oplus \ker T_2$, consequently, $\alpha_T = \alpha_{T_1} + \alpha_{T_2}$. Similarly, $\beta_T = \beta_{T_1} + \beta_{T_2}$ and hence

$$\text{ind}\, T_1 \oplus T_2 = \text{ind}\, T_1 + \text{ind}\, T_2.$$

Let $\mathcal{H}_1 \oplus \mathcal{H}_2 = \mathcal{H} = \mathcal{H}'_1 \oplus \mathcal{H}'_2$ be any two direct sum decompositions of the Hilbert space $\mathcal{H}$. Write the operator T as

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} : \mathcal{H}_1 \oplus \mathcal{H}_2 \to \mathcal{H}'_1 \oplus \mathcal{H}'_2$$

with respect to this decomposition.

Lemma 3.1. *If in the decomposition of T as above $T_{22} : \mathcal{H}_2 \to \mathcal{H}'_2$ is invertible and $\tilde{T} = T_{11} - T_{12}T_{22}^{-1}T_{21} : \mathcal{H}_1 \to \mathcal{H}'_1$ then $\alpha_{\tilde{T}} = \alpha_T, \beta_{\tilde{T}} = \beta_T$ and $indT = \alpha_{\tilde{T}} - \beta_{\tilde{T}}$.*

Proof. The proof is a sort of row reduction

$$\begin{aligned}
&\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \to \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & T_{22}^{-1} \end{bmatrix} \\
&= \begin{bmatrix} T_{11} & T_{12}T_{22}^{-1} \\ T_{21} & I \end{bmatrix} \to \begin{bmatrix} 1 & -T_{12}T_{22}^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} T_{11} & T_{12}T_{22}^{-1} \\ T_{21} & 1 \end{bmatrix} \\
&= \begin{bmatrix} T_{11} - T_{12}T_{22}^{-1}T_{21} & 0 \\ T_{21} & 1 \end{bmatrix} \to \begin{bmatrix} T_{11} - T_{12}T_{22}^{-1}T_{21} & 0 \\ T_{21} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -T_{21} & T_{22} \end{bmatrix} \\
&= \begin{bmatrix} \tilde{T} & 0 \\ 0 & T_{22} \end{bmatrix}
\end{aligned}$$

Thus, we obtain Invertible operators L and L' such that

$$LTL' = \tilde{T} \oplus T_{22}$$

Since T_{22} is invertible, $\ker LTL' = \ker \tilde{T} \oplus \{0\}$ and it follows that

$$\alpha_T = \alpha_{LTL'} = \alpha_{\tilde{T}}$$

Similarly,

$$\beta_T = \beta_{LTL'} = \beta_{\tilde{T}}$$

This completes the proof of the Lemma. □

Most of the basic properties of index are contained in the following theorem.

Theorem 3.2. *The index of a Fredholm operator is*

(i) locally constant

(ii) invariant under compact pertabution

(iii) a homomorphism, that is, if S, T are any two Fredholm Operators then ST is Fredholm and

$$ind\, ST = ind\, S + ind T$$

Proof. Let $T : \mathcal{H} \to \mathcal{H}$ be any Fredholm operatoar. Decompose the Hilbert space $\mathcal{H}$ as

$$\ker T \oplus (\ker T)^{\perp} = \mathcal{H} = \ker T^{*} \oplus (\ker T^{*})^{\perp}$$

Write T as a 2×2 block matrix with respect to this decomposition. Since T is Fredholom ran T is closed and it follows that $(\ker T^{*})^{\perp} = \mathrm{cl}(\mathrm{ran}\ T) = \mathrm{ran}\ T$. Hence the operator $T_{22} : (\ker T)^{\perp} \to (\ker T^{*})^{\perp} = \mathrm{ran}\ T$ is invertible. If S is any other Fredholm operator such that $\|S - T\| < \epsilon$ and S is written as a 2×2 block matrix with respect to the same decomposition of $\mathcal{H}$ as above then S_{22} is invertible for sufficiently small ϵ. By the preceeding lemma,

$$\mathrm{ind}\, S = \alpha_{\tilde{S}} - \beta_{\tilde{S}}.$$

But the operator $\tilde{S} : \ker T \to \ker T^{*}$ is finite dimensional. Therefore,

$$\alpha_{\tilde{S}} - \beta_{\tilde{S}} = \alpha_T - \beta_T = \mathrm{ind}\, T$$

Thus $\mathrm{ind}\, T$ is locally constant.

To prove (iii), note that the map,

$$\psi : t \to \mathrm{ind}(T + tK), K \text{ in } \mathcal{C}(\mathcal{H})$$

is locally constant, therefore constant on any connected set. In particular,

$$\mathrm{ind}\, T = \psi(0) = \psi(1) = \mathrm{ind}(T + K).$$

To prove (iii), note that

$$ST \oplus I = LQ_{\epsilon}L',$$

where,

$$Q_{\epsilon} = \begin{bmatrix} S & 0 \\ \epsilon I & T \end{bmatrix}, L = \begin{bmatrix} I & -\epsilon^{-1}S \\ 0 & I \end{bmatrix} \text{ and } L' = \begin{bmatrix} T & \epsilon^{-1}I \\ -\epsilon I & 0 \end{bmatrix}$$

The two operators S and T are Fredholm and hence $S \oplus T$ is Fredholm. Since L and L' are invertible, for sufficiently small ϵ the operator $ST \oplus I$ is Fredholm and hence ST is Fredholm. By part (i) index is locally constant. Therefore, for sufficiently small ϵ,

$$\mathrm{ind} ST = \mathrm{ind}(ST \oplus I) = \mathrm{ind}\, Q_{\epsilon} = \mathrm{ind}(S \oplus T) = \mathrm{ind}\, S + \mathrm{ind}\, T.$$

This completes the proof of the theorem. □

Example 3.3. *Let $U_+ : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$ be the shift operator,*

$$U_+(a_0, \ldots, a_n, \ldots) = (0, a_0, \ldots, a_n, \ldots)$$

Clearly, ran U_+ is closed and $\ker U_+ = \{0\}$. A simple computation shows that $\dim \ker U_+^* = 1$. Thus,

$$\operatorname{ind} U_+ = -1 \text{ and } \operatorname{ind} U_+^* = 1.$$

Since index is a homomorphism, it follows that

$$\operatorname{ind} U_+^n = -n \text{ and ind } U_+^{*n} = n.$$

The basic spectral properties of a compact operator are contained in the following theorem.

Theorem 3.4. *If K is a compact operator on the Hilbert space $\mathcal{H}$ then $\sigma(K)$ is countable with 0 the only possible limit point. If λ is a nonzero point in $\sigma(K)$ then λ is an eigen value of finite multiplicity and $\bar{\lambda}$ is an eigen value of K^* with the same multiplicity.*

Proof. If λ is a nonzero complex number then $-\lambda I$ is invertible and hence $K - \lambda$ is Fredholm and $\operatorname{ind}(K - \lambda) = 0$. Therefore, if λ is in $\sigma(K)$ then $\ker(T - \lambda) \neq \{0\}$ and hence λ is an eigen value of K of finite multiplicity. Moreover, since $\operatorname{ind}(K - \lambda) = 0$, we see that $\bar{\lambda}$ is an eigenvalue of K^* of the same multiplicity.

Let $\{\lambda_n\}$ be a sequence a distinct eigenvalues of K with corresponding eigen vector $\{f_n\}$. If $\mathcal{M}_n = \operatorname{span}\{f_1, \ldots, f_n\}$ then $\mathcal{M}_1 \subsetneqq \mathcal{M}_2 \subsetneqq, \ldots$, since the eigen vectors corresponding to distinct eigenvalues are linearly independent. Let g_n be a unit vector in $\mathcal{M}_n$ orthogonal to $\mathcal{M}_{n-1}$. For any h in $\mathcal{H}$,

$$h = \sum (h, g_n) g_n + g_0, g_0 \perp g_n \quad \forall_n$$

Since $\|h\| = \sum |(h, g_n)|^2 + \|g_0\|^2$, it follows that $(h, g_n) \to 0$. Therefore, the sequence $g_n \to 0$ weakly and hence $K g_n \to 0$ in norm. Since $g_n \in \mathcal{M}_n$, there exists scalars α_k such that $g_n = \sum_{k=1}^n \alpha_k f_k$,

$$\begin{aligned} K g_n &= \sum_{k=1}^n \alpha_k K f_k = \sum_{k=1}^n \alpha_k \lambda_k f_k = \lambda_n \sum_{k=1}^n \alpha_k f_k + \sum_{k=1}^n \alpha_k (\lambda_k - \lambda_n) f_k \\ &= \lambda_n g_n + h_n, h_n \epsilon \mathcal{M}_{n-1}. \end{aligned}$$

Therefore,

$$\lim_{n \to \infty} |\lambda_n|^2 \leq \lim_{n \to \infty} (|\lambda_n|^2 \|g_n\|^2 + \|h_n\|^2) = \lim_{n \to \infty} \|K g_n\|^2 = 0.$$

$\square$

2 Chapter 2: Ext(x) as a Semigroup with Identity

4 Extensions and Essential Unitary Operators

While classifying essentially normal operators is our main goal, it turns out that to solve our specific problem it is useful to consider a related problem of a more general nature. First, observe that if $\mathcal{S}_T$ is the C^*-algebra generated by the essentially normal operator T, the compact operators $\mathcal{C}(\mathcal{H})$ and the identity operator I on the Hilbert space $\mathcal{H}$ then $\mathcal{S}_T/\mathcal{C}(\mathcal{H})$ is isomorphic to the C^*-algebra generated by 1 and $\pi(T)$ in the Calkin algebra $\mathcal{U}(\mathcal{H})$. Since T is essentially normal, it follows that $\mathcal{S}/\mathcal{C}(\mathcal{H})$ is commutative and we have

$$\begin{array}{ccc} \mathcal{S}_T & & C(\sigma_{\mathcal{U}(\mathcal{H})}(\pi(T))) = C(\sigma_{\text{ess}}(T)) \\ \pi\Big\downarrow & & \simeq \Big\uparrow\ \Gamma_{\mathcal{S}_T/\mathcal{C}(\mathcal{H})} \\ \mathcal{S}_T/\mathcal{C}(\mathcal{H}) & = & \mathcal{S}_T/\mathcal{C}(\mathcal{H}) \subseteq \mathcal{U}(\mathcal{H}) \end{array}$$

where, $\Gamma_{\mathcal{S}_T/\mathcal{C}(\mathcal{H})}$ is the Gelfand map and we have an extension, that is

$$0 \to \mathcal{C}(\mathcal{H}) \to \mathcal{S}_T \xrightarrow{\varphi_T} C(\sigma_{\text{ess}}(T)) \to 0$$

is exact. Conversely, if $\mathcal{S}$ is any C^*-algebra of operators on the Hilbert space $\mathcal{H}$ containing compact operators, that is $\mathcal{C}(\mathcal{H}) \subseteq \mathcal{S} \subseteq \mathcal{L}(\mathcal{H})$ and X is any compact subset of the complex plane $\mathbb{C}$ such that

$$0 \to \mathcal{C}(\mathcal{H}) \xrightarrow{i} \mathcal{S} \xrightarrow{\varphi} C(X) \to 0.$$

is exact then for any T in $\mathcal{S}, \varphi(TT^* - T^*T) = 0$ and it follows that T is essentially normal. Fix any T in $\mathcal{S}$ such that $\varphi(T) = \text{id}|_X$. Let $\mathcal{S}_T$ be the C^*-algebra generated by the operator T, the compact operators and the identity on $\mathcal{H}$. Now, $\varphi(\mathcal{S}_T)$ is a C^*-subalgebra of $C(X)$ containing the identity function and therefore must be all of $C(X)$. If S is any operator in $\mathcal{S}$ then there is always an operator S' in $\mathcal{S}_T$ such that $\varphi(S) = \varphi(S')$ so that $\varphi(S - S') = 0, S - S'$ is compact and hence S is in $\mathcal{S}_T$.

We have shown that there is a natural correspondence between essentially normal operators T with essential spectrum, a compact set $X \subseteq \mathbb{C}$ and extensions of $\mathcal{C}(\mathcal{H})$ by $C(X)$. We now relate unitary equivalence modulo the compacts of essentially normal operators to extensions. If $(\mathcal{S}_1, \varphi_1)$ and $(\mathcal{S}_2, \varphi_2)$ are two extensions corresponding to equivalent essentially normal operators T_1 and T_2, that is, $U^*T_2U = T_1 + K$ for some unitary operator U and compact operator K then $U^*\mathcal{S}_2U = \mathcal{S}_1$ by continuity of the map $T \mapsto U^*TU$ and $\varphi_2(T) = \varphi_1(U^*TU)$ for all T in $\mathcal{S}_2$.

Definition 4.1. Two extensions $(\mathcal{S}_1, \varphi_1)$ and $(\mathcal{S}_2, \varphi_2)$ are *equivalent* if there exists a unitary operator U such that $U^*\mathcal{S}_2U = \mathcal{S}_2$ and $\varphi_2(T) = \varphi_1(U^*TU)$.

Thus, if the essentially normal operators T_1 and T_2 are equivalent modulo the compacts then the corresponding extensions are equivalent. Conversely, if the extensions are equivalent then

$$\varphi_1(U^*T_2U) = \varphi_2(T_2) = \mathrm{id}|_X = \varphi_1(T_1)$$

and we see that $U^*T_2U - T_1$ is compact.

The classification problem for essentially normal operators and for extensions of $\mathcal{C}(\mathcal{H})$ by $C(X)$ are identical for any compact subset X of $\mathcal{C}$. The extension point of view of course has many advantages. For any compact metrizable space X, let $\mathrm{Ext}(X)$ denote the *equivalence classes of the extensions* of $\mathcal{C}(\mathcal{H})$ by $C(X)$, if X is a compact subset of the complex plane $\mathcal{C}$ then $\mathrm{Ext}(X)$ is just the *equivalence classes of essentially normal operators N* with $\sigma_{\mathrm{ess}}(N) = X$. Note that if Δ is a subset of the real line and if S is any operator such that $\pi(S)$ is normal with spectrum Δ then $\pi(S)$ is self adjoint,

$$\pi(S - S^*) = 0 \Rightarrow S = \mathrm{Re}\, S + \text{compact}.$$

By the Weyl–von Neumann Theorem any two of these operators are equivalent modulo the compacts or in other words, $\mathrm{Ext}(\Delta) = 0$, for $\Delta \subseteq R$.

Proposition 4.2. *If X and Y are homeomorphic then there is bijection from Ext(X) to Ext(Y).*

Proof. If $p : X \to Y$ is any homeomorphism, then the map $p^* : C(Y) \to C(X)$ defined by $f \mapsto f \circ p$ for f in $C(\mathrm{Y})$ is an isomorphism. If $(\mathcal{S}, \varphi)$ is an extension of $\mathcal{C}(\mathcal{H})$ by $C(Y)$ then $(\mathcal{S}, p^*\varphi)$ is an extension of $\mathcal{C}(\mathcal{H})$ by $C(X)$. If $(\mathcal{S}_1, \varphi_1)$ and $(\mathcal{S}_2, \varphi_2)$ are two equivalent extensions of $\mathcal{C}(\mathcal{H})$ by $C(Y)$ and U is the unitary operator implementing this unitary equivalence then $p^*\varphi_1(U^*TU) = p^*\varphi_2(T)$ and the two extensions $(\mathcal{S}_1, p^*\varphi_1)$ and $(\mathcal{S}_2, p^*\varphi_2)$ are equivalent. This completes the proof. $\square$

In particular, if T is an essentially normal operator with essential spectrum homeomorphic to a subset of the real line then T is in $\mathcal{N} + \mathcal{C}$.

What about essentially normal operators with essential spectrum homeomorphic to the unit circle $\mathbb{T}$. The next theorem shows that $\mathrm{Ext}(\mathbb{T}) = \mathbb{Z}$.

