Contractivity, Complete Contractivity and Curvature inequalities

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Declaration

I hereby declare that the work reported in this thesis is entirely original and has been carried out by me under the supervision of Professor Gadadhar Misra at the Department of Mathematics, Indian Institute of Science, Bangalore. I further declare that this work has not been the basis for the award of any degree, diploma, fellowship, associateship or similar title of any University or Institution.

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Professor Gadadhar Misra
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DEDICATED TO MY PARENTS, ELDER BROTHER AND MY WIFE ANJALI

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Abstract

Let $\|\cdot\|_{\mathbf{A}}$ be a norm on $\mathbb{C}^m$ given by the formula $\|(z_1,\ldots,z_m)\|_{\mathbf{A}} = \|z_1A_1+\cdots+z_mA_m\|_{\rm op}$ for some choice of an m-tuple of $n\times n$ linearly independent matrices $\mathbf{A}=(A_1,\ldots,A_m)$. Let $\Omega_{\mathbf{A}}\subset\mathbb{C}^m$ be the unit ball with respect to the norm $\|\cdot\|_{\mathbf{A}}$. Given $p\times q$ matrices $V_1,\ldots,V_m$ and a function $f\in\mathcal{O}(\Omega_{\mathbf{A}})$, the algebra of function holomorphic on an open set U containing the closed unit ball $\bar{\Omega}_{\mathbf{A}}$ define

$$\rho_V(f) := \begin{pmatrix} f(w)I_p & \sum_{i=1}^m \partial_i f(w)V_i \\ 0 & f(w)I_q \end{pmatrix},$$

$w\in\Omega_{\mathbf{A}}$. Clearly, ρ_V defines an algebra homomorphism. We study contractivity (resp. complete contractivity) of such homomorphisms.

The homomorphism ρ_V induces a linear map $L_V : (\mathbb{C}^m, \|\cdot\|_{\mathbf{A}}^*) \to \mathcal{M}_{p\times q}(\mathbb{C})$,

$$L_V(w) = w_1V_1+\cdots+w_mV_m.$$

The contractivity (resp. complete contractivity) of the homomorphism ρ_V determines the contractivity (resp. complete contractivity) of the linear map L_V and vice-versa. It is known that contractive homomorphisms of the disc and the bi-disc algebra are completely contractive, thanks to the dilation theorems of B. Sz.-Nagy and Ando respectively. However, examples of contractive homomorphisms ρ_V of the (Euclidean) ball algebra which are not completely contractive was given by G. Misra.

From the work of V. Paulsen and E. Ricard, it follows that if $m\geq 3$ and $\mathbb{B}$ is any ball in $\mathbb{C}^m$ with respect to some norm, say $\|\cdot\|_{\mathbb{B}}$, then there exists a contractive linear map $L : (\mathbb{C}^m, \|\cdot\|_{\mathbb{B}}^*) \to \mathcal{B}(\mathcal{H})$ which is not complete contractive. The characterization of those balls in $\mathbb{C}^2$ for which contractive linear maps are always completely contractive remained open. We answer this question for balls of the form $\Omega_{\mathbf{A}}$ in $\mathbb{C}^2$.

The class of homomorphisms of the form ρ_V arise from localization of operators in the Cowen-Douglas class of Ω. The (complete) contractivity of a homomorphism in this class naturally produces inequalities for the curvature of the corresponding Cowen-Douglas bundle. This connection and some of its very interesting consequences are discussed.

Contents

Chapter 1

Introduction

In 1936 von Neumann (see [29, Chapter 1, Corollary 1.2]) proved that if T is a bounded linear operator on a separable complex Hilbert space $\mathcal{H}$, then

$$\|p(T)\| \le \|p\|_{\infty,\mathbb{D}} := \sup\{|p(z)| : |z| < 1\}$$

if and only if $\|T\| \le 1$. The original proof of this inequality is intricate. A couple of decades later, Sz.-Nazy (see [29, Chapter 4 , Theorem 4.3]) proved that a bounded linear operator T admits a unitary (power) dilation if and only there exists a unitary operator U on a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ such that

$$P_{\mathcal{H}}\, p(U)_{|\mathcal{H}} = p(T),$$

for all polynomials p. The existence of such a dilation may be established by actually constructing a unitary operator U, dilating T. This construction is due to Schaffer [33]. Clearly, the von Neumann inequality follows from the existence of a power dilation via the spectral theorem for unitary operators.

The von Neumann inequality says that the homomorphism ρ_T induced by T on the polynomial ring $P[z]$ by the rule $\rho_T(p) = p(T)$ is contractive. The homomorphism ρ_T therefore extends to the closure of the polynomial ring $\mathcal{P}[z]$ with respect to the sup norm $\|p\|_{\infty,\mathbb{D}}$. This is the disc algebra which consists of all continuous functions on the closed unit disc $\bar{\mathbb{D}}$, which are holomorphic on the open unit disc $\mathbb{D}$.

Over the years many questions related to the von Neumann inequality have been studied. Typically, these questions involve replacing, the polynomial ring $(P[z], \|\cdot\|_{\infty,\mathbb{D}})$ with some other ring of functions. For instance, the rings of rational functions $\mathrm{Rat}(\Omega)$ with poles off $\bar{\Omega}$ on some open, bounded, connected subset of $\mathbb{C}$, equipped with the supremum norm on Ω.

Suppose T is an operator with $\sigma(T) \subseteq \bar{\Omega}$. Set $r(T) := p(T)q(T)^{-1}$, for $r \in \mathrm{Rat}(\Omega)$. Since q does not vanish on $\bar{\Omega}$ and $\sigma(T) \subseteq \bar{\Omega}$, it follows that $r(T)$ is well-defined. It is natural to ask, prompted by the inequality of von Neumann, when the homomorphism ρ_T, defined by the rule $\rho_T(r) = r(T)$, is contractive on $\mathrm{Rat}(\Omega)$. There is no good answer to this question, in general. Also, in this more general setting, let us say that a homomorphism $\rho_T : \mathrm{Rat}(\Omega) \to \mathcal{B}(\mathcal{H})$ admits a normal dilation if there exists a normal operator $N : \mathcal{K} \to \mathcal{K}$, $\mathcal{H} \subseteq \mathcal{K}$ and $\sigma(N) \subseteq \partial\Omega$ such that

$$P_{\mathcal{H}} r(N)_{|\mathcal{H}} = r(T).$$

Clearly, if there exists a normal dilation, then it would follow that $\|r(N)\|_{\mathrm{op}} \leq \|r\|_{\infty,\Omega}$ making the homomorphism $r \mapsto r(N)$ contractive. However, in most cases, the converse fails.

In 1984, J. Agler (see [3]) proved that if Ω is the annulus $\mathbb{A}$, then every contractive homomorphism of the ring $\mathrm{Rat}(\mathbb{A})$ admits a normal dilation. Recently, M. Dristchell and S. McCullough [13] have shown that for a domain of connectivity ≥ 2 the converse statement is false in general.

In the very fundamental work of Arveson [1,2], he studied the normal dilation in detail and showed that the existence of a normal dilation is equivalent to complete contractivity of the homomorphism ρ_T :

Let $R = (\!(r_{ij})\!)\,, r_{ij} \in \mathrm{Rat}(\Omega)$ be a matrix valued rational function. Let

$$\|R\| = \sup\{\|\,(\!(r_{ij}(z))\!)\,\|_{\mathrm{op}} : z \in \Omega\}.$$

Define $R(T)$ naturally to be the operator $(\!(r_{ij}(T))\!)$. The homomorphism ρ_T is said to be completely contractive if $\|R(T)\| \leq \|R\|_{\infty,\Omega}$ for all $R \in \mathrm{Rat}(\Omega) \otimes \mathcal{M}_k(\mathbb{C}), k = 1, 2, \ldots, n, \ldots$.

A deep theorem proved by Arveson says that T has a normal dilation if and only if ρ_T is completely contractive. Clearly, if ρ_T is completely contractive, then it is contractive. The dilation theorems due to Sz.-Nazy and Agler give the non-trivial converse. Thus for the case of the disc and the annulus algebras contractive homomorphisms are always completely contractive.

Most of these notions apply to the rings of polynomials in more than one variable, or even to the ring of holomorphic functions, in a neighborhood of $\bar{\Omega}$, where Ω is some open bounded connected subset of $\mathbb{C}^m$. Indeed, the theorem of Arveson remains valid in this more general setting.

The first dilation theorem for a commuting pair of contractions was proved by Ando. He showed that if T_1, T_2 are a pair of commuting contractions, then there exists a pair of commuting unitaries, which dilate T_1, T_2 simultanously, that is,

$$P_{\mathcal{H}}(p(U_1, U_2))|_{\mathcal{H}} = p(T_1, T_2)$$

(see [29, Chapter 5, Theorem 5.5]). In otherwords, every contractive homomorphisms of the bi-disc algebra is completely contractive. The only other dilation theorem, in the multi-variable context is due to Agler and Young which is for the Symmetrized bi-disc [4]. Soon after, Parrott showed that there are three commuting contractions for which it is impossible to find commuting unitaries dilating them. This naturally leads to the question, in view of Arveson's theorem, for which function algebras $\mathcal{A}(\Omega)$, all contractive homomorphisms must be necessarily completely contractive. At the moment, this is known to be true of the disc, bi-disc, Symmetrized bi-disc and the annulus algebras. Counter examples are known for domains of connectivity ≥ 2 and the ball algebra and any balls in $\mathbb{C}^m$, $m \geq 3$, as we will explain below.

Neither Ando's proof of the existence of a unitary dilation for a pair of commuting contractions, nor the counter example to such an existence theorem due to Parrott involved the notion of complete contractivity directly. However, G. Misra in the papers [23], [24] and [25] began the study of Parrott like examples, comparing the norm and the cb-norm, on domains $\Omega \subset \mathbb{C}^m$ other than the tri-disc. This was further studied in depth by V. Paulsen [30], where he showed that the question of contractive vs completely contractive for Parrott like homomorphisms ρ_V includes the question of contractive vs completely contractive for linear maps L_V from some finite dimensional Banach space X to $\mathcal{M}_n(\mathbb{C})$. The counter examples we mentioned in the previous paragraph were found by him for such linear maps for $m \geq 5$. Such examples were found for $m = 3, 4$ later by E. Ricard leaving the question of what happens when $m = 2$ open. This is the question we answer in this thesis. We point out that the results of Paulsen used deep ideas from geometry of finite dimensional Banach spaces. In contrast, our results are elementary in nature, although the computations, at times, are somewhat involved.

1.1 Preliminaries

Let Ω be a bounded domain (open connected set) in $\mathbb{C}^m$ and $\mathcal{O}(\Omega)$ be the algebra of bounded functions holomorphic in some neighborhood of $\overline{\Omega}$. We equip the algebra $\mathcal{O}(\Omega)$ with the sup norm, that is,

$$\|f\|_\infty = \sup_{z \in \Omega} |f(z)|, f \in \mathcal{O}(\Omega).$$

For $i = 1, \ldots, m$ and any choice of V_i in $\mathcal{M}_{p,q}(\mathbb{C})$, let $T_i = \begin{pmatrix} w_i I_p & V_i \\ 0 & w_i I_q \end{pmatrix}$, $w = (w_1, \ldots, w_m) \in \Omega$. The m-tuple $T = (T_1, \ldots, T_m)$ of linear transformations on $\mathbb{C}^{p+q}$ is commuting and defines a homomorphism $\rho_V : \mathcal{O}(\Omega) \to \mathcal{M}_{p+q}(\mathbb{C})$ given by the formula

$$\rho_V(f) := f(T_1, \ldots, T_m) = \left(\begin{smallmatrix} f(w)I_p & \sum_{i=1}^m \partial_i f(w) V_i \\ 0 & f(w)I_q \end{smallmatrix}\right), f \in \mathcal{O}(\Omega),$$

where V denotes the m-tuple $(V_1, \ldots, V_m)$. The homomorphism ρ_V induces the linear map $L_V : \mathbb{C}^m \to \mathcal{M}_{p,q}(\mathbb{C})$ given by the formula

$$L_V(z) = z_1 V_1 + \cdots + z_m V_m.$$

For v in $\mathbb{C}^m$,

$$\mathcal{C}_{\Omega,w}(v) := \sup\{|\sum v_i \partial_i f(w)| : f \in \mathcal{O}(\Omega), f(w) = 0, \|f\|_\infty \leq 1\}$$

defines a norm on $\mathbb{C}^m$. It is the Carathéodory norm of Ω at w. We see that $\|\rho_V\| \leq 1$ if and only if $\|L_V\|_{(\mathbb{C}^m, \mathcal{C}_{\Omega,w}) \to (\mathcal{M}_{p,q}, \|\cdot\|_{\text{op}})} \leq 1$ (here $\|\cdot\|_{1,2}$ denotes the operator norm from $(X, \|\cdot\|_1)$ to $(Y, \|\cdot\|_2)$). We can say a little more after tensoring with $\mathcal{M}_k$. Let $\rho_V^{(k)}$ be the operator $\rho_V \otimes I_k : \mathcal{O}(\Omega) \otimes \mathcal{M}_k \to (\mathcal{M}_{p+q}(\mathbb{C}) \otimes \mathcal{M}_k, \|\cdot\|_{\text{op}})$, where for $F \in \mathcal{O}(\Omega) \otimes \mathcal{M}_k$. We define $\|F\| = \sup_{z \in \Omega} \|(\!(f_{ij}(z))\!)\|$, $f_{ij} \in \mathcal{O}(\Omega)$. Similarly, set $L_V^{(k)} := L_V \otimes I_k$. Now, we have $\|\rho_V^{(k)}\| \leq 1$ if and only if $\|L_V^{(k)}\|_{(\mathbb{C}^m \otimes \mathcal{M}_k, \|\cdot\|_k^*) \to (\mathcal{M}_k \otimes \mathcal{M}_{p+q}, \|\cdot\|_{\text{op}})} \leq 1$ (cf. [7, Proposition 2.1] and [30, Proposition 3.5]).

Here we study homomorphisms ρ_V defined on $\mathcal{O}(\Omega_{\mathbf{A}})$, where $\Omega_{\mathbf{A}}$ is a bounded domain of the form

$$\Omega_{\mathbf{A}} \ := \ \{(z_1, z_2, \ldots, z_m) : \|z_1 A_1 + \cdots + z_m A_m\|_{\text{op}} < 1\}$$

for some choice of a linearly independent set of $n \times n$ matrices $\{A_1, \ldots, A_m\}$.

By definition, $\Omega_{\mathbf{A}}$ is the unit ball obtained via an isometric embedding into $(\mathcal{M}_n, \|\cdot\|_{\text{op}})$. It is therefore a unit ball in $\mathbb{C}^m$ with respect to some norm, say, $\|\cdot\|_{\mathbf{A}}$. It also has a natural operator space structure obtained via this embedding. However, it is possible to pick different isometric embeddings of a $(\mathbb{C}^m, \|\cdot\|_{\mathbf{A}})$ into the operators on some Hilbert space. Whether these different (isometric) embeddings give the same operator space structure is an interesting question on its own right. Picking $A_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $A_2 = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ gives the embedding of the Euclidean ball as the space of "row vectors", while if we pick the transpose of A_1 and A_2, we would be embedding it as the space of column vectors. As is well known, these two embeddings give rise to different operator space structures leading to an example of a contractive homomorphism on the ball algebra which is not completely contractive. While for any n in $\mathbb{N}$, if we pick $A_1 = I_{2n}, A_2 = \left(\begin{smallmatrix} 0 & B \\ 0 & 0 \end{smallmatrix}\right)$, then they determine the same norm (independent of n) on $\mathbb{C}^2$ as long as $\|B\| = 1$. However, the operator space structure is also independent of n, which we show in Chapter 3. Following the example of the ball, even if we pick the new pair to be $A_1 = I_{2n}$ and $A_2 = \left(\begin{smallmatrix} 0 & B \\ 0 & 0 \end{smallmatrix}\right)^{\text{t}}$, the operator space we obtain remains the same. The ball $\Omega_{\mathbf{A}}$, in this case, is the set $\{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2| < 1\}$. We see that it has several distinct isometric embeddings into $\mathcal{M}_{2n}(\mathbb{C})$. Surprisingly, all of these give the same operator space structure on $(\mathbb{C}^2, \|\cdot\|_{\mathbf{A}})$. Therefore, unlike the case of the Euclidean ball, we have to find some other way of showing the existence of distinct operator space structures on this normed space, which we do in Chapter 3.

From the work of V. Paulsen and E. Ricard (cf. [30], [28]), it follows that if $m \geq 3$ and $\mathbb{B}$ is any ball in $\mathbb{C}^m$, then there exists a contractive linear map which is not completely contractive. It is known that contractive homomorphisms of the disc and the bi-disc algebra are completely contractive, thanks to the dilation theorem of B. Sz.-Nagy and Ando. However, an example of a contractive homomorphism of the (Euclidean) ball algebra which is not completely contractive was given in [23, 24]. The characterization of those balls in $\mathbb{C}^2$ for which "contractive linear maps are always completely contractive" remained open. We answer this question in Chapter 5 for domains of the form $\Omega_{\mathbf{A}}$, $\mathbf{A} = (A_1, A_2)$ in $\mathbb{C}^2 \otimes \mathcal{M}_2(\mathbb{C})$. Along the way we obtain some interesting applications for domains of the form $\Omega_{\mathbf{A}}$ in $\mathbb{C}^m, m \in \mathbb{N}$.

The (linear) polynomial $P_{\mathbf{A}}$ defined by the rule

$$P_{\mathbf{A}}(z_1, z_2, \ldots, z_m) = z_1 A_1 + z_2 A_2 + \cdots + z_m A_m,$$

maps the ball $\Omega_{\mathbf{A}}$ into $(\mathcal{M}_n(\mathbb{C}), \|\cdot\|_{\mathrm{op}})_1$ by definition. We develop several methods to determine when $\|L_V\| \leq 1$. We recall that $\|L_V\| \leq 1$ if and only if $\|\rho_V\| \leq 1$. We show that $\|L_V\| \leq \|L_V^{(n)}(P_{\mathbf{A}})\|$. Finding a V such that

$$\|L_V\|_{(\mathbb{C}^m, \mathcal{C}_{\Omega,w}) \to (\mathcal{M}_{p,q}(\mathbb{C}), \|\cdot\|_{\mathrm{op}})} \leq 1$$

for which $\|L_V^{(n)}(P_{\mathbf{A}})\|_{\mathrm{op}} > 1$ gives an example of a contractive homomorphism on $\mathcal{O}(\Omega_{\mathbf{A}})$ which is not completely contractive. However, finding such a V is far from obvious, as we will see.

Furthermore, in Chapter 2, we show that for homomorphisms of our class, the property "contractivity implies complete contractivity", remains unaffected under bi-holomorphic equivalence. Thus we describe some natural bi-holomorphic, actually linear, equivalence for domains of the form $\Omega_{\mathbf{A}}$ and work with a convenient representative from each equivalence class. We give a list of such representatives for the class of domains $\Omega_{\mathbf{A}}$ in Chapter 2.

The class of homomorphisms of the form ρ_V arise from localization of operators in the Cowen-Douglas class of Ω. The (complete) contractivity of a homomorphism in this class naturally produces inequalities for the curvature of the corresponding Cowen-Douglas bundle (cf. [24, Theorem5.2]). This connection and some of its very interesting consequences are discussed in Chapter 4.

In the paper [27], Parrott showed that if U_1 and U_2 are a pair of non-commuting unitaries then the homomorphism ρ_V, $V=(I,U_1,U_2)$, is contractive on the tri-disc algebra $\mathcal{A}(\mathbb{D}^3)$ which is not completely contractive. Equivalently, he shows that there does not exist commuting unitaries dilating the commuting contractions $\left(\begin{smallmatrix} I_2 & 0\\ 0 & I_2\end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & U_1\\ 0 & 0\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & U_2\\ 0 & 0\end{smallmatrix}\right)$. This shows that Ando's theorem does not generalize to $m>2$. The Parrott examples were further studied in a series of papers [23–26, 30].

Let $\Omega^*_{\mathbf{A}}$ be the unit ball for the dual norm $\|\cdot\|^*_{\mathbf{A}}$. We point out that contractive homomomorphisms of our class are completely contractive for $\mathcal{O}(\Omega_{\mathbf{A}})$ if and only if it is true for $\mathcal{O}(\Omega^*_{\mathbf{A}})$ (see [26] and [30]).

Let $P_{\mathbf{A}}:\Omega_{\mathbf{A}}\to(\mathcal{M}_n(\mathbb{C}))_1$ be the matrix valued polynomial on $\Omega_{\mathbf{A}}$ defined by $P_{\mathbf{A}}(z_1,z_2,\ldots,z_m)=z_1A_1+z_2A_2+\cdots+z_mA_m$, where $(\mathcal{M}_n(\mathbb{C}))_1$ is the matrix unit ball with respect to the operator norm. For $(z_1,z_2,\ldots,z_m)$ in $\Omega_{\mathbf{A}}$, the norm

$$\|P_{\mathbf{A}}\|_\infty := \sup_{(z_1,\ldots,z_m)\in\Omega_{\mathbf{A}}}\|P_{\mathbf{A}}(z_1,\ldots,z_m)\|_{\mathrm{op}}$$

is at most 1 by definition of the polynomial $P_{\mathbf{A}}$. We say that $P_{\mathbf{A}}$ is a defining function for $\Omega_{\mathbf{A}}$. As we have indicated earlier, we detect the failure of complete contractivity by checking if $\|\rho_V(P_{\mathbf{A}})\|\le 1$ or not. Often, one works with a defining function which is assumed to be smooth. Our defining function takes values in $\mathcal{M}_n(\mathbb{C})$, it is holomorphic, indeed, it is a linear map.

For $(\alpha,\beta)\in\mathbb{B}\times\mathbb{B}$, define $p^{(\alpha,\beta)}_{\mathbf{A}}:\Omega_{\mathbf{A}}\to\mathbb{D}$ to be the linear map

$$p^{(\alpha,\beta)}_{\mathbf{A}}(z_1,\ldots,z_m)=\langle P_{\mathbf{A}}(z_1,\ldots,z_m)\alpha,\beta\rangle\,.$$

The sup norm $\|p^{(\alpha,\beta)}_{\mathbf{A}}\|_\infty$ on $\Omega_{\mathbf{A}}$, for any pair of vectors (α,β) in $\mathbb{B}\times\mathbb{B}$, is at most 1 by definition. Let $\wp^{(\alpha,\beta)}_{\mathbf{A}}$ denote the set of linear functions $\{p^{(\alpha,\beta)}_{\mathbf{A}}:(\alpha,\beta)\in\mathbb{B}\times\mathbb{B}\}$. Let $V=(V_1,\ldots,V_m)$, $V_i\in\mathcal{M}_{p,q}$, and $\rho_V:\wp^{(\alpha,\beta)}_{\mathbf{A}}\longmapsto\mathcal{B}(\mathbb{C}^p\oplus\mathbb{C}^q)\cong\mathcal{B}(\mathbb{C}^{p+q})$ be the homomorphism defined by

$$\rho_V(p^{(\alpha,\beta)}_{\mathbf{A}})=\begin{pmatrix} p^{(\alpha,\beta)}_{\mathbf{A}}(0)I_p & \partial_1 p^{(\alpha,\beta)}_{\mathbf{A}}(0)V_1+\ldots+\partial_m p^{(\alpha,\beta)}_{\mathbf{A}}(0)V_m\\ 0 & p^{(\alpha,\beta)}_{\mathbf{A}}(0)I_q\end{pmatrix},\ p^{(\alpha,\beta)}_{\mathbf{A}}\in\wp^{(\alpha,\beta)}_{\mathbf{A}}.$$

Lemma 1.1. $\sup_{\|\alpha\|=\|\beta\|=1}\|\rho_V(p^{(\alpha,\beta)}_{\mathbf{A}})\|\le\|\rho^{(n)}_V(P_{\mathbf{A}})\|$.

Proof. The proof is a straightforward computation:

$$\begin{aligned}
\sup_{\|\alpha\|=\|\beta\|=1} \|\rho_V(p_{\mathbf{A}}^{(\alpha,\beta)})\| &= \sup_{\|\alpha\|=\|\beta\|=1} \| \partial_1 p_{\mathbf{A}}^{(\alpha,\beta)}(0)V_1 + \cdots + \partial_m p_{\mathbf{A}}^{(\alpha,\beta)}(0)V_m \| \\
&= \sup_{\|\alpha\|=\|\beta\|=1} \| \langle A_1\alpha,\beta\rangle V_1 + \cdots + \langle A_m\alpha,\beta\rangle V_m \| \\
&= \sup_{\|\alpha\|=\|\beta\|=\|u\|=\|v\|=1} |\langle A_1\alpha,\beta\rangle\langle V_1u,v\rangle + \cdots + \langle A_m\alpha,\beta\rangle\langle V_m u,v\rangle| \\
&= \sup_{\|\alpha\|=\|\beta\|=\|u\|=\|v\|=1} |\langle (A_1\otimes V_1 + \cdots + A_m\otimes V_m)\alpha\otimes u, \beta\otimes v\rangle| \\
&= \sup_{\|\alpha\|=\|\beta\|=\|u\|=\|v\|=1} |\langle \rho_V^{(n)}(P_{\mathbf{A}})\alpha\otimes u, \beta\otimes v\rangle| \\
&\leq \|\rho_V^{(n)}(P_{\mathbf{A}})\|. \qquad (1.1)
\end{aligned}$$

This completes the proof. □

Since $p_{\mathbf{A}}^{(\alpha,\beta)}$ is linear, the derivative $Dp_{\mathbf{A}}^{(\alpha,\beta)}(0) = p_{\mathbf{A}}^{(\alpha,\beta)}$. The set of vectors

$$\{(\langle A_1\alpha,\beta\rangle,\ldots,\langle A_m\alpha,\beta\rangle) : \alpha,\beta\in\mathbb{B}^2\} \subseteq \mathbb{C}^m$$

is a subset of the dual unit ball $\Omega_{\mathbf{A}}^*$ by definition. We will not distinguish between this set of vectors and the set of linear maps $\wp_{\mathbf{A}}^{(\alpha,\beta)}$ induced by them.

The linear map L_V, $V=(V_1,\ldots,V_m)$, is contractive if and only if

$$\sup_{(\lambda_1,\ldots,\lambda_m)\in\Omega_{\mathbf{A}}^*} \|\lambda_1V_1+\cdots+\lambda_mV_m\|_{\mathrm{op}} = \sup_{(\lambda_1,\ldots,\lambda_m)\in\Omega_{\mathbf{A}}^*}\sup_{\|u\|_2=1} \|\lambda_1V_1u+\cdots+\lambda_mV_mu\|_2 \leq \|(\lambda_1,\ldots,\lambda_m)\|_{\mathbf{A}}^*.$$

Or, equivalently,

$$\sup_{(\lambda_1,\ldots,\lambda_m)\in\Omega_{\mathbf{A}}^*}\sup_{\|u\|_2=1=\|v\|_2} |\lambda_1\langle V_1u,v\rangle+\cdots+\lambda_m\langle V_mu,v\rangle| \leq \|(\lambda_1,\ldots,\lambda_m)\|_{\mathbf{A}}^*,$$

that is, $\|L_V\|\leq 1$ if and only if $(\langle V_1u,v\rangle,\ldots,\langle V_mu,v\rangle)$ is in $\Omega_{\mathbf{A}}$ for every pair of unit vectors u and v. We find that

$$\begin{aligned}
\sup_{\|u\|_2=1=\|v\|_2} \|(\langle V_1u,v\rangle,\ldots,\langle V_mu,v\rangle)\|_{\mathbf{A}}^2 &= \|\langle V_1u,v\rangle A_1+\cdots+\langle V_mu,v\rangle A_m)\|_{\mathrm{op}}^2 \\
&= \sup_{\|u\|_2=1=\|v\|_2}\sup_{\|\alpha\|_2=1=\|\beta\|_2} |\langle \sum_{j=1}^m \langle A_j\alpha,\beta\rangle V_ju,v\rangle\ |^2 \\
&= \sup_{\|u\|_2=1}\sup_{\|\alpha\|_2=1=\|\beta\|_2} \|\sum_{j=1}^m \langle A_j\alpha,\beta\rangle V_ju\ \|_2^2 \\
&= \sup_{\|u\|_2=1}\sup_{\|\alpha\|=\|\beta\|=1} \|L_V\big((\langle A_1\alpha,\beta\rangle,\ldots,\langle A_m\alpha,\beta\rangle)\big).u\|_2^2 \\
&= \sup_{\|\alpha\|=\|\beta\|=1} \|L_V\big((\langle A_1\alpha,\beta\rangle,\ldots,\langle A_m\alpha,\beta\rangle)\big)\|_{\mathrm{op}}
\end{aligned} \qquad (1.2)$$

We have seen that $\{(\langle V_1 u, v\rangle, \dots, \langle V_m u, v\rangle) : \|u\|_2 \le 1, \|v\|_2 \le 1\} \subseteq \Omega_{\mathbf{A}}$ for any fixed but arbitrary m tuple V for which L_V is contractive. However, it is not clear if there is a collection of contractive homomorphisms which produce all of $\Omega_{\mathbf{A}}$. Similarly, the set $\{(\langle A_1\alpha, \beta\rangle, \dots, \langle A_m\alpha, \beta\rangle) : \|\alpha\|_2 \le 1, \|\beta\|_2 \le 1\} \subseteq \Omega^*_{\mathbf{A}}$. Again, we don't know if for some choice of $\mathbf{A}$ equality occurs.

Thus we have shown that L_V is contractive if and only if it is contractive on the set $\wp_{\mathbf{A}}^{(\alpha,\beta)}$. However, as we have pointed out earlier, L_V is contractive if and only if the homomorphism ρ_V is contractive. Similarly, L_V is contractive on the set $\wp_{\mathbf{A}}^{(\alpha,\beta)}$ if and only if the restriction $\rho_{V_{|\wp_{\mathbf{A}}^{(\alpha,\beta)}}}$ of the homomorphism ρ_V to $\wp_{\mathbf{A}}^{(\alpha,\beta)}$ is contractive. Therefore we have proved the following.

Proposition 1.2. *The following conditions are equivalent.*

(i) $\|\rho_V\| = \sup_{\|p\|_\infty \le 1}\{\|\rho_V(p)\| : p \in \mathcal{O}(\Omega_{\mathbf{A}}),\, p(0) = 0\} \le 1$

(ii) $\sup_{\|\alpha\|=\|\beta\|=1}\{\|\rho_V(p_{\mathbf{A}}^{(\alpha,\beta)})\| : p_{\mathbf{A}}^{(\alpha,\beta)} \in \wp_{\mathbf{A}}^{(\alpha,\beta)}\} \le 1$

(iii) $\|L_V\|_{(\mathbb{C}^m, \|\cdot\|^*_{\mathbf{A}}) \to (\mathcal{M}_n, \|\cdot\|_{\mathrm{op}})} \le 1$

(iv) $\sup_{\|\alpha\|=\|\beta\|=1} \|L_V\big((\langle A_1\alpha, \beta\rangle, \dots, \langle A_m\alpha, \beta\rangle)\big)\|_{\mathrm{op}} \le 1$

Corollary 1.3. *If* $\|\rho_V^{(n)}(P_{\mathbf{A}})\| \le 1$ *then* ρ_V *is contractive.*

Proof. It is enough to check the contractivity of the restriction of ρ_V to the set $\wp_{\mathbf{A}}^{(\alpha,\beta)}$. On this set, as we have shown in Lemma 1.1, the norm of ρ_V is bounded above by $\|\rho_V^{(n)}(P_{\mathbf{A}})\|$. □

Remark 1.4. This proposition says that checking the contractivity of ρ_V on the algebra $\mathcal{O}(\Omega_{\mathbf{A}})$ may be reduced to checking it on $\wp_{\mathbf{A}}^{(\alpha,\beta)}$. Thus this class of homomorphisms $\wp_{\mathbf{A}}^{(\alpha,\beta)}$ serves as a class of "Test functions". Apart from this, for this class of homomorphisms ρ_V, we have the property $\|\rho_V\| \le \|\rho_V^{(n)}(P_{\mathbf{A}})\|$. This inequality often happens to be strict making it possible to construct examples of contractive homomorphisms which are not completely contractive.

Choosing $\mathbf{A} = \left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)\right)$, we see that $\Omega_{\mathbf{A}}$ defines the Euclidean ball in $\mathbb{C}^2$. Choose $V_1 = (v_{11}\ v_{12}), V_2 = (v_{21}\ v_{22})$. We will prove that

$$\sup_{\|\alpha\|=\|\beta\|=1} \|\rho_V(p_{\mathbf{A}}^{(\alpha,\beta)})\| < \|\rho_V(P_{\mathbf{A}})\|_{\mathrm{op}}.$$

This example, of a contractive homomorphism of the ball algebra which is not completely contractive, was found in [23, 24].

Theorem 1.5. *For $\Omega_{\mathbf{A}} = \mathbb{B}^2$, we have*

$$\sup_{\|\alpha\|=\|\beta\|=1} \|\rho_V(p_{\mathbf{A}}^{(\alpha,\beta)})\| < \|\rho_V(P_{\mathbf{A}})\|_{\rm op}.$$

Proof. By definition of ρ_V, we have

$$\begin{aligned}
\sup_{\|\alpha\|=\|\beta\|=1} \|\rho_V(p_{\mathbf{A}}^{(\alpha,\beta)})\|^2 &= \sup_{\|\alpha\|=\|\beta\|=\|u\|=\|v\|=1} |\langle A_1\alpha,\beta\rangle\langle V_1u,v\rangle + \langle A_2\alpha,\beta\rangle\langle V_2u,v\rangle|^2 \\
&= \sup_{\|\alpha\|=\|\beta\|=\|u\|=1} |\alpha_1(v_{11}u_1 + v_{12}u_2) + \alpha_2(v_{21}u_1 + v_{22}u_2)|^2|\beta_1|^2 \\
&= \sup_{\|\alpha\|=\|u\|=1} |\alpha_1(v_{11}u_1 + v_{12}u_2) + \alpha_2(v_{21}u_1 + v_{22}u_2)|^2 \\
&= \sup_{\|u\|=1} |v_{11}u_1 + v_{12}u_2|^2 + |v_{21}u_1 + v_{22}u_2|^2 \\
&= \left\| \left(\begin{smallmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{smallmatrix}\right) \right\|_{\rm op}^2.
\end{aligned}$$

On the other hand, we have

$$\|\rho_V(P_{\mathbf{A}})\|_{\rm op}^2 = \|V_1\|^2 + \|V_2\|^2$$

where $\|V_1\|^2 = |v_{11}|^2 + |v_{12}|^2, \|V_2\|^2 = |v_{21}|^2 + |v_{22}|^2$. It follows that

$$\left\| \left(\begin{smallmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{smallmatrix}\right) \right\|_{\rm op}^2 < \|V_1\|^2 + \|V_2\|^2.$$

Hence we have

$$\sup_{\|\alpha\|=\|\beta\|=1} \|\rho_V(p_{\mathbf{A}}^{(\alpha,\beta)})\| < \|\rho_V(P_{\mathbf{A}})\|_{\rm op}.$$

□

Remark 1.6. It is therefore natural to ask which of the domains $\Omega_{\mathbf{A}} \subset \mathbb{C}^2$ has the property

$$\sup_{\|\alpha\|=\|\beta\|=1} \|\rho_V(p_{\mathbf{A}}^{(\alpha,\beta)})\| < \|\rho_V^{(n)}(P_{\mathbf{A}})\|.$$

If the answer is affirmative, then there is a possibility of producing an example of a contractive homomorphism of $\mathcal{O}(\Omega_{\mathbf{A}})$ which is not completely contractive. However, as we will see, unlike the case of the Euclidean ball, this requires lot more work in general.

If a normed linear space $(\mathbb{C}^m, \|\cdot\|_{\mathbf{A}})$ admits only one operator space structure, then every contractive linear map from $(\mathbb{C}^m, \|\cdot\|_{\mathbf{A}})$ into $\mathcal{M}_k(\mathbb{C})$, $k \in \mathbb{N}$ must be completely contractive. As before, for some linearly independent set of $n \times n$ matrices $\{A_1, \ldots, A_m\}$, setting

$$\|(z_1, \cdots, z_m)\|_{\mathbf{A}} := \|z_1A_1 + \cdots + z_mA_m\|_{\rm op},$$

we obtain an m-dimensional normed linear space $\mathbf{V_A}$. This makes the map

$$(z_1, \ldots, z_m) \to z_1 A_1 + \cdots + z_m A_m$$

an isometry from $\mathbf{V_A}$ into $(\mathcal{M}_n, \|\cdot\|_{\mathrm{op}})$. Therefore, $\mathbf{V_A}$ inherits an operator space structure from $\mathcal{M}_n$. Similarly we can think of $\mathbf{V_{A^t}}$ as an operator space via the isometric embedding

$$(z_1, \ldots, z_m) \to z_1 A_1^{\mathrm{t}} + \cdots + z_m A_m^{\mathrm{t}}$$

into $\mathcal{M}_n(\mathbb{C})$, where $\mathbf{A}^{\mathrm{t}} = (A_1^{\mathrm{t}}, \ldots, A_m^{\mathrm{t}})$ is obtained by taking the transpose.

Let $\mathbf{A} = \left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \right)$. The norm it determines on $\mathbf{V_A} (\cong \mathbb{C}^2)$ is the ℓ_2 norm. Note that $P_{\mathbf{A}} : \mathbf{V_A} \to \mathcal{M}_2(\mathbb{C})$ defines a linear isometric embedding. Suppose $V = (\!(\mathbf{v}_{ij})\!) \in \mathcal{M}_k(\mathbf{V_A})$, where $\mathbf{v}_{ij} \in \mathbf{V_A}$. Define $P_{\mathbf{A}}^{(k)} := P_{\mathbf{A}} \otimes I_k : \mathcal{M}_k(\mathbf{V_A}) \to \mathcal{M}_k(\mathcal{M}_2(\mathbb{C}))$ by $P_{\mathbf{A}}^{(k)}(V) = (\!(P_{\mathbf{A}}(\mathbf{v}_{ij}))\!)$. Let $\mathbf{v}_{ij} = (v_{ij}^1 \ v_{ij}^2), i, j = 1, \ldots, k$, then

$$P_{\mathbf{A}}^{(2)}(V) = \left(\begin{smallmatrix} V_1 & V_2 \\ 0 & 0 \end{smallmatrix}\right),$$

where $V_1 = (\!(v_{ij}^1)\!)$ and $V_2 = (\!(v_{ij}^2)\!)$. Similarly if we take $\mathbf{A}^t = \left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) \right)$ then $\mathbf{V_{A^t}}$ becomes an operator space. Therefore we have

$$P_{\mathbf{A}^{\mathrm{t}}}^{(2)}(V) = \left(\begin{smallmatrix} V_1 & 0 \\ V_2 & 0 \end{smallmatrix}\right).$$

For the record, the norm of $P_{\mathbf{A}}^{(2)}(V)$ and $P_{\mathbf{A}^{\mathrm{t}}}^{(2)}(V)$ are given in the following lemma.

Lemma 1.7. *If* $\mathbf{v}_1 = (v_{11} \ v_{12}), \mathbf{v}_2 = (v_{21} \ v_{22})$, *then*

$$\left\| \left(\begin{smallmatrix} \mathbf{v}_1 & \mathbf{v}_2 \\ 0 & 0 \end{smallmatrix}\right) \right\|^2 = \|\mathbf{v}_1\|^2 + \|\mathbf{v}_2\|^2 = |v_{11}|^2 + |v_{12}|^2 + |v_{21}|^2 + |v_{22}|^2$$

and

$$\left\| \left(\begin{smallmatrix} \mathbf{v}_1 & 0 \\ \mathbf{v}_2 & 0 \end{smallmatrix}\right) \right\|^2 = \left\| \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix} \right\|_{\mathrm{op}}^2.$$

Consequently, for this choice of V, the norms $\|P_{\mathbf{A}}^{(2)}(V)\|$ and $\|P_{\mathbf{A}^{\mathrm{t}}}^{(2)}(V)\|$ are not equal. The existence of two distinct operator space structures on $\mathbf{V_A}$ follows from this.

However, most of the time, this trick doesn't work, that is, the operator space structures induced by $\mathbf{A}$ and $\mathbf{A}^{\mathrm{t}}$ are completely isometric. In that situation, the following algorithm is adopted, which involves a careful "case by case" analysis. Fix $\mathbf{v}_1 = (v, 0), \mathbf{v}_2 = (0, w)$. Let $L_{(\mathbf{v}_1, \mathbf{v}_2)} : (\mathbb{C}^2, \| \cdot \|_{\Omega_{\mathbf{A}}}^*) \to (\mathbb{C}^2, \| \cdot \|_2)$ be the linear map $(z_1, z_2) \mapsto (z_1 v, z_2 w)$.

(i) For β in $\mathbb{C}^2$, and $\mathbf{v}_1 = (v, 0)$, $\mathbf{v}_2 = (0, w)$ as above, let

$$g_{(v,w)}(\beta) := \{1 - |v|^2 \|A_1^* \beta\|^2 - |w|^2 \|A_2^* \beta\|^2 + |vw|^2 (\|A_1^* \beta\|^2 \|A_2^* \beta\|^2 - |\langle A_1 A_2^* \beta, \beta \rangle|^2)\}.$$

We show that $L_{(\mathbf{v}_1, \mathbf{v}_2)} : (\mathbb{C}^2, \| \cdot \|_{\Omega_{\mathbf{A}}}^*) \to (\mathbb{C}^2, \| \cdot \|_2)$ is contractive if and only if $|v|^2 \leq \frac{1}{\|A_1^*\|^2}$ and (v, w) is in $\mathcal{E} := \{(v, w) : \inf_{\beta, \|\beta\|_2 = 1} g_{(v,w)}(\beta) \geq 0\}$.

(ii) We then show that there exists $\mathbf{v}_1, \mathbf{v}_2$ for which $L_{(\mathbf{v}_1,\mathbf{v}_2)}$ is contractive while $L^{(2)}_{(\mathbf{v}_1,\mathbf{v}_2)}(P_{\mathbf{A}})$ is not contractive. Therefore, this contractive linear map, namely, $L_{(\mathbf{v}_1,\mathbf{v}_2)}$ cannot be completely contractive.

(iii) The contractivity of $L_{(\mathbf{v}_1,\mathbf{v}_2)}(P_{\mathbf{A}})$ is shown to be equivalent to the condition

$$\inf_\beta\{1 - |v|^2\|A_1^*\beta\|^2 - |w|^2\|A_2^*\beta\|^2 : \|\beta\|_2 = 1\} \geq 0.$$

(iv) There exists $\beta \in \mathbb{C}^2$ such that either $(A_2^* - \mu A_1^*)\beta = 0$ or $(A_1^* - \nu A_2^*)\beta = 0$ for some μ, ν in $\mathbb{C}$. The set

$$\mathcal{B} := \{\beta : \|\beta\|_2 = 1,\, (A_2^* - \mu A_1^*)\beta = 0 \,\text{or}\, (A_1^* - \nu A_2^*)\beta = 0 \,\text{for some}\, \mu, \nu \in \mathbb{C}\}$$

of these vectors is non-empty.