Theorem 4.3. *If $\pi(T)$ is a unitary then T is a compact perturbation of a unitary operator, a shift of multiplicity n or the adjoint of the shift of multiplicity n, according as ind $T = 0$, ind $T = -n < 0$ or ind $T = n > 0$.*

Proof. If $\pi(T)$ is unitary then T^*T-I is compact. Multiplying by $((T^*T)^{1/2}+I)^{-1}$, we see that $(T^*T)^{1/2}-I$ is also compact. If $T=W|T|$ is the polar decomposition for T then $T=W+K$ for some compact operator T. If $\operatorname{ind} T=n\leq 0$ then $\operatorname{ind} W=\operatorname{ind} T=n\leq 0$ and $\dim\ker W\leq \dim[\operatorname{ran} W]^\perp$. Choose a partial isometry L with initial space $\ker W$ and final space contained in $[\operatorname{ran} W]^\perp$. The operator $V=W+L$ is an isometry. We can now apply the Wold–von Neumann decomposition to the isometry V to obtain an unitary operator U and a unilateral shift S of some multiplicity such that $V=U\oplus S$. Since

$$\operatorname{ind} T=\operatorname{ind} V=\operatorname{ind} U+\operatorname{ind} S=\operatorname{ind} S.$$

it follows that ind $S=n$, which in turn implies S is a shift of multiplicity n. If ind $T=0$ then $T=U+K$ for some compact operator K. However, if ind $T<0$, then $T=U\oplus S+$ compact with S a shift of multiplicity n. To obtain the desired result, we would have to show $U\oplus S\sim S+$ compact. This is indeed correct as will be seen in the next lecture. Assuming this result to be called, 'Absorption Lemma', for the moment, the proof of the theorem is complete for operators T with ind $T<0$. For an operator T with ind $T>0$, apply the preceding method to the adjoint operator T^*. This completes the proof. □

We have already seen that the problem of classifying essentially normal operators is equivalent to that of classifying extensions of $\mathcal{C}(\mathcal{H})$ by $C(X)$ for compact sets $X\subseteq\mathbb{C}$. We now introduce yet another way of looking at the same problem. Let $(\mathcal{S},\varphi)$ be the extension

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{C}(\mathcal{H}) & \hookrightarrow & \mathcal{S} & \xrightarrow{\varphi} & C(X) & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow\tau & & \\ 0 & \longrightarrow & \mathcal{C}(\mathcal{H}) & \hookrightarrow & \mathcal{L}(\mathcal{H}) & \xrightarrow{\pi} & \mathcal{U}(\mathcal{H}) & \longrightarrow & 0 \end{array}$$

The map τ is determined by $\tau\varphi(T)=\pi(T)$. It is easily verified that $\tau: C(X)\to\mathcal{U}(\mathcal{H})$ is a unital *-monomorphism. On the other hand, given a unital *-monomorphism $\tau: C(X)\to\mathcal{U}(\mathcal{H})$, define $\mathcal{S}=\pi^{-1}[\operatorname{Im}\tau]$ and $\varphi=\tau^{-1}\circ\pi$. The pair $(\mathcal{S},\varphi)$ obtained in this manner is an extension of $\mathcal{C}(\mathcal{H})$ by $C(X)$. Given an essentially normal operator T, we obtain the associated extension $(\mathcal{S}_T,\varphi_T)$ which in turn gives rise to the unital *-monomorphism $\tau: C(X)\to\mathcal{U}(\mathcal{H})$, what is the relationship of τ to the operator T? Note that $\varphi(p(T))=p$ for a polynomial the relationship of τ to the operator T? Note that $\varphi(p(T))=p$ for a polynomial and therefore, $\tau\varphi(p(T))=\pi(p(T))=p(\pi(T))$. Thus the map τ is just the functional calculus for the operator $\pi(T)$. If we start with a unital *-monomorphism $\tau: C(X)\to\mathcal{U}(\mathcal{H})$ then by taking T to be any operator such that $\pi(T)=\tau(\mathrm{id}|_X)$, we obtain an essentially normal operator with essential spectrum X. How do we define equivalence for the unital *-monomorphisms $\tau : C(X) \to \mathcal{U}(\mathcal{H})$? If we start with two equivalent essentially normal operators T_1, T_2 and obtain the corresponding unital *-monomorphisms τ_1, τ_2 then for f in $C(X)$

$$\begin{aligned}\tau_1(p) = f(\pi(T_1)) &= f(\pi(U^*T_2U + K)) \\ &= f(\pi(U^*)\pi(T_2)\pi(U)) = (\alpha_U)(\tau_2(p)).\end{aligned}$$

Definition 4.4. Any two unital *-monomorphisms $\tau_k : C(X) \to \mathcal{U}(\mathcal{H}), k = 1, 2$ are *equivalent* if $\tau_1 = (\alpha_U)\tau_2$.

In the preceeding paragraph, we have seen that equivalent extensions give rise to equivalent operators. The converse statement for X a compact subset of $\mathbb{C}$ is easily verified. Thus the classification problem for essentially normal operators is again identical to that of classifying unital *-monomorphisms $\tau : C(X) \to \mathcal{U}(\mathcal{H})$. We will for the rest of these notes work only with these objects and occasionally use essentially normal operators for motivating certain definitions. The equivalence class $[\tau]$ of a *-monomorphisim τ will be called an extension, and $\mathrm{Ext}(X)$ will be the set of all such equivalence classes for fixed compact metrizable space X. Some times we will write $[\tau_x]$, to emphasize that $[\tau_x]$ is an element of $\mathrm{Ext}(X)$.

Since our main problem is to study normal elements in the Calkin algebra, it would seem that the correct notion of equivalence is some what weaker. Define two extensions $[\tau_1]$ and $[\tau_1]$ to be *weakly equivalent* if there is an essentially unitary operator T such that $\pi(T)\tau_1(f)\pi(T)^* = \tau_2(f)$ for all f in $C(X)$. Weak equivalence, perhaps is the more natural equivalence in our setting. We will however show that the weaker notion of equivalence is actually equivalent to the equivalence we have defined for unital *-monomorphism $\tau : C(X) \to \mathcal{U}(\mathcal{H})$. This is no longer true if we consider unital *-monomorphisms of non abelian C^*-algebras.

5 Absorption Lemma

In this section, we will prove the absorption lemma, which was used in proving $\mathrm{Ext}(\mathbb{T}) = \mathbb{Z}$. Keeping later developments in mind, we prove a little more than the absorption lemma.

Lemma 5.1. *Let $\underline{T} = (T_1, \ldots, T_n)$ be such that $(\pi(T_1), \ldots, \pi(T_n))$ is a commuting family of normal elements on $\mathcal{U}(\mathcal{H})$ and let $\underline{\lambda}$ in $\mathbb{C}^n$ be in the joint essential spectrum of $\underline{T}$. Given $\epsilon > 0$ and a finite dimensional subspace $\mathcal{M} \subseteq \mathcal{H}$, there exists a nonzero vector φ in $\mathcal{M}^\perp$ such that*

$$\|(T_m - \lambda_m)\varphi\| < \epsilon \text{ for all } m.$$

Proof. Let $S = \sum(T_m - \lambda_m)^*(T_m - \lambda_m)$ and $\mathcal{U}_{\underline{T}} \subseteq \mathcal{U}(\mathcal{H})$ be the C^*-algebra generated by $\pi(T_1), \ldots, \pi(T_n)$. Since there exists a unital *-homomorphism $\rho : \mathcal{U}_{\underline{T}} \to \mathbb{C}$ such that

$$\rho(\pi(T_m)) = \lambda_m \text{ for all } m$$

and $\pi(S)$ is on $\mathcal{U}_{\underline{T}}$, it follows that $\rho(\pi(S)) = 0$. Hence, $0 \in \sigma_{\text{ess}}$ and $\chi_{[0,\epsilon)}(S)$ is a projection of infinite rank. If (ran $\chi_{[0,\epsilon)}) \cap (\mathcal{M}^\perp \backslash \{0\}) = \emptyset$ then the projection $P_{\mathcal{M}}$ would map the infinite dimensional space ran $\chi_{[0,\epsilon)}$ injectively into the finite dimensional space $\mathcal{M}$. This contradiction guarantees the existence of

$$\varphi \in \mathcal{M}^\perp \backslash \{0\} \cap \text{ran } \chi_{[0,\epsilon)}(S), \|\varphi\| = 1$$

To complete the proof, note that

$$\epsilon > \langle S\varphi, \varphi \rangle \geq \sum \|(T_m - \lambda_m)\varphi\|^2.$$

□

Theorem 5.2. *Let $\{T_n\}_{n=1}^\infty$ be a family of operators on $\mathcal{H}$ such that $\{\pi(T_n)\}$ is a commuting family of normal elements in $\mathcal{U}(\mathcal{H})$. If $\underline{\lambda}^{(r)} \in \sigma_{\text{ess}}(\underline{T})$ for $r = 1, 2, \ldots$ then there exists an orthonormal sequence $\{\psi_r\}_{n=1}^\infty$ in $\mathcal{H}$ such that*

$$T_m = \begin{bmatrix} D_m & 0 \\ 0 & R_m \end{bmatrix} + K_m, K_m \in \mathcal{C}(\mathcal{H})$$

The decomposition of T_m is with respect to the subspaces $\mathcal{M} =$ Clos Span $\{\psi_r\}$ and $\mathcal{M}^\perp$. The operator D_m in $\mathcal{L}(\mathcal{M})$ is diagonal and $D_m\psi_r = \lambda_m^{(r)}\psi_r$.

Proof. First consider the case of self adjoint operators $\{T_m\}$. Construct an orthonormal sequence $\{\psi_r\}$ such that

$$\|T_m\psi_r - \lambda_k^{(r)}\psi_r\| < (1/2)^r \quad m \leq r$$

If $\{\psi_1, \ldots, \psi_{r-1}\}$ are pairwise orthogonal and satisfy the inequality above then apply the lemma with $n = r, \underline{\lambda} = \underline{\lambda}^{(n)}, \mathcal{M} = \text{Span}\{\psi_1, \ldots, \psi_{r-1}\}$ and $\epsilon = (1/2)^{r-1}$ to obtain ψ_r as desired. Now, decompose each T_m with respect to $\mathcal{M}$ and $\mathcal{M}^\perp$ as

$$T_m = \begin{bmatrix} X_m & Y_m^* \\ Y_m & R_m \end{bmatrix}$$

and note that

$$T_m - D_m \oplus R_m = \begin{bmatrix} X_m - D_m & Y_m^* \\ Y_m & 0 \end{bmatrix},$$

$$\begin{aligned} \|(X_m - D_m)\psi_r\|^2 + \|Y_m\psi_r\|^2 &= \|(T_m - D_m)\psi_r\|^2 \\ &= \|T_m\psi_r - \lambda_m^{(r)}\psi_r\|^2 < ((1/2)^r)^2. \end{aligned}$$

It follows that, $X_m - D_m$ and Y_m are both Hilbert–Schmidt and hence $T_m - D_m \oplus R_m$ is compact. To complete the proof in the general case, apply this technique to the sequence $\{\mathrm{Re}\, T_m, \mathrm{Im}\, T_m\}$. □

Corollary 5.3. *(Absorption lemma). If T is essentially normal and N is normal with essential spectrum contained in that of T then $T \oplus N \sim T$.*

Proof. Let the sequence $\lambda^{(r)}$ be dense in $\sigma_{\text{ess}}(T)$, isolated points being counted infinitely often. The theorem implies that

$$T = D \oplus R + K$$

for some compact operator K and D is diagonal with eigen values $\lambda^{(r)}$. The operators T and D have the same essential spectrum and hence

$$\sigma_{\text{ess}}(T) = \sigma_{\text{ess}}(D) = \sigma_{\text{ess}}(D \oplus N)$$

By the Wely–von Neumann theorem, $D \oplus N$ is equivalent to D and therefore, $T \oplus N$ is equivalent to T. This completes the proof. □

In a different direction, theorem 5.2 can be used to show that the two notions of equivalence (strong and weak) we have introduced are in fact the same.

Proposition 5.4. *Weakly equivalent extensions are equivalent.*

Proof. Let $\tau_k : C(X) \to \mathcal{U}(\mathcal{H}_k), k = 1, 2$ be weakly equivalent *-monomorphisms. Any unitary map $U : \mathcal{H}_2 \to \mathcal{H}_1$, induces an isomorphism $\alpha_u : \mathcal{U}(\mathcal{H}_2) \to \mathcal{U}(\mathcal{H}_1)$ and $\alpha_u \tau_2$ is strongly equivalent to τ_2. Thus, we need to only show that τ_1 and $\alpha_u \tau_2$ are strongly equivalent. Therefore, we may assume $\mathcal{H}_1 = \mathcal{H} = \mathcal{H}_2$ without loss of generality.

If $\tau, \tau' : C(X) \to \mathcal{U}(\mathcal{H})$ are weakly equivalent *-monomorphisms then there exists an operator S such that $\pi(S)$ is unitary and

$$(\alpha_S \tau)(f) = \pi(S)\tau(f)\pi(S^*) = \tau'(f).$$

If $\pi(S')$ is any other unitary element in $\mathcal{U}(\mathcal{H})$ commuting with $\mathrm{Im}\, \tau$ then

$$\begin{aligned}(\alpha_{ss'}\tau)(f) &= \pi(S)\pi(S')\tau(f)\pi(S'^*)\pi(S^*) \\ &= \pi(S)\tau(f)\pi(S^*) = \tau'(f)\end{aligned}$$

If S' can be chosen such that SS' is a compact perturbation of an unitary then τ would be strongly equivalent to τ'. The fact that $SS' = U + K$ for some unitary U and compact K in turn would follow from showing

$$\mathrm{ind}\, SS' = 0, \text{ that is, } \mathrm{ind}\, S' = -\mathrm{ind}\, S.$$

We now establish the existence of the operator S'. Let $\{f_1, \ldots, f_m, \ldots\}$ be dense in $C(X)$ and $\underline{\lambda} = (\lambda_1, \ldots, \lambda_m, \ldots)$ be in the joint spectrum of $(\tau(f_1), \ldots, \tau(f_m), \ldots)$. Fix operators $T^{f_1}, \ldots, T^{f_m}, \ldots$ such that $\pi(T^{f_m}) = \tau(f_m)$ for $m = 1, 2, \ldots$. Apply theorem 5.2 with $\lambda^{(r)} = \underline{\lambda}$ for all r to obtain an orthonormal sequence ψ_r such that

$$T^{f_m} = \lambda_m I \oplus R_m + K_m$$

where K_m is compact and the decomposition of T^{f_m} is with respect to $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^{\perp}, \mathcal{M} = $ clso span $\{\psi_r\}$. Let U_+ be the shift operator on $\mathcal{M}$ and define

$$U_+^{(n)} = \begin{cases} U_+^n & n \leq 0 \\ U_+^{*n} & n > 0 \end{cases}$$

Define the operator

$$S'_{(n)} = \begin{cases} \mathrm{Id} & \text{on } \mathcal{M}^{\perp} \\ U_+^{(n)} & \text{on } \mathcal{M} \end{cases}$$

and note that $S'_{(n)}$ is essentially unitary, ind $S'_{(n)} = n$. To verify that $\pi(S'_{(n)})$ commutes with $\operatorname{Im} \tau$, observe that

$$\begin{aligned} [S'_{(n)}, T^{f_m}] &= S'_{(n)} T^{f_m} - T^{f_m} S'_{(n)} \\ &= \lambda_m U_+^{(n)} \oplus R_m - \lambda_m U_+^{(n)} \oplus R_m + \text{compact} \\ &= \text{compact} \end{aligned}$$

Thus,

$$\pi(S_{(n)})\tau(f_m) = \pi(S_{(n)} T^{f_m}) = \pi(T^{f_m} S_{(n)}) = \tau(f_m)\pi(S_{(n)})$$

Since $\pi(S_{(n)})$ commutes with dense subset of $\operatorname{Im} \tau$, it follows that $\pi(S_{(n)})$ commutes with all of $\operatorname{Im} \tau$ and the proof is complete. $\square$

6 Splitting

Given a *-monomorphism $\tau : C(X) \to \mathcal{U}(\mathcal{H})$ and $f \in C(X)$, write T' for any operator T in $\mathcal{L}(\mathcal{H})$ such that $\pi(T') = \tau(f)$, it will be always understood that T' is determined only up to simultaneous unitary equivalence modulo the compacts. If T is in $\mathcal{L}(\mathcal{H})$ and E is a projection in $\mathcal{L}(\mathcal{H})$ then write T_E for the operator $ET|_{EH}$ in $\mathcal{L}(E\mathcal{H})$.

Lemma 6.1. *Suppose $\tau_e : C(X) \to \mathcal{U}(\mathcal{H})$ is a *-monomorphism with $\tau_e(1) = e \neq 1$, where e is a projection in the Calkln algebra $\mathcal{U}(\mathcal{H})$.*

(a) There exists a projection E in $\mathcal{L}(\mathcal{H})$ such that $\pi(E) = e$.

*(b) There exists a unital *-monomorphism $\tau_{e,E} : C(X) \to \mathcal{U}(E\mathcal{H})$ such that*

$$\tau_{e,E}(f) = \pi(T^f_E), \textit{where } \pi(T^f) = \tau_e(f).$$

(c) If F is another projection such that $\pi(F) = e$, then $[\tau_{e,E}] = [\tau_{e,F}]$.

Proof. (a) First, if $\pi(T) = e$ then $\pi(T - T^*) = 0$ which implies $T = \operatorname{Re} T+$ compact. Thus $\operatorname{Re} T$ is a self adjoint lifting of e and $\sigma_{\text{ess}}(\operatorname{Re} T) = \sigma(e)$. We can now perturb $\operatorname{Re} T$ by a compact operator so as to obtain a self adjoint operator E such that $\sigma_{\text{ess}}(E) = \sigma(E) = \sigma_{\text{ess}}(\operatorname{Re} T)$. The operator E then would be a projection.

(b) Note, $\tau_e(1) = e$ implies that $\tau_e(f) = \tau_e(1 \cdot f \cdot 1) = e\tau_e(f)e$, that is, $\pi(T^f - ET^fE) = 0$. Thus, the map $\tau_{e,E} : f \mapsto \pi(T^f_E)$ is well defined. Note, $\tau_e(1) = e$ also implies that the projection e commutes with $\operatorname{Im} \tau_e$ and therefore $\pi(T^fE - ET^f) = 0$. If we decompose the operator T^f with respect to E and $I - E$, then the off diagonal entries are compact. Thus, the map $\tau_{e,E} : f \mapsto \pi(T^f_E)$ is *-homomorphism.

(c) Let U and V be isometries on $\mathcal{H}$ such that $UU^* = E\mathcal{H}$ and $VV^* = F\mathcal{H}$. Define $\tilde{\tau}_{e,E} : C(X) \to \mathcal{U}(\mathcal{H})$ and $\tilde{\tau}_{e,F} : C(X) \to \mathcal{U}(\mathcal{H})$ by

$$\tilde{\tau}_{e,E}(f) = \pi(U^*T^f_EU) \quad \text{and} \quad \tilde{\tau}_{e,F}(f) = \pi(V^*T^f_FV).$$

It follows that $[\tilde{\tau}_{e,E}] = [\tau_{e,E}]$ and $[\tilde{\tau}_{e,F}] = [\tau_{e,F}]$. We will show that $\tilde{\tau}_{e,E}$ is weakly equivalent to $\tilde{\tau}_{e,F}$. Observe that

$$\begin{aligned}\pi(V^*U)\tilde{\tau}_{e,\tilde{E}}(U^*V) &= \pi(V^*U)\pi(U^*T^f_EU)\pi(U^*V)\\ &= \pi(V^*UU^*T^f_EUU^*V) = \pi(V^*ET^f_EEV)\\ &= \pi(V^*ET^fEV) = \pi(V^*FT^fFV) = \tilde{\tau}_{e,F}.\end{aligned}$$

In the last but one equality, we have used the fact that E and F differ by a compact operator. Finally note that,

$$U^*VV^*U = U^*FU = U^*(E + \text{compact})U = I + \text{compact}$$

and similarly, $V^*UU^*V = 1+$compact. Thus, the operator V^*U is essentially unitary and the proof of the lemma is complete. □

If $\mathcal{Z}$ is a separable abelian C^*-subalgebra of the Calkin algebra $\mathcal{U}(\mathcal{H})$ and $\Gamma_{\mathcal{Z}} : \mathcal{Z} \to C(\tilde{X})$ is the Gelfand map,

$$\begin{array}{lcl} C(X) & \overset{p^*}{\dashrightarrow} & C(\tilde{X}) \\ \tau \downarrow & & \uparrow \Gamma_{\mathcal{Z}} \\ \operatorname{Im} \tau & \hookrightarrow & \mathcal{Z} \subseteq \mathcal{U}(\mathcal{H}) \end{array}$$

then $\Gamma_{\mathcal{Z}} \circ \tau$ is an injection of $C(X)$ into $C(\tilde{X})$ and is induced by a continuous surjection $p : \tilde{X} \twoheadrightarrow X$, that is,

$$\begin{aligned} \tilde{\Gamma}_{\mathcal{Z}} \circ \tau(f) &= p(f) \\ \tau(f) &= \Gamma_{\mathcal{Z}}^{-1} p(f) \\ \tau &= p \cdot (\Gamma_{\mathcal{Z}}^{-1}) \end{aligned}$$

In particular, if e is an projection in $\mathcal{U}(\mathcal{H})$ commuting with $\operatorname{Im} \tau$ and the algebra $\mathcal{Z} = C^*[\operatorname{Im} \tau, e]$ then it is possible to split the extension $[\tau]$ with respect to certain subsets of X. In the following, we make this precise.

Since e is a projection in $\mathcal{Z}$ and $\Gamma_{\mathcal{Z}}$ is the Gelfand map, it follows that there exists a clopen subset X_1 of $\tilde{X}$ and the characteristic function χ_{x_1} of the set X_1 maps to e under the Gelfand map $\Gamma_{\mathcal{Z}}$. Let $X_2 = \tilde{X} \backslash X_1$. Thus, $\tilde{X} = X_1 \sqcup X_2$ is the disjoint union of the two sets X_1 and X_2. We claim that the map $p : \tilde{X} \to X$ is one to one on X_1 and on X_2, therefore p is a homeomorphism on these sets. In fact, χ_{x_1} together with $\Gamma_{\mathcal{Z}}(\operatorname{Im} \tau)$ must separate points of $\tilde{X}$. However on X_1, the function χ_{x_1} can not distinguish any points, therefore all the points in X_1 must be separated by $\Gamma_{\mathcal{Z}}(\operatorname{Im} \tau)$. But if $p(x) = p(y)$, for any two points in X_1 then they are not separated. The fact that p is one to one on X_2 follows similarly. Note that p^* is an injective map by construction and therefore the map $p : \tilde{X} \to X$ must be surjective. In particular, $p(\tilde{X}_1) \cup p(\tilde{X}_2) = X$.

We identify $\tilde{X}_1$ with the closed subset $X_1 = p(\tilde{X}_1)$ of X and similarly, $\tilde{X}_2$ is identified with the closed subset $X_2 = p(\tilde{X}_2)$ of X such that $X_1 \cup X_2 = X$.

Now, if $\tau_e^d : C(X) \to \mathcal{U}(\mathcal{H})$ is the *-homomorphism defined by, $\tau_e^d(f) = e\tau(f)$ then τ_e^d is not unital and

$$\ker \tau_e^d = \{f \in C(X) : f|_{X_1} = 0\}, \text{ for some closed subset } X_1 \text{ of } X.$$

Define, τ_e to be the *-monomorphism induced by τ_e^d from $C(X_1) \simeq C(X)/(\ker \tau_e^d)$ into the Calkin algebra $\mathcal{U}(\mathcal{H})$. The map τ_e is not unital, in fact, $\tau_e(1) = e$. By applying the preceeding lemma, we obtain the extension $[\tau_{e,E}]$, which depends only on the class e and not on the representative E. We now collect what we have said so far plus a little more in the following lemma.

Lemma 6.2. *If e in $\mathcal{U}(\mathcal{H})$ commutes with $\operatorname{Im} \tau, \mathcal{Z} = C^*[\operatorname{Im} \tau, e]$ and $\Gamma_{\mathcal{Z}} : \mathcal{Z} \to C(\tilde{X})$ be the Gelfand map then*

$$\tilde{X} = \tilde{X}_1 \sqcup \tilde{X}_2, \textit{ there exists a continuous surjection } p : \tilde{X} \twoheadrightarrow X,$$

which is injective on both $\tilde{X}_1$ and $\tilde{X}_2$. If $\tau_e^d = e\tau$, then $C(X)/(\ker \tau_e^d) = C(X_1)$ and if $\tau_{1-e}^d = (1-e)\tau$, then $C(X)/(\ker \tau_{1-e}^d) = C(X_2)$ where $X_1 = p(\tilde{X}_1)$ and $X_2 = p(\tilde{X}_2)$.

Proof. We have already proved the first part in the preceeding discussion. To prove the second half, note that

$$0 = \tau_e^d(f) \Leftrightarrow 0 = e\tau(f) \Leftrightarrow 0 = \Gamma_{\mathcal{Z}}(e\tau(f)) \Leftrightarrow$$
$$0 = \chi_{\tilde{x}_1} p^*(f) \Leftrightarrow 0 = (f \circ p)\chi_{\tilde{x}_1} \Leftrightarrow 0 = f(X_1)$$

This completes the proof of the lemma. □

Hence τ_e^d induces a *-monomorphism $\tau_e : C(X_1) \to \mathcal{U}(\mathcal{H})$, which is not unital, indeed $\tau_e(1) = e$. However, Lemma 6.1 allows us to choose a unital *-monomorphism $\tau_1 = T_{e,E} : C(X_1) \to \mathcal{U}(\mathcal{H})$. Similarly, by considering the *-homomorphism $\tau_{1-e}^d : C(X) \to \mathcal{U}(\mathcal{H})$, we obtain the map $\tau_2 = \tau_{1-e,I-E}$.

Definition 6.3. Given a function f in $C(X_1 \sqcup X_2)$ and extensions $[\tau_{x_1}]$ and $[\tau_{x_2}]$, define the map $\tau_{x_1} \sqcup \tau_{x_2} : C(X_1 \sqcup X_2) \to \mathcal{U}(\mathcal{H}_1) \oplus \mathcal{U}(\mathcal{H}_2) \hookrightarrow \mathcal{U}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ by

$$\tau_{x_1} \sqcup \tau_{x_2}(f) = \tau_{x_1}(f|_{x_1}) \oplus \tau_{x_2}(f|_{x_2}).$$

Now, we claim that

$$p_*([\tau_1 \sqcup \tau_2]) = [\tau] = (i_{x_1,x}) * [\tau_1] + (i_{x_2,x}) * [\tau_2].$$

First for f in $C(X)$,

$$\begin{aligned}(\tau_1 \sqcup \tau_2)(f \circ p) &= \tau_1(f \circ p|_{\tilde{x}_1}) \oplus \tau_2(f \circ p|_{\tilde{x}_2}) \\ &= \tau_1(f \circ i_{x_1,x}) \oplus \tau_2(f \circ i_{x_2,x})\end{aligned}$$

Thus, we get

$$p_*([\tau_1 \sqcup \tau_2]) = (i_{x_1,x}) * [\tau_1] + (i_{x_2,x}) * [\tau_2].$$

Secondly for f in $C(X)$,

$$\begin{aligned}[\tau_1(i_{x_1,x})^* + \tau_2(i_{x_2,x})^*](f) &= \tau_1(f \circ i_{x_1,x}) \oplus \tau_2(i_{x_2,x}) \oplus \Re_X \\ &= \tau_1(f|_{x_1}) \oplus \tau_2(f|_{x_2}) \oplus \Re_X \\ &= e\tau(f) \oplus (1-e)\tau(f) \oplus \Re_X \\ &= \tau(f) \oplus \Re_X\end{aligned}$$

Thus, we get

$$[\tau] = (i_{x_1,x}) * [\tau_1] + (i_{x_2,x}) * [\tau_2].$$

There is a rather pretty way of saying all this in metrical language. Note that $\tau_1(f) = ET^f|_{E\mathcal{H}}$, where $\pi(T^f) = \tau(f)$ and that in the matrix decomposition of T^f with respect to $E\mathcal{H}$ and $(I-E)\mathcal{H}$ the off diagonal entries are compact. For f in $C(X_1 \sqcup X_2)$, we see that

$$
\begin{aligned}
(\tau_1 \sqcup \tau_2)(f) &= \tau_1(f|_{x_1}) \oplus \tau_2(f|_{x_2}) \\
&= \pi[ET^f|_{E\mathcal{H}}] \oplus \pi[(I-E)T^f|_{(I-E)\mathcal{H}}] \\
&= \pi[ET^f|_{E\mathcal{H}} \oplus (I-E)T^f|_{(I-E)\mathcal{H}}] = \pi(T^f) = \tau(f)
\end{aligned}
$$

What we have done is to simultaniously obtain a direct sum decomposition of the operators T^f modulo the compacts.

For any two extensions $[\tau_{x_1}]$ and $[\tau_{x_2}]$ in $\mathrm{Ext}(X_1)$ and $\mathrm{Ext}(X_2)$ respectively, define the two maps $\beta : \mathrm{Ext}(X_1) \oplus \mathrm{Ext}(X_2) \to \mathrm{Ext}(X)$ by

$$\beta[[\tau_{x_1}], [\tau_{x_2}]] = (i_{x_1,x})_*[\tau_{x_1}] + (i_{x_2,x})_*[\tau_{x_2}]$$

and $\lambda : \mathrm{Ext}(X_1) \oplus \mathrm{Ext}(X_2) \to \mathrm{Ext}(X_1 \sqcup X_2)$ by

$$\lambda[[\tau_{x_1}], [\tau_{x_2}]] = [\tau_{x_1} \sqcup \tau_{x_2}].$$

Note that the class of $\tau_{X_1} \sqcup \tau_{X_2}$ depends only on the class of τ_{X_1} and τ_{X_2}. What we have shown above is that $p_*\lambda = \beta$. Now we make the following definition.

Definition 6.4. An extension $[\tau_X]$ is said to split with respect to a closed cover $\{X_1, X_2\}$ of X if it is in the range of β or equivalently in the range of $p_*\lambda$.

Proposition 6.5. *If $\{X_1, X_2\}$ is a closed cover of X then the natural map $p : X_1 \sqcup X_2 \twoheadrightarrow X$ is a continuous surjection and the operation $\sqcup$ induces an isomorphism $\lambda : Ext(X_1) \oplus Ext(X_2) \to Ext(X_1 \sqcup X_2)$.*

Proof. Let χ be the characteristic function of X_1 and $e = p(\chi)$. If $[\tau]$ is an extension in $\mathrm{Ext}(X_1 \sqcup X_2)$ then as in Lemma 6.2, we obtain two extensions $[\tau_1] = [\tau_{e,E}]$ and $[\tau_2] = [\tau_{1-e,I-E}]$ which depend only on the projection e and $[\tau_1 \sqcup \tau_2] = [\tau]$. Define, $\mu : \mathrm{Ext}(X_1 \sqcup X_2) \to \mathrm{Ext}(X_1) \oplus \mathrm{Ext}(X_2)$ by $\mu([\tau]) = ([\tau_1], [\tau_2])$. It is clear that $\lambda \circ \mu = \mathrm{id}$ on $\mathrm{Ext}(X_1 \sqcup X_2)$. To show that $\mu \circ \lambda = \mathrm{id}$ on $\mathrm{Ext}(X_1) \oplus \mathrm{Ext}(X_2)$, let E be the projection onto $\mathcal{H}_1$ in $\mathcal{L}(\mathcal{H}_l \oplus \mathcal{H}_2)$ and in view of Lemma 6.1, we may use this projection to define the map μ, if we do that then $\mu \circ \lambda$ obviously the identity map.

Finally, we remark that if X is the union of two disjoint closed sets X_1 and X_2 then X is actually equal to $X_1 \sqcup X_2$ and the map p_* is just the identity map. Since λ is just seen to be an isomorphism, it follows that $p_* \circ \lambda$ is an isomorphism, in other words, every extension in such a space splits. $\square$

7 Uniqueness of the Trivial Class

The main goal in this section is to show that $\mathrm{Ext}(X)$ is an abelian semigroup with an identity for any compact metric space X, the fact that $\mathrm{Ext}(X)$ is a group will be established much later.

First note that if T_1 and T_2 are two essentially normal operators with fixed essential spectrum X then $T = T_1 \oplus T_2$ is also essentially normal with essential spectrum equal to X and the class of T depends only on those of T_1 and T_2. Thus, for $X \subseteq \mathbb{C}$, we may define addition in $\mathrm{Ext}(X)$ by

$$[T_1] + [T_2] = [T_1 \oplus T_2]$$

If τ_1 and τ_2 are the unital *-monomorphisms corresponding to the operators T_1 and T_2 then we have defined the sum $\tau_1 + \tau_2$ by the functional calculus for $T_1 \oplus T_2$

$$(\tau_1 + \tau_2)(f) = f(\pi(T_1 \oplus T_2)).$$

However, if ρ is the map determined by the diagram

$$\begin{array}{ccc} \mathcal{L}(\mathcal{H}_1) \oplus \mathcal{L}(\mathcal{H}_2) & \hookrightarrow & \mathcal{L}(\mathcal{H}_1 \oplus \mathcal{H}_2) \\ \pi \oplus \pi \downarrow & & \downarrow \pi \\ \mathcal{U}(\mathcal{H}_1) \oplus \mathcal{U}(\mathcal{H}_2) & \dashrightarrow & \mathcal{U}(\mathcal{H}_1 \oplus \mathcal{H}_2) \end{array}$$

then

$$(\tau_1 + \tau_2)(f) = f(\pi(T_1 \oplus T_2)) = \rho(f(\pi(T_1)) \oplus f(\pi(T_2))) = \rho(\tau_1(f) \oplus \tau_2(f)).$$

For any compact metrizable space X, we define the sum $\tau_1 + \tau_2$ by the formula

$$(\tau_1 + \tau_2)(f) = \rho(\tau_1(f) \oplus \tau_2(f)), \text{ for } f \text{ in } C(X).$$

Now $\tau_1 + \tau_2$ is a unital *-monomorphism form $C(X)$ into $\mathcal{U}(\mathcal{H}_1) \oplus \mathcal{U}(\mathcal{H}_2)$, however and we can think of $\mathcal{U}(\mathcal{H}_1) \oplus \mathcal{U}(\mathcal{H}_2)$ as a sub algebra of $\mathcal{U}(\mathcal{H}_1 \oplus \mathcal{H}_2)$. We also note that the class $[\tau_1 + \tau_2]$ depends only on the class of τ_1 and τ_2 and is therefore well defined as an element of $\mathrm{Ext}(X)$. The sum we have defined in $\mathrm{Ext}(X)$, obviously makes it into an abelian semigroup.

What would be the identity element in $\mathrm{Ext}(X)$? Again, we examine an essentially normal operator to answer this question. If an essentially normal operator N with essential spectrum $\sigma_{\mathrm{ess}}(N) = X$ is in the class $\mathcal{N} + \mathcal{C}$ then the absorption lemma implies that for any essentially normal operator T with $\sigma_{\mathrm{ess}}(T) = X$, the operator $T \oplus N$ is equivalent to T. Thus for $X \subseteq \mathbb{C}$ and any operator N in $\mathcal{N} \oplus \mathcal{C}$ we have,

$$[T] = [T \oplus N] = [T] + [N].$$

This amounts to saying that the class $[N], N$ in $\mathcal{N}+\mathcal{C}$ acts as the identity in $\mathrm{Ext}(X)$, for this reason, operators in $\mathcal{N} \oplus \mathcal{C}$ will be called trivial. For $X \subseteq \mathbb{C}$, the Weyl–von Neumann theorem states that the class of any such operator must be uniquely determined. As we have pointed out, we can compactly perturb normal operator N to obtain N' such that $\sigma(N') = X = \sigma_{\text{ess}}(N')$. If $\Re_x$ is the associated unital *-monomorphism, $\Re_x : C(X) \to \mathcal{U}(\mathcal{H})$ then the diagram

$$\begin{array}{ccc} & & \mathcal{L}(\mathcal{H}) \\ & \Re_0 \nearrow & \downarrow \pi \\ \Re_x : & C(X) \to & \mathcal{U}(\mathcal{H}) \end{array}$$

is commutative, where $\Re_0$ is defined by $\Re_0(f) = f(N')$.

Definition 7.1. For any compact metrizable space X, a unital *-monomorphism $\Re : C(X) \to \mathcal{U}(\mathcal{H})$ is *trivial* if we can find a unital *-monomorphism $\Re_0 : C(X) \to \mathcal{L}(\mathcal{H})$ such that $\Re = \pi \circ \Re_0$. We say that $\Re_0$ *trivializes* $\Re$ and that $\Re$ lifts to $\mathcal{L}(\mathcal{H})$.

We next show that the class $[\tau]$ of a trivial element in $\mathrm{Ext}(X)$ is uniquely determined. This is a generalisation of the Weyl–von Neumann Theorem. But, we will first need a lemma on projections.

Lemma 7.2. *If $\mathcal{U}$ is an abelian C^*-algebra generated by countably many projections then the maximal ideal space $\mathcal{M}$ of $\mathcal{U}$ Is totally disconnected, that is $\mathcal{M}$ has a basis of Clopen sets. Moreover, $\mathcal{U}$ has a single self adjoint generator.*

Proof. Let e_n be the projections generating the algebra $\mathcal{U}$. There exists Clopen sets U_n such that $\Gamma_{\mathcal{U}}(e_n) = \chi_{U_n}$, where $\Gamma_{\mathcal{U}}$ is the Gelfand map. The U_n's separate points since the e_n's generate the algebra $\mathcal{U}$. Consider the map

$$\gamma : \mathcal{M} \to \{0,1\}^{\mathbb{N}}, \quad \gamma(x) = (\chi_{U_n}(x)).$$

Then, γ is a homeomorphism onto a compact subset of $\{0,1\}^N$. The map

$$\varphi : \{0,1\}^{\mathbb{N}} \to [0,1], \varphi(a_n) = \sum_0^{\infty} \frac{2}{3^n} a_n$$

is a one to one map of $\{0,1\}^{\mathbb{N}}$ onto the Cantor set. The map $h = \varphi \circ \gamma$ is one to one and by Stone–Weirstrass theorem, the C^*-algebra generated by h is isomorphic to $C(\mathcal{M})$ and the proof is complete. $\square$

The following theorem establishing the unity of the trivial element is a generalisation of the Weyl–von Neumann theorem.