In the last chapter we show that there exists a $\lambda > 0$, say λ_0, such that $(v, \lambda_0 v)$ is in $\mathcal{E}$ with the property:

$g_{(v,\lambda_0 v)}(\beta'') > g_{(v,\lambda_0 v)}(\beta') > g_{(v,\lambda_0 v)}(\beta)$ or $g_{(v,\lambda_0 v)}(\beta') > g_{(v,\lambda_0 v)}(\beta'') > g_{(v,\lambda_0 v)}(\beta)$ whenever $\beta', \beta'' \in \mathcal{B}$.

Also, we then prove that there exists a v ($|v| < \frac{1}{\|A_1^*\|}$, this is necessary for contractivity), say v_0, such that $(v_0, \lambda v_0)$ is in $\mathcal{E}_0 := \{(v,w) : \inf_\beta g_{(v,w)}(\beta) = 0\}$.

Hence there exists a v_0, λ_0 and β_0 such that

$$1 - |v_0|^2\|A_1^*\beta_0\|^2 - |\lambda_0 v_0|^2\|A_2^*\beta_0\|^2 + \lambda_0^2|v_0|^4(\|A_1^*\beta_0\|^2\|A_2^*\beta_0\|^2 - |\langle A_1A_2^*\beta_0, \beta_0\rangle|^2) = 0$$

which is equivalent to $\|L_{(\mathbf{v}_1,\mathbf{v}_2)}(P_{\mathbf{A}})\| > 1$.

We now discuss the relationship of homomorphisms of the form ρ_V with m tuple of operators T in the Cowen-Douglas class $B_1(\Omega)$, $\Omega \subseteq \mathbb{C}^m$. In the papers [10] and [12], it is shown that the operator T can be realized as the adjoint of the commuting tuple $\boldsymbol{M} = (M_1, \ldots M_m)$ of multiplication operators defined by the coordinate functions on a reproducing kernel Hilbert space $(\mathcal{H}, K)$ consisting of holomorphic functions on $\Omega^* := \{\bar{w} : w \in \Omega\}$. It then follows that the joint kernel $\cap_{i=1}^m \ker(M_i - w_i)^*$ is spanned the vector K_w. We think of $w \mapsto K_w$ as a frame for a anti-holomorphic line bundle $\mathcal{L}_{\mathbf{M}}$ on Ω. The Hermitian metric of this line bundle is $K_w(w)$ on the fiber at w.

Fix an operator T in the Cowen-Douglas class $B_1(\Omega)$. This is the same as fixing a Hilbert space $\mathcal{H}$ of holomorphic functions on Ω and a positive definite kernel K, which is holomorphic in the first variable and anti-holomorphic in the second, on Ω. Then the operator T is unitarily equivalent to $\boldsymbol{M}^*$. Let $\mathcal{N}(w) = \cap_{i=1}^m \ker(M_i - w_i)^{*2}$ and let $N_i(w)$

be the commuting tuple of finite dimensional operators obtained by restricting M_i^* to $\mathcal{N}(w)$, $i = 1, \ldots, m$. The commuting tuple $N(w)$ is of the form

$$\left(\begin{pmatrix} \bar{w}_1 & \mathbf{v}_1 \\ 0 & \bar{w}_1 I \end{pmatrix}, \ldots, \begin{pmatrix} \bar{w}_m & \mathbf{v}_m \\ 0 & \bar{w}_m I \end{pmatrix} \right),$$

it is the localization of T at w. These pairwise commuting operators induce a homomorphism ρ_V except that the $\mathbf{v}_1, \ldots, \mathbf{v}_m$ are of size $1 \times m$. (It is also possible to add several rows of zeros to each of these vectors making them $m \times m$ matrices.) It is easy to show that the $(m+1)$ dimensional space $\mathcal{N}(w)$ is spanned by the vectors $\{K_w, \bar{\partial}_1 K_w, \ldots, \bar{\partial}_m K_w\}$. It therefore has a natural inner product, which it inherits from the Hilbert space $\mathcal{H}$, namely, $\langle \bar{\partial}_i K_w, \bar{\partial}_j K_w \rangle = (\partial_j \bar{\partial}_i K_w)(w)$, $i, j = 0, 1, \ldots, m$, where $\bar{\partial}_0 K_w := K_w$. The curvature of the line bundle is a $(1,1)$ form given by the formula $\sum_{i,j=1}^m \frac{\partial^2}{\partial w_i \partial \bar{w}_j} \log \|\gamma(w)\|^2 dw_i \wedge d\bar{w}_j$. We let $\mathcal{K}(w)$ denote the matrix of the coefficients of the curvature $(1,1)$ form.

There is a close relationship between the operators $N_1(w), \ldots, N_m(w)$ and the curvature $\mathcal{K}(w)$, namely,

$$-\big(\!\!\big(\operatorname{tr} N_i(w) N_j(w)^*\big)\!\!\big)^{\mathrm{t}} = \mathcal{K}(w)^{-1}.$$

This relationship was derived in [10] for $m = 1$ and in [11] for $m = 2$.

Suppose K is a positive definite kernel. Then for any natural number n, the kernel $K^n(z, w)$, the point-wise product of the kernel K, is positive definite. This is no longer true if we replace the natural numbers n by positive real numbers λ. However, we show that $\big(\!\!\big((\partial_j \bar{\partial}_i K^\lambda)(w, w) \big)\!\!\big)_{i,j=0}^m$ is positive definite for all positive real numbers λ and therefore it defines an inner product on the space $\mathcal{N}^{(\lambda)}(w)$, the linear span of the vectors $\{K^\lambda(\cdot, w), \bar{\partial}_1 K^\lambda(\cdot, w), \ldots, \bar{\partial}_m K^\lambda(\cdot, w)\}$. We conclude that the first order jet bundle determined by these vectors possesses a non-degenerate Hermitian inner product. The Hermitian metric induced on the jet bundle of order k by the kernel K^λ ($\lambda > 0$) need not be non-degenerate in general for $k > 1$.

We define, for $i = 1, \ldots, m$, the operators $N_i^{(\lambda)}(w)$ on $\mathcal{N}^{(\lambda)}(w)$ by the rule

$$\big(N_i^{(\lambda)}(w) - \bar{w}_i I_{m+1}\big)(\partial_i K^\lambda)(\cdot, w) = \begin{cases} K^\lambda(\cdot, w) & \text{if } i \neq 0 \\ 0 & \text{if } i = 0 \end{cases}.$$

These are pairwise commuting nilpotent operators. However, they need not be the localization of some operator in $B_1(\Omega)$ unless λ is a natural number. We study the contractivity (resp. complete contractivity) properties of the homomorphism induced by the operators $N^{(\lambda)}$ starting with a fixed operator T in $B_1(\Omega)$. The contractivity properties of the homomorphism induced by the localization operators is equivalent to a curvature inequality. We study the Bergman kernel of the matrix unit ball and some of its open subsets. This provides examples to show that the curvature inequality does not necessarily imply the stronger inequality $\|p(T)\| \leq \|p\|_\infty$ for all polynomials p in m variables.

Chapter 2

Biholomorphic equivalence

2.1 Linear equivalence

We describe a natural class of domains in $\mathbb{C}^m$ which admit an isometric embedding into the normed linear space $(\mathcal{M}_n(\mathbb{C}), \|\cdot\|_{\mathrm{op}})$, where $\|\cdot\|_{\mathrm{op}}$ denotes the operator norm on the space of $n \times n$ complex matrices. For any m-tuple of matrices $\mathbf{A} = (A_1, \ldots, A_m)$ in $\mathbb{C}^m \otimes \mathcal{M}_n(\mathbb{C})$, let

$$\Omega_{\mathbf{A}} \;:=\; \{(w_1, w_2, \ldots, w_m) : \|w_1 A_1 + \cdots + w_m A_m\|_{\mathrm{op}} < 1\}.$$

Clearly, $\Omega_{\mathbf{A}} = (\mathbb{C}^m, \|\cdot\|_{\mathbf{A}})_1$ is the unit ball in $\mathbb{C}^m$ with respect to some norm. Similarly, let $\Omega_{\tilde{\mathbf{A}}}$ be the ball in $\mathbb{C}^m$ defined by the m-tuple of matrices $\tilde{\mathbf{A}} = (\tilde{A}_1, \ldots, \tilde{A}_m)$ in $\mathbb{C}^m \otimes \mathcal{M}_n(\mathbb{C})$, that is,

$$\Omega_{\tilde{\mathbf{A}}} \;:=\; \{(z_1, z_2, \ldots, z_m) : \|z_1 \tilde{A}_1 + \cdots + z_m \tilde{A}_m\|_{\mathrm{op}} < 1\}.$$

Again, $\Omega_{\tilde{\mathbf{A}}} = (\mathbb{C}^m, \|\cdot\|_{\tilde{\mathbf{A}}})_1$ with respect to some norm $\|\cdot\|_{\tilde{\mathbf{A}}}$.

Proposition 2.1. *The two domains $\Omega_{\tilde{\mathbf{A}}}$ and $\Omega_{\mathbf{A}}$ are bi-holomorphic via an invertible linear map $R : \mathbb{C}^m \to \mathbb{C}^m$ if and only if $(R \otimes I)(\mathbf{A}) = \tilde{\mathbf{A}}$.*

Proof. Suppose $\Omega_{\tilde{\mathbf{A}}}$ is biholomorphic to $\Omega_{\mathbf{A}}$. Let $e_1, \ldots, e_m$ be the standard basis for $\mathbb{C}^m$. Let $R : \mathbb{C}^m \to \mathbb{C}^m$ be a linear map. Set $w := Rz$, $z \in \mathbb{C}^m$. Since $\Omega_{\tilde{\mathbf{A}}}$ is biholomorphic to $\Omega_{\mathbf{A}}$, via the invertible linear map R, it follows that

$$\begin{aligned}
(R \otimes I)(\mathbf{A}) &= \big(R \otimes I\big)(e_1 \otimes A_1 + \ldots + e_m \otimes A_m) \\
&= R(e_1) \otimes A_1 + \ldots + R(e_m) \otimes A_m \\
&= (R_{11} e_1 + \ldots + R_{m1} e_m) \otimes A_1 + \ldots + (R_{1m} e_1 + \ldots + R_{mm} e_m) \otimes A_m \\
&= e_1 \otimes (R_{11} A_1 + \ldots + R_{1m} A_m) + \ldots + e_m \otimes (R_{m1} A_1 + \ldots + R_{mm} A_m) \\
&= e_1 \otimes \tilde{A}_1 + \ldots + e_m \otimes \tilde{A}_m,
\end{aligned}$$

where $\tilde{A}_i = \sum_{j=1}^m R_{ij}A_j$. This shows that $(R \otimes I)(\mathbf{A}) = \tilde{\mathbf{A}}$.

Conversely, assume that $(R \otimes I)(\mathbf{A}) = \tilde{\mathbf{A}}$ for some invertible linear map $R : \mathbb{C}^m \to \mathbb{C}^m$, that is, $\tilde{A}_i = \sum_{j=1}^m R_{ij}A_j$. Now, $\sum_{i=1}^m z_i \tilde{A}_i = \sum_{i=1}^m z_i \sum_{j=1}^m R_{ij}A_j = \sum_{j=1}^m (\sum_{i=1}^m R_{ij}z_i)A_j = \sum_{j=1}^m w_j A_j$, where $\sum_{i=1}^m R_{ij}z_i = w_j$. Since R is invertible, it follows that $\Omega_{\tilde{\mathbf{A}}}$ is bi-holomorphic to $\Omega_{\mathbf{A}}$ via the linear map R. $\square$

This Proposition prompts the following Definition.

Definition 2.2. The m-tuple of matrices $\mathbf{A} = (A_1, \dots, A_m)$ is equivalent to another m-tuple of matrices $\tilde{\mathbf{A}} = (\tilde{A}_1, \dots, \tilde{A}_m)$ if there exist a invertible linear map $R : \mathbb{C}^m \to \mathbb{C}^m$ such that $(R \otimes I)(\mathbf{A}) = \tilde{\mathbf{A}}$. Thus $\mathbf{A}$ and $\tilde{\mathbf{A}}$ belong to the same equivalence class if and only if $\tilde{\mathbf{A}}_i$ is in the span of $\{A_1, \dots, A_m\}$ for each i.

2.1.1 Examples

Example 2.3. Let $\mathbb{D}^2 = \{(z_1, z_2) : \max\{|z_1|, |z_2|\} < 1\}$. Then $\mathbb{D}^2$ is of the form $\Omega_{\mathbf{A}}$, where $\mathbf{A} = \left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)\right)$.

Pick a, b, c, d in $\mathbb{C}$ with the property that $\det \left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right) \neq 0$. Then the pair of 2×2 matrices $\tilde{\mathbf{A}} = \left(\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right), \left(\begin{smallmatrix} c & 0 \\ 0 & d \end{smallmatrix}\right)\right)$ defines a domain $\Omega_{\tilde{\mathbf{A}}}$ in $\mathbb{C}^2$ bi-holomorphic to $\Omega_{\mathbf{A}}$ via the linear map $R = \left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)$.

Example 2.4. The Euclidean ball $\mathbb{B}^2 = \{(z_1, z_2) : |z_1|^2 + |z_2|^2 < 1\}$ is determined by the pair $\mathbf{A} = \left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)\right)$, which is equivalent to $\tilde{\mathbf{A}} = \left(\left(\begin{smallmatrix} a & c \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} b & d \\ 0 & 0 \end{smallmatrix}\right)\right)$ for any choice of a, b, c and d in $\mathbb{C}$ with $\det \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \neq 0$.

The biholomorphic equivalent copy of the ball $\mathbb{B}^2$ is the ellipsoid:

$$\Omega_{\tilde{\mathbf{A}}} = \{(z_1, z_2) : |(a + b)z_1|^2 + |(c + d)z_2|^2 < 1\}.$$

Example 2.5. Let $\Omega_{\mathbf{A}} = \{(z_1, z_2) : |z_1|^2 + |z_2| < 1\}$, where $\mathbf{A} = \left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)\right)$. The domain $\Omega_{\mathbf{A}}$ is biholomorphic $\Omega_{\tilde{\mathbf{A}}}$ for any pair

$\tilde{\mathbf{A}} = \left(\left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right), \left(\begin{smallmatrix} c & d \\ 0 & c \end{smallmatrix}\right)\right)$, $a, b, c, d \in \mathbb{C}$ with $\det \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \neq 0$.

Corollary 2.6. *A domain $\Omega_{\mathbf{A}}$ in $\mathbb{C}^2$ is bi-holomorphic to $\Omega_{\tilde{\mathbf{A}}}$, where $\tilde{A}_1 = p \left(\begin{smallmatrix} d_1 & 0 \\ 0 & d_2 \end{smallmatrix}\right) + q \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, $\tilde{A}_2 = r \left(\begin{smallmatrix} d_1 & 0 \\ 0 & d_2 \end{smallmatrix}\right) + s \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and the equivalence is implemented via the linear map $R = \left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right)$, which is assumed to be invertible.*

Proof. Let $\Omega_{\mathbf{A}}$ be a domain in $\mathbb{C}^2$ determined by some pair of 2×2 matrices, say $\mathbf{A} = (A_1, A_2)$. Clearly, if U, V are unitaries on $\mathbb{C}^2$, then the pair (UA_1V, UA_2V) determines the same set $\Omega_{\mathbf{A}}$. So, we may assume without loss of generality that $\mathbf{A}$ is of the form $\left(\left(\begin{smallmatrix} d_1 & 0 \\ 0 & d_2 \end{smallmatrix}\right), \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right)$. The proof is completed by appealing to Proposition 2.1. $\square$

As a consequence of the above corollary, we will prove the following corollary.

Corollary 2.7. *Let $\mathbb{A}_1$ be of the form $\begin{pmatrix} 1 & 0 \\ 0 & d_2 \end{pmatrix}$ or $\begin{pmatrix} d_1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\mathbb{A}_2$ be of the form $\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}, \begin{pmatrix} 1 & b \\ c & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & b \\ c & 1 \end{pmatrix}$ with one of b or c positive real. Any domain of the form $\Omega_{\mathbf{A}}$ in $\mathbb{C}^2$ is bi-holomorphically equivalent to $\Omega_{\mathbb{A}}$.*

Proof. Since d_1 and d_2 are not simultaneously zero, we let $p = \frac{1}{d_1}$ or $p = \frac{1}{d_2}$. If we choose $q = 0$, then we have $\mathbb{A}_1 = \begin{pmatrix} 1 & 0 \\ 0 & d_2 \end{pmatrix}, \begin{pmatrix} d_1 & 0 \\ 0 & 1 \end{pmatrix}$.

Now, suppose $d_2 \neq 0$. Choose $r = -\frac{d}{d_2}s$ then there are two possibilities.

(i) If $\det \begin{pmatrix} d_1 & a \\ d_2 & d \end{pmatrix} = 0$, then we will get $\mathbb{A}_2 = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$.

(ii) If $\det \begin{pmatrix} d_1 & a \\ d_2 & d \end{pmatrix} \neq 0$, that is, $s = \frac{1}{d_1 d - d_2 a}$, then $A_2 = \begin{pmatrix} 1 & b \\ c & 0 \end{pmatrix}$.

Similarly, if we assume $d_1 \neq 0$, then we may assume $A_2 = \begin{pmatrix} 0 & b \\ c & 1 \end{pmatrix}$.

If we conjugate $\mathbb{A}_1, \mathbb{A}_2$ by a diagonal unitary $U = \begin{pmatrix} \exp(i\theta) & 0 \\ 0 & \exp(i\phi) \end{pmatrix}$, then we may assume one of b or c is positive real. □

Example 2.8. The upper triangular matrices in the unit ball of $\mathcal{M}_2$ corresponds to $\mathbf{A} = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right)$, that is, $\Omega_{\mathbf{A}} = \{(z_1, z_2, z_3) : \| \begin{pmatrix} z_1 & z_2 \\ 0 & z_3 \end{pmatrix} \| < 1\}$.

This domain is biholomorphic to $\Omega_{\tilde{\mathbf{A}}}$, where $\tilde{\mathbf{A}} = \left(\begin{pmatrix} a_1 & b_1 \\ 0 & c_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ 0 & c_2 \end{pmatrix}, \begin{pmatrix} a_3 & b_3 \\ 0 & c_3 \end{pmatrix}\right)$ for any choice of $a_i, b_i, c_i \in \mathbb{C}$, $i = 1, 2, 3$, with $\det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \neq 0$.

Example 2.9. The unit ball in $(\mathcal{M}_2, \| \cdot \|_{\text{op}})$ corresponds to the choice:

$$\mathbf{A} = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right).$$

Pick a_i, b_i, c_i, d_i, $i = 1, 2, 3, 4$, in $\mathbb{C}$ such that the determinant of $R = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{pmatrix}$ is not zero and let

$$\tilde{\mathbf{A}} = \left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}, \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix}, \begin{pmatrix} a_4 & b_4 \\ c_4 & d_4 \end{pmatrix}\right).$$

Then the unit ball in $(\mathcal{M}_2, \| \cdot \|_{\text{op}})$ is biholomorphic to $\Omega_{\tilde{\mathbf{A}}}$ via the linear map induced by R on $\mathbb{C}^4$.

2.2 Carath*é*odory norm and contractive Homomorphisms

We recall the definition of Carathéodory norm from Jarnicki and Pflug(cf. [19]).

Definition 2.10. For any $\mathbf{v} \in \mathbb{C}^m$ and Ω a domain in $\mathbb{C}^m$, the Carathéodory norm $\mathcal{C}_{\Omega,w}(\mathbf{v})$ of the vector $\mathbf{v}$ at $w \in \Omega$ is defined to be the extremal quantity

$$\sup_f \{|f'(w)\mathbf{v}| : f \in \mathrm{Hol}(\Omega, \mathbb{D}), f(w) = 0\}.$$

Let $\mathbb{B}$ be the open unit ball in $\mathbb{C}^m$ with respect to some norm, say $\|\cdot\|_{\mathbb{B}}$ in $\mathbb{C}^m$. Thus $\mathbb{B}$ is the open set

$$(\mathbb{C}^m, \|\cdot\|_{\mathbb{B}})_1 = \{(z_1, \ldots, z_m) \in \mathbb{C}^m : \|(z_1, \ldots, z_m)\|_{\mathbb{B}} < 1\}.$$

Proposition 2.11. *For any holomorphic function $f : \mathbb{B} \to \mathbb{D}$ with $f(0) = 0$ and a vector $\mathbf{v} \in \mathbb{C}^m$, we have*

$$|\sum_{i=1}^m (\partial_i f(0)) v_i| \le \|\mathbf{v}\|_{\mathbb{B}}.$$

Proof. Let $g_{\mathbf{v}} : \mathbb{D} \to \mathbb{B}$ be the holomorphic function defined by

$$g_{\mathbf{v}}(\lambda) = \lambda \frac{\mathbf{v}}{\|\mathbf{v}\|_{\mathbb{B}}}, \lambda \in \mathbb{D}.$$

The Schwartz Lemma for the unit disc now applies to the function $f \circ g_{\mathbf{v}}$ and gives

$$1 \ge |(f \circ g_{\mathbf{v}})'(0)| = |f'(g_{\mathbf{v}}(0)) g_{\mathbf{v}}'(0)| = |f'(0) \frac{\mathbf{v}}{\|\mathbf{v}\|_{\mathbb{B}}}| = \frac{|f'(0)\mathbf{v}|}{\|\mathbf{v}\|_{\mathbb{B}}}$$

completing the proof. □

Let $\Omega \subset \mathbb{C}^m$ be an open bounded and connected set, $w \in \Omega$. Let

$$\mathcal{D}_{\Omega,w} = \{f'(w) : f \in \mathrm{Hol}(\Omega, \mathbb{D}), f(w) = 0\} \subseteq \mathbb{C}^m.$$

The Proposition merely says that $\mathcal{D}_{\mathbb{B},0}$ is a subset of the dual unit ball $(\mathbb{C}^m, \|\cdot\|_{\mathbb{B}}^*)_1$.