Theorem 7.3. *If X is a compact metric space then there exists a trivial extension $\Re_x$ in Ext(X). Any two trivial extensions are equivalent.*

Proof. Let $\{x_n\}$ be a dense set in X, where each isolated point x_n is counted infinitely often. Take $\mathcal{H} = \ell^2(\mathbb{N})$ and define $\Re_x : C(X) \to \mathcal{U}(\mathcal{H})$ by

$$\Re_x(f) = \pi[\mathrm{diag}(f(x_n))],$$

where, diag$(f(x_n))$ is the diagonal operator with respect to the standard orthonormal basis in $\ell^2(\mathbb{N})$. The map $\Re_x$ is obviously a *-homomorphism that factors through π. If $\Re_x(f) = 0$ then diag$(f(x_n))$ is compact. Therefore, $f(x_n) \to 0$ and it follows that $f \equiv 0$. Thus, $\Re_x$ is a *-monomorphism.

We next show that any two *-monomorphism of this type are equivalent. Let $\Re'_X$ be a *-monomorphism corresponding to another such sequence $\{y_n\}$. It is easy to show that there exists a permutation ν of $\mathbb{N}$ such that $d(x_n, y_{\nu(n)}) \to 0$ where, d is the metric on X. Let U be the unitary operator, which sends e_n to $e_{\nu(n)}$. We have,

$$U(\mathrm{diag}(f(x_n)))e_n = f(x_n)e_{\nu(n)}$$

$$(\mathrm{diag}(f(y_n))Ue_n = f(y_{\nu(n)})e_{\nu(n)},$$

which implies

$$U\mathrm{diag}(f(x_n)) - \mathrm{diag}(f(y_n))U = \mathrm{diag}(f(x_n) - f(y_{\nu(n)})) = \text{compact}.$$

Therefore, $\Re_X$ and $\Re'_X$ are equivalent.

Finally, we show that any trivial map is equivalent to the one we have described. If $\Re : C(X) \to \mathcal{U}(\mathcal{H})$ is trivial then there is a trivializing map $\Re_0 : C(X) \to \mathcal{L}(\mathcal{H})$ such that $\pi \circ \Re_0 = \Re$.

Let $\mathbb{E}$ be the spectral measure on X such that

$$\Re(f) = \int f \, \mathrm{d}\mathbb{E}.$$

If U_n is a basis of open sets in X and $\mathcal{Z}_0$ is the C^*-algebra generated by $\mathbb{E}(U_n)$ then by the preceeding lemma, $\mathcal{Z}_0$ has a single self adjoint generator H. Let $\Gamma_{\mathcal{Z}_0} : \mathcal{Z}_0 \to C(\tilde{X}_0)$ be the Gelfand map, where, $\tilde{X}_0 = \sigma(H)$.

Let $\epsilon > 0$ and f be a continuous function on X. There exists a finite number of U_n covering X such that

$$|f(x) - f(x')| < \epsilon, \text{ if } x, x' \text{ are in } U_n$$

Fix x_n in U_n and note that

$$\|f - \sum f(x_n)\chi_{U_n}\| < \epsilon.$$

It follows that $\operatorname{Im}\Re_0 \subseteq \mathcal{Z}_0$ and that the map $\Gamma_{\mathcal{Z}_0}\Re_0 : C(X) \to C(\tilde{X}_0)$ is injective, therefore, it is induced by a surjection

$$p_0 : \tilde{X}_0 \to X,$$

$$\begin{array}{ccc} p_0^* : C(X) & \dashrightarrow & C(\tilde{X}_0) \\ \Re_0 \downarrow & & \uparrow \Gamma_{\mathcal{Z}_0} \\ \operatorname{Im}\Re_0 & \hookrightarrow & \mathcal{Z}_0 \subseteq \mathcal{L}(\mathcal{H}) \end{array}$$

Thus, $\Gamma_{\mathcal{Z}_0}\Re_0 = p_0^* : C(X) \to C(\tilde{X}_0)$ is given by

$$\Re_0(f) = \Gamma_{\mathcal{Z}_0}^{-1} p_0^*(f) = \Gamma_{\mathcal{Z}_0}^{-1}(f \circ p_0) = f \circ p_0(H).$$

However, by Weyl's theorem, there exists a diagonal operator D with $\sigma_{\text{ess}}(D) = \sigma_{\text{ess}}(H)$ such that $H - D$ is compact. We may further assume that $\sigma(D) = \sigma_{\text{ess}}(D) = \tilde{X}_0$. Since $\pi(f \circ p_0) = (f \circ p_0)(\pi(D))$, it follows that

$$\Re_0(f) = f \circ p_0(H) = f \circ p_0(D + K) = f \circ p_0(D) + K_f.$$

If $D = \operatorname{diag}(\lambda_n)$ then $\{\lambda_n\}$ is dense in $\tilde{X}_0$ and consequently, $\{x_n = p_0(\lambda_n)\}$ is dense in X. Moreover,

$$\Re(f) = \pi\Re_0(f) = \pi(f \circ p_0(D)) = \pi\operatorname{diag}(f(x_n))$$

for all f in $C(X)$. Therefore, $\Re_0$ arises as above from the sequence $\{x_n\}$. This completes the proof. $\square$

The proof of the following corollary is contained in that of the theorem. The converse statement will be proved later.

Corollary 7.4. *The image of a trivial map is contained in a C^*-algebra generated by projections.*

8 Identity for Ext(X)

We have seen that the class of a trivial map

$$\begin{array}{ccc} & & \mathcal{L}(\mathcal{H}) \\ & \overset{\Re_0}{\nearrow} & \downarrow \pi \\ \Re : & C(X) & \to \mathcal{U}(\mathcal{H}) \end{array}$$

is uniquely determined. We now show that the class $[\Re]$ acts as the ideality in the abelian semi group Ext(X).

Theorem 8.1. *If $[\tau]$ is any extension in Ext(X) and $\Re : C(X) \to \mathcal{U}(\mathcal{H})$ is trivial then $[\tau] + [\Re] = [\tau]$.*

Proof. Let $\{X_r\}$ be a countable dense set in X and $\lambda_m^{(r)} = f_m(x_r)$. If $\{f_m\}$ is dense in $C(X)$ and T^{f_m} in $\mathcal{L}(\mathcal{H})$ is chosen such that $\pi(T^{f_m}) = \tau(f_m)$ then T^{f_m} is essentially normal,

$$[T^{f_m}, [T^{f_n}]^*] \in \mathcal{C}(\mathcal{H}) \text{ for all } m, n \text{ and}$$
$$\lambda^{(r)} = \{\lambda_m^{(r)} = f_m(x_r) | m \geq 1\} \in \sigma_{\text{ess}}(T^{f_m})_{m \geq 1}.$$

Let $\mathcal{M} = \text{Clos Span}\{\psi_r\}$ as in Lemma 5.2, $E = P_{\mathcal{M}}$ be the projection onto $\mathcal{M}$ and obtain the decomposition

$$T^{f_m} = \begin{bmatrix} D_m & 0 \\ 0 & R_m \end{bmatrix} + K_m, \quad K_m \in \mathcal{C}(\mathcal{H}) \cdots\cdots\cdots (*)$$

as in that lemma. Since (*) holds for a dense set it follows that for any f in $C(X)$ and T^f in $\mathcal{L}(\mathcal{H})$ satisfying $\pi(T^f) = \tau(f)$ we have

$$T^f = \begin{bmatrix} S_f & 0 \\ 0 & R_f \end{bmatrix} + K_f, \quad K_f \in \mathcal{C}(\mathcal{H})$$

The fact that S_f and R_f are determined upto a compact operator implies the maps

$$\tau_1 : f \to S_f \quad \text{and} \quad \tau_2 : f \to R_f.$$

are well defined. The off diagonal entries in T^f are compact therefore, both τ_1 and τ_2 are homomorphisms. Furthermore, in obtaining the decomposition (*) by using the $\lambda^{(r)}$ twice in succession, the operator R_m itself can be written as $R_m = D_m \oplus R'_m$.

In particular,

$$\sigma_{\text{ess}}(R_m) = \sigma_{\text{ess}}(T^{f_m}) = \sigma_{\text{ess}}(D_m).$$

Therefore,

$$\pi(R_m) = \pi(T^{f_m}) = \|f_m\|_\infty$$

and similarly

$$\|\pi(D_m)\| = \|\pi(T^{f_m})\| = \|f_m\|$$

Both the maps τ_1 and τ_2 are thus *-monomorphisms.

If $\Re_0 : X \to \mathcal{L}(\mathcal{H})$ is the *-monomorphism $f \to \text{diag} f(x_r)$ then τ_1 and $\pi \circ \Re_0$ agree on a dense set therefore, $\tau_1 = \pi \circ \Re_0 = \Re$ is the trivial map. Since $\tau = \tau_1 + \tau_2$ by construction it follows that

$$\tau + \Re = \tau_1 + \tau_2 + \Re = \Re + \tau_2 + \Re = \tau_2 + \Re = \tau_1 + \tau_2 = \tau$$

This completes the proof. $\square$

If $\Re : C(X) \to \mathcal{U}(\mathcal{H})$ is trivial with trivializing map $\Re_0 : C(X) \to \mathcal{L}(\mathcal{H})$ then for any invertible f in $C(X), \tau(f)$ is invertible and $\operatorname{ind}(\tau(f)) = \operatorname{ind} \Re_0(f) = 0$.

Since $\tau_0(f)$ is a normal operator. Let $\pi^1(X)$ be the first cohomotopy group of X. Define the map $\gamma_X : \mathrm{Ext}(X) \to \mathrm{Hom}(\pi^1(X), \mathcal{U})$ by

$$(\gamma_X[\tau])([f]) = \operatorname{ind} \tau(f).$$

The map γ_X is well defined. The relation

$$\operatorname{ind} \tau(fg) = \operatorname{ind} \tau(f)\tau(g) = \operatorname{ind} \tau(f) + \operatorname{ind} \tau(g)$$

shows that $\gamma_X[\tau] : \pi^1(X) \to \mathcal{U}$ is a homomorphism.

Finally,

$$\operatorname{ind}(\tau_1 + \tau_2)(f) = \operatorname{ind}(\tau_1(f) \oplus \tau_2(f)) = \operatorname{ind} \tau_1(f) + \operatorname{ind} \tau_2(f)$$

Shows that γ_X is a homomorphism. We ask, whether γ_X is injective. An affirmative answer will characterize the trivial maps. In 1972, Brown, Douglas and Fillmore showed that for X a compact subset of the complex plane $\mathbb{C}$, the map γ_X is, in fact, injective.

Corollary 8.2. *Ext is a covariant function from compact metrizable spaces to abelian semigroups.*

Proof. Given a continuous function $p : X \to Y$ and an extension $\tau : C(X) \to \mathcal{U}(\mathcal{H})$, we may define $p_*\tau$ in a natural way by

$$(p_*\tau)(f) = \tau(f \circ p), f \in C(Y)$$

The map, $p_*\tau : C(Y) \to \mathcal{U}(\mathcal{H})$ is easily seen to be a *-homomorphism, which is injective if p is surjective. In general the kernel of this map may be non-trivial. To eliminate this kernel and obtain a *-monomorphism, define

$$(p_*\tau)(f) = \tau(p \circ f) + \Re_Y(f),$$

$\Re_Y : C(Y) \to \mathcal{U}(\mathcal{H})$ is any trivial map. However, while $p_*\tau$ is not well defined, it determines a well defined map $p_* : \mathrm{Ext}(X) \to \mathrm{Ext}(Y)$ by

$$p_*([\tau]) = [p_*\tau],$$

where we have used a fixed but arbitrary trivial map in defining $p_*\tau$. Since $p_*\tau$ and $p_*\tau'$ are equivalent if and only if τ and τ' are equivalent, it follows that the map, $p_* : \mathrm{Ext}(X) \to \mathrm{Ext}(Y)$ is well defined. If p is surjective then

$\tau \circ p^*$ is a *-monomorphism, by the preceeding theorem $\tau \circ p^*$ and $\tau \circ p^* + \Re_Y$ determine the same class in $\mathrm{Ext}(Y)$. Clearly, p_* preserves the semigroup structure,

$$(\mathrm{id}|_x) = \mathrm{id}|_{\mathrm{Ext}(X)}$$

and for any $q : Y \to Z$ continuous

$$(qp)_* = q_* p_*$$

Thus, Ext is a covariant function. This completes the proof. □

It was shown in the proof of the Theorem 8.2 that if τ is a trivial map then $\mathrm{Im}\,\tau$ is contained in an abelian C^*-algebra generated by projections. The following theorem establishing the converse leads naturally to the concept of splitting.

Theorem 8.3. *If $\tau : C(X) \to \mathcal{U}(\mathcal{H})$ is a *-monomorphism with $Im\,\tau$ contained in a abelian C^*-algebra $\mathcal{Z}$ generated by countable projections then τ is trivial.*

Proof. The abelian C^*-algebren $\mathcal{Z}$ generated by countable projections is *-isomorphic to $C(\tilde{X})$, for $\tilde{X}$ a subset of $\mathbb{R}$. If $\Gamma_{\mathcal{Z}} : \mathcal{Z} \to C(\tilde{X})$ is the *-isomorphism,

$$\begin{array}{ccc} C(X) & \xrightarrow{p^*} & C(\tilde{X}) \\ \tau \downarrow & & \uparrow \Gamma_{\mathcal{Z}} \\ \mathrm{Im}\,\tau & \hookrightarrow & \mathcal{Z} \subseteq \mathcal{U}(\mathcal{H}) \end{array}$$

then $\Gamma_{\mathcal{Z}} \circ \tau$ is injective and is induced by a subjective continuous map, $p : \tilde{X} \to X$.

Thus,

$$\Gamma_{\mathcal{Z}} \circ \tau = p^* \text{ or } \tau = \Gamma_{\mathcal{Z}}^{-1} \circ p^* = p_*(\Gamma_{\mathcal{Z}}^{-1}).$$

But $\Gamma_{\mathcal{Z}}^{-1}$ is trivial since $\tilde{X}$ is a subset of $\mathbb{R}$ therefore, τ is also trivial. □

Chapter 3: The Mayer–Vietrois Sequence

9 First Splitting Lemma

In this section, we prove the first splitting lemma which is the first step in the iterated splitting argument.

Proposition 9.1. *Let X and Y be compact metrizable spaces and $q : X \to Y$ be a continuous surjection. Let $[\tau]$ in Ext(X) be any extension such that $[\Re] = q_*[\tau]$ is trivial in Ext(Y) and let $\Re_0$ be the trivializing map. The map $\Re_0 : C(Y) \to \mathcal{L}(\mathcal{H})$ is induced by a spectral measure $\mathbb{E}$ on Y, that is,*

$$\Re_0(f) = \int f \, \mathrm{d}\mathbb{E}$$

if C is any closed subset of Y such that $q|_{q-1(\partial C)}$ is one–one then the projection $\pi(\mathbb{E}(C))$ commutes with Im τ.

Proof. If f is any continuous function on X then $f \circ q^{-1}$ is well defined and continuous on ∂C, hence extends to a continuous function f_1 on all of Y. The function $g = f - f_1 \circ q$ is continuous on X and vanishes on $q^{-1}(\partial C)$. There exists a continuous function G on X vanishing in a neighbourhood of $q^{-1}(\partial C)$ such that $\|g - G\| < \epsilon$, where $\epsilon > 0$ is arbitrary. Any such function G is a sum of two functions f_2 and f_3 such that Supp$f_2 \subseteq q^{-1}(\text{int } C)$ and Supp$f_3 \subseteq q^{-1}(Y \backslash C)$. Hence, any function f in $C(X)$ can be approximated by a function of the form $f_1 \circ q + f_2 + f_3$. Therefore, it is enough to check that the projection $e = \pi(\mathbb{E}(C))$ commutes with each of

$$\tau(f_1 \circ q), \tau(f_2) \text{ and } \tau(f_3).$$

Note that,

$$\Re = (\tau q^*)(f_1) = \tau(f_1 \circ q) = \pi \int f_1 \, \mathrm{d}\mathbb{E},$$

and hence

$$\begin{aligned}\pi(\mathbb{E}(C))\tau(f_1 \circ q) &= \pi(\mathbb{E}(C) \int f_1 \, \mathrm{d}\mathbb{E}) \\ &= \pi \int \chi_C f_1 \, \mathrm{d}\mathbb{E} = \tau(f_1 \circ q)\pi(\mathbb{E}(C)).\end{aligned}$$

Since the set $K = \text{supp} f_2$ is a compact subset of X, it follows that we can find a continuous function h on Y, which is one on $q(K)$ and supp h is contained in int C. Further,

$$f_2 = f_2(h \circ q) = (h \circ q) f_2,$$

and hence

$$\begin{aligned}\pi(\mathbb{E}(C))\tau(f_2) &= \pi(\mathbb{E}(C))\tau((h \circ q) f_2) \\ &= \pi(\mathbb{E}(C))\tau((h \circ q)\tau(f_2)) \\ &= \pi(\mathbb{E}(C) \int h \, \mathrm{d}\mathbb{E})\tau(f_2)\end{aligned}$$

$$
\begin{aligned}
&= \pi\Big(\int \chi_C h \; \mathrm{d}\mathbb{E}\Big)\tau(f_2) \\
&= \pi\Big(\int h \; \mathrm{d}\mathbb{E}\Big)\tau(f_2) \\
&= \tau((h \circ q) f_2) = \tau(f_2)
\end{aligned}
$$

Similarly,

$$\tau(f_2)\pi(\mathbb{E}(C)) = \tau(f_2).$$

The proof that $\tau(f_3)$ commutes with $\pi(\mathbb{E}(C))$ is identical and the proof of the theorem is complete. □

Remark 9.2. If f is any continuous function on X such that $f \circ q^{-1}(C) = 0$ then supp$f \subseteq q^{-1}(\text{int } C)$ and as in the proof of the theorem,

$$\tau(f) = \pi(\mathbb{E}(C))\tau(f),$$

that is, the function f is in $\ker \tau^d_{1-e}$, where the projection $e = \pi(\mathbb{E}(C))$. If $\tilde{X} = X_1 \cup X_2$ is the maximal ideal space of the C^*-algebra $\mathcal{Z}$ generated by $\operatorname{Im} \tau$ and e then $X_1 \subseteq q^{-1}(C)$. Similarly, if f is any continuous function vanishing on $q^{-1}(Y \backslash \text{int } C)$ then f is in $\ker \tau^d_e$ and $X_2 \subseteq q^{-1}(Y \backslash \text{int } C)$.