Corollary 2.12. *The set $\mathcal{D}_{\mathbb{B},0}$ is the dual unit ball $(\mathbb{C}^m, \|\cdot\|_{\mathbb{B}}^*)_1$.*

Proof. Clearly, any $l \in (\mathbb{C}^m, \|\cdot\|_{\mathbb{B}}^*)_1$ defines a holomorphic function $l : \mathbb{B} \to \mathbb{D}$ with $l(0) = 0$ and $l'(0) = l$. □

The set $\mathcal{D}_{\Omega,w}$ is the unit ball with respect to some norm(cf. [30, Proposition 3.1] [7, Theorem 1.1]). Except when Ω is the ball with respect to some norm and $w = 0$, describing the set $\mathcal{D}_{\Omega,w}$ appears to be a hard problem.

The set $\mathcal{D}_{\Omega,w}$ is determined by calculating the Carathéodory norm for the domain Ω. It is the unit ball with respect to the norm dual to the Carathéodory norm. The explicit form of the Carathéodory norm is known, for instance, in the case of the annulus in $\mathbb{C}$ (cf. [19]).

Let $f : \Omega \to \tilde{\Omega}$ be a holomorphic map. Define the push forward $f_*(\mathbf{v})$ of a vector $\mathbf{v}$ under the function f to be the vector $f'(w)\mathbf{v}$.

Lemma 2.13. $\mathcal{C}_{\tilde{\Omega},f(w)}(f_*(\mathbf{v})) \le \mathcal{C}_{\Omega,w}(\mathbf{v})$.

Proof. The proof is straightforward:

$$\begin{aligned}\mathcal{C}_{\tilde{\Omega},f(w)}(f_*(\mathbf{v})) &= \sup\{|g'(f(w))f_*(\mathbf{v})| : g \in \mathrm{Hol}(\tilde{\Omega},\mathbb{D}), g(f(w)) = 0\}\\ &= \sup\{|g'(f(w))f'(w)(\mathbf{v})| : g \in \mathrm{Hol}(\tilde{\Omega},\mathbb{D}), g(f(w)) = 0\}\\ &= \sup\{|(g\circ f)'(w)\mathbf{v}| : g \in \mathrm{Hol}(\tilde{\Omega},\mathbb{D}), g(f(w)) = 0\}\\ &\le \sup\{|h'(w)\mathbf{v}| : h \in \mathrm{Hol}(\Omega,\mathbb{D}), h(w) = 0\}.\end{aligned}$$

□

Corollary 2.14. *Suppose* $\mathbf{v} \in \mathbb{C}^m$ *with* $\mathcal{C}_{\Omega,w}(\mathbf{v}) \le 1$. *Then for any holomorphic function* $F : \Omega \to (\mathcal{M}_k)_1$ *with* $F(w) = 0$, *we have* $\mathcal{C}_{(\mathcal{M}_k)_1,0}(F_*(\mathbf{v})) \le \mathcal{C}_{\Omega,w}(\mathbf{v}) \le 1$.

If we pick $\mathbf{v} = (v_1, \dots, v_m) \in \mathbb{C}^m$ with $\mathcal{C}_{\Omega,w}(\mathbf{v}) \le 1$, then the commuting tuple

$$N(\mathbf{v}, w) := \left(\begin{pmatrix} w_1 & v_1 \\ 0 & w_1\end{pmatrix}, \cdots, \begin{pmatrix} w_m & v_m \\ 0 & w_m\end{pmatrix}\right)$$

defines a contractive homomorphism of the algebra $\mathcal{O}(\Omega)$. This homomorphism is then completely contractive. To prove this, notice that the induced homomorphism ρ is given by the formula

$$\rho_{\mathbf{v}}(f) = \begin{pmatrix} f(w) & f'(w)\mathbf{v} \\ 0 & f(w)\end{pmatrix}, f \in \mathcal{O}(\Omega).$$

We may assume $f(w) = 0$ without loss of generality (cf. [23, Lemma 3.3] [30, Lemma 4.1]). Hence contractivity of $\rho_{\mathbf{v}}$ amounts to

$$\|\rho_{\mathbf{v}}\| = \sup_f\{|f'(w)\mathbf{v}| : f \in \mathrm{Hol}(\Omega,\mathbb{D}), f(w) = 0\} = \mathcal{C}_{\Omega,w}(\mathbf{v}) \le 1.$$

Let $F : \Omega \to \mathcal{M}_k$ be a holomorphic function. Now, we have

$$\rho^{(k)}_{\mathbf{v}}(F) := (\rho_{\mathbf{v}}(F_{ij})) = \begin{pmatrix} F(w) & F'(w)\mathbf{v} \\ 0 & F(w)\end{pmatrix}.$$

We may again assume, without loss of generality, that $F(w) = 0$ (cf. [23, Lemma 3.3]). Hence we have (by the norm decreasing property of the Carathéodory norm) that

$$\begin{aligned}\|\rho^{(k)}_{\mathbf{v}}\| &= \sup_F\{\|F'(w)\mathbf{v}\|_{\mathrm{op}} : F \in \mathrm{Hol}(\Omega, (\mathcal{M}_k)_1), F(w) = 0\}\\ &= \sup_F\{\mathcal{C}_{(\mathcal{M}_k)_1,0}(F_*(\mathbf{v})) : F \in \mathrm{Hol}(\Omega, (\mathcal{M}_k)_1), F(w) = 0\}\\ &\le \mathcal{C}_{\Omega,w}(\mathbf{v}).\end{aligned}$$

This shows that $\|\rho^{(k)}_{\mathbf{v}}\| \le 1$ whenever $\|\rho_{\mathbf{v}}\| = \mathcal{C}_{\Omega,w}(\mathbf{v}) \le 1$. Here we have made essential use of the two properties: (i) the Carathéodory norm decreases under holomorphic maps and (ii) $\mathcal{C}_{(\mathcal{M}_k)_1,0} = \|\cdot\|_{\mathrm{op}}$.

Before we proceed any further, we note that the set

$$\mathcal{D}^{(k)}_{\Omega,w} := \{DF(w) : F \in \mathrm{Hol}(\Omega, (\mathcal{M}_k)_1), F(w) = 0\} \subseteq \mathbb{C}^m \otimes \mathcal{M}_k$$

is the unit ball in $\mathbb{C}^m \otimes \mathcal{M}_k$ with respect to some norm, say, $\|\cdot\|^*_k$ (cf. [30]). Thus contractivity of $\rho^{(k)}_{\mathbf{v}}$, in this case, is the same as the contractivity of the linear map

$$L^{(k)}_{N(\mathbf{v},w)} : (\mathbb{C}^m \otimes \mathcal{M}_k, \|\cdot\|^*_k) \to (\mathcal{M}_k, \|\cdot\|_{\mathrm{op}})$$

given by the formula $L^{(k)}_{N(\mathbf{v},w)}(\Theta) = v_1\Theta_1 + \cdots + v_m\Theta_m$, where $\Theta = (\Theta_1, \ldots, \Theta_m) \in \mathbb{C}^m \otimes \mathcal{M}_k$. We have shown that $L^{(k)}_{N(\mathbf{v},w)}$ is contractive for all $k > 1$, without knowing anything about the norm $(\mathbb{C}^m \otimes \mathcal{M}_k, \|\cdot\|^*_k)$, as long as we know it is contractive for $k = 1$.

What happens if we pick $V = (V_1, \ldots, V_m)$ in $\mathbb{C}^m \otimes \mathcal{M}_{p,q}$ and consider the homomorphism induced by the commuting tuple of matrices:

$$N(V, w) := \left(\begin{pmatrix} w_1 I_p & V_1 \\ 0 & w_1 I_q \end{pmatrix}, \cdots, \begin{pmatrix} w_m I_p & V_m \\ 0 & w_m I_q \end{pmatrix} \right).$$

As before, the contractivity of the induced homomorphism is the requirement that

$$\sup\{\| \begin{pmatrix} f(w) & f'(w)V \\ 0 & f(w) \end{pmatrix} \| : f \in \mathrm{Hol}(\Omega, \mathbb{D}), f(w) = 0\} \le 1,$$

where $f'(w)V = (\partial_1 f)(w)V_1 + \cdots + (\partial_m f)(w)V_m$. Again we assume, without loss of generality, that $f(w) = 0$. Therefore the contractivity of ρ_V is equivalent to the contractivity of the linear operator

$$L_{N(V,w)} : (\mathbb{C}^m, \mathcal{C}_{\Omega,w})^* \to (\mathcal{M}_{p,q}, \|\cdot\|_{\mathrm{op}}),$$

where $L_{N(V,w)}(\theta) = V_1\theta_1 + \cdots + V_m\theta_m$ for $\theta \in \mathbb{C}^m$. For a holomorphic function $F : \Omega \to \mathcal{M}_k$, we have

$$\rho^{(k)}_V(F) := (\rho_V(F_{ij})) = \begin{pmatrix} F(w)\otimes I & F'(w)V \\ 0 & F(w)\otimes I \end{pmatrix},$$

where $F'(w)V = (\partial_1 F)(w) \otimes V_1 + \cdots + (\partial_m F)(w) \otimes V_m$.

For one final time, assume $F(w) = 0$, without loss of generality (cf. [23, Lemma 3.3]). Hence

$$\|\rho^{(k)}_V\| := \sup_F \{\|F'(w)V\|_{\mathrm{op}} : F \in \mathrm{Hol}(\Omega, (\mathcal{M}_k)_1), F(w) = 0\}.$$

This is the norm of the linear operator

$$L^{(k)}_{N(V,w)} : (\mathbb{C}^m \otimes \mathcal{M}_k, \|\cdot\|^*_k) \to (\mathcal{M}_k \otimes \mathcal{M}_{p,q}, \|\cdot\|_{\mathrm{op}})$$

given by the formula $L^{(k)}_{N(V,w)}(\Theta) = \Theta_1 \otimes V_1 + \cdots + \Theta_m \otimes V_m$ for $\Theta = (\Theta_1, \ldots, \Theta_m) \in \mathbb{C}^m \otimes \mathcal{M}_k$. Clearly $L^{(1)}_{N(V,w)} = L_{N(V,w)}$, which we have already encountered. We will attempt to determine whether $\|L_{N(V,w)}\| \le 1$ implies that $\|L^{(k)}_{N(V,w)}\| \le 1$ for $k > 1$.

When V is in $\mathbb{C}^m \otimes \mathbb{C}$, this is easily done as we have seen, by using the two basic properties of the Carathéodory norm listed above. However, in general, it is the following question: Given that

$$\|\theta_1 \otimes V_1 + \cdots + \theta_m \otimes V_m\| \leq 1$$

for $\theta = (\theta_1, \ldots, \theta_m) \in \mathcal{D}_{\Omega,w}$, does it follow that

$$\|\Theta_1 \otimes V_1 + \cdots + \Theta_m \otimes V_m\| \leq 1$$

for $\Theta = (\Theta_1, \ldots, \Theta_m) \in \mathcal{D}^{(k)}_{\Omega,w}$ for all $k > 1$?

2.2.1 Invariance of $L^{(k)}_{N(V,w)}, k \geq 1$ under bi-holomorphic maps

Let $\varphi : \widetilde{\Omega} \to \Omega$ be the bi-holomorphic map with $\varphi(w) = z$. The linear map $D\varphi(w) : (\mathbb{C}^m, \mathcal{C}_{\widetilde{\Omega},w}) \to (\mathbb{C}^m, \mathcal{C}_{\Omega,z})$ is a contraction by definition. Since φ is invertible, $D\varphi^{-1}(z) : (\mathbb{C}^m, \mathcal{C}_{\Omega,z}) \to (\mathbb{C}^m, \mathcal{C}_{\widetilde{\Omega},w})$ is also a contraction. However, since $D\varphi^{-1}(z) = D\varphi(w)^{-1}$, it follows that $D\varphi(w)$ must be an isometry. The map $F \to F \circ \varphi$ is a bijection from $\mathrm{Hol}_z(\Omega, (\mathcal{M}_k)_1)$ onto $\mathrm{Hol}_w(\widetilde{\Omega}, (\mathcal{M}_k)_1)$. Therefore for each w in $\widetilde{\Omega}$ and a bi-holomorphic φ from $\widetilde{\Omega}$ to Ω such that $\varphi(w) = z$ we have

$$\{DF(z) : F \in \mathrm{Hol}(\Omega, (\mathcal{M}_k)_1), F(z) = 0\} = \{DF\circ\varphi(w) : F \in \mathrm{Hol}(\widetilde{\Omega}, (\mathcal{M}_k)_1), F\circ\varphi(w) = 0\}.$$

Set $D\varphi(w) := \begin{pmatrix} \varphi_{11} & \varphi_{12} & \cdots\varphi_{1m} \\ \vdots & \vdots & \vdots \\ \varphi_{m1} & \varphi_{m2} & \cdots\varphi_{mm} \end{pmatrix}$ and $DF(z) = (A_1, \ldots, A_m)$. By the chain rule we have

$$DF(\varphi(w))D\varphi(w) = (\varphi_{11}A_1 + \cdots + \varphi_{m1}A_m, \ldots, \varphi_{1m}A_1 + \cdots + \varphi_{mm}A_m).$$

Thus $D\varphi(w) \otimes I_k$ maps $(\mathbb{C}^m \otimes \mathcal{M}_k, \|\cdot\|^*_{\Omega,k})$ onto $(\mathbb{C}^m \otimes \mathcal{M}_k, \|\cdot\|^*_{\widetilde{\Omega},k})$. Since $D\varphi(w)$ is an isometry, it follows that $D\varphi(w) \otimes I_k$ is an isometry with respect to the two norms $\|\cdot\|^*_{\Omega,k}$ and $\|\cdot\|^*_{\widetilde{\Omega},k}$. Let $L_V : (\mathbb{C}^m, \mathcal{C}_{\widetilde{\Omega},w})^* \to (\mathcal{M}_{p,q}, \|\cdot\|_{\mathrm{op}})$ be the linear map induced by $V = (V_1^{\mathrm{t}}, \ldots, V_m^{\mathrm{t}})^{\mathrm{t}}$, where $\mathcal{C}_{\widetilde{\Omega},w}$ is the Carathéodory norm of $\widetilde{\Omega}$ at a fixed but arbitrary $w \in \widetilde{\Omega}$. The linear map L_V (which depends on the point w in Ω) is contractive if and only if $L_{D\varphi(w).V}$ is contractive, where $D\varphi(w).V = (D\varphi(w) \otimes I)(V)$.

Lemma 2.15.

$$\| L_V \|_{(\mathbb{C}^m, \mathcal{C}_{\widetilde{\Omega},w})^* \to (\mathcal{M}_{p,q}, \|\cdot\|_{\mathrm{op}})} \leq 1$$

if and only if

$$\| L_{(D\varphi(w)\otimes I)(V)} \|_{(\mathbb{C}^m, \mathcal{C}_{\Omega,z})^* \to (\mathcal{M}_{p,q}, \|\cdot\|_{\mathrm{op}})} \leq 1.$$

Proof. Let $q : \widetilde{\Omega} \longmapsto \mathbb{D}$ be a holomorphic map with $q(w) = 0$ and $\|q\|_{\infty,\mathbb{D}} \leq 1$, then $Dq(w)$ is in $\mathcal{D}^1_{\widetilde{\Omega},w}$. Thus $\| L_V \|_{(\mathbb{C}^m, \mathcal{C}_{\widetilde{\Omega},w})^* \to (\mathcal{M}_{p,q}, \|\cdot\|_{\rm op})} \leq 1$ is equivalent to $|\langle Dq(w), V\rangle| \leq 1$. Similarly, if $p : \Omega \longmapsto \mathbb{D}$ is a holomorphic map with $p(z) = 0$, then $Dp(z)$ is also in $\mathcal{D}^1_{\Omega,z}$. Consider the commutative diagram

$$\begin{array}{ccc} \widetilde{\Omega} & \xrightarrow{\varphi} & \Omega \\ & \searrow_q & \downarrow_p \\ & & \mathbb{D}. \end{array}$$

Thus $q = p \circ \varphi$. We therefore have $Dq(w) = Dp(\varphi(w))D\varphi(w)$. Hence, we have

$$\begin{aligned} \|L_V\| &= \sup_{Dq(w)\in \mathcal{D}^1_{\widetilde{\Omega},w}} \||\langle Dq(w), V\rangle| \\ &= \sup_{Dq(w)\in \mathcal{D}^1_{\widetilde{\Omega},w}} |\langle Dp(\varphi(w))D\varphi(w), V\rangle| \\ &= \sup_{Dp(z)\in \mathcal{D}^1_{\Omega,z}} |\langle (\partial_1 p(\varphi(w)), \ldots, \partial_m p(\varphi(w))) \begin{pmatrix} \varphi_{11} & \varphi_{12} & \cdots\varphi_{1m} \\ \vdots & \vdots & \vdots \\ \varphi_{m1} & \varphi_{m2} & \cdots\varphi_{mm} \end{pmatrix}, V\rangle| \\ &= \sup_{Dp(z)\in \mathcal{D}^1_{\Omega,z}} |\langle Dp(\varphi(w)), (D\varphi(w) \otimes I)(V)\rangle| \end{aligned}$$

completing the proof. $\square$

The following corollary is a consequence of Lemma 2.15.

Corollary 2.16.

$$\| L_V \|_{(\mathbb{C}^m, \|\cdot\|_{\widetilde{\Omega}})^* \to (\mathcal{M}_{p,q}, \|\cdot\|_{\rm op})} \leq 1$$

if and only if

$$\| L_{(D\varphi(w)\otimes I)(V)} \|_{(\mathbb{C}^m, \|\cdot\|_{\Omega})^* \to (\mathcal{M}_{p,q}, \|\cdot\|_{\rm op})} \leq 1.$$

Proof. We know that $\mathcal{C}_{\widetilde{\Omega},0}(v) = \|v\|_{\widetilde{\Omega}}$ and $\mathcal{C}_{\Omega,0}(v) = \|v\|_{\Omega}$. Using Lemma (2.15) we get the desired result. $\square$

We have indicated that contractivity of $\rho_V|^{(k)}$ is the same as contractivity of the linear map

$$L^{(k)}_{N(V,w)} : (\mathbb{C}^m \otimes \mathcal{M}_k, \|\cdot\|^*_k) \to (\mathcal{M}_k \otimes \mathcal{M}_{p,q}, \|\cdot\|_{\rm op}).$$

We recall from [18, Theorem 2] that for Z, W in the matrix ball $(\mathcal{M}_k)_1$ and $\mathbf{u} \in \mathbb{C}^{k\times k}$, we have

$$D\psi_W(Z) \cdot \mathbf{u} = (I - WW^*)^{\frac{1}{2}}(I - ZW^*)^{-1}\mathbf{u}(I - W^*Z)^{-1}(I - W^*W)^{\frac{1}{2}}.$$

The following Proposition characterizes the contractivity of $\rho^{(k)}$ (cf. [23, Lemma 3.3]).

Proposition 2.17. $\|\rho_V^{(k)}(P_{\mathbf{A}})\| \le 1$ *if and only if*

$$\sup_{z\in\Omega_{\mathbf{A}}} \|((I-P_{\mathbf{A}}(z)P_{\mathbf{A}}(z)^*)^{-\frac{1}{2}}\otimes I_n)(A_1\otimes V_1+\cdots+A_m\otimes V_m)((I-P_{\mathbf{A}}(z)^*P_{\mathbf{A}}(z))^{-\frac{1}{2}}\otimes I_n)\| \le 1.$$

The homomorphism $\rho_V^{(k)}$ is contractive if and only if $\rho_{(D\varphi(w)\otimes I)(V)}^{(k)}$ is contractive for all w in $\widetilde{\Omega}$.

Proposition 2.18.
$$\|L_{N(V,w)}^{(k)}\|_{(\mathbb{C}^m\otimes\mathcal{M}_k,\|\cdot\|_{\widetilde{\Omega},k}^*)\to(\mathcal{M}_k\otimes\mathcal{M}_{p,q},\|\cdot\|_{\mathrm{op}})} \le 1$$
if and only if
$$\|L_{N((D\varphi(w)\otimes I)(V),z)}^{(k)}\|_{(\mathbb{C}^m\otimes\mathcal{M}_k,\|\cdot\|_{\Omega,k}^*)\to(\mathcal{M}_k\otimes\mathcal{M}_{p,q},\|\cdot\|_{\mathrm{op}})} \le 1.$$

Proof. Let $P:\widetilde{\Omega}\to(\mathcal{M}_k(\mathbb{C}))_1$ be a matrix valued polynomial on $\widetilde{\Omega}$ which is of the form $P(w)=w_1P_1+\cdots+w_mP_m$. Then it is easy to see that $DP(w)$ is in $\mathcal{D}_{\widetilde{\Omega},w}^k$. By Proposition 2.17 we have $\|L_{N(V,w)}^{(k)}\|_{(\mathbb{C}^m\otimes\mathcal{M}_k,\|\cdot\|_{\widetilde{\Omega},k}^*)\to(\mathcal{M}_k\otimes\mathcal{M}_{p,q},\|\cdot\|_{\mathrm{op}})} \le 1$ if and only if

$$\|(I-P(w)P(w)^*)^{-\frac{1}{2}}(P_1\otimes V_1+\cdots+P_m\otimes V_m)(I-P(w)^*P(w))^{-\frac{1}{2}}\| \le 1.$$

Similarly, if $Q:\Omega\to(\mathcal{M}_k(\mathbb{C}))_1$ is a matrix valued polynomial on Ω which is of the form $Q(z)=z_1Q_1+\cdots+z_mQ_m$, then $DQ(z)$ is in $\mathcal{D}_{\Omega,z}^k$. Consider the commutative diagram

$$\begin{array}{ccc} \widetilde{\Omega} & \xrightarrow{\varphi} & \Omega \\ & \searrow_P & \downarrow_Q \\ & & (\mathcal{M}_k)_1. \end{array}$$

Thus $P=Q\circ\varphi$. Hence we have $DP(w)=DQ(\varphi(w))D\varphi(w)$. Hence we have

$$\begin{aligned} &\|L_{N(V,w)}^{(k)}\|_{(\mathbb{C}^m\otimes\mathcal{M}_k,\|\cdot\|_{\widetilde{\Omega},k}^*)\to(\mathcal{M}_k\otimes\mathcal{M}_{p,q},\|\cdot\|_{\mathrm{op}})} \\ &\quad= (I-P(w)P(w)^*)^{-\frac{1}{2}}(P_1\otimes V_1+\cdots+P_m\otimes V_m)(I-P(w)^*P(w))^{-\frac{1}{2}} \\ &\quad= r\big((\varphi_{11}Q_1+\cdots+\varphi_{m1}Q_m)\otimes V_1+\cdots+(\varphi_{1m}Q_1+\cdots+\varphi_{mm}Q_m)\otimes V_m\big)s \\ &\quad= r(Q_1\otimes(\varphi_{11}V_1+\cdots+\varphi_{1m}V_m)+\cdots+Q_m\otimes(\varphi_{m1}V_1+\cdots+\varphi_{mm}V_m))s \\ &\quad= \|L_{N((D\varphi(w)\otimes I)(V),z)}^{(k)}\|_{(\mathbb{C}^m\otimes\mathcal{M}_k,\|\cdot\|_{\Omega,k}^*)\to(\mathcal{M}_k\otimes\mathcal{M}_{p,q},\|\cdot\|_{\mathrm{op}})}, \end{aligned}$$

where $r=(I-Q(\varphi(w))Q(\varphi(w))^*)^{-\frac{1}{2}}$ and $s=(I-Q(\varphi(w))^*Q(\varphi(w)))^{-\frac{1}{2}}$. This completes the proof. □

We have seen that if $\widetilde{\Omega}$ and Ω are bi-holomorphic then $\|\rho_V^{(k)}\| = \|\rho_{(D\varphi(w)\otimes I)(V)}^{(k)}\|$. Evidently we have the following corollary.

Corollary 2.19. *Suppose $\varphi : \widetilde{\Omega} \to \Omega$ with $\varphi(0) = 0$, is a bi-holomorphic map. Then every contractive linear map from $(\mathbb{C}^m, \|\cdot\|_\Omega)$ to $\mathcal{M}_n(\mathbb{C})$ is completely contractive if and only if every contractive linear map of $(\mathbb{C}^m, \|\cdot\|_{\tilde{\Omega}})$ to $\mathcal{M}_n(\mathbb{C})$ is completely contractive.*

Proof. From Lemmas 2.15 and Proposition 2.18 it follows that every contractive linear map of $(\mathbb{C}^m, \|\cdot\|^*_{\Omega})$ is completely contractive if and only if every contractive linear map of $(\mathbb{C}^m, \|\cdot\|^*_{\tilde{\Omega}})$ is also completely contractive. This property does not change if we replace a ball with the dual ball completing the proof. □

This means in finding contractive homomorphisms which are not completely contractive, we need not distinguish between balls which are bi-holomorphically equivalent ball (with 0 as a fixed point).

Example 2.20. In Example 2.3 we have seen that $\mathbb{D}^2$ is bi-holomorphic to $\Omega_{\tilde{\mathbf{A}}}$ via the linear map $R = \left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)$, where $\tilde{\mathbf{A}} = \left(\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right), \left(\begin{smallmatrix} c & 0 \\ 0 & d \end{smallmatrix}\right)\right)$. By Corollary 2.19 we see that $\alpha_{\Omega_{\tilde{\mathbf{A}}}} = 1$.

Chapter 3

Contractivity and complete contractivity – some examples

3.1 Dual norm computation

We have discussed the class of domains $\Omega_{\mathbf{A}} = \{(z_1, z_2) : \|z_1 A_1 + z_2 A_2\|_{\mathrm{op}} < 1\}$ in $\mathbb{C}^2$, where $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & d_2 \end{pmatrix}$ or $\begin{pmatrix} d_1 & 0 \\ 0 & 1 \end{pmatrix}$ and A_2 is one of $\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}, \begin{pmatrix} 1 & b \\ c & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & b \\ c & 1 \end{pmatrix}$ with $b \in \mathbb{R}^+$. We have seen that the contractivity of the homomorphism ρ_V is equivalent to the contractivity of the linear map $L_V : (\mathbb{C}^2, \|\cdot\|^*_{\Omega_{\mathbf{A}}}) \to (\mathbb{C}^2, \|\cdot\|_2)$. Thus if we know the norm $\|\cdot\|^*_{\Omega_{\mathbf{A}}}$ dual to the norm $\|\cdot\|_{\Omega_{\mathbf{A}}}$, then we may be able to compute the norm of L_V. However, computing the dual norm $\|\cdot\|^*_{\Omega_{\mathbf{A}}}$ appears to be a hard problem. The ℓ^∞_2 and ℓ^2_2 unit balls are of the form $\Omega_{\mathbf{A}}$, and one knows the duals of these norms. Let X be the two dimensional normed linear space with respect to the norm

$$\|(x, y)\| = \frac{|y| + \sqrt{|y|^2 + 4|x|^2}}{2}, (x, y) \in X.$$

The unit ball with respect to the norm is equal to $\Omega_{\mathbf{A}} = \{(x, y) : |y| + |x|^2 < 1\}$, where $\mathbf{A} = (A_1, A_2)$ with $A_1 = I_2, A_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Therefore $\{I_2, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\}$ forms a basis of X. It is also easy to see that $\{\frac{1}{2} I_2, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\}$ forms a dual basis of X. We recall the definition of annihilator from Rudin [32]. Suppose $\widetilde{Y}$ is a Banach space, M is a subspace of $\widetilde{Y}$ and N is a subspace of $\widetilde{Y}^*$. Neither M nor N are assuned to be closed.

Definition 3.1. The annihilators of $M^\perp$ and ${}^\perp N$ are defined as follows:

$$M^\perp = \{\tilde{y}^* \in \widetilde{Y}^* : \langle \tilde{y}, \tilde{y}^* \rangle = 0 \,\forall\, \tilde{y} \in M\},$$

$${}^\perp N = \{\tilde{y} \in \widetilde{Y} : \langle \tilde{y}, \tilde{y}^* \rangle = 0 \,\forall\, \tilde{y}^* \in N\}.$$

If M is assumed to be a closed subspace of $\widetilde{Y}$, then the quotient $\widetilde{Y}/M$ is also a Banach space, with respect to the quotient norm. The duals of M and of $\widetilde{Y}/M$ can be describe using the annihilator $M^{\perp}$ of M. The following Theorem (cf. [32, Theorem 4.9]) describes this relation explicitly.

Theorem 3.2. *Let M be a closed subspace of a Banach space $\widetilde{Y}$. The Hahn-Banach theorem ensures that each m^* in M^* extends to a linear functional $\tilde{y}^* \in \widetilde{Y}^*$. Define*

$$\sigma m^* = \tilde{y}^* + M^{\perp}.$$

Then σ is an isometric isomorphism of M^* onto $\widetilde{Y}^*/M^{\perp}$.

This Theorem is used in the first computation of the dual of the normed linear space X. We also obtain the dual norm by a direct computation.

Theorem 3.3. *The dual norm of X is given by*

$$\|(\alpha,\beta)\| = \begin{cases} \frac{|\alpha|^2+4|\beta|^2}{4|\beta|} & \text{if } |\beta| \geq \frac{|\alpha|}{2}; \\ |\alpha| & \text{if } |\beta| \leq \frac{|\alpha|}{2}. \end{cases}$$

Proof. (First proof) Let X^* be the dual of X. It is easy to verify that the annihilator of X is spanned by the elements of the form $\{\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)\}$, that is, $X^{\perp} = span\{\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)\}$. The set $\{\frac{1}{2}I_2, \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)\}$ forms a basis of X^*. We extends elements of X^* to linear functional on $(\mathcal{M}_2(\mathbb{C}), \|\cdot\|_{\text{op}})$. From Theorem 3.2, it follows that

$$\|(\alpha,\beta)\|^2 = \inf_{a,b} \left\| \begin{pmatrix} \frac{\alpha}{2}+a & b \\ \beta & \frac{\alpha}{2}-a \end{pmatrix} \right\|_{\text{tr}}^2,$$

where $\|\cdot\|_{\text{tr}}$ denote the trace norm. Now,

$$\begin{aligned} &\inf_{a,b} \left\| \begin{pmatrix} \frac{\alpha}{2}+a & b \\ \beta & \frac{\alpha}{2}-a \end{pmatrix} \right\|_{\text{tr}}^2 \\ &= \inf_{a,b}\{\frac{|\alpha|^2}{2} + 2|a|^2 + |b|^2 + |\beta|^2 + 2\Big|\frac{\alpha^2}{4} - a^2 - b\beta\Big|\} \\ &= \inf_{a,b}\{\frac{|\alpha|^2}{2} + 2|a|^2 + |b|^2 + |\beta|^2 + 2\Big||\frac{|\alpha|^2}{4} - b\beta| - |a|^2\Big|\}. \end{aligned}$$

To compute this infimum, we consider the two cases $\left|\frac{|\alpha|^2}{4} - b\beta\right| \geq |a|^2$ and $|a|^2 \geq \left|\frac{|\alpha|^2}{4} - b\beta\right|$. In either case, we have

$$\begin{aligned} \inf_{a,b} \left\| \begin{pmatrix} \frac{\alpha}{2}+a & b \\ \beta & \frac{\alpha}{2}-a \end{pmatrix} \right\|_{\text{tr}}^2 &= \inf_{b}\{\frac{|\alpha|^2}{2} + |b|^2 + |\beta|^2 + 2\Big|\frac{|\alpha|^2}{4} - b\beta\Big|\} \\ &= \inf_{b}\{\frac{|\alpha|^2}{2} + |b|^2 + |\beta|^2 + 2\Big|\frac{|\alpha|^2}{4} - |b\beta|\Big|\}. \end{aligned} \tag{3.1}$$

Let

$$M = \inf_{b, |b| \leq \frac{|\alpha|^2}{4|\beta|}} \{\frac{|\alpha|^2}{2} + |b|^2 + |\beta|^2 + 2\Big|\frac{|\alpha|^2}{4} - |b\beta|\Big|\}$$

and

$$N = \inf_{b, |b| \geq \frac{|\alpha|^2}{4|\beta|}} \{\frac{|\alpha|^2}{2} + |b|^2 + |\beta|^2 + 2\Big|\frac{|\alpha|^2}{4} - |b\beta|\Big|\}.$$

Now,

$$\begin{aligned} M &= \inf_{b, |b| \leq \frac{|\alpha|^2}{4|\beta|}} \{\frac{|\alpha|^2}{2} + |b|^2 + |\beta|^2 + 2\Big|\frac{|\alpha|^2}{4} - |b\beta|\Big|\} \\ &= \inf_{b, |b| \leq \frac{|\alpha|^2}{4|\beta|}} \{|\alpha|^2 + (|b| - |\beta|)^2\}. \end{aligned} \tag{3.2}$$

If $\frac{|\alpha|^2}{4|\beta|} \geq |\beta|$, then we can take $|b| = |\beta|$ in Equation (3.2) and the infimum is $|\alpha|^2$. If $\frac{|\alpha|^2}{4|\beta|} \leq |\beta|$, then the largest value $|b|$ can take is $\frac{|\alpha|^2}{4|\beta|}$. In this case, $M = (\frac{|\alpha|^2+4|\beta|^2}{4|\beta|})^2$. Therefore,

$$M = \begin{cases} (\frac{|\alpha|^2+4|\beta|^2}{4|\beta|})^2 & \text{if } |\beta| \geq \frac{|\alpha|}{2}; \\ |\alpha|^2 & \text{if } |\beta| \leq \frac{|\alpha|}{2}. \end{cases}$$

$$\begin{aligned} N &= \inf_{b, |b| \geq \frac{|\alpha|^2}{4|\beta|}} \{\frac{|\alpha|^2}{2} + |b|^2 + |\beta|^2 + 2\Big|\frac{|\alpha|^2}{4} - |b\beta|\Big|\} \\ &= \inf_{b, |b| \geq \frac{|\alpha|^2}{4|\beta|}} \{(|b| + |\beta|)^2\} \\ &= (\frac{|\alpha|^2 + 4|\beta|^2}{4|\beta|})^2. \end{aligned} \tag{3.3}$$

If $|\beta| \leq \frac{|\alpha|}{2}$, we have

$$\begin{aligned} \inf_{a,b} \Big\| \begin{pmatrix} \frac{\alpha}{2}+a & b \\ \beta & \frac{\alpha}{2}-a \end{pmatrix} \Big\|_{\mathrm{tr}}^2 &= \min\{|\alpha|^2, (\frac{|\alpha|^2 + 4|\beta|^2}{4|\beta|})^2\} \\ &= |\alpha|^2 \end{aligned}$$

and if $|\beta| \geq \frac{|\alpha|}{2}$, we have

$$\inf_{a,b} \Big\| \begin{pmatrix} \frac{\alpha}{2}+a & b \\ \beta & \frac{\alpha}{2}-a \end{pmatrix} \Big\|_{\mathrm{tr}}^2 = (\frac{|\alpha|^2 + 4|\beta|^2}{4|\beta|})^2.$$

Hence

$$\|(\alpha, \beta)\| = \begin{cases} \frac{|\alpha|^2+4|\beta|^2}{4|\beta|} & \text{if } |\beta| \geq \frac{|\alpha|}{2}; \\ |\alpha| & \text{if } |\beta| \leq \frac{|\alpha|}{2}. \end{cases}$$

This completes the proof.

(Second proof) Let $f_{\alpha,\beta} : X \to \mathbb{C}$ be a linear functional defined by $f_{\alpha,\beta}(x,y) = \alpha x + \beta y$. Now,

$$\begin{aligned}
\|f_{\alpha,\beta}\|^2 &= \sup_{|x|^2+|y|=1} |\alpha x + \beta y|^2 \\
&= \sup_{|x|^2+|y|=1} (|\alpha||x| + |\beta||y|)^2 \\
&= \sup_{|x|^2+|y|=1} (|\alpha||x| + |\beta|(1-|x|^2))^2 \\
&= \sup_{0\le|x|^2\le1} |\alpha x|^2 + |\beta|^2(1-|x|^2)^2 + 2|\alpha||x||\beta|(1-|x|^2).
\end{aligned}$$

Note that the supremum has to be taken with $|x|^2 \le 1-|y|$. Let $g(|x|) = |\alpha||x|+|\beta|(1-|x|^2)$. The derivative of g with respect $|x|$ is equal to $g'(|x|) = |\alpha| - 2|\beta||x|$. Now, $g'(|x|) = 0$ is equivalent to $|x| = \frac{|\alpha|}{2|\beta|}$. Also $g''(|x|) = -2|\beta|$ which is less than zero. Therefore, the supremum of g is attained at $|x| = \frac{|\alpha|}{2|\beta|}$. If $|\beta| \le \frac{|\alpha|}{2}$, then the derivative does not vanish for x with $0 \le |x|^2 \le 1$. In fact, the derivative is positive in the range $0 \le |x|^2 \le 1$ and the supremum is attained at $|x| = 1$. Therefore, we have two cases:

(a).

$$\|(\alpha,\beta)\| = \frac{|\alpha|^2+4|\beta|^2}{4|\beta|} \quad \text{if } |\beta| \ge \frac{|\alpha|}{2}\ .$$

(b).

$$\|(\alpha,\beta)\| = |\alpha| \quad \text{if } |\beta| \le \frac{|\alpha|}{2}\ .$$

This completes the proof.

□

3.2 Contractivity and complete contractivity

Let $\Omega_{\mathbf{A}} = \{(z_1,z_2) : \|z_1A_1 + z_2A_2\|_{\text{op}} < 1\}$ in $\mathbb{C}^2$, where $\mathbf{A} = (A_1, A_2)$ and $A_1 = I_2, A_2 = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. We now describe contractivity of the special class of homomorphisms $\rho_V : \mathcal{O}(\Omega_{\mathbf{A}}) \to \mathcal{M}_3(\mathbb{C})$ induced by a pair of the form $(\left(\begin{smallmatrix} w_1 & \mathbf{v}_1 \\ 0 & w_1 I_2 \end{smallmatrix}\right), \left(\begin{smallmatrix} w_2 & \mathbf{v}_2 \\ 0 & w_2 I_2 \end{smallmatrix}\right))$, where $\mathbf{v}_i \in \mathbb{C}^2$ for $i = 1, 2$. We have seen that $\| \rho_V \|_{\mathcal{O}(\Omega_{\mathbf{A}})\to\mathcal{M}_3(\mathbb{C})} \le 1$ if and only if $\| L_V \|_{(\mathbb{C}^2, \|\cdot\|^*_{\Omega_{\mathbf{A}}})\to(\mathbb{C}^2,\|\cdot\|_2)} \le 1$ if and only if $\|L_V^*\|_{(\mathbb{C}^2,\|\cdot\|_2)\to(\mathbb{C}^2,\|\cdot\|_{\Omega_{\mathbf{A}}})} \le 1$. Let $L_V : (\mathbb{C}^2, \|\cdot\|^*_{\Omega_{\mathbf{A}}}) \to (\mathbb{C}^2, \|\cdot\|_2)$ be the linear map induced by the pair $\mathbf{v}_1$ and $\mathbf{v}_2$. The matrix representing $L_V^* : (\mathbb{C}^2, \|\cdot\|_2) \to (\mathbb{C}^2, \|\cdot\|_{\Omega_{\mathbf{A}}})$ is of the form $\left(\begin{smallmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{smallmatrix}\right), V := \left(\begin{smallmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{smallmatrix}\right)$, where $\mathbf{v}_1 = (v_{11}\ v_{12})$ and $\mathbf{v}_2 = (v_{21}\ v_{22})$. The following theorem provides a characterization of contractivity of the homomorphism ρ_V in terms of V.

Theorem 3.4. *$\| L_V \|_{(\mathbb{C}^2, \|\cdot\|^*_{\Omega_{\mathbf{A}}}) \to (\mathbb{C}^2, \|\cdot\|_2)} \leq 1$ if and only if $\|L_V^*\|_{(\mathbb{C}^2, \|\cdot\|_2) \to (\mathbb{C}^2, \|\cdot\|_{\Omega_{\mathbf{A}}})} \leq 1$ if and only if*

$$\left(\|\mathbf{v}_1\|^2 + \frac{\|\mathbf{v}_2\|^2}{4} + \sqrt{(\|\mathbf{v}_1\|^2 - \frac{\|\mathbf{v}_2\|^2}{4})^2 + |\langle \mathbf{v}_1, \mathbf{v}_2 \rangle|^2} \right)^2 \leq 4\sqrt{(\|\mathbf{v}_1\|^2 - \frac{\|\mathbf{v}_2\|^2}{4})^2 + |\langle \mathbf{v}_1, \mathbf{v}_2 \rangle|^2}.$$

Proof. (First proof) Suppose $\|L_V^*\|_{(\mathbb{C}^2, \|\cdot\|_2) \to (\mathbb{C}^2, \|\cdot\|_{\Omega_{\mathbf{A}}})} \leq 1$. Since the matrix representation of L_V^* is of the form $\left(\begin{smallmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{smallmatrix}\right)$. We have $\left(\begin{smallmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{smallmatrix}\right)\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in (\mathbb{C}^2, \|\cdot\|_{\Omega_{\mathbf{A}}})$. We are interested in the expression

$$\begin{aligned} (v_{11}x + v_{12}y)A_1 + (v_{21}x + v_{22}y)A_2 &= \left(\begin{smallmatrix} v_{11}x+v_{12}y & v_{21}x+v_{22}y \\ 0 & v_{11}x+v_{12}y \end{smallmatrix}\right) \\ &= xA_1^{'} + yA_2^{'}, \end{aligned} \tag{3.4}$$

where $A_1^{'} = \left(\begin{smallmatrix} v_{11} & v_{21} \\ 0 & v_{11} \end{smallmatrix}\right)$ and $A_2^{'} = \left(\begin{smallmatrix} v_{12} & v_{22} \\ 0 & v_{12} \end{smallmatrix}\right)$. Thus $\|L_V^*\|_{(\mathbb{C}^2, \|\cdot\|_2) \to (\mathbb{C}^2, \|\cdot\|_{\Omega_{\mathbf{A}}})} \leq 1$ if and only if $\sup_{|x|^2+|y|^2=1} \|xA_1^{'} + yA_2^{'}\|^2_{\text{op}} \leq 1$.

$$\begin{aligned} \sup_{|x|^2+|y|^2=1} \|xA_1^{'} + yA_2^{'}\|^2 &= \sup_{|x|^2+|y|^2=1} \sup_{\|\alpha\|_2=\|\beta\|_2=1} |\langle (xA_1^{'} + yA_2^{'})\alpha, \beta \rangle|^2 \\ &= \sup_{|x|^2+|y|^2=1} \sup_{\|\alpha\|_2=\|\beta\|_2=1} |x\langle A_1^{'}\alpha, \beta\rangle + y\langle A_2^{'}\alpha, \beta\rangle|^2 \\ &= \sup_{\|\alpha\|_2=\|\beta\|_2=1} (|\langle A_1^{'}\alpha, \beta\rangle|^2 + |\langle A_2^{'}\alpha, \beta\rangle|^2). \end{aligned} \tag{3.5}$$