Lemma 9.3. *(First Splitting Lemma). If $X = A \cup B$ with $A \cap B = \{x_0\}$ then $\beta : Ext(A) \oplus Ext(B) \to Ext(X)$ is an isomorphism.*

Proof. Define a continuous function $q : X \to [-1, 1]$ by

$$q(x) = \begin{cases} d(x_0, x)/(d(x_0, x) + d(x, a)) & x \in A \\ -d(x_0, x)/(d(x_0, x) + d(x, b)) & x \in B \end{cases}$$

where a in $A \backslash \{x_0\}$ is arbitrary and b in $B \backslash \{x_0\}$ is arbitrary. Let C be the closed interval $[0, 1]$. If $[\tau]$ is any extension in $\operatorname{Ext}(X)$ then $q_*([\tau])$ is trivial and $\pi(\mathbb{E}(C))$ commutes with $\operatorname{Im} \tau$, thus, $[\tau]$ splits into $[\tau_1]$ and $[\tau_2]$ with respect to some closed cover $\{X_1, X_2\}$ of X, that is,

$$[\tau] = i_{1*}[\tau_1] + i_2[\tau_2],$$

where $i_k : X_k \to X,\ k = 1, 2$ is the inclusion map. However, as pointed out in the preceeding remark

$$X_1 \subseteq q^{-1}(Y \backslash \text{int } C) \subseteq B$$

$$X_2 \subseteq q^{-1}(C) = A.$$

Since $i_1 = i_{B,X} \circ i_{x_1,B}$ and $i_2 = i_{A,X} \circ i_{X_2,A}$, it follows that

$$\begin{aligned}[\tau] &= (i_{B,X})_*((i_{X_1,B})_*[\tau_1]) + (i_{A,X})_*((i_{X_2,A})_*[\tau_2]) \\ &= (i_{B,X})_*[\tau_1'] + (i_{A,X})_*[\tau_2']\end{aligned}$$

Therefore, β is surjective.

The map $r : X \to A$,

$$r(x) = \begin{cases} x & x \in A \\ x_0 & x \in B. \end{cases}$$

is a retraction of X onto A and let $s : X \to B$ be a similar retraction of X onto B. Note that,

$$\begin{aligned}&((r_*, s_*)\beta)([\tau_1], [\tau_2]) \\ &= (r_* i_{1*}[\tau_1] + r_* i_{2*}[\tau_1], \beta_* i_{1*}[\tau_1] + s_* i_{2*}[\tau_2]).\end{aligned}$$

But $r_* i_{1*}[\tau_1] = (r \circ i_1)_*[\tau_1] = [\tau_1]$ and $r_* i_{2*}[\tau_2]$ is trivial.

Similarly, $s_* i_{1*}[\tau_1]$ is trivial and $s_* i_{2*}[\tau_2] = [\tau_2]$. Therefore, $(r_*, i_*)\beta = \mathrm{id}|_{\mathrm{Ext}(A) \oplus \mathrm{Ext}(B)}$ and β is injective. $\square$

10 If X/A is Totally Disconnected then $(i_{A,X})_*$ is Surjective

Lemma 10.1. *If A is a closed subset of a compact metrizable space X such that X/A is totally disconnected then*

(a) $X \backslash A$ can be written as the disjoint union of clopen sets such that diam $X_n \to 0$,

(b) the function $r : X \to A$ defined by $r|_A = \mathrm{id}$ and $r(x) = a_n$ for all x in X_n, where a_n in ∂A is chosen such that dist$(a_n, X_n) =$ dist$(\partial A, X_n)$, is a retraction.

Proof. Recall that a totally disconnected metric space has a basis of clopen sets. Let $q : X \to X/A$ be the quotient map and $\{U_n : n \geq 1\}$ be a decreasing neighbourhood basis of $q(A)$ consisting of clopen sets U_n in X/A. The set $q^{-1}(U_n^c)$ is homeomorphic to U_n^c. Since U_n^c is a clopen set in a totally disconnected space, it follows that, U_n^c is itself is totally disconnected and hence so is $q^{-1}(U_n^c)$. Therefore, there exists a finite cover of $q^{-1}(U_n^c)$ by clopen sets $F_{n,k}, 1 \leq k \leq m_n$, which can be chosen to have the additional properties

(i) diam $F_{n,k} \le 1/n$ for all k

(ii) $F_{n,k} \cap F_{n,k'} = \emptyset$ for $k \ne k'$

Any enumeration $\{X_n\}$ of $\{F_{n,k} : n \ge 1, 1 \le k \le m_n\}$ would satisfy (a). Since $\mathrm{dist}(X_n, A) \to 0$, its follows that the map r in (b) is continuous and the proof of the lemma is complete. $\square$

Lemma 10.2. *Let $T \in \mathcal{L}(\mathcal{H})$ be self adjoint and F_n be orthogonal projections, $F = \sum_{n=1}^{\infty} F_n$. If there exist scalars λ_n such that $TF_n - \lambda_n F_n$ is compact for all n then there exist projections $F'_n \subseteq F_n$ such that $F_n - F'_n$ is of finite rank and*

(i) $T = \oplus T_n + K,\ K$ compact.

(ii) $T_n = \lambda_n I + K_n,\ K_n$ compact for $n \ge 1$ with respect to the decomposition $\mathcal{H} = \oplus_{n \ge 0} F'_n \mathcal{H}$, where $F'_0 = I - \sum_1^{\infty} F'_n$. In fact, if $T(m)$ is any commuting family of self adjoint operators such that $T(m)F_n - \lambda_n(m)F_n$ is compact for all m, n then (a) and (b) hold simultaneously for all m.

Proof. If $TF_n - \lambda_n F_n$ is compact then $TF_n - F_nT$ and $F_nTF_n - \lambda_n F_n$ are also compact. Note that,

$$\begin{aligned}
\|[T, F_n^{(k)}]\| &= \|TF_n^{(k)} - F_n^{(k)}T\| \\
&= \|TF_nF_n^{(k)} - F_n^{(k)}F_nT\| \\
&= \|TF_nF_n^{(k)} + F_nTF_n^{(k)} + \|F_n^{(k)}TF_n - F_nTF_n^{(k)} \\
&\quad -F_n^{(k)}TF_n - F_n^{(k)}F_nT\| \\
&\le \|[T, F_n]F_n^{(k)}\| + \|F_nTF_n^{(k)} - F_n^{(k)}TF_n\| + \|F_n^{(k)}[T, F_n]\|
\end{aligned}$$

and

$$\begin{aligned}
F_nTF_n^{(k)} - F_n^{(k)}TF_n &= F_nTF_nF_n^{(k)} - F_n^{(k)}F_nTF_n \\
&= [F_nTF_n, F_n^{(k)}] \\
&= [\lambda_n F_n + K, F'_n] \\
&= [K, F_n^{(k)}],
\end{aligned}$$

Now, if $F_n^{(k)}$ is any projection such that $F_n^{(k)} \subseteq F_n$ and $F_n^{(k)} \to 0$ strongly then $\|[T, F_n^{(k)}]\| \to 0$ as $k \to \infty$. Therefore, there exists projections $F'_n \subseteq F_n$ of finite codimension such that $\|[T, F'_n]\| \le 1/n^2$. Consider the matrix of the operator T with respect to the decomposition,

$$\mathcal{H} = \oplus_{n \ge 0} F'_n \mathcal{H}, F'_0 = 1 - \sum_{n \ge 1} F'_n.$$

All entries above the diagonal are compact, since

$$F'_k T F'_n = F'_k[T, F'_n]F'_n \text{ for all } n > k \geq 0,$$

and similarly all entries below the diagonal are also compact. We have $\| F'_k T F_n\| < 1/n^2, k \neq n$ and hence the operator formed by these entries is compact. Therefore,

$$T = \oplus_{n \geq 0} T_n + \text{ compact},$$

where $T_n = F'_n T|_{F'_n \mathcal{H}}$. Moreover,

$$T_n = I_{F_n \mathcal{H}} + \text{ compact, for } n \geq 1.$$

If $T(m)$ is commuting family of self adjoint operators satisfying the hypothesis of the lemma then

$$\|[T(m), F_n^{(k)}] \to 0 \text{ as } k \to \infty, \text{ for all } m, n.$$

Hence for each n, there exists a projection F'_n of finite codimension in F_n such that

$$\|[T(m), F'_n]\| \leq 1/n^2 \text{ for all } m \leq n.$$

In the decomposition of $T(m)$ all entries above the diagonal are compact. The difference is that, except for a finite number of entries with $n < m$, we have

$$\|[F'_k T(m) F'_n\| \leq 1/n^2 \text{ for all } m \leq n.$$

However, the operator formed by the entries above the diagonal is still compact. Since $T(m)$ is self adjoint the operator formed by entries below the diagonal is also compact. The proof in this case is completed as before. □

Remark 10.3. If $T = \lambda I +$ compact, is any self adjoint operator with $\sigma_{\text{ess}}(T) = \lambda$ and χ_ϵ is the characteristic function $\chi_{[\lambda - \epsilon, \lambda + \epsilon]}$ then

(a) $\chi_\epsilon(T) = I -$ compact.

(b) $\chi_\epsilon(T)\mathcal{H}$ is a reducing subspace for T.

(c) $T = \lambda \chi_\epsilon(T) + K + T(I - \chi_\epsilon(T))$, where K is compact and $\|K\| < \epsilon$.

Remark 10.4. If $\{e_n\}$ is a family of commuting projections in the Calkin algebra $\mathcal{U}(\mathcal{H})$, then there exist orthogonal projection F_n on $\mathcal{H}$ such that $\pi(E_n) = e_n$. To see this, note that the C^*-algebra generated by the family $\{e_n\}$ is isomorphic to $C(X)$, where X is totally disconnected. The inverse of the Gelfand map Γ^{-1} is a *-monomorphism, which must be trivial since X is totally disconnected. Let Γ_0 be the trivializing map. If $E_n = \Gamma_0 \Gamma(e_n)$ then E_n's are orthogonal projections and $\pi(E_n) = e_n$.

Theorem 10.5. *If A is a closed subset of X such that X/A is totally disconnected then the map $i_* : Ext(A) \to Ext(X)$ induced by the inclusion map $i : A \to X$ is an isomorphism.*

Proof. Let $X_1, \ldots, X_n, \ldots$ be the clopen sets and $r : X \to A$ be the retraction of Lemma 10.1. Since r is a retraction, $r \circ i = \mathrm{id}|_A$, and it follows that $r_* i_* = \mathrm{id}|_{\mathrm{Ext}(A)}$. We will show that $i_* r_* = \mathrm{id}|_{\mathrm{Ext}(X)}$. Fix a *-monomorphism $\tau : C(X) \to \mathcal{U}(\mathcal{H})$. We claim that there exist mutually orthogonal projections E_n such that $\pi(E_n) = \tau(\chi_n)$, where χ_n is the characteristic function of χ_n and

(1) $\tau(g \circ r) = \tau(g \circ r)\pi(E_0) + \pi(\sum_{n=1}^{\infty} g(a_n)E_n)$ for all $g \in C(A)$.

(2) $\tau' : g \to \tau(g \circ r)\pi(E_0)$ is a *-monomorphism, where $E_0 = I - \displaystyle\sum_{n=1}^{\infty} E_n$.

To establish (1), it is enough to find projections E_n such that (1) holds for a sequence $\{g_m\}$ dense in $C_{\mathbb{R}}(A)$. Choose self adjoint operators H^{gm} such that $\pi(H^{g_m}) = \tau(g_m \circ r)$. Since $\tau(\chi_n)$ is a family of commuting projections in $\mathcal{U}(\mathcal{H})$, it follows by Remark 10.4 – that there exist mutually orthogonal projections F_n such that $\pi(F_n) = \tau(\chi_n)$. Note that

$$\pi(H^{g_m}F_n - g_m(a_n)F_n) = \tau((g_m \circ r)\chi_n - g_m(a_n)\chi_n) = 0,$$

so that $H_m^g F_n - g_m(a_n)F_n$ is compact for all m, n and Lemma 10.2 applies. We have,

$$H^{g_m} = \oplus_{n \geq 0} H_n^{g_m} + K_m, K_m \text{ compact},$$

where $H_n^{g_m} = F_n' H^{g_m}|_{F_n'\mathcal{H}}$ and

$$H_n^{g_m} = g_m(a_n)I_n + K_{mn},\ K_{mn} \text{ compact},$$

where I_n is the identity on $F_n'\mathcal{H}$.

Now, apply Remark 10.3 to obtain projection $F_n^{(m)}$ of finite codimension in F_n' such that

$$H_n^{g_m} = g_m(a_n)F_n^{(m)} + K_n^{(m)} + H_n^{g_m}(I - F_n^{(m)})$$

with $K_n^{(m)}$ compact and $\|K_n^{(m)}\| \to 0$ as $n \to \infty$. For each n, let F_n'' be the projection on the intersection of the ranges of $F_n^{(1)}, \ldots, F_n^{(n)}$. Then F_n'' is of finite dimension in F_n and

$$H^{g_m} = H_m' \oplus g_m(a_n)F_n'' + \text{compact},\ n \geq 1$$

with respect to the decomposition $\mathcal{H} = (I - \sum_{n\geq 1} F_n'')\mathcal{H} \oplus \sum_{n\geq 1} F_n''\mathcal{H}$. Finally, let E_n be any projection of codimension 1 in F_n'' and $E_0 = I - \sum_{n\geq 1} E_n$. Then $\pi(E_n) = \tau(X_n)$ for $n \geq 1$ and (1) is satisfied for all g_m in $\mathbb{C}_{\mathbb{R}}(A)$ and hence for all g in $C(A)$.

To see that (2) is satisfied, observe first that $\tau(g \circ r)$ commutes with $\pi(E_0)$ for $g = g_m$ by construction and hence for all g. Therefore τ' is a *-homomorphism. The final dropping down from F_n'' to E_n implies that the spectrum of $\tau(g_m \circ r)\pi(E_0)$ contains the cluster points of $\{g_m(a_n)\}$, and hence that

$$\overline{\lim}_{n\to\infty}|g_m(a_n)| \leq \|\tau(g_m \circ r)\pi(E_0)\|.$$

It follows that this relation holds for all real g. Let $k \in \ker \tau'$ be real valued. Then $k(a_n) \to 0$, so $f = \sum k(a_n)\chi_n$ and $h = k \circ r - f$ are continuous. But $h = 0$ outside A, so $h = 0$ on ∂A. Since $f = 0$ on ∂A it follows that k vanishes there and $k(a_n) = 0$ for all n. Then from (1) it follows that $k = 0$.

To complete the proof, define $\mu : C(X) \to \mathcal{L}(I - E_0)\mathcal{H}$ as follows. The decomposition $f = (f - f \circ r) + f$ shows that $C(X)$ is the linear direct sum of the ideal $\mathcal{S}(A)$ of functions vanishing on A and the subalgebra $r^*C(A)$. Since X_n is totally disconnected, there exist $\mu_n : C(X_n) \to \mathcal{L}(E_n\mathcal{H})$ such that $\tau|_{C(X_n)} = \pi\mu_n$. Let $\mu_1 = \sum \mu_n$. Define the map $\mu_0 : r^*C(A) \to \mathcal{L}(I - E_0)\mathcal{H}$ by

$$\mu_0(g \circ r) = \sum_{n\geq 1} g(a_n)E_n.$$

Let $\mu(f) = \mu_1(f - f \circ r) + \mu_0(f \circ r)$. Note that,

$$\pi(0 + \mu_1) = \tau|_{\mathcal{S}(A)},$$

where 0 is the zero map into $\mathcal{L}(E_0\mathcal{H})$. This map is *-linear; in order that it be a homomorphism, it is necessary and sufficient that

$$\mu_1((g \circ r)f) = \mu_2(g \circ r)\mu_1(f)$$

for all f in $\mathcal{S}(A), g \in C(A)$. For f in $\mathcal{S}(A)$, the expansion $f = \sum f\chi_n$ converges in norm, so by linearity and continuity it is enough to verify this relation for f's satisfying $\sum f\chi_n = f$. Then $(g \circ r)f = g(a_n)f$, So

$$\mu_1((g \circ r)f) = g(a_n)\mu_1(f);$$

on the other hand, $\mu_1(f) = \mu_1(f)\mu_1(\chi_n) = \mu_1(f)E_n$. So

$$\begin{aligned}\mu(g \circ r)\mu_1(f) &= \mu_0(g \circ r)E_n\mu_1(f)\\ &= f(a_n)E_n\mu_1(f) = g(a_n)\mu_1(f).\end{aligned}$$

We now show that τ is equivalent to $\tau \circ r^* \circ i^* + \mu$, that is,

$$\tau(f) \sim \tau(f \circ r) \oplus \pi\mu(f).$$

The claims (1) and (2) imply that $r_*[\tau]$ is equivalent to $[\tau']$. If $U : \mathcal{H} \to E_0\mathcal{H}$ implements this equivalence, then $U \oplus I : \mathcal{H} \oplus (I - E_0)\mathcal{H} \to E_0\mathcal{H} \oplus (I - E_0)\mathcal{H}$ converts the relation $\tau(f) \sim \tau(f \circ r) \oplus \pi\mu(f)$, which is equivalent to

$$\tau(f) \sim 0 \oplus \pi\mu_1(f), f \in \mathcal{S}(A)$$

$$\tau(g \circ r) \sim \tau(g \circ r) \oplus \pi\mu_0(g \circ r), g \in C(A)$$

into

$$\tau(f) \sim 0 \oplus \pi\mu_1(f), f \in \mathcal{S}(A)$$

$$\tau(f) \sim \tau(g \circ r)\pi(E_0) \oplus \pi\mu_0(g \circ r), g \in C(A).$$

The last relations are equalities and the proof is complete. □

11 Ext$(A) \to$ Ext$(X) \to$ Ext(X/A) is Exact

In this section, we would prove the following theorem.