Now,

$$\langle A_1^{'}\alpha, \beta \rangle = v_{11}\alpha_1\overline{\beta_1} + v_{11}\alpha_2\overline{\beta_2} + v_{21}\alpha_2\overline{\beta_1} = v_{11}\langle \alpha, \beta \rangle + v_{21}\alpha_2\overline{\beta_1}.$$

Similarly we have $\langle A_2^{'}\alpha, \beta \rangle = v_{12}\langle \alpha, \beta \rangle + v_{22}\alpha_2\overline{\beta_1}$. Putting the value of $\langle A_1^{'}\alpha, \beta \rangle$ and $\langle A_2^{'}\alpha, \beta \rangle$ in Equation (3.5) we have

$$\begin{aligned} \sup_{\|\alpha\|_2=\|\beta\|_2=1} (|\langle A_1^{'}\alpha, \beta\rangle|^2 + |\langle A_2^{'}\alpha, \beta\rangle|^2) = \sup_{\|\alpha\|_2=\|\beta\|_2=1} & \|\mathbf{v}_1\|^2 ||\langle \alpha, \beta \rangle|^2 + \|\mathbf{v}_2\|^2 |\alpha_2\overline{\beta_1}|^2 \\ & + 2Re\, \langle \mathbf{v}_1, \mathbf{v}_2 \rangle\, \langle \alpha, \beta \rangle \overline{\alpha_2}\beta_1, \end{aligned}$$

where $\|\mathbf{v}_1\|^2 = |v_{11}|^2 + |v_{12}|^2$,$\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = (v_{11}\bar{v}_{21} + v_{12}\bar{v}_{22})$ and $\|\mathbf{v}_2\|^2 = |\bar{v}_{21}|^2 + |\bar{v}_{22}|^2$. Choosing $\alpha' = (\alpha_1 \exp(i\theta), \alpha_2 \exp(-i\theta)), \beta' = (\beta_1 \exp(i\theta), \beta_2 \exp(-i\theta))$ we have

$$\begin{aligned} & \sup_{\|\alpha\|_2=\|\beta\|_2=1} (|\langle A_1^{'}\alpha, \beta\rangle|^2 + |\langle A_2^{'}\alpha, \beta\rangle|^2) \\ &= \sup_{\|\alpha\|_2=\|\beta\|_2=1} \|\mathbf{v}_1\|^2 ||\langle \alpha, \beta \rangle|^2 + \|\mathbf{v}_2\|^2 |\alpha_2\overline{\beta_1}|^2 + 2|\, \langle \mathbf{v}_1, \mathbf{v}_2 \rangle\, \langle \alpha, \beta \rangle \alpha_2\overline{\beta_1}| \\ &= \sup_{\|\alpha\|_2=\|\beta\|_2=1} \|\mathbf{v}_1\|^2 ||\langle U\alpha, U\beta \rangle|^2 + \|\mathbf{v}_2\|^2 |(U\alpha)_2\overline{(U\beta)_1}|^2 + 2|\, \langle \mathbf{v}_1, \mathbf{v}_2 \rangle\, \langle U\alpha, U\beta \rangle (U\alpha)_2\overline{(U\beta)_1}| \\ &= \sup_{||\beta||=1, U} \|\mathbf{v}_1\|^2 |\langle Ue_2, U\beta \rangle|^2 + \|\mathbf{v}_2\|^2 |(Ue_2)_2\overline{(U\beta)_1}|^2 \\ & \quad +2|\langle \mathbf{v}_1, \mathbf{v}_2 \rangle \langle Ue_2, U\beta \rangle (Ue_2)_2\overline{(U\beta)_1}|, \end{aligned} \tag{3.6}$$

where $U : \mathbb{C}^2 \to \mathbb{C}^2$ is a unitary which is of the form $U = \left(\begin{smallmatrix} a & -\bar{b} \\ b & \bar{a} \end{smallmatrix}\right)$ with $|a|^2 + |b|^2 = 1$ and $e_2 = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$. Let $\alpha = \left(\begin{smallmatrix} \alpha_1 \\ \alpha_2 \end{smallmatrix}\right)$ and $\beta = \left(\begin{smallmatrix} \beta_1 \\ \beta_2 \end{smallmatrix}\right)$, then $U\alpha = \left(\begin{smallmatrix} a & -\bar{b} \\ b & \bar{a} \end{smallmatrix}\right)\left(\begin{smallmatrix} \alpha_1 \\ \alpha_2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} a\alpha_1 - \bar{b}\alpha_2 \\ b\alpha_1 + \bar{a}\alpha_2 \end{smallmatrix}\right)$. Hence $(U\alpha)_2 = b\alpha_1 + \bar{a}\alpha_2$ and $\overline{(U\beta)_1} = (\bar{a}\bar{\beta}_1 - b\bar{\beta}_2)$. Thus we have

$$\begin{aligned} (U\alpha)_2\overline{(U\beta)_1} &= \bar{a}b\alpha_1\bar{\beta}_1 + \bar{a}^2\alpha_2\bar{\beta}_1 - b^2\alpha_1\bar{\beta}_2 - \bar{a}b\alpha_2\bar{\beta}_2 \\ &= \langle \left(\begin{smallmatrix} \bar{a}b & \bar{a}^2 \\ -b^2 & -\bar{a}b \end{smallmatrix}\right)\left(\begin{smallmatrix} \alpha_1 \\ \alpha_2 \end{smallmatrix}\right), \left(\begin{smallmatrix} \beta_1 \\ \beta_2 \end{smallmatrix}\right)\rangle. \end{aligned}$$

In particular, we have $(Ue_2)_2\overline{(U\beta)_1} = \bar{a}^2\bar{\beta}_1 - \bar{a}b\bar{\beta}_2$.

We claim that $\sup_U |(Ue_2)_2\overline{(U\beta)_1}| = \frac{(1+|\beta_1|)}{2}$.

In order to prove the claim, it is sufficient to observe that

$$\begin{aligned} \sup_U |(Ue_2)_2\overline{(U\beta)_1}|^2 &= \sup_{|a|^2+|b|^2=1} |\overline{a}^2\overline{\beta_1} - \overline{a}b\overline{\beta_2}|^2 \\ &= \sup |(\cos t)^2\, \cos\psi\, \exp i(-2\theta - x) - \cos t\, \sin t\, \sin\psi \exp i(-\theta - y + \phi)|^2 \\ &= \sup(\cos t)^4\, (\cos\psi)^2 + \frac{\sin 2t^2}{4}\, (\sin\psi)^2 \\ &- (\cos t)^2\, \cos\psi\, \sin 2t\, \sin\psi \cos(\theta + x + \phi - y) \end{aligned} \tag{3.7}$$

where $a = \cos t \exp i\theta, b = \sin t \exp i\phi, \beta_1 = \cos\psi \exp ix$ and $\beta_2 = \sin\psi \exp iy$. If we choose $\theta + x + \phi = y$, then the right hand side of (3.7) is

$$\sup_t((\cos t)^2\, \cos\psi + \frac{\sin 2t}{2}\, \sin\psi)^2.$$

Let $f(t) = \big((\cos t)^2\, \cos\psi + \frac{\sin 2t}{2}\, \sin\psi\big)$. The derivative of f with respect to t is $f'(t) = -\sin 2t \cos\psi + \cos 2t \sin\psi$. If we assume $0 = f'(t)$, then we have $\psi - 2t = n\pi$, where $n \in Z$. Also, $f''(t) = -2\cos 2t\, \cos\psi - 2\sin 2t\, \sin\psi = -2\cos(\psi - 2t)$. If $\psi - 2t = 2n\pi$, then $f''(t) \leq 0$. Therefore, we conclude that the maximum value of $f(t)$ is achieved at $\psi - 2t = 2n\pi$ and the maximum value of $f(t)$ is equal to $\frac{(1+|\beta_1|)}{2}$. This proves the claim. Putting $\sup_U |(Ue_2)_2\overline{(U\beta)_1}| = \frac{(1+|\beta_1|)}{2}$ in Equation (3.6) we have

$$\sup_{\|\beta\|=1} \|\mathbf{v}_1\|^2|\beta_2|^2 + \frac{\|\mathbf{v}_2\|^2}{4}(1 + |\beta_1|)^2 + |\langle \mathbf{v}_1, \mathbf{v}_2\rangle|(1 + |\beta_1|)|\beta_2|. \tag{3.8}$$

Let $\beta_1 = \cos t \exp(ix_2)$, $\beta_2 = \sin t \exp(ix_1)$ then the Equation (3.8) simplifies to the equation $ax^2 + cxy + by^2$, where $a = \|\mathbf{v}_1\|^2, b = \frac{\|\mathbf{v}_2\|^2}{4}, c = |\langle \mathbf{v}_1, \mathbf{v}_2\rangle|, y = 1 + \cos t, x = \sin t$ and supremum is taken over $x^2 + y^2 = 2y$, that is, $x^2 + (y - 1)^2 = 1$. Also, note that

$$\begin{aligned} \sup_{x^2+(y-1)^2=1} \|ax^2 + cxy + by^2\| &= \sup_{x^2+(y-1)^2=1} \| \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)^{\mathrm{t}} \begin{pmatrix} a & c/2 \\ c/2 & b \end{pmatrix} \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \| \\ &= \sup_{x^2+(y-1)^2=1} \| \left(\begin{smallmatrix} x \\ y-1+1 \end{smallmatrix}\right)^{\mathrm{t}} \begin{pmatrix} a & c/2 \\ c/2 & b \end{pmatrix} \left(\begin{smallmatrix} x \\ y-1+1 \end{smallmatrix}\right) \| \\ &= \sup_{x^2+w^2=1} \| \left(\begin{smallmatrix} x \\ w+1 \end{smallmatrix}\right)^{\mathrm{t}} \begin{pmatrix} a & c/2 \\ c/2 & b \end{pmatrix} \left(\begin{smallmatrix} x \\ w+1 \end{smallmatrix}\right) \|, \end{aligned} \tag{3.9}$$

where $(y-1)=w$. Let $u=\left(\begin{smallmatrix} x \\ w \end{smallmatrix}\right)$, then $\left(\begin{smallmatrix} x \\ w+1 \end{smallmatrix}\right)=\left(\begin{smallmatrix} x \\ w \end{smallmatrix}\right)+\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)=u+e_2$. Also, let $T=\begin{pmatrix} a & c/2 \\ c/2 & b \end{pmatrix}$, then T is self-adjoint matrix. Since T is self-adjoint matrix, there exist a unitary matrix U such that $U^{-1}TU=D$, where $D=\left(\begin{smallmatrix} c_1 & 0 \\ 0 & c_2 \end{smallmatrix}\right)$. Therefore, Equation (3.9) is equivalent to

$$\begin{aligned}
\sup_{\|u\|_2=1}\|(u+e_2)^{\mathrm{tr}}T(u+e_2)\| &= \sup_{\|u\|_2=1}\|(u+e_2)^{\mathrm{tr}}U^{-1}TU(u+e_2)\| \\
&= \sup_{\|u\|_2=1} = \|u^{\mathrm{tr}}Du+e_2^{\mathrm{tr}}Du+u^{\mathrm{tr}}De_2+e_2^{\mathrm{tr}}De_2\| \\
&= \sup_{x^2+w^2=1} c_1x^2+c_2w^2+2c_2w+c_2 \\
&= \sup_{w} c_1(1-w^2)+c_2w^2+2c_2w+c_2 \\
&= \sup_{w}(c_2-c_1)w^2+2c_2w+c_2+c_1. \qquad (3.10)
\end{aligned}$$

Suppose $g(w)=\sup_w(c_2-c_1)w^2+2c_2w+c_2+c_1$. The derivative of g with respect to w is $g'(w)=2(c_2-c_1)w+2c_2$. Now, $0=g'(w)$ implies that $w=\frac{-c_2}{(c_2-c_1)}$. Also, $g''(w)=2(c_2-c_1)$ and $g''(w)\leq 0$ if $c_1>c_2$. Therefore, the maximum value of $g(w)$ is achieved at $w=\frac{-c_2}{(c_2-c_1)}$ and is equal to $g(w)=\frac{c_1^2}{(c_1-c_2)}$. The eigen values of $\begin{pmatrix} a & c/2 \\ c/2 & b \end{pmatrix}$ are equal to $c_1=\frac{\left(a+b+\sqrt{(a-b)^2+c^2}\right)}{2}$, $c_2=\frac{\left(a+b-\sqrt{(a-b)^2+c^2}\right)}{2}$. Therefore, from Equation (3.10) we have

$$\begin{aligned}
g(w) &= \frac{\left(a+b+\sqrt{(a-b)^2+c^2}\right)^2}{4\sqrt{(a-b)^2+c^2}} \\
&= \frac{\left(\|\mathbf{v}_1\|^2+\frac{\|\mathbf{v}_2\|^2}{4}+\sqrt{(\|\mathbf{v}_1\|^2-\frac{\|\mathbf{v}_2\|^2}{4})^2+|\langle\mathbf{v}_1,\mathbf{v}_2\rangle|^2}\right)^2}{4\sqrt{(\|\mathbf{v}_1\|^2-\frac{\|\mathbf{v}_2\|^2}{4})^2+|\langle\mathbf{v}_1,\mathbf{v}_2\rangle|^2}}. \qquad (3.11)
\end{aligned}$$

Hence $\|L_V\|^2\leq 1$ if and only if $\frac{\left(\|\mathbf{v}_1\|^2+\frac{\|\mathbf{v}_2\|^2}{4}+\sqrt{(\|\mathbf{v}_1\|^2-\frac{\|\mathbf{v}_2\|^2}{4})^2+|\langle\mathbf{v}_1,\mathbf{v}_2\rangle|^2}\right)^2}{4\sqrt{(\|\mathbf{v}_1\|^2-\frac{\|\mathbf{v}_2\|^2}{4})^2+|\langle\mathbf{v}_1,\mathbf{v}_2\rangle|^2}}\leq 1$ which is equivalent to

$$\left(\|\mathbf{v}_1\|^2+\frac{\|\mathbf{v}_2\|^2}{4}+\sqrt{(\|\mathbf{v}_1\|^2-\frac{\|\mathbf{v}_2\|^2}{4})^2+|\langle\mathbf{v}_1,\mathbf{v}_2\rangle|^2}\right)^2\leq 4\sqrt{(\|\mathbf{v}_1\|^2-\frac{\|\mathbf{v}_2\|^2}{4})^2+|\langle\mathbf{v}_1,\mathbf{v}_2\rangle|^2}.$$

(Second proof) Suppose $\| L_V \|_{(\mathbb{C}^2,\|\cdot\|^*_{\Omega_{\mathbf{A}}})\to(\mathbb{C}^2,\|\cdot\|_2)}\leq 1$. Since the matrix representation of L_V^* is of the form $\left(\begin{smallmatrix} v_{11} & v_{21} \\ v_{12} & v_{22} \end{smallmatrix}\right)$. Therefore, we have $\| L_V \|_{(\mathbb{C}^2,\|\cdot\|^*_{\Omega_{\mathbf{A}}})\to(\mathbb{C}^2,\|\cdot\|_2)}\leq 1$ if and only if $\sup_{\|(\alpha,\beta)\|=1}|v_{11}\alpha+v_{21}\beta|^2+|v_{12}\alpha+v_{22}\beta|^2\leq 1$. Now,

$$\sup_{\|(\alpha,\beta)\|=1}|v_{11}\alpha+v_{21}\beta|^2+|v_{12}\alpha+v_{22}\beta|^2=\sup_{\|(\alpha,\beta)\|=1}\|\mathbf{v}_1\|^2|\alpha|^2+\|\mathbf{v}_2\|^2|\beta|^2+2\Re\langle\mathbf{v}_1,\mathbf{v}_2\rangle\alpha\bar{\beta}, \qquad (3.12)$$

where $\|\mathbf{v}_1\|^2 = |v_{11}|^2 + |v_{12}|^2$,$\langle \mathbf{v}_1, \mathbf{v}_2\rangle = (v_{11}\bar{v}_{21} + v_{12}\bar{v}_{22})$ and $\|\mathbf{v}_2\|^2 = |\bar{v}_{21}|^2 + |\bar{v}_{22}|^2$. Choosing α and β in such a way that $\Re\langle \mathbf{v}_1, \mathbf{v}_2\rangle\alpha\bar{\beta} = |\langle \mathbf{v}_1, \mathbf{v}_2\rangle\alpha\bar{\beta}|$. Hence from Equation (3.12) we have

$$\sup_{\|(\alpha,\beta)\|=1} |v_{11}\alpha + v_{21}\beta|^2 + |v_{12}\alpha + v_{22}\beta|^2 = \sup_{\|(\alpha,\beta)\|=1} \|\mathbf{v}_1\|^2|\alpha|^2 + \|\mathbf{v}_2\|^2|\beta|^2 + 2|\langle \mathbf{v}_1, \mathbf{v}_2\rangle\alpha\bar{\beta}|. \tag{3.13}$$

We have seen in previous section that if $|\beta| \leq \frac{|\alpha|}{2}$, then $\|(\alpha, \beta)\| = |\alpha|$. Therefore, $\|(\alpha, \beta)\| = 1$ implies that $|\beta| \leq \frac{1}{2}$. For this case, we have

$$\sup_{\|(\alpha,\beta)\|=1} |v_{11}\alpha + v_{21}\beta|^2 + |v_{12}\alpha + v_{22}\beta|^2 = \|\mathbf{v}_1\|^2 + \frac{\|\mathbf{v}_2\|^2}{4} + \Re\langle \mathbf{v}_1, \mathbf{v}_2\rangle.$$

Also, if $|\beta| \geq \frac{1}{2}$, then $\|(\alpha, \beta)\| = \frac{|\alpha|^2+4|\beta|^2}{4|\beta|}$. Hence $\|(\alpha, \beta)\| = 1$ implies that $|\alpha|^2 + (2|\beta| - 1)^2 = 1$. Setting $x = |\alpha|$ and $y = 2|\beta| - 1$ we have $|\beta| = \frac{y+1}{2}$. Therefore, we have

$$\sup_{x^2+y^2=1} x^2\|\mathbf{v}_1\|^2 + \|\mathbf{v}_2\|^2\frac{(y+1)^2}{4} + |\langle \mathbf{v}_1, \mathbf{v}_2\rangle x(y+1)|.$$

Hence from Equation (3.11) we have

$$\sup_{\|(\alpha,\beta)\|=1} |v_{11}\alpha + v_{21}\beta|^2 + |v_{12}\alpha + v_{22}\beta|^2 = \frac{\left(\|\mathbf{v}_1\|^2 + \frac{\|\mathbf{v}_2\|^2}{4} + \sqrt{(\|\mathbf{v}_1\|^2 - \frac{\|\mathbf{v}_2\|^2}{4})^2 + |\langle \mathbf{v}_1, \mathbf{v}_2\rangle|^2}\right)^2}{4\sqrt{(\|\mathbf{v}_1\|^2 - \frac{\|\mathbf{v}_2\|^2}{4})^2 + |\langle \mathbf{v}_1, \mathbf{v}_2\rangle|^2}}.$$

Therefore, we have

$$\begin{aligned}\|L_V\|^2 &= \max\{\frac{\left(\|\mathbf{v}_1\|^2 + \frac{\|\mathbf{v}_2\|^2}{4} + \sqrt{(\|\mathbf{v}_1\|^2 - \frac{\|\mathbf{v}_2\|^2}{4})^2 + |\langle \mathbf{v}_1, \mathbf{v}_2\rangle|^2}\right)^2}{4\sqrt{(\|\mathbf{v}_1\|^2 - \frac{\|\mathbf{v}_2\|^2}{4})^2 + |\langle \mathbf{v}_1, \mathbf{v}_2\rangle|^2}}, \|\mathbf{v}_1\|^2 + \frac{\|\mathbf{v}_2\|^2}{4} + \Re\langle \mathbf{v}_1, \mathbf{v}_2\rangle\}\\ &= \frac{\left(\|\mathbf{v}_1\|^2 + \frac{\|\mathbf{v}_2\|^2}{4} + \sqrt{(\|\mathbf{v}_1\|^2 - \frac{\|\mathbf{v}_2\|^2}{4})^2 + |\langle \mathbf{v}_1, \mathbf{v}_2\rangle|^2}\right)^2}{4\sqrt{(\|\mathbf{v}_1\|^2 - \frac{\|\mathbf{v}_2\|^2}{4})^2 + |\langle \mathbf{v}_1, \mathbf{v}_2\rangle|^2}}\end{aligned}$$

This completes the second proof. $\square$

Let $P_{\mathbf{A}} : \Omega_{\mathbf{A}} \to (\mathcal{M}_2)_1$ be the matrix valued polynomial on $\Omega_{\mathbf{A}}$ defined earlier. The contractivity of $\rho_V^{(2)}$ is equivalent to $\|A_1 \otimes \mathbf{v}_1 + A_2 \otimes \mathbf{v}_2\| \leq 1$. Now, we will compute $\|A_1 \otimes \mathbf{v}_1 + A_2 \otimes \mathbf{v}_2\|$.