Theorem 11.1. *If A is a closed subset of X, then*

$$Ext(A) \xrightarrow{i_*} Ext(X) \xrightarrow{q_*} Ext(X/A)$$

is exact, where $i : X \to A$ is the inclusion map and $q : X \to X/A$ is the quotient map.

Note that, $q \circ i$ is a constant map and hence $(q \circ f)_*([\tau_A])$ is always trivial. Therefore, im $i_* \subseteq \ker q_*$ and the other inclusion is a consequence of the following Proposition, where we add enough projections to $\operatorname{Im} \tau$ and obtain the C^*-algebra $\mathcal{Z}$ such that part of it's maximal ideal space is totally disconnected. Thus, we are able to apply Theorem 10.5.

Let $q : X \to Y$ be a continuous surjection, B be a closed subset of Y such that $q|_{q^{-1}(Y \setminus B)}$ is injective and let $A = q^{-1}(B) \subseteq X$. We have,

$$\begin{array}{ccc} A & \overset{i}{\hookrightarrow} & X \\ q' \uparrow & & q \ \uparrow \\ B & \overset{j}{\hookrightarrow} & Y \end{array}$$

where $q' = q|_A$ and i, j are the inclusion maps.

Proposition 11.2. $\ker q_* \subseteq i_*(\ker q'_*)$.

Proof. Let $\tau : C(X) \to \mathcal{U}(\mathcal{H})$ be any *-monomorphism such that $[\tau]$ is in $\ker q_*$, that is, $\Re = q_*([\tau])$ is trivial, so that

$$\Re(g) = \tau(g \circ q) = \pi \int_Y g \; \mathrm{d}\mathbb{E}, \quad g \in C(Y)$$

for some projection valued measure $\mathbb{E}$ on Y. Let $\{U_n\}$ be a basis of open sets for $X \backslash A$ such that $\mathrm{cl}\, U_n$ is disjoint from A for all n and let $C_n = q(\mathrm{cl}\; U_n)$. If $\mathcal{Z}$ is the C^*-algebra generated by $\mathrm{Im}\, \tau$ and all projections $e_n = \pi E_n$, where $E_n = \mathbb{E}(C_n)$, then $\mathcal{Z}$ is commutative by Theorem 9.1. Thus, we have

$$\begin{array}{ccccc} C(Y) \overset{q^*}{\to} & C(X) & \overset{p^*}{\to} & C(\tilde{X}) \\ & \tau \downarrow & & \uparrow \Gamma_{\mathcal{Z}} \\ & \mathrm{Im}\, \tau & \hookrightarrow & \mathcal{Z} \subseteq \mathcal{U}(\mathcal{H}) \end{array}$$

where the map p^* is induced by a continuous surjection $p : \bar{X} \to X$ and $\Gamma_{\mathcal{Z}}$ is the Gelfand map such that $p_*([\tilde{\tau}]) = [\tau]$, where $\tilde{\tau} = \Gamma_{\mathcal{Z}}^{-1}$. Let $\tilde{A} = p^{-1}(A) \subseteq \tilde{X}$. We claim that

(1) p is homeomorphism on $\tilde{A}$.

(2) $\tilde{X} \backslash \tilde{A}$ is totally disconnected.

For any arbitrary but fixed $\tilde{x}$ in $\tilde{X}$, if $x = p(\tilde{x}) \notin \mathrm{cl}\; U_n$ then $y = g \circ p(\tilde{x}) \notin C_n$. There exists a function $g \in C(Y)$ with $g(y) = 1$ and $g = 0$ on C_n and therefore,

$$\Re(g) e_n = \pi \int g \chi_{C_n} \; \mathrm{d}\mathbb{E}_n = 0.$$

On the other hand, if $\tilde{\tau}(\chi_n) = e_n$, where χ_n is the characteristic function of $\tilde{C}_n \subseteq \tilde{X}$ then we have

$$0 = (\Gamma_{\mathcal{Z}}(\Re(g)) e_n)(\tilde{x}) = ((g \circ q \circ p)\chi_n)(\tilde{x}) = g(y)\chi_n(\tilde{x}) = \chi_n(\tilde{x}),$$

which implies that $\chi_n(\tilde{x}) = 0$.

In particular, if $\tilde{x} \in \tilde{A}$ then $\chi_n(\tilde{x}) = 0$ for all n. Since $\mathcal{Z}$ is generated by the projections e_n and $\mathrm{Im}\, \tau$, it follows that $\mathrm{Im}\, \tau$ must separate points of $\tilde{X}$, which completes the proof of (1).

Proof of (2) is similar. Let $\tilde{x}$ in $\tilde{X}$ be such that $p(\tilde{x})$ is in U_n, $qp(\tilde{x}) = y \in \mathrm{int}\, C_n \subseteq C_n$. There exists a continuous function g on Y with $g(y) = 1$ and $\mathrm{supp}\; g$ contained in C_n. Since $g = g\chi_{C_n}$, we have

$$\begin{aligned} e_n \Re(g) &= [\pi \mathbb{E}(C_n)][\pi \int g \; \mathrm{d}\mathbb{E}] \\ &= \pi \int g \chi_{C_n} \; \mathrm{d}\mathbb{E} = \pi \int g \; \mathrm{d}\mathbb{E} = \Re(g), \end{aligned}$$

which implies

$$\Gamma_{\mathcal{Z}}((1-e_n)\Re(g))(\tilde{x}) = 0.$$

But, $\Gamma_{\mathcal{Z}}(\Re(g))(\tilde{x}) = g(y)$ and hence $\chi_n(\tilde{x}) = 1$. Thus, $p(\tilde{x}) \in U_n$ implies that $\tilde{x} \in \tilde{C}_n$, that is, $p^{-1}(U_n) \subseteq \tilde{C}_n$. Let $\tilde{x}_1, \tilde{x}_2$ be any two points in $\tilde{X}\backslash\tilde{A}$ such that $p(\tilde{x}_1) \neq p(\tilde{x}_2)$. Since the U_n's form a basis for $X\backslash A$, it follows that $p(\tilde{x}_1) \in U_n$ and $p(\tilde{x}_2) \notin \text{cl}\, U_n$ for some n. So, there exists e_n such that

$$(\Gamma_{\mathcal{Z}}(e_n))(\tilde{x}_1) = 1 \quad \text{and} \quad (\Gamma_{\mathcal{Z}}(e_n))(\tilde{x}_2) = 0,$$

that is, there exists a clopen set $\tilde{C}_n$ such that $\tilde{x}_1 \in \tilde{C}_n$ and $\tilde{x}_2 \notin \tilde{C}_n$ and hence $\tilde{x}_1$ and $\tilde{x}_2$ are distinguished by a clopen set. If $p(\tilde{x}_1) = p(\tilde{x}_2)$ then

$$\Gamma_{\mathcal{Z}}(\tau(f)) = f \circ p$$

and $\operatorname{Im} \tau$ can not distinguish these two points so that either they are separated by a clopen set or they are equal. We have shown that $\tilde{X}\backslash\tilde{A}$ has a basis of consisting of clopen sets and hence it is totally disconnected. The quotient space $\tilde{X}/\tilde{A}$ is obtained by identifying $\tilde{A}$ to a single point is also totally disconnected. This completes the proof of (2).

Recall that if is $\tilde{i} : \tilde{A} \to \tilde{X}$ is the inclusion map then $\tilde{i}_*$ is a surjection by Theorem 10.5. Since the map $p : \tilde{X} \to X$ is a homeomorphism on $\tilde{A}$, it follows that $(\tilde{i} \circ (p|_{\tilde{A}})^{-1})_*$ is also surjective. Therefore, $[\tilde{\tau}] = (\tilde{i} \circ (p|_{\tilde{A}})^{-1})_*[\tau']$, for some $[\tau']$ in $\operatorname{Ext}(A)$. Since $p_*([\tilde{\tau}]) = [\tau]$, it follows that

$$[\tau] = p_*([\tilde{\tau}]) = (p_*\tilde{i}_*(p|_{\tilde{A}})_*^{-1})[\tau'] = (i_{A,x})_*[\tau'],$$

where $i_{A,x} = p \circ \tilde{i} \circ (p|_{\tilde{A}})^{-1} : A \to X$ is the inclusion map. To complete the proof of the Proposition, we have to show that $q'_*([\tau']) = 0$.

If $\mathcal{Z}'$ is the commutative C^*-algebra generated by $\operatorname{Im} q_*[\tau] \subseteq \operatorname{Im} \tau$ and the projections e_n then $\mathcal{Z}'$ is isomorphic to $C(\tilde{Y})$. We obtain the commutative diagram

$$\begin{tikzcd}
& \tilde{X} \arrow[r, "\tilde{q}"] \arrow[d, "p"] & \tilde{Y} \arrow[d, "\tilde{p}"'] & \\
& X \arrow[r, "q"] & Y & \\
A \arrow[ur, "i"] \arrow[uur, "k"] \arrow[rrr, "q'"] & & & B \arrow[ul, "j"'] \arrow[uul, "l"']
\end{tikzcd}$$

where $k = \tilde{i} \circ (p|_{\tilde{A}})^{-1}, \ell = \tilde{j} \circ (p|_{\tilde{B}})^{-1}$ and note that $\tilde{q} \circ k = \ell \circ q'$. However, $\tilde{q}_* k_*([\tau'])$ is trivial, since $\mathcal{Z}'$ is contained in the algebra generated by the projections e_n so $(\ell \circ q')_*[\tau']$ is also trivial. We can again show that the map $\tilde{p}$ is a homeomorphism on $\tilde{p}^{-1}(B)$ and that $\tilde{Y} \backslash \tilde{p}^{-1}(B)$ is totally disconnected. We apply Theorem 10.5 one more time to infer that $q'_*[\tau']$ is trivial. The proof is complete. $\square$

12 Mayer–Vietoris Sequence

Let X_1 and X_2 be closed subsets of X such that $X_1 \cup X_2 = X$ and let $A = X_1 \cap X_2, \mathrm{Ext}_1(A)$ be the group of invertible elements in $\mathrm{Ext}(A)$ and $i_k : A \to X_k,\ j_k : X_k \to X$ for $k = 1, 2$ be inclusion maps. Define the map $\alpha : \mathrm{Ext}_1(A) \to \mathrm{Ext}(X_1) \oplus \mathrm{Ext}(X_2)$ by

$$\alpha([\tau_A]) = [i_{1*}([\tau_A]),\ i_{2*}(-[\tau_A])].$$

Theorem 12.1. *(Mayer–Vietoris). The sequence*

$$Ext_1(A) \xrightarrow{\alpha} Ext(X_1) \oplus Ext(X_2) \xrightarrow{\beta} Ext(X)$$

is exact, that is, $\mathrm{Im}\,\alpha = \ker \beta$.

Proof. We have the commutative diagram

$$\begin{array}{ccc} X_1 \sqcup X_2 & \xrightarrow{q} & X \\ j \uparrow & & \uparrow i \\ A \sqcup A & \xrightarrow{q'} & A \end{array}$$

where i, j are inclusion maps. If $\beta([\tau_1], [\tau_2]) = 0$ then

$$q_*([\tau_1] \sqcup [\tau_2]) = \beta([\tau_1], [\tau_2]) = 0.$$

Proposition 11.2 guarantees the existence of $[\tau_{A \sqcup A}]$ in $\mathrm{Ext}(A \sqcup A)$ such that

$$j_*[\tau_{A \sqcup A}] = [\tau_1] \sqcup [\tau_2] \text{ and } q'_*[\tau_{A \sqcup A}] = 0$$

Since $\tau_{A \sqcup A}$ is in $\mathrm{Ext}(A \sqcup A)$, it follows that

$$\tau_{A \sqcup A} = \tau'_1 \sqcup \tau''_2 \text{ for some } [\tau'_A] \sqcup [\tau''_A] \text{ in } \mathrm{Ext}(A).$$

Note that,

$$0 = q'_*(\tau_{A \sqcup A}) = q'_*([\tau'_A] \cup [\tau''_A]) = [\tau'_A] + [\tau''_A].$$

So, $[\tau'_A]$ and $[\tau''_A]$ are invertible and $-[\tau'_A] = [\tau''_A]$.

For f in $C(X_1 \sqcup X_2)$, we have

$$\begin{aligned}\tau'_A i_1^*(f|_{X_1}) &\oplus \Re_{X_1}(f|_{X_1}) + \tau''_A i_1^*(f|_{X_1}) \oplus \Re_{X_2}(f|_{X_2}) \\ &= \tau'_A(f|_A) + \tau''_A(f|_A) + \Re_{X_1}(f|_{X_1}) + \Re_{X_2}(f|_{X_2}) \\ &= \tau'_A(f|_A) + \tau''_A(f|_A) + \Re_{X_1 \sqcup X_2}(f).\end{aligned}$$

It follows that

$$i_{1*}[\tau'_A] \sqcup i_{2*}[\tau''_A] = j_*([\tau'_A] \sqcup [\tau''_A]).$$

However, if $\mu : \mathrm{Ext}(X_1 \sqcup X_2) \to \mathrm{Ext}(X_1) \oplus \mathrm{Ext}(X_2)$ is the isomorphism discussed in Section 6.

$$\begin{aligned}\alpha[\tau'_A] &= (i_{1*}[\tau'_A], i_{2*}(-[\tau'_A])) = (i_{1*}[\tau'_A], i_{2*}[\tau''_A]) \\ &= \mu(i_{1*}[\tau'_A] \sqcup i_{2*}[\tau''_A]) = \mu i_*([\tau'_A] \sqcup [\tau''_A]) \\ &= \mu i_*[\tau'_{A \sqcup A}] = \mu([\tau_1] \sqcup [\tau_2]) = ([\tau_1], [\tau_2]).\end{aligned}$$

This shows that $\ker \beta \subseteq \operatorname{Im} \alpha$ and completes the proof. □

Corollary 12.2. *Let X be of any compact metric space. If Ext(X) is a group then Ext(B) is also group for any closed subset B of X.*

Proof. Take $X_1 = B$ and $X_2 = X$. Let $[\tau]$ be any extension in $\mathrm{Ext}(B)$. Since $\mathrm{Ext}(X)$ is a group, $j_{1*}[\tau]$ is invertible and

$$\beta([\tau], -j_{i*}[\tau]) = 0.$$

There exists $\tau' \in \mathrm{Ext}(B)$ such that $\alpha([\tau']) = ([\tau'], i_{2*}(-[\tau'])) = ([\tau], -j_{1*}[\tau])$. □

Chapter 4: Determination of Ext(x) as a Group for Planar Sets

13 The Second Splitting Lemma

The first splitting lemma allowed us to split every extension $[\tau]$ in $\mathrm{Ext}(X)$, where $X = A \cup B$ and $A \cap B = \{x_0\}$, with respect to the closed cover $\{A, B\}$ of X. However, we will actually need a stronger form of splitting, one that allow any extension $[\tau]$ in $\mathrm{Ext}(X)$ to split with respect to the closed cover $\{A, B\}$ of X such that $A \cap B$ is homeomorphic to a closed interval rather than a point. The precise statement is given by Corollary at the end of this section.

Lemma 13.1. *If H is a self adjoint operator on $\mathcal{H}$, $\mathcal{M}$ is a finite dimensional subspace and $\epsilon > \varnothing$ then there exist a finite dimensional subspace $\mathcal{M}' \supseteq \mathcal{M}$ and a compact self adjoint operator K such that $H + K$ is reduced by $\mathcal{M}'$ and $\|K\| < \varepsilon$.*

Proof. Let $\{\Delta_i\}$ be a decomposition of $\sigma(H)$ into a finite number of Borel sets of diameter less than ϵ and let

$$\mathcal{M}' = \sum \mathbb{E}(\Delta_i)\mathcal{M}.$$

where $\mathbb{E}$ the spectral resolution of H. If E_i is the projection $\mathbb{E}(\Delta_i)\mathcal{M}, E = \sum E_i$ and

$$K = -(I - E)HE - EH(I - E)$$

then $H + K$ commutes with E and K is compact, since E is finite rank. To complete the proof, we will have to show

$$\|(I - E)HE\| < \epsilon.$$

Fix $\lambda_i \in \Delta_i$; then

$$(I - E)HE = (I - E)\sum HE_i = (I - E)\sum (H - \lambda_i)E_i,$$

$$\begin{aligned}\|(I - E)HE\| &\leq \|\sum (H - \lambda_i)E_i\| \\ &\leq \max \|(H - \lambda_i)E_i\| \\ &\leq \max \|(H - \lambda_i)\mathbb{E}(\Delta_i)\| < \epsilon\end{aligned}$$

Since $\sum (H - \lambda_i)E$ is essentially an orthogonal sum. □

Definition 13.2. An operator matrix $A_{i,j}$ is n-diagonal if $A_{ij} = 0$ for $|i-j| > n$.

Lemma 13.3. *For any compact self adjoint operators $H_0, H_1, \ldots$ on $\mathcal{H}$ there exist compact self adjoint operators $K_0, K_1, \ldots$ on $\mathcal{H}$ and a decomposition $\mathcal{H} = \oplus_{k \geq \emptyset}\mathcal{H}_k$ into finite dimensional subspaces relative to which the operator matrix for $H_0 + K_0$ is diagonal and that for $H_n + K_n$ is $(n + 1)$ diagonal, $n \geq 1$.*

Proof. Fix an orthonormal basis $\{\varphi_{ij} | 0 < j < i < \infty\}$ for $\mathcal{H}$.

Step 1: Choose a finite dimensional subspace $\mathcal{M}_{00}$ containing φ_{00} and a compact self adjoint operator K_{00}, $\|K_{00}\| < 1$ such that $H_0 + K_{00}$ is reduced by $\mathcal{M}_{00}$.