Theorem 3.5. $\|A_1 \otimes \mathbf{v}_1 + A_2 \otimes \mathbf{v}_2\| \leq 1$ *if and only if*

$$2\|\mathbf{v}_1\|^2 + \|\mathbf{v}_2\|^2 + \sqrt{\|\mathbf{v}_2\|^4 - 4|\langle \mathbf{v}_1, \mathbf{v}_2\rangle|^2} \leq 2.$$

Proof. Suppose $\|A_1 \otimes \mathbf{v}_1 + A_2 \otimes \mathbf{v}_2\| \leq 1$. Now,

$$\begin{aligned}
\|A_1 \otimes \mathbf{v}_1 + A_2 \otimes \mathbf{v}_2\|^2 &= \|(v_{11}A_1 + v_{21}A_2, v_{12}A_1 + v_{22}A_2)\|^2 \\
&= \left\|(v_{11}A_1 + v_{21}A_2, v_{12}A_1 + v_{22}A_2)\begin{pmatrix}(v_{11}A_1 + v_{21}A_2)^* \\ (v_{12}A_1 + v_{22}A_2)^*\end{pmatrix}\right\| \\
&= \|\|\mathbf{v}_1\|^2 A_1A_1^* + \langle \mathbf{v}_1, \mathbf{v}_2\rangle A_1A_2^* + \langle \mathbf{v}_2, \mathbf{v}_1\rangle A_2A_1^* + \|\mathbf{v}_2\|^2 A_2A_2^*\| \\
&= \left\|\begin{pmatrix}\|\mathbf{v}_1\|^2 + \|\mathbf{v}_2\|^2 & \langle \mathbf{v}_2, \mathbf{v}_1\rangle \\ \langle \mathbf{v}_1, \mathbf{v}_2\rangle & \|\mathbf{v}_1\|^2\end{pmatrix}\right\|.
\end{aligned}$$

Let $\tilde{C} = \begin{pmatrix}\|\mathbf{v}_1\|^2 + \|\mathbf{v}_2\|^2 & \langle \mathbf{v}_2, \mathbf{v}_1\rangle \\ \langle \mathbf{v}_1, \mathbf{v}_2\rangle & \|\mathbf{v}_1\|^2\end{pmatrix}$. $\tilde{C}$ is a self-adjoint matrix. The norm of $\tilde{C}$ is

$$\frac{2\|\mathbf{v}_1\|^2 + \|\mathbf{v}_2\|^2 + \sqrt{\|\mathbf{v}_2\|^4 - 4|\langle \mathbf{v}_1, \mathbf{v}_2\rangle|^2}}{2}.$$

Hence $\|A_1 \otimes \mathbf{v}_1 + A_2 \otimes \mathbf{v}_2\| \leq 1$ is equivalent to the inequality

$$2\|\mathbf{v}_1\|^2 + \|\mathbf{v}_2\|^2 + \sqrt{\|\mathbf{v}_2\|^4 - 4|\langle \mathbf{v}_1, \mathbf{v}_2\rangle|^2} \leq 2.$$

This completes the proof. □

As a consequence of this Theorem, we have the following corollary.

Corollary 3.6. *There exists a contractive homomorphism of $\mathcal{O}(\Omega_{\mathbf{A}})$ which is not complete contractive.*

Proof. Let $\mathbf{v}_1 = (\frac{1}{\sqrt{2}}, 0)$ and $\mathbf{v}_2 = (0, 1)$. Then it follows from Theorem 3.4 that $\|\rho_V\| \leq 1$ for this pair $(\mathbf{v}_1, \mathbf{v}_2)$. Also, we have $\|\rho_V^{(2)}(P_{\mathbf{A}})\| > 1$. This completes the proof. □

Let $\Omega_{\mathbf{A}} = \{(z_1, z_2, z_3) : \|z_1A_1 + z_2A_2 + z_3A_3\| < 1\} \subset \mathbb{C}^3$, where $A_1 = \left(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right)$, $A_2 = \left(\begin{smallmatrix}0 & 1\\ 0 & 0\end{smallmatrix}\right)$ and $A_3 = \left(\begin{smallmatrix}0 & 0\\ 0 & 1\end{smallmatrix}\right)$. The matrix representing $L_V^* : (\mathbb{C}^3, \|\cdot\|_2) \to (\mathbb{C}^3, \|\cdot\|_{\Omega_{\mathbf{A}}})$ is of the form $\left(\begin{smallmatrix}v_{11} & 0 & 0\\ 0 & v_{22} & 0\\ 0 & 0 & v_{33}\end{smallmatrix}\right)$, $V := \left(\begin{smallmatrix}\mathbf{v}_1\\ \mathbf{v}_2\\ \mathbf{v}_3\end{smallmatrix}\right)$, where $\mathbf{v}_1 = (v_{11}, 0, 0)$, $\mathbf{v}_2 = (0, v_{22}, 0)$ and $\mathbf{v}_3 = (0, 0, v_{33})$.

Theorem 3.7. $\|\rho_V^*\|_{(\mathbb{C}^3, \|\cdot\|_2) \to (\mathbb{C}^3, \|\cdot\|_{\mathbf{A}})} \leq 1$ *if and only if* $|v_{11}|^2(1 - |v_{33}|^2) \geq (|v_{22}|^2 - |v_{33}|^2)$.

Proof. Suppose $\|\rho_V^*\|_{(\mathbb{C}^3, \|\cdot\|_2) \to (\mathbb{C}^3, \|\cdot\|_{\mathbf{A}})} \leq 1$. Note that $\|\rho_V^*\|_{(\mathbb{C}^3, \|\cdot\|_2) \to (\mathbb{C}^3, \|\cdot\|_{\mathbf{A}})} \leq 1$ is equivalent to

$$\inf_{\beta} \det \begin{pmatrix} 1 - |\beta_1|^2|v_{11}|^2 & 0 & 0 \\ 0 & 1 - |\beta_1|^2|v_{22}|^2 & -\beta_2\bar{\beta}_1 v_{22}\bar{v}_{33} \\ 0 & -\bar{\beta}_2\beta_1\bar{v}_{22}v_{33} & 1 - |\beta_2 v_{33}|^2 \end{pmatrix} \geq 0$$

with $|v_{11}|^2 \leq 1, |v_{22}|^2 \leq 1, |v_{33}|^2 \leq 1$, where $\sum_{i=1}^{3} |\beta_i|^2 = 1$. Now,

$$\begin{aligned}
&\inf_{\beta} \det \begin{pmatrix} 1-|\beta_1|^2|v_{11}|^2 & 0 & 0 \\ 0 & 1-|\beta_1|^2|v_{22}|^2 & -\beta_2\bar{\beta}_1 v_{22}\bar{v}_{33} \\ 0 & -\bar{\beta}_2\beta_1\bar{v}_{22}v_{33} & 1-|\beta_2 v_{33}|^2 \end{pmatrix} \\
&= \inf_{\beta}(1-|\beta_1|^2|v_{11}|^2)\{(1-|\beta_1|^2|v_{22}|^2)(1-|\beta_2|^2|v_{33}|^2) - |\beta_2|^2|v_{33}|^2|\beta_1|^2|v_{22}|^2\}. \qquad (3.14)
\end{aligned}$$

Putting $|\beta|^2 = r$ in Equation (3.14) we have

$$\inf_{0\leq r\leq 1}\{1 - r(|v_{11}|^2 + |v_{22}|^2 - |v_{11}|^2|v_{33}|^2) - (1-r)|v_{33}|^2 + r^2|v_{11}|^2(|v_{22}|^2 - |v_{33}|^2\}.$$

Let

$$f(r) = \inf_{0\leq r\leq 1}\{1 - r(|v_{11}|^2 + |v_{22}|^2 - |v_{11}|^2|v_{33}|^2) - (1-r)|v_{33}|^2 + r^2|v_{11}|^2(|v_{22}|^2 - |v_{33}|^2)\}.$$

The derivative of f with respect r is equal to

$$f'(r) = -(|v_{11}|^2 + |v_{22}|^2 - |v_{33}|^2 - |v_{11}|^2|v_{33}|^2) + 2r|v_{11}|^2(|v_{22}|^2 - |v_{33}|^2).$$

Now, if $f'(r) = 0$, then we have

$$r = \frac{(|v_{11}|^2 + |v_{22}|^2 - |v_{33}|^2 - |v_{11}|^2|v_{33}|^2)}{|v_{11}|^2(|v_{22}|^2 - |v_{33}|^2)}.$$

Also, $f''(r) = |v_{11}|^2(|v_{22}|^2 - |v_{33}|^2)$. If $|v_{22}|^2 > |v_{33}|^2$ then $f''(r) > 0$. Therefore, the infimum is equal to

$$|v_{11}|^2(1 - |v_{33}|^2) - (|v_{22}|^2 - |v_{33}|^2).$$

Hence $\|\rho_V^*\|_{(\mathbb{C}^3,\|\cdot\|_2)\to(\mathbb{C}^3,\|\cdot\|_{\mathbf{A}})} \leq 1$ if and only if

$$|v_{11}|^2(1 - |v_{33}|^2) \geq (|v_{22}|^2 - |v_{33}|^2).$$

This completes the proof. □

Let $P_{\mathbf{A}} : \Omega_{\mathbf{A}} \to (\mathcal{M}_2)_1$ be the matrix valued polynomial on $\Omega_{\mathbf{A}}$. Now we want to estimate the norm of $\|A_1 \otimes \mathbf{v}_1 + A_2 \otimes \mathbf{v}_2 + A_3 \otimes \mathbf{v}_3\|$.

Theorem 3.8. $\|A_1 \otimes \mathbf{v}_1 + A_2 \otimes \mathbf{v}_2 + A_3 \otimes \mathbf{v}_3\| \leq 1$ *if and only if* $|v_{11}|^2 + |v_{22}|^2 \leq 1$ *and* $|v_{33}|^2 \leq 1$.

Proof. Assume that $\|A_1 \otimes \mathbf{v}_1 + A_2 \otimes \mathbf{v}_2 + A_3 \otimes \mathbf{v}_3\| \leq 1$. Now,

$$\begin{aligned}
\|A_1 \otimes \mathbf{v}_1 + A_2 \otimes \mathbf{v}_2 + A_3 \otimes \mathbf{v}_3\|^2 &= \|(v_{11}A_1, v_{22}A_2, v_{33}A_3\|^2 \\
&= \left\| \begin{pmatrix} |v_{11}|^2 + |v_{22}|^2 & 0 \\ 0 & |v_{33}|^2 \end{pmatrix} \right\| \\
&= \max\{|v_{11}|^2 + |v_{22}|^2, |v_{33}|^2\}.
\end{aligned}$$

Therefore, $\|A_1 \otimes \mathbf{v}_1 + A_2 \otimes \mathbf{v}_2 + A_3 \otimes \mathbf{v}_3\| \leq 1$ implies that $|v_{11}|^2 + |v_{22}|^2 \leq 1$ and $|v_{33}|^2 \leq 1$. This completes the proof. □

As before we will prove the following corollary.

Corollary 3.9. *There exists a contractive homomorphism of $\mathcal{O}(\Omega_{\mathbf{A}})$ which is not complete contractive.*

Proof. Let $\mathbf{v}_1 = (\frac{1}{2}, 0, 0), \mathbf{v}_2 = (0, 1, 0)$ and $\mathbf{v}_3 = (0, 0, 1)$. Then it follows from the Theorem 3.7 that $\|\rho_V\| \leq 1$. Also, we have $\|\rho_V^{(2)}(P_{\mathbf{A}})\| > 1$. This completes the proof. $\square$

Chapter 4

Operator spaces

We recall the notion of an operator space. We describe two distinguished operator spaces, namely, the MIN and MAX operator spaces. These two operator spaces have played an important role in the development of operator theory in the recent past.

Definition 4.1. (cf. [29, Chapter 13, 14]) An abstract operator space is a linear space $\mathbf{V}$ together with a family of norms $\|\cdot\|_k$ defined on $\mathcal{M}_k(V)$, $k = 1, 2, 3, \ldots$, where $\|\cdot\|_1$ is simply a norm on the linear space $\mathbf{V}$. These norms are required to satisfy the following compatibility conditions:

1. $\|T \oplus S\|_{p+q} = \max\{\|T\|_p, \|S\|_q\}$ and
2. $\|ASB\|_q \leq \|A\|_{\mathrm{op}}\|S\|_p\|B\|_{\mathrm{op}}$

for all $S \in \mathcal{M}_q(\mathbf{V}), T \in \mathcal{M}_p(\mathbf{V})$ and $A \in \mathcal{M}_{qp}(\mathbb{C}), B \in \mathcal{M}_{pq}(\mathbb{C})$.

Two such operator spaces $(\mathbf{V}, \|\cdot\|_k)$ and $(\mathbf{W}, \|\cdot\|_k)$ are said to be completely isometric if there is a linear bijection $T : \mathbf{V} \to \mathbf{W}$ such that $T \otimes I_k : (\mathcal{M}_k(\mathbf{V}), \|\cdot\|_k) \to (\mathcal{M}_k(\mathbf{W}), \|\cdot\|_k)$ is an isometry for every $k \in \mathbb{N}$. Here we have identified $\mathcal{M}_k(\mathbf{V})$ with $\mathbf{V} \otimes \mathcal{M}_k$ in the usual manner. We note that a normed linear space $(\mathbf{V}, \|\cdot\|)$ admits an operator space structure if and only if there is an isometric embedding of it into the algebra of operators $\mathcal{B}(\mathcal{H})$ on some Hilbert space $\mathcal{H}$. This is the well-known theorem of Ruan (cf. [14]). Here we study "different" operator space structures on a finite dimensional normed linear space which admit an isometric embedding onto a subspace of $\mathcal{M}_n(\mathbb{C})$. Let $\mathbf{A} = (A_1, \ldots, A_m)$, where $A_1, \ldots, A_m$ are $n \times n$ linearly independent matrices. For $(z_1, \ldots, z_m)$ in $\mathbb{C}^m$, set $\|(z_1, \ldots, z_m)\|_{\mathbf{A}} = \|z_1A_1 + \cdots + z_mA_m\|_{\mathrm{op}}$. Let $\mathbf{V}_{\mathbf{A}}$ be the m-dimensional normed linear space with respect to the norm $\|(z_1, \cdots, z_m)\|_{\mathbf{A}}$. This makes the map

$$(z_1, \ldots, z_m) \to z_1A_1 + \cdots + z_mA_m$$

an isometry from $\mathbf{V_A}$ into $(\mathcal{M}_n, \|\cdot\|_{\rm op})$. Therefore, $\mathbf{V_A}$ inherits an operator space structure from $\mathcal{M}_n$. Recall that $\Omega_{\mathbf{A}}$ is the unit ball with respect to the norm $\|(z_1, \cdots, z_m)\|_{\mathbf{A}}$. Let $\mathbf{V}_{\mathbf{A}^{\rm t}}$ be the normed linear space obtained by using the transpose, namely, $\mathbf{A}^{\rm t} := (A_1^{\rm t}, \ldots A_m^{\rm t})$. By definition, $\mathbf{V}_{\mathbf{A}^{\rm t}}$ has an isometric embedding into $\mathcal{M}_n$ giving it an operator space structure. Let $\Omega_{\mathbf{A}^{\rm t}}$ be the unit ball with respect to the norm $\|\cdot\|_{\mathbf{A}^{\rm t}}$. Evidently, the two normed linear spaces $(\mathbf{V_A}, \|\cdot\|_{\mathbf{A}})$ and $(\mathbf{V}_{\mathbf{A}^{\rm t}}, \|\cdot\|_{\mathbf{A}^{\rm t}})$ are isometric. We ask if the operator space structures they inherit from $\mathcal{M}_n(\mathbb{C})$ via the embedding involving the map induced by $\mathbf{A}$ and $\mathbf{A}^{\rm t}$ are isometric. In case these operator space structures are isometric, what are other possible operator space structures on $(\mathbf{V_A}, \|\cdot\|_{\mathbf{A}})$? We answer this question, after recalling the notions of MIN and MAX operator spaces and a measure of their distance, namely, $\alpha(\mathbf{V})$ following [29, Chapter 14]).

Definition 4.2. The MIN operator structure $MIN(\mathbf{V})$ on a (finite dimensional) normed linear space is obtained by isometrically embedding $\mathbf{V}$ in the C^* algebra $C\big((\mathbf{V}^*)_1\big)$, of continuous functions on the unit ball $(\mathbf{V}^*)_1$ of the dual space. Thus for $(\!(v_{ij})\!)$ in $\mathcal{M}_k(\mathbf{V})$, we set

$$\|(\!(v_{ij})\!)\|_{MIN} = \|(\!(\widehat{v_{ij}})\!)\| = \sup\{\|(\!(f(v_{ij}))\!)\| : f \in (\mathbf{V}^*)_1\},$$

where the norm of a scalar matrix $(\!(f(v_{ij}))\!)$ in $\mathcal{M}_k$ is the operator norm.

For an arbitrary $k \times k$ matrix over $\mathbf{V}$, we simply write $\|(\!(v_{ij})\!)\|_{MIN(\mathbf{V})}$ to denote its norm in $\mathcal{M}_k(\mathbf{V})$. This is the minimal way in which we represent the normed space as an operator space. However, it is not difficult to create a “maximal” representation. We shall denote it by $MAX(\mathbf{V})$.

Definition 4.3. The matrix normed space $MAX(\mathbf{V})$ is defined by setting

$$\|(\!(v_{ij})\!)\|_{MAX} = \sup\{\|(\!(T(v_{ij}))\!)\| : T : \mathbf{V} \to B(\mathcal{H})\},$$

and the supremum is taken over all isometries T and all Hilbert spaces $\mathcal{H}$.

It is easy to verify that every operator space structure on a normed linear space $\mathbf{V}$ lies between $MIN(\mathbf{V})$ and $MAX(\mathbf{V})$. To aid the understanding of the extent to which the two operator space structures $MIN(\mathbf{V})$ and $MAX(\mathbf{V})$ differ, Paulsen introduced the constant $\alpha(\mathbf{V})$ (cf. [29, Chapter 14]), which we recall below.

Definition 4.4. The constant $\alpha(\mathbf{V})$ is defined by setting

$$\alpha(\mathbf{V}) = \sup\{\|(\!(v_{ij})\!)\|_{MAX} : \|(\!(v_{ij})\!)\|_{MIN} \le 1,\ (\!(v_{ij})\!) \in \mathcal{M}_k(\mathbf{V}), k \in \mathbb{N}\}.$$

Thus $\alpha(\mathbf{V}) = 1$ if and only if the identity map is a complete isometry from $MIN(\mathbf{V})$ to $MAX(\mathbf{V})$. Equivalently, we conclude that there exist a unique operator space structure on $\mathbf{V}$ whenever $\alpha(\mathbf{V})$ is 1. Therefore, those normed linear spaces for which $\alpha(\mathbf{V}) = 1$ are rather special. Unfortunately, there aren't too many of them. The known examples are the $(\mathbb{C}, |\cdot|)$ and $(\mathbb{C}, \|\cdot\|_\infty)$ (resp. the unit ball in $\mathbb{C}^2$ with respect to the ℓ_1 norm). Indeed, Paulsen has shown that $\alpha(\mathbf{V}) > 1$ whenever $\dim(\mathbf{V}) \geq 5$. Following this, Eric Ricard (cf. [28] [30]) has shown that $\alpha(\mathbf{V}) > 1$ for $\dim(\mathbf{V}) \geq 3$. This leaves the question open for normed linear spaces whose dimension is 2. This is the question we address here in some special cases.

4.1 Operator norm calculation

The operator norm of the block matrix $S = \left(\begin{smallmatrix} \alpha I_m & B \\ 0 & \alpha I_n \end{smallmatrix}\right)$, where B is an $m \times n$ matrix and $\alpha \in \mathbb{C}$ is not hard to compute (cf. [23, Lemma 2.1]). This computation easily extends to an operator T of the form $T = \left(\begin{smallmatrix} \alpha_1 I_m & B_{m\times n} \\ 0 & \alpha_2 I_n \end{smallmatrix}\right)$, where $\alpha_1 \neq \alpha_2$ are in $\mathbb{C}$. Here we provide the straightforward computation following [23, Lemma 2.1].

Lemma 4.5.

$$\|T\|_{\text{op}} = \frac{(|\alpha_2|^2 + \|B\|^2 + |\alpha_1|^2) + \sqrt{(|\alpha_2|^2 + \|B\|^2 - |\alpha_1|^2)^2 + 4\|B\|^2|\alpha_1|^2}}{2}.$$

Proof. Note that $\det\left(\begin{smallmatrix} A & X \\ C & D \end{smallmatrix}\right) = \det D \det(A - XD^{-1}C)$ and

$$TT^* = |\alpha_1|^2 I_{m+n} + \left(\begin{smallmatrix} BB^* & \bar{\alpha}_2 B \\ \alpha_2 B^* & (|\alpha_2|^2 - |\alpha_1|^2) I_m \end{smallmatrix}\right).$$

For $x \in \mathbb{C}$, we have

$$\begin{aligned}
\det\left(\begin{smallmatrix} BB^* - xI_n & \bar{\alpha}_2 B \\ \alpha_2 B^* & (a-x) I_m \end{smallmatrix}\right) &= \det\{BB^* - xI_m - \frac{|\alpha_2|^2}{(a-x)} BB^*\} \det\{(a-x)I_n\} \\
&= (-1)^m \left(\det\{BB^* + \frac{(a-x)x}{|\alpha_1|^2 + x} I_n\}\right) (a-x)^{n-m} (|\alpha_1|^2 + x)^m \\
&= (-1)^n \left(\det(U\{BB^* + \frac{(a-x)x}{|\alpha_1|^2 + x} I_n\}U^*)\right) (a-x)^{n-m} (|\alpha_1|^2 + x)^m,
\end{aligned}$$

where U is a unitary which makes BB^* into a diagonal matrix D_1 and $(|\alpha_2|^2 - |\alpha_1|^2) = a$. Thus the maximum eigenvalue of $\left(\begin{smallmatrix} BB^* & \bar{\alpha}_2 B \\ \alpha_2 B^* & (|\alpha_2|^2 - |\alpha_1|^2) I_m \end{smallmatrix}\right)$ is

$$x = \frac{(|\alpha_2|^2 + \|B\|^2 - |\alpha_1|^2) + \sqrt{(|\alpha_2|^2 + \|B\|^2 - |\alpha_1|^2)^2 + 4\|B\|^2|\alpha_1|^2}}{2}.$$

Using the spectral mapping theorem, the norm of TT^* is

$$\frac{(|\alpha_2|^2+\|B\|^2+|\alpha_1|^2)+\sqrt{(|\alpha_2|^2+\|B\|^2-|\alpha_1|^2)^2+4\|B\|^2|\alpha_1|^2}}{2}.$$

□

Corollary 4.6. $\left\| \begin{pmatrix} \alpha_1 I_m & B_{m\times n} \\ 0 & \alpha_2 I_n \end{pmatrix} \right\|_{\rm op} \leq 1$ *if and only if* $\|B\|^2 \leq (1-|\alpha_1|^2)(1-|\alpha_2|^2)$.