Step 2: Choose a finite dimensional subspace $\mathcal{M}_{10}$ containing $(\mathcal{M}_{00}+\varphi_{10})$ and compact K_{10}, $\|K_{10}\| < 1/4$ such that $(H_0 + K_{00}) + K_{10}$ is reduced by both $\mathcal{M}_{00}$ and $\mathcal{M}_{10}$ (apply Lemma 13.1 to $(H_0 + K_0)|_{\mathcal{M}_{00}^{\perp}}$). Choose a finite dimensional subspace $\mathcal{M}_{11}$ containing $(\mathcal{M}_{10} + \varphi_{11})$ and compact $K_{11}, \| K_{11}\| < 1/4$ such that $H_1 + K_{11}$ is reduced by $\mathcal{M}_{11}$.

Iteration of this procedure making n applications of Lemma 13.1 at the nth step, produces finite dimensional subspaces $\mathcal{M}_{ij}$ and compact operators $K_{ij},\ 0 \leq j \leq i < \infty$ such that

(i) $\varphi_{ij} \in \mathcal{M}_{ij}$.

(ii) $\|K_{ij}\| < 1/(i+1)^2$

(iii) $H_n + \sum_{m=n}^{\infty} K_{mn}$ is reduced by $\mathcal{M}_{mn}, m \geq n$.

(iv) $\mathcal{M}_{ij} \subseteq \mathcal{M}_{i,j+1}$ and $\mathcal{M}_{ij} \subseteq \mathcal{M}_{i+1,0}$

The operator $K_n = \sum_{m=n}^{\infty} K_{mn}$ is compact and put $\mathcal{H}_k = \mathcal{M}_{k,0} \ominus \mathcal{M}_{k-1,0}$. Then $\mathcal{H} = \oplus_{k\geq 0}\mathcal{H}_k$ by (i) and (iv) and this decomposition reduces $H_0 + K_0$ by (iii). Moreover,

$$\mathcal{H}_k \subseteq \mathcal{M}_{k,0} \subseteq \mathcal{M}_{k+n,n} \subseteq \mathcal{M}_{k+n+1,0}$$

imply that

$$\mathcal{H}_k \subseteq \mathcal{M}_{k+n,n} \subseteq \mathcal{M}_0 \oplus \mathcal{M}_1 \oplus, \cdots, \oplus \mathcal{M}_{k+n+1}$$

and hence that

$$(H_n + K_n)\mathcal{H}_k \subseteq \mathcal{H}_0 \oplus \cdots \oplus \mathcal{H}_{k+n+1}$$

for all k and n. Since $H_n + K_n$ is self adjoint it follows that it is $(n+1)$ diagonal. $\square$

Theorem 13.4. *For any self adjoint elements $h_0, h_1, \ldots$ of $\mathcal{U}(\mathcal{H})$ such that h_0 commutes with all h_n, there exist $c \in \mathcal{U}(\mathcal{H}), 0 \leq c \leq 1$ such that, c commutes with all h_n for $n \geq 0$ and*

(a) $cf(h_0) = f(h_0)$ for all continuous f vanishing on $[\frac{1}{2}, \infty)$.

(b) $cf(h_0) = 0$ for all continuous f vanishing on $[-\infty, \frac{1}{2})$.

Proof. By Lemma 13.3, there exist self adjoint operators H_n with $\pi(H_n) = h_n$ and decomposition $\mathcal{H} = \oplus_{k\geq 0}\mathcal{H}_k$ into finite dimensional subspaces relative to which $H_0 = \oplus_{k\geq 0} H_{0k}$ is diagonal and H_n is $(n+1)$-diagonal, $n \geq 1$. Construct a sequence of continuous functions $\varphi_k : \mathbb{R} \to [0,1]$ such that

(i) $\{\varphi_k\}$ decreases to the characteristic function of $(-\infty, \frac{1}{2}]$ and vanishes on $[2, \infty)$.

(ii) $\|(\varphi_k - \varphi_{k+1}\|_\infty \to 0$

(iii) $\lim_{k\to\infty} \|[\varphi_k(H_0), H_n]|_{\mathcal{H}_k}\| = 0, n \geq 1.$

Then with $C = \oplus_{k\geq 0}\varphi_k(H_{0k})$, it follows that $c = \pi(C)$ has the required properties.

Obviously, $0 \leq C \leq I$ and $[C, H_0]$ is compact. To see that $[C, H_n]$ is compact observe; if S is diagonal and T is n-diagonal with respect to the decomposition $\mathcal{H} = \oplus\mathcal{H}_k$ then $[S, T]$ is compact if and only if

$$\|[S, T]|_{\mathcal{H}_k}\| \to 0 \text{ as } k \to \infty.$$

For,

$$[S, T] = \oplus_{k\geq 0}[S, T]_{\mathcal{H}_{(n+1)k+q}}, 0 \leq q \leq n$$

and each of these is essentially an orthogonal sum.

$$[C, H_n]|_{\mathcal{H}_k} = [C - \varphi_k(H_0), H_n]|_{\mathcal{H}_k} + [\varphi_k(H_0), H_n]|_{\mathcal{H}_k},$$

the second term tends to zero by (iii) and the first term is dominated in norm by

$$2\|H_n\|\mathrm{Sup}\{\|\varphi_j - \varphi_k\|_\infty : k - n - 1 \leq j \leq k + n + 1\}$$

(since H_n is $(n + 1)$-diagonal) which tends to zero by (ii).

Moreover, if f vanishes on $[\frac{1}{2}, \infty)$ then $\varphi_k f = f$ for all k, so that $Cf(H_0) = f(H_0)$; if f vanishes on $(-\infty, \frac{1}{2}]$ then $\|\varphi_k f\|_\infty \to 0$ and therefore $Cf(H_0)$ is compact.

To construct the sequence $\{\varphi_k\}$, let f_j be continuous,

$$f_j(x) = \begin{cases} 1 & (-\infty, 1/2] \\ 0 & (1/2 + 1/j, \infty) \\ -jx + \frac{j}{2} + 1 & \text{otherwise} \end{cases}$$

Then f_j decreases to the characteristic function of $(-\infty, 1/2]$ and $\|f_j - f_{j+1}\|_\infty \to 0$. Since $[H_0, H_n]$ is compact, it follows that $[f(H_0), H_n]$ is compact for all continuous f, and hence that $\|[f(H_0), H_n]|_{\mathcal{H}_k}\| \to 0$. Choose $N_1 < N_2 < \ldots$, such that

$$\|[f_j(H_0), H_n]|_{\mathcal{H}_k}\| \leq \frac{1}{j} \quad \text{for } k \geq N_j \text{ and } n \leq j.$$

The sequence $\{\varphi_k\}$ defined by $\varphi_k = f_1$ for $k < N_1$ and $\varphi_k = f_j$ for $N_j \leq k < N_{j+1}$ then has the required properties and the proof is complete. □

We postpone the proof of the following important corollary to section 16.

Lemma 13.5. *(Second Splitting Lemma). Let $X \subseteq [0,1] \times [0,1]$ be a closed set containing $\{(1/2, y) : 0 \leq y \leq 1\}$. Let $A = X \cap [0,1/2] \times [0,1]$ and $B = X \cap [1/2,1] \times [0,1]$. If $[\tau]$ in Ext(X) is any extension then $[\tau]$ splits with respect to A and B.*

14 Projective Limits

Let $p_n : X_{n+1} \to X_n$ be continuous. The projective limit of (X_n, p_n) is a space X together with projection maps $\pi_n : X \to X_n$ such that

$$p_n \circ \pi_{n+1} = \pi_n \text{ for all } n \text{.....................}(*)$$

and if Y is another space together with projection maps q_n satisfying (*) we require that

$$\begin{tikzcd} & X \arrow[d, "\pi_n"] \\ \pi'_n : Y \arrow[ur, "\Phi"] \arrow[r] & X_n \end{tikzcd}$$

is commutative, that is, there exists a unique continuous map φ and $\pi'_n = \pi_n \circ \varphi$. Thus we may take the projective limit to be

$$X = \varprojlim(X_n, p_n) = \{x \in \prod X_n : p_n(x_{n+1}) = x_n\}$$

and π_n is defined by $\pi_n(x) = x_n$. All this we can do for Groups (semi groups) and homomorphisms. In particular, if $p_n : X_{n+1} \to X_n$ is continuous then $p_{n*} : \mathrm{Ext}(X_{n+1}) \to \mathrm{Ext}(X_n)$ and $\varprojlim(\mathrm{Ext}X_n, p_{n*})$ is defined as above. There is always a natural map $P : \mathrm{Ext}(\varprojlim(X_n, p_n)) \to \varprojlim(\mathrm{Ext}(X_n), p_{n*})$ defined by

$$P(\tau_X) = \tau_X \circ \pi_n^*, \pi_n : X \to X_n$$

To see that $[P(\tau_X)]$ is in $\varprojlim(\mathrm{Ext}(X_n), p_{n*})$, note that

$$\tau_X(p_n \circ \pi_{n+1})^* = \tau_X p_n^*$$

and that $p_{n*} : (\mathrm{Ext}(X_n), p_{n*}) \to \mathrm{Ext}(X_n)$ with

$$p_{n*}\pi_{n+1*}([\tau_X]) = \pi_{n*}[\tau_X].$$

The map P may, in general, may have a nn trivial kernel. However, for our purposes, it is important to show that the map P is surjective, while this is true, we prove it under some what restrictive hypothesis.

Theorem 14.1. *If (X_n, p_n) is an inverse system with $p_n : X_{n+1} \to X_n$ surjective then the induced map*

$$P : \mathit{Ext}(\varprojlim X_n) \to \varprojlim \mathit{Ext}(X_n)$$

is also surjective.

Proof. Let $([\tau_n])_{n\geq 0}$ be in $\varprojlim \mathrm{Ext}(X_n)$, that is,

$$p_{n*}[\tau_{n+1}] = [\tau_n],$$

we claim, τ_{n+1} can be chosen in such a way that the following diagram

$$\begin{tikzcd} & C(X_{n+1}) \arrow[d, "\tau_{n+1}"] \\ \tau_n : C(X_n) \arrow[ur, "p_n^*"] \arrow[r] & \mathcal{U}(\mathcal{H}) \end{tikzcd}$$

is commutative, that is,

$$\tau_{n+1} \circ p_n^* = \tau_n \quad \text{for all } n.$$

We proceed inductively. Let τ_1 be arbitrary. Given $p_{i*}[\tau_2] = [\tau_1]$, for any τ_2 in $[\tau_2]$, the map

$$g \longrightarrow \tau_2(g \circ p_1), g \in C(X_1)$$

is equivalent to τ_1. Let α_U be the automorphism inducing this equivalence.

Define,

$$\tau_2' = \alpha_U \tau_2$$

note that

$$(\tau_2' p_1^*)(g) = (\alpha_U \tau_2 p_1^*)(g) = \alpha_U \tau_2(g \circ p_1) = \tau_1(g).$$

Let $\mathcal{P} = \cup_{n\geq 1} \pi_n^* \{C(X_n)\}$, where $\pi_n : \varprojlim X_n \to X_n$ is the projection. It is easy to verify that $\mathcal{P}$ is a dense subalgebra of $C(X)$ via the Stone–Weirstrass Theorem. The extensions $[\tau_n]$ determine a map $\tau : \mathcal{P} \to \mathcal{U}(\mathcal{H})$,

$$\tau(g \circ \pi_n) = \tau_n(g).$$

Since $\tau_{n+1} p_n^* = \tau_n$, it follows that

$$\tau(g \circ p_n \circ \pi_{n+1}) = \tau_{n+1}(g \circ p_n) = \tau_n(g) = \tau(g \circ \pi_n).$$

Thus, the map τ is a well defined *-monomorphism. By construction, $P([\tau]) = (p_{n*}[\tau])_{n\geq 0} = ([\tau_n])_{n\geq 0}$. The proof is complete. □

We end the discussion of inverse limits with a very useful lemma. Let X be any compact metrizable space and $[\tau]$ any extension in $\mathrm{Ext}(X)$. For any multi index $\varepsilon : \{1, \dots, n\} \to \{a, b\}$ of size n, let

$$X_n = \bigsqcup_{k, |\varepsilon_k| = n} X_{\varepsilon_k} \text{ and } \tau_n = \bigsqcup_{k, |\varepsilon_k| = n} \tau_{\varepsilon_k} \text{ and } n \geq 0,$$

where $X_0 = X$ and $\tau_0 = \tau$. If $p_n : X_{n+1} \to X_n$ is the natural map then $p_{n*} : \mathrm{Ext}(X_{n+1}) \to \mathrm{Ext}(X_n)$ is surjective if and only if each $\tau_{\varepsilon_k} |\varepsilon_k| = n$ splits.

Lemma 14.2. *(Iterated Splitting) Let $X_n == \sqcup_{k, |\varepsilon_k| = n} X_{\varepsilon_k}$ be a closed cover X, and $p_n : X_{n+1} \to X_n$ be the natural map. If the diameter of components in X_n goes to zero as $n \to \infty$ and p_{n*} is surjective then Ext(X) is trivial.*

Proof. Note that if the diameter of components in X_n goes to zero then $\lim\limits_{\leftarrow} X_n$ is totally disconnected and $\mathrm{Ext}(\lim\limits_{\leftarrow} X_n) = 0$. By the preceeding theorem,

$$P : \mathrm{Ext}(\lim_{\leftarrow} X_n) \to \lim_{\leftarrow} \mathrm{Ext}(X_n)$$

is surjective. Since each $p_{n*} : \mathrm{Ext}(X_{n+1}) \to \mathrm{Ext}(X_n)$ is surjective, there is a surjection

$$\tilde{P} : \lim_{\leftarrow} \mathrm{Ext}(X_n) \to \mathrm{Ext}(X)$$

and it follows that $\mathrm{Ext}(X) = 0$. □

Remark 14.3. Note that when p_{n*} is not surjective the method of the lemma can be applied to any $[\tau] \in \mathrm{Ext}(X)$ for which $\tau_{\varepsilon_k}, |\varepsilon_k| = n$ splits for all n to infer that $[\tau]$ is trivial.

15 Ext(X) is a Group

In this section, we will show that $\mathrm{Ext}(X)$ is a group for any compact metric space X. First, we will establish that $\mathrm{Ext}([0, 1]^{\mathbb{N}}) = \{0\}$. In particular, it would follow that $\mathrm{Ext}([0, 1]^{\mathbb{N}})$ is a group and hence we would have shown that $\mathrm{Ext}(\Lambda)$ is a group for any closed subset Λ of $[0, 1]^{\mathbb{N}}$ by Corollary 12.2. Any compact metric space X is homomorphic to a closed subset of $[0, 1]^{\mathbb{N}}$, therefore, $\mathrm{Ext}(X)$ is seen to be a group for any such X. Before showing that $\mathrm{Ext}([0, 1]^{\mathbb{N}})$ is a group, we prove the second splitting lemma.

Lemma 15.1 (Second Splitting Lemma)**.** *Let $X \subseteq [0, 1] \times [0, 1]$ be a closed set containing $\{(1/2, y) : 0 \leq y \leq 1\}$. Let $A = X \cap [0, 1/2] \times [0, 1]$ and $B = X \cap [1/2, 1] \times [0, 1]$. If $[\tau]$ in Ext(X) is any extension then $[\tau]$ splits with respect to A and B.*

Proof. Since $X \subset \mathbb{C}$, there exists an essentially normal operator N such that $\sigma_{\text{ess}}(N) = X$ and $\tau(f) = f(\pi(N))$. Let $\pi(N) = n = h_0 + ih,$ n is normal and h_0, h_1 are self adjoint in $\mathcal{U}(\mathcal{H})$. Since,

$$\sigma(1/2 + ih) \subseteq \{1/2\} \times I \subseteq X,$$

it follows that

$$\eta(f) = f(1/2 + ih)$$

is a well defined *-homomorphism of $C(X)$. If $\tau' = \tau + \eta$, then τ' is equivalent to τ. There exists c in $\mathcal{U}(\mathcal{H}),\ 0 \leq c \leq 1$, commuting with h_0 and h such that

$$cf(h_0) = 0,\ \text{for all continous functions vanishing on } [-\infty, 1/2]$$

$$cf(h_0) = f(h_0),\ \text{for all continuous functions vanishing on } [1/2, \infty].$$

Let $e = \begin{bmatrix} c & (c(1-c))^{1/2} \\ (c(1-c))^{1/2} & 1-c \end{bmatrix}$ be in $\mathcal{M}_2(\mathcal{U}(\mathcal{H}))$. Identifying $\operatorname{Im}\tau'$ as $\operatorname{diag}(\tau(f), \eta(f))$ in $\mathcal{M}_2(\mathcal{U}(\mathcal{H}))$, we claim that e commutes with $\operatorname{Im}\tau'$. This amounts to verifying

$$(c(1-c))^{1/2}\eta(f) = \tau(f)(c(1-c))^{1/2}.$$

Let

$$f_1(x) = \begin{cases} 0 & x \in [1/2, \infty] \\ x - 1/2 & x \in [0, 1/2] \\ -1/2 & x \in [-\infty, 0] \end{cases}$$

and let

$$f_2(x) = \begin{cases} 1/2 & x \in [1, \infty] \\ x - 1/2 & x \in [1/2, 1] \\ 0 & x \in [-\infty, 1/2] \end{cases}$$

Note that, $(f_1 + f_2)(x) = x - 1/2$ for x in $[0, 1]$, and that

$$\begin{aligned}(h_0 + ih)c^n &= c^n(h_0 - 1/2 + ih + 1/2) \\ &= c^n(h_0 - 1/2) + c^n(ih + 1/2) \\ &= c^n(f_1(h_0)) + f_2(h_0) + c^n(ih + 1/2) \\ &= f_1(h_0) + c^n(ih + 1/2).\end{aligned}$$

Thus, for any polynomial g, we have

$$\tau(f)g(c) = g(1)f_1(h_0) + g(c)\eta(f).$$

Since the function $(x(x-1))^{1/2}$ can be approximated by polynomials vanishing at 1, it follows that

$$\tau(f)(c(1-c))^{1/2} = (c(1-c))^{1/2}\eta(f).$$

Therefore, the projection e commutes with $\operatorname{Im}\tau'$. Define,

$$\tau_e^d(f) = e\tau'(f).$$

By Lemma 6.2, if $\mathcal{I}_{X_1}$ and $\mathcal{I}_{X_1}$ are the ideals corresponding to $\ker \tau_e^d$ and $\ker \tau_{1-e}^d$ then there are unital *-monomorphism τ_1 and τ_2 such that

$$[\tau_k] \in \operatorname{Ext}(X_k), k = 1, 2$$

and

$$i_{1*}[\tau_1] + i_{2*}[\tau_2] = [\tau'] = [\tau].$$

Let f in $C[0,1]$ be such that f vanishes precisely on $[0, 1/2]$. Define a continuous function g on X by setting $g(z) = f(x)$. We have

$$\tau_e^d(g) = e\tau'(g) = eg(n \oplus (1/2 + ih)) = e(g(n) \oplus f(1/2)) = e(f(h_0) \oplus 0) = 0.$$

So, g is in $\ker\ \tau_e^d$. Since the zero set of $g = B$, it follows that $X_1 \subseteq B$. Similarly, it can be shown that $X_2 \subseteq A$. We have shown that τ splits with respect to A and B and the proof is complete. $\square$

To use the technique of the proof of the second splitting lemma for the Hilbert cube $[0,1]^{\mathbb{N}}$ and show that $\operatorname{Ext}([0,1]^{\mathbb{N}})$ is a group, we have to discuss the infinite sum of extensions. While, the sum of two extensions was defined as

$$(\tau_1 + \tau_2)(f) = \pi(T_1^f \oplus T_2^f),$$

there is no obvious way to define the sum of infinitely many extensions, for the simple reason that an infinite sum of compact operators need not be compact. However, some times, it is possible to define such a sum.

Let $X_1, \ldots, X_m, \ldots$ be closed subsets of X such that diameter of X_m converges to zero and $\cup X_m = X$. If there exists operators T_m^f and $x_m \in X_m$ such that $\pi(T_m^f) = \tau_m(f|_{X_m})$ and $\|T_m^f - f(x_m)\| \to 0$ then define the extension $\tau = \sum_{m\geq 0}(i_{x_m,x})_*(\tau_m) : C(X) \to \mathcal{U}(\oplus_{m\geq 0}\mathcal{H}_m)$ by

$$\tau(f) = \pi(\oplus_{m\geq 0}T_m^f) \text{ for } f \text{ in } C(\overline{\cup X_m}).$$

For any other choice of points y_m in X_m and operators S_m^f, we have

$$\|T_m^f - S_m^f\| \leq [\|T_m^f - f(x_m)\| + \| - S_m^f + f(y_m)\| + \|f(x_m) - f(y_m)\|] \to 0.$$

Therefore, $\oplus_{m\geq 0}T_m^f - \oplus_{m\geq 0}S_m^f$ is compact and τ is well defined. Since,

$$\|\tau_m(f|_{X_m}) - f(x_m)\| = \|f|_{X_m} - f(x_m)\|_\infty \to 0,$$

it follows that the required T_m^f exists. The following observation will be necessary for us.

Remark 15.2. If $q : X \to Y$ is continuous, $Y_n = q(X_n)$, and $p_n : X \to Y_n$ is the restriction of q and $q' = q|_{\overline{\cup X_n}}$ then $q'_*(\sum i_{n*}\tau_n) = \sum q_{n*}(\tau_n)$.

Theorem 15.3. *Ext*$([0,1)^{\mathbb{N}}) = 0$.

Proof. Let $X_a = [0,\frac{1}{2}] \times [0,1]^{\mathbb{N}}, X_b = [\frac{1}{2},1] \times [0,1]^{\mathbb{N}}, X_1 = X_a \sqcup X_b$ and $p_0 : X_a \sqcup X_b \to [0,1]^{\mathbb{N}} = X_0$. If $[\tau]$ in Ext(X) is any extension and $\tau(x_i) = h_i$ then the map $\eta_0 : C(X_a) \to \mathcal{U}(\mathcal{H})$ defined by

$$\eta_0(f) = f(\frac{1}{2}, h_1, \ldots)$$

is a *-homomorphism. As in the proof of the second splitting lemma $\tau_0 + \eta_0$ splits that is, there exists $[\tau_a]$ in Ext(X_a) and $[\tau_b]$ in Ext(X_b) such that

$$p_{0*}[\tau_a \sqcup \tau_b] = [\tau_0 + \eta_0], p_0 : X_a \sqcup X_b \to X_0.$$

Since both X_a and X_b are homeomorphic to X_0, we may iterate this procedure to obtain

(1) $X_1, \ldots, X_n, \ldots$ such that the maximum diameter of the 2^n components in X_n goes to zero as $n \to \infty$.

(2) if $\tau_n = \bigsqcup_{|\varepsilon_k|=n} \tau_{\varepsilon_k}$ and $\eta_n = \bigsqcup_{|\varepsilon_k|=n} \eta_{\varepsilon_k}$ then $p_{n*}[\tau_{n+1}] = [\tau_n] + [\eta_n]$, where $p_n : X_{n+1} \to X_n$.

Since diameter of components in X_n goes to zero, we may define the infinite sum

$$\tau'_n = \tau_n + \eta_n + p_{n*}(\eta_{n+1}) + p_{n*} \circ p_{n+1*}(\eta_{n+1}) + \cdots.$$

We claim that $(\tau'_n)_{n\geq 0}$ is in $\varprojlim(\text{Ext}(X_n))$. Note,

$$\begin{aligned} p_{n*}(\tau'_{n+1}) &= p_{n*}(\tau_{n+1}) + p_{n*}(\eta_{n+1}) + p_{n*}p_{n+1*}(\eta_{n+1}) + \cdots \\ &= \tau_n + \eta_n + p_{n*}(\eta_{n+1}) + p_{n*}p_{n+1*}(\eta_{n+1}) + \cdots = \tau_n, \end{aligned}$$

by Remark 15.2. Therefore, $([\tau'_n])_{n\geq 0}$ is an element of *displaystylelim*$_{\leftarrow}$Ext(X_n). The map $p_n : X_{n+1} \to X_n$ is surjective. An application of Lemma 14.2 shows that $[\tau'_0]$ is trivial, or in other words, $[\tau]$ is invertible. ∎

Corollary 15.4. *Ext*$([0,1]^{\mathbb{N}})$ *is a group.*

Corollary 15.5. *Ext*(X) *is a group for any compact metric space* X.

16 γ_X is Injective

We have now all the ingredients to prove that the map γ_X is injective for $X \subseteq \mathbb{C}$. Injectivity of γ_X is established by showing that any extension $[\tau]$ in $\ker \gamma_X$ splits into $[\tau_1]$ and $[\tau_2]$ with respect to some closed cover $\{X_1, X_2\}$ of X such that $[\tau_k]$ is in $\ker \gamma_{X_k}$ for $k = 1, 2$. We iterates this procedure and apply Remark 14.3, to see that $[\tau] = 0$. The inductive step in this argument depends on injectivity of $\gamma_{[0,1]/A}$, where $A \subseteq [0, 1]$ is an arbitrary closed subset. The injectivity of the map γ, in this special case, in turn depends on the following lemma.

Lemma 16.1. *Let $\tau : C(X) \to \mathcal{U}(\mathcal{H})$ be a *-monomorphism such that the extension $[\tau]$ is in $\ker \gamma_X$. If τ admits a splitting into $[\tau_1]$ and $[\tau_2]$ with respect to a closed cover $\{X_1, X_2\}$ of X, $X_1 \cap X_2 = \{x_0\}$ then $[\tau_k]$ is in $\ker \gamma_{X_k}$ for $k = 1, 2$.*

Proof. If T^f is any operator such that $\tau(f) = \pi(T^f)$ and τ splits then there exist operators $T_1^{f_1}$ and $T_2^{f_2}$ inducing *-monomorphisms

$$\tau_1 : f_1 \to \pi(T_1^{f_1}), f_1 \in C(X_1) \text{ and } \tau_2 : f_2 \to \pi(T_2^{f_2}), f \in C(X_2).$$

$$\tau(f) = \pi \begin{bmatrix} T_1^{f|_{X_1}} & 0 \\ 0 & T_2^{f|_{X_2}} \end{bmatrix}, f \in C(X)$$

Let $f_1 : X_1 \to \mathbb{C}\backslash\{0\}$ be continuous, define $f : X \to C\backslash\{0\}$ by

$$f(x) = \begin{cases} f_1(x) & x \in X_1 \\ f_1(X_0) & x \in X_2 \end{cases}$$

Note that,

$$\tau_(f) = \pi \begin{bmatrix} T_1^{f_1} & 0 \\ 0 & T_2^C \end{bmatrix},$$

where C is the constant function $C(x) = f_1(x_0) \neq 0$ for all x in X_2 and ind $\tau_2(C) = 0$. Thus,

$$(\gamma_{X_1}[\tau_1])(f_1) = \operatorname{ind} \tau_1(f_1) = \operatorname{ind} T_1^{f_1} = \operatorname{ind} \tau(f) = 0$$

This completes the proof. □

Proposition 16.2. *If $X = [0, 1]/A$, where A is some closed subset of $[0, 1]$, then γ_x is injective.*

Proof. By considering components in the compliment of A, it is easy to see that X is the union of a sequence X_n of closed subsets, each homeomorphic to a circle or an interval with diameter $X_n \to 0$ and there is a x_0 in X such that $X_m \cap X_n = \{x_0\}$ for all $m \neq n$. Moreover, X can be regarded as a subset of $\mathbb{C}$.

Note that, γ_{x_m} is injective since each X_m is homomorphic to a circle or an interval. Let $\tau : C(X) \to \mathcal{U}(\mathcal{H})$ be a *-monomorphism, $[\tau] \in \ker \gamma_x$ and let $Y_n \cup_{m>n} X_m$. Since X_1 and Y_1 intersect in a single point, $[\tau]$ splits into $[\tau_1]$ and $[\tau_{Y_1}]$ with respect to the closed cover $\{X_1, Y_1\}$ of X by the first splitting lemma. If we write Y_1 as $X_2 \cup Y_2$, the extension $[\tau_{Y_1}]$ will again split into $[\tau_2]$ and $[\tau_{Y_2}]$ with respect to the closed cover $\{X_2, Y_2\}$ of Y_2. Continuing, we obtain

$$\tau(f) = \tau_1(f|_{x_1}) \oplus \cdots \oplus \tau_n(f|_{x_n}) \oplus \tau_{y_n}(f|y_n).$$

Each $[\tau_k]$ is in $\ker \gamma_{x_k}$ by the preceeding lemma and therefore it is trivial. Let f be any function in $C(X)$ which is constant on Y_n for some n, these functions are dense in $C(X)$. Let

$$\tau_0(f) = \tau_{10}(f|_{x_1}) \oplus \cdots \oplus \tau_{n0}(f|_{x_n}) \oplus f(x_0),$$

where τ_{k0} is the trivializing map for τ_k. Since τ_0 is defined on a dense subset, it has a continuous extension to $C(X)$. But, $\pi\tau_0 = \tau$ on a dense set and hence $\pi\tau_0 = \tau$ on all of $C(X)$. Therefore, τ is trivial and the proof is complete. □

Just as we needed Lemma 16.1 to prove injectivity of γ in this special case, we would need the following lemma to prove injectivity of γ in general.

Lemma 16.3. *If $X \subseteq \mathbb{C}$, X_1 and X_2 are the intersections of X with the closed half planes determined by a straight line L and β : Ext$(X_1) \oplus$ Ext$(X_2) \to$ Ext(X) then*

$$\ker \gamma_X \beta \subseteq \ker \gamma_{X_1} \oplus \ker \gamma_{X_2}.$$

Proof. Let $g : X_1 \to \mathbb{C}\backslash\{0\}$, define $g' : X_1 \cup L \to \mathbb{C}\backslash\{0\}$ by extending g linearly while taking care to avoid the origin, if necessary. Let $p : X \to X_1 \cup L$ be the map

$$p(x) = \begin{cases} \text{Proj onto } L, & x \in X_2 \\ x, & x \in X_1 \end{cases}$$

which is continuous. Finally, let $f = g' \circ p$. The function $f|_{X_2} = g'|_L$ is null homotopic. If $[\tau]$ is $\ker \gamma_X \beta$ then

$$\text{ind}(\tau_1(f|_{X_1}) \oplus \tau_2(f|_{X_2})) = 0.$$

For the particular function f constructed above $f|_{X_2}$ is null homotopic and $f|_{X_1} = g$ and hence

$$\operatorname{ind}\tau_1(g) = 0.$$

Therefore, $[\tau_1]$ is in $\ker\gamma_{x_1}$. Similarly, we can show $[\tau_2]$ is in $\ker\gamma_{x_2}$ and the proof is complete. $\square$

Theorem 16.4. *Let $X \subseteq \mathbb{C}$, X_1 and X_2 be the intersections of X with the closed half planes determined by a straight line L. If $[\tau]$ is any extension in* $\ker\gamma_x$ *then τ splits into $[\tau_1]$ and $[\tau_2]$ with respect to the closed cover $\{X_1, X_2\}$ of X. Furthermore, $[\tau_1]$ is in* $\ker\gamma_{x_1}$ *and $[\tau_2]$ is in* $\ker\gamma_{x_2}$.

Proof. We will need the following diagram and inclusion maps. Let J be any compact interval containing $X \cap L$. Consider

$$\begin{array}{ccccc} X_1 \cup J & \xrightarrow{i'_1} & X \cup J & \xleftarrow{i'_2} & X_2 \cup J \\ \uparrow j_1 & & \uparrow j & & \uparrow j_2 \\ X_1 & \overrightarrow{i_1} & X & \overleftarrow{i_2} & X_2 \end{array}$$

where i, j are the inclusion maps. If $[\tau]$ is in $ker\gamma_x$ then $j_*[\tau]$ is in $\ker\gamma_{X\cup J}$. By the second splitting lemma, there exists $\tau_{X_1\cup J}$ and $\tau_{X_2\cup J}$ such that

$$j_*[\tau] = i'_{1*}[\tau_{X_1\cup J}] + i'_{2*}[\tau_{X_2\cup J}].$$

By the preceeding lemma, $\tau_{X_1\cup J} \in \ker\gamma_{X_1\cup J}$ and $\tau_{X_2\cup J} \in \ker\gamma_{X_2\cup J}$. Let $q_{k*} : X_k \cup J \to X_k \cup J/X_k,\ k = 1, 2$ be the quotient map. Again, $q_{k*}[\tau_{X_k\cup J}]$ is in the $\ker\gamma_{X_k\cup J},\ k = 1, 2$. However, $X_k \cup J/X_k,\ k = 1, 2$, homeomorphic to $J/X_1 \cap X_2$ and γ is injective on such spaces. Therefore, $q_{k*}[\tau_{X_k\cup J}],\ k = 1, 2$ is trivial. But

$$\operatorname{Ext}(X_k) \to \operatorname{Ext}(X_k \cup J) \to \operatorname{Ext}(X_k \cup J/X_k),\ k = 1, 2$$

is exact, so there exist $[\tau_{X_k}]$ such that $j_{k*}[\tau_{X_k}] = [\tau_{X_k\cup J}]$. The proof is completed by showing

$$\beta([\tau_{x_1}], [\tau_{x_2}]) = [\tau].$$

Since it is easy to $j_*\beta([\tau_{x_1}], [\tau_{x_2}]) = j_*[\tau]$, it is enough to show j_* is injective. But, the Mayer–Vietoris sequence

$$\operatorname{Ext}(X \cap J) \to \operatorname{Ext}(X) \oplus \operatorname{Ext}(J) \to \operatorname{Ext}(X \cup J).$$

is exact and $\operatorname{Ext}(X \cap J) = \{0\}$ and hence j_* is injective. The last statement in the theorem is merely the previous lemma, so the proof is complete. $\square$

As explained in the first paragraph, this allows us to apply iterated splitting argument, establishing the injectivity of γ.

Corollary 16.5. *For any closed subset X of the complex plane $\mathbb{C}$, the map γ : Ext$(X) \to$ Hom$(\pi^1(X), Z)$ is injective.*

Concluding Remarks

The following theorem, determines the essential unitary equivalence classes of essential normal operators with essential spectrum $X \subseteq \mathbb{C}$.

Theorem 16.6. *Two essentially normal operators T_1 and T_2 are essentially equivalent if and only if $\sigma_{\text{ess}}(T_1) = \sigma_{\text{ess}}(T_2) = X$ and*

$$\textit{ind}(T_1 - \lambda) = \textit{ind}(T_2 - \lambda) \textit{ for } \lambda \textit{ in } \mathbb{C}\backslash X.$$

Proof. If we let $\mathbb{C}\backslash X = O_\infty \cup O_1 \cup \ldots$ denote the components, where O_∞ is the unbounded one, then $\pi^1(X)$ is the free abelian group with one generator $[O_i]$ for each bounded component. If $[\tau]$ is the extension corresponding to an essentially normal operator then the map $\gamma_X([\tau])$ is defined by $[O_i] \to n_i$, where $n_i = \text{ind}(T - \lambda_i)$ for some λ_i in O_i and proof of the theorem is complete. □

It is actually possible to show that γ_X is a surjective mapping [cf. 1]. Therefore the equivalence classes of essentially normal operators with essential spectrum X is obtained by prescribing arbitrary integers for the bounding components of $\mathbb{C}\backslash X$.

Finally, note that for $X \subseteq \mathbb{C}$,

$$\text{Ext}(X) \simeq \text{Hom}(\pi^1(X), \mathbb{Z}).$$

References

[1] L. G. Brown, R. G. Douglas and P. A. Fillmore, Unitary equivlence moduló the compact operators and extensions of C^*-algebras, Proc. Conf. on Operator Theory, Springer Lecture Notes 345 (1973), 58 – 128.

[2] L. G. Brown, R. G. Douglas and P. A. Fillmore, Extensions of C^*-algebras, and K-homology, Ann. Math., (2) 105 (1977), 265 – 324.

[3] R. G. Douglas, Banach algebra techniques in operator theory, Academic Press, New York, 1972.

[4] R. G. Douglas, C^*-algebra extensions and K-homology, Princeton University Press, 1980.