Proof. From Lemma 4.5, it follows that $\left\| \begin{pmatrix} \alpha_1 I_m & B_{m\times n} \\ 0 & \alpha_2 I_n \end{pmatrix} \right\| \leq 1$ if and only if

$$\frac{(|\alpha_2|^2+\|B\|^2+|\alpha_1|^2)+\sqrt{(|\alpha_2|^2+\|B\|^2-|\alpha_1|^2)^2+4\|B\|^2|\alpha_1|^2}}{2} \leq 1$$

which is clearly equivalent to

$$\|B\|^2 \leq (1-|\alpha_1|^2)(1-|\alpha_2|^2).$$

This completes the proof. □

Let $A_1 = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}$ with $z_1, z_2 \in \mathbb{C}$. The norm computation in the preceding Corollary shows that $\|z_1 A_1 + z_2 A_2\|_{\rm op} \leq 1$ if and only if

$$|z_2\beta|^2 \leq (1-|z_1\alpha_1|^2)(1-|z_1\alpha_2|^2).$$

However, we can say a little more. For B_1, B_2 in $\mathcal{M}_{m+n}(\mathbb{C})$ of the form $B_1 = \begin{pmatrix} \alpha_1 I_m & 0 \\ 0 & \alpha_2 I_m \end{pmatrix}$ and $B_2 = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}$, set $\|(z_1,z_2)\|_{\mathbf{B}} := \|z_1 B_1 + z_2 B_2\|_{\rm op}$. Proof of the following lemma is the same as that of the corollary.

Lemma 4.7. $\|(z_1,z_2)\|_{\mathbf{B}} \leq 1$ *if and only if* $|z_2|^2\|B\|^2 \leq (1-|z_1\alpha_1|^2)(1-|z_1\alpha_2|^2)$.

Remark 4.8. Let $\widetilde{\mathbf{B}}$ be another pair $(\widetilde{B}_1, \widetilde{B}_2)$ in $\mathbb{C}^2 \otimes \mathcal{M}_{m+n}(\mathbb{C})$ with $B_1 = \widetilde{B}_1$ and $\|B_2\| = \|\widetilde{B}_2\|$. It then follows that $\|\cdot\|_{\mathbf{B}} = \|\cdot\|_{\widetilde{\mathbf{B}}}$. In conclusion, we have shown that the normed linear space $(V_{\mathbf{A}}, \|\cdot\|_{\mathbf{A}})$, where $\mathbf{A} = \left(\begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}, \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}\right)$ has several different isometric embedding into $\mathcal{M}_{m+n}(\mathbb{C}), m, n \in \mathbb{N}$. To see this simply pick any B with $\|B\| = |\beta|$. However, it is not clear if any of the embedding give rise to distinct operator space.

Let $\mathbf{V_B}$ be the two dimensional linear space with respect to the norm $\|(z_1,z_2)\|_{\mathbf{B}}$. Since $B_1, B_2 \in \mathcal{M}_{m+n}(\mathbb{C})$, we embed $\mathbf{V_B}$ into $\mathcal{M}_{m+n}(\mathbb{C})$ isometrically. Therefore we think of $\mathbf{V_B}$ as an operator space. Note that $P_{\mathbf{B}} : \mathbf{V_B} \to \mathcal{M}_{m+n}(\mathbb{C})$ defines a linear isometric embedding into $\mathcal{M}_{m+n}(\mathbb{C})$. Suppose $V = (\!(\mathbf{v}_{ij})\!) \in \mathcal{M}_k(\mathbf{V_B})$, where $\mathbf{v}_{ij} \in \mathbf{V_B}$. We define $P^{(k)}_{\mathbf{B}} : P_{\mathbf{B}} \otimes I_k : \mathcal{M}_k(\mathbf{V_B}) \to \mathcal{M}_k(\mathcal{M}_{m+n}(\mathbb{C}))$ by $P^{(k)}_{\mathbf{B}}(V) = (\!(P_{\mathbf{B}}(\mathbf{v}_{ij}))\!)$. Let $\mathbf{v}_{ij} = (v^1_{ij}\ v^2_{ij})$ then

$$P^{(k)}_{\mathbf{B}}(V) = \begin{pmatrix} \alpha_1 V_1 \otimes I_m & V_2 \otimes B \\ 0 & \alpha_2 V_1 \otimes I_n \end{pmatrix},$$

where $V_1 = (\!(v^1_{ij})\!)$ and $V_2 = (\!(v^2_{ij})\!)$. The cb-norm of $P_{\mathbf{B}}$ is defined to be $\sup_k \|P^{(k)}_{\mathbf{B}}\|$. To compute $\|P^{(k)}_{\mathbf{B}}\|$ norm, we will need the following Lemma (cf. [9, Theorem 1.3.3]).

Lemma 4.9. *Let $M = \left(\begin{smallmatrix} A & X \\ X^* & C \end{smallmatrix}\right)$ be the block decomposition of M in $\mathcal{M}_{m+n}(\mathbb{C})$. Assume that A, C are positive definite. The following properties hold:*

1. *M is positive-definite if and only if $A - XC^{-1}X^*$ is positive-definite.*

2. *If C is positive-definite, then M is positive semi-definite if and only if $A - XC^{-1}X^*$ is positive semi-definite.*

Recall that a closed (resp. open) subset Ω of $\mathbb{C}^m$ is the closed (resp. open) unit ball in some norm if and only if it is bounded, balanced, convex, and absorbing . If we set

$$\|x\| = \inf\{x : t^{-1}x \in \Omega, t > 0\},$$

then $\|\cdot\|$ is a norm on $\mathbb{C}^m$ and Ω is the closed (resp. open) unit ball with respect to this norm (cf. [32, Theorem 1.34]). Now we will compute the norm $\|P^{(k)}_{\mathbf{B}}\|$. Let U, V be unitaries that diagonalize $V_1V_1^*$ and BB^*, that is, $D_1 := UV_1V_1^*U^* = \begin{pmatrix} |d_1|^2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & |d_n|^2 \end{pmatrix}$ and $D_2 : VBB^*V^* = \begin{pmatrix} |c_1|^2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & |\beta|^2 \end{pmatrix}$.

Theorem 4.10. $\left\|P^{(k)}_{\mathbf{B}}\right\| < 1$ *if and only if* $I_n - |\alpha_2|^2 D_1 > 0$,

$$I_n - |\alpha_1|^2 D_1 - |\beta|^2 UV_2V_2^*U^* - |\alpha_2\beta|^2 UV_2V_1^*U^*(I_n - |\alpha_2|^2D_1)^{-1}UV_1V_2^*U^* > 0,$$

where $\|B\| = |\beta|$.

Proof. Let $S = \left(\begin{smallmatrix} \alpha_1V_1\otimes I_m & V_2\otimes B \\ 0 & \alpha_2V_1\otimes I_m \end{smallmatrix}\right)$. Note that $\det\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) = \det D \det(A - BD^{-1}C)$ and

$$SS^* = \left(\begin{smallmatrix} |\alpha_1|^2V_1V_1^*\otimes I_m + V_2V_2^*\otimes BB^* & \bar{\alpha}_2V_2V_1^*\otimes B \\ \alpha_2V_1V_2^*\otimes B^* & |\alpha_2|^2V_1V_1^*\otimes I_m \end{smallmatrix}\right).$$

If $\|S\| < 1$ then we have $I_{2n+2m} - SS^* > 0$ and conversely. Hence $I_{2n+2m} - SS^* > 0$ is equivalent to the condition that

$$\left(\begin{smallmatrix} I_n\otimes I_m - |\alpha_1|^2V_1V_1^*\otimes I_m + V_2V_2^*\otimes BB^* & -\bar{\alpha}_2V_2V_1^*\otimes B \\ -\alpha_2V_1V_2^*\otimes B^* & I_n\otimes I_m - |\alpha_2|^2V_1V_1^*\otimes I_m \end{smallmatrix}\right) > 0. \tag{4.1}$$

Let $\widetilde{U} = \left(\begin{smallmatrix} U\otimes V & 0 \\ 0 & U\otimes V \end{smallmatrix}\right)$. Then it is easy to see that $\widetilde{U}$ is an unitary. Multiply both side of the Equation (4.1) by $\widetilde{U}$ and $\widetilde{U}^*$ we have

$$\begin{pmatrix} I_n\otimes I_m - |\alpha_1|^2D_1\otimes I_m + UV_2V_2^*U^*\otimes D_2 & -\bar{\alpha}_2UV_2V_1^*U^*\otimes VBV^* \\ -\alpha_2UV_1V_2^*U^*\otimes VB^*V^* & I_n\otimes I_m - |\alpha_2|^2D_1\otimes I_m \end{pmatrix} > 0. \tag{4.2}$$

Using Lemma 4.9 together with the Equation (4.2) we have $I_n - |\alpha_2|^2D_1 > 0$ and

$$I_n - |\alpha_1|^2 D_1 - |\beta|^2 UV_2V_2^*U^* - |\alpha_2\beta|^2 UV_2V_1^*U^*(I_n - |\alpha_2|^2D_1)^{-1}UV_1V_2^*U^* > 0.$$

The converse statement is easily verified. $\square$

Let $A_1 = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}$ and $\mathbf{A} = (A_1, A_2)$. We can think of $\mathbf{V_A}$ as an operator space via the linear isometric embedding $P_{\mathbf{A}} : \mathbf{V_A} \to \mathcal{M}_2$. Therefore, we have

$$P_{\mathbf{A}}^{(k)}(V) = \begin{pmatrix} \alpha_1 V_1 & \beta V_2 \\ 0 & \alpha_2 V_1 \end{pmatrix}.$$

Taking B to be the scalar operator β and $m = n = 1$, we have:

Theorem 4.11. *$\left\|P_{\mathbf{A}}^{(k)}\right\| < 1$ if and only if $I_n - |\alpha_2|^2 D_1 > 0$,*

$$I_n - |\alpha_1|^2 D_1 - |\beta|^2 U V_2 V_2^* U^* - |\alpha_2 \beta|^2 U V_2 V_1^* U^* (I_n - |\alpha_2|^2 D_1)^{-1} U V_1 V_2^* U^* > 0$$

where $U V_1 V_1^* U^* = D_1$.

Remark 4.12. Theorems 4.10 and 4.11 together imply that $\|P_{\mathbf{A}}^{(k)}\| = \|P_{\mathbf{B}}^{(k)}\|, k = 1, 2, \ldots$. Therefore, the operator space structure on $\mathbf{V_A}$ obtained via $P_{\mathbf{B}}$ is independent of B.

4.2 Domains in $\mathbb{C}^2$

In this section we study the class domains $\Omega_{\mathbf{A}} = \{(z_1, z_2) : \|z_1 A_1 + z_2 A_2\|_{\text{op}} < 1\}$ in $\mathbb{C}^2$, where $A_1 = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, A_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\mathbf{A} = (A_1, A_2)$. We may assume without loss of generality, as shown in Chapter 1 that $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & d_2 \end{pmatrix}$ or $A_1 = \begin{pmatrix} d_1 & 0 \\ 0 & 1 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}, \begin{pmatrix} 1 & b \\ c & 0 \end{pmatrix}$ or $A_2 = \begin{pmatrix} 0 & b \\ c & 1 \end{pmatrix}$ with $b \in \mathbb{R}^+$.

Consider the case: $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & d_2 \end{pmatrix}, A_2 = \begin{pmatrix} 1 & b \\ c & 0 \end{pmatrix}$ and $\mathbf{A} = (A_1, A_2)$. In particular, for $m = 2, n = 2$, we can give an operator space structure on the normed linear space $\mathbf{V_A}$ via the linear isometric embedding $P_{\mathbf{A}} : \mathbf{V_A} \to \mathcal{M}_2$. For $V = (\!(\mathbf{v}_{ij})\!) \in \mathcal{M}_k(\mathbf{V_A})$, we have

$$P_{\mathbf{A}}^{(k)}(V) = \begin{pmatrix} V_3 & bV_2 \\ cV_2 & d_2 V_1 \end{pmatrix},$$

where $\mathbf{v}_{ij} \in \mathbf{V_A}$ and $V_1 = (\!(v_{ij}^1)\!), V_2 = (\!(v_{ij}^2)\!), V_3 = V_1 + V_2$. Similarly we can think $\mathbf{V}_{\mathbf{A}^{\mathrm{t}}}$ as an operator space via the linear isometric embedding $P_{\mathbf{A}^{\mathrm{t}}} : \mathbf{V}_{\mathbf{A}^{\mathrm{t}}} \to \mathcal{M}_2$, where $\mathbf{A}^{\mathrm{t}} = (A_1^{\mathrm{t}}, A_2^{\mathrm{t}})$. Therefore, we have

$$P_{\mathbf{A}^{\mathrm{t}}}^{(k)}(V) = \begin{pmatrix} V_3 & cV_2 \\ bV_2 & d_2 V_1 \end{pmatrix}.$$

Therefore, it is natural to ask if these two operator space structure are same. The following theorem says, in particular, that $\|P_{\mathbf{A}}^{(2)}(V)\|_{\text{op}} \neq \|P_{\mathbf{A}^{\mathrm{t}}}^{(2)}(V)\|_{\text{op}}$ if and only if $b \neq |c|$ and $1 \neq |d_2|$, for some V in $\mathcal{M}_2(\mathbf{V_A})$.

Theorem 4.13. *For $V_1 = \begin{pmatrix} v_{11} & v_{12} \\ 0 & 0 \end{pmatrix}$ and $V_2 = \begin{pmatrix} v_{21} & v_{22} \\ 0 & 0 \end{pmatrix}, V = (V_1, V_2)$ we have $\|P_{\mathbf{A}}^{(2)}(V)\|_{\text{op}} = \|P_{\mathbf{A}^{\mathrm{t}}}^{(2)}(V)\|_{\text{op}}$ if and only if either $1 = |d_2|$ or $b = |c|$.*

Proof. Note that

$$\begin{aligned}\|P_{\mathbf{A}}^{(2)}(V)\|_{\text{op}}^2 &= \left\| \begin{pmatrix} V_3 & bV_2 \\ cV_2 & d_2V_1 \end{pmatrix} \begin{pmatrix} V_3^* & \bar{c}V_2^* \\ bV_2^* & \bar{d}_2V_1^* \end{pmatrix} \right\|_{\text{op}} \\ &= \left\| \begin{pmatrix} V_3V_3^*+b^2V_2V_2^* & \bar{c}V_3V_2^*+b\bar{d}_2V_2V_1^* \\ cV_2V_3^*+bd_2V_1V_2^* & |c|^2V_2V_2^*+|\bar{d}_2|^2V_1V_1^* \end{pmatrix} \right\|_{\text{op}}. \end{aligned} \tag{4.3}$$

Similarly we have

$$\|P_{\mathbf{A}^{\mathrm{t}}}^{(2)}(V)\|_{\text{op}}^2 = \left\| \begin{pmatrix} V_3V_3^*+|c|^2V_2V_2^* & bV_3V_2^*+c\bar{d}_2V_2V_1^* \\ bV_2V_3^*+\bar{c}d_2V_1V_2^* & b^2V_2V_2^*+|\bar{d}_2|^2V_1V_1^* \end{pmatrix} \right\|_{\text{op}}. \tag{4.4}$$

We first assume that $\|P_{\mathbf{A}}^{(2)}(V)\|_{\text{op}}^2 = \|P_{\mathbf{A}^{\mathrm{t}}}^{(2)}(V)\|_{\text{op}}^2$. Therefore, the above condition is equivalent to

$$\left\| \begin{pmatrix} V_3V_3^*+b^2V_2V_2^* & \bar{c}V_3V_2^*+b\bar{d}_2V_2V_1^* \\ cV_2V_3^*+bd_2V_1V_2^* & |c|^2V_2V_2^*+|\bar{d}_2|^2V_1V_1^* \end{pmatrix} \right\|_{\text{op}} = \left\| \begin{pmatrix} V_3V_3^*+|c|^2V_2V_2^* & bV_3V_2^*+c\bar{d}_2V_2V_1^* \\ bV_2V_3^*+\bar{c}d_2V_1V_2^* & b^2V_2V_2^*+|\bar{d}_2|^2V_1V_1^* \end{pmatrix} \right\|_{\text{op}}. \tag{4.5}$$

Putting $\mathbf{v}_1 = (v_{11},\ v_{12}), \mathbf{v}_2 = (v_{21},\ v_{22})$ and $\mathbf{v}_3 = \mathbf{v}_1 + \mathbf{v}_2$ in Equation (4.5) we have

$$\left\| \begin{pmatrix} \|\mathbf{v}_3\|^2+b^2\|\mathbf{v}_2\|^2 & \bar{c}\langle\mathbf{v}_3,\mathbf{v}_2\rangle+b\bar{d}_2\langle\mathbf{v}_2,\mathbf{v}_1\rangle \\ c\langle\mathbf{v}_2,\mathbf{v}_3\rangle+bd_2\langle\mathbf{v}_1,\mathbf{v}_2\rangle & |c|^2\|\mathbf{v}_2\|^2+|\bar{d}_2|^2\|\mathbf{v}_1\|^2 \end{pmatrix} \right\|_{\text{op}} = \left\| \begin{pmatrix} \|\mathbf{v}_3\|^2+|c|^2\|\mathbf{v}_2\|^2 & b\langle\mathbf{v}_3,\mathbf{v}_2\rangle+c\bar{d}_2\langle\mathbf{v}_2,\mathbf{v}_1\rangle \\ b\langle\mathbf{v}_2,\mathbf{v}_3\rangle+\bar{c}d_2\langle\mathbf{v}_1,\mathbf{v}_2\rangle & b^2\|\mathbf{v}_2\|^2+|\bar{d}_2|^2\|\mathbf{v}_1\|^2 \end{pmatrix} \right\|_{\text{op}}. \tag{4.6}$$

The maximum eigenvalue of $\begin{pmatrix} \|\mathbf{v}_3\|^2+b^2\|\mathbf{v}_2\|^2 & \bar{c}\langle\mathbf{v}_3,\mathbf{v}_2\rangle+b\bar{d}_2\langle\mathbf{v}_2,\mathbf{v}_1\rangle \\ c\langle\mathbf{v}_2,\mathbf{v}_3\rangle+bd_2\langle\mathbf{v}_1,\mathbf{v}_2\rangle & |c|^2\|\mathbf{v}_2\|^2+|\bar{d}_2|^2\|\mathbf{v}_1\|^2 \end{pmatrix}$ is

$$x = \frac{p_1 + \sqrt{p_1^2 - 4q_1}}{2},$$

where $p_1 = \|\mathbf{v}_3\|^2 + b^2\|\mathbf{v}_2\|^2 + |c|^2\|\mathbf{v}_2\|^2 + |\bar{d}_2|^2\|\mathbf{v}_1\|^2, q_1 = (\|\mathbf{v}_3\|^2 + b^2\|\mathbf{v}_2\|^2)(|c|^2\|\mathbf{v}_2\|^2 + |\bar{d}_2|^2\|\mathbf{v}_1\|^2) - |\bar{c}\langle\mathbf{v}_3, \mathbf{v}_2\rangle + b\bar{d}_2\langle\mathbf{v}_2, \mathbf{v}_1\rangle|^2$.

Similarly the maximum eigenvalue of $\begin{pmatrix} \|\mathbf{v}_3\|^2+|c|^2\|\mathbf{v}_2\|^2 & b\langle\mathbf{v}_3,\mathbf{v}_2\rangle+c\bar{d}_2\langle\mathbf{v}_2,\mathbf{v}_1\rangle \\ b\langle\mathbf{v}_2,\mathbf{v}_3\rangle+\bar{c}d_2\langle\mathbf{v}_1,\mathbf{v}_2\rangle & b^2\|\mathbf{v}_2\|^2+|\bar{d}_2|^2\|\mathbf{v}_1\|^2 \end{pmatrix}$ is

$$y = \frac{p_1 + \sqrt{p_1^2 - 4q_2}}{2},$$

where $q_2 = (\|\mathbf{v}_3\|^2 + |c|^2\|\mathbf{v}_2\|^2)(b^2\|\mathbf{v}_2\|^2 + |\bar{d}_2|^2\|\mathbf{v}_1\|^2) - |b\langle\mathbf{v}_3, \mathbf{v}_2\rangle + c\bar{d}_2\langle\mathbf{v}_2, \mathbf{v}_1\rangle|^2$. Since we are assuming $x = y$, it follows that $q_1 = q_2$. This simplifies to the equation

$$(|c|^2 - b^2)\{(\|\mathbf{v}_3\|\|\mathbf{v}_2\|^2 - |\langle\mathbf{v}_3, \mathbf{v}_2\rangle|^2) - |\bar{d}_2|^2(\|\mathbf{v}_2\|\|\mathbf{v}_1\|^2 - |\langle\mathbf{v}_2, \mathbf{v}_1\rangle|^2)\} = 0. \tag{4.7}$$

Since $\mathbf{v}_3 = \mathbf{v}_1 + \mathbf{v}_2$, we have $(\|\mathbf{v}_3\|\|\mathbf{v}_2\|^2 - |\langle\mathbf{v}_3, \mathbf{v}_2\rangle|^2) = (\|\mathbf{v}_2\|\|\mathbf{v}_1\|^2 - |\langle\mathbf{v}_2, \mathbf{v}_1\rangle|^2)$. Hence either $b = |c|$ or $1 = |d_2|$.

Conversely if either $b = |c|$ or $1 = |d_2|$, then $q_1 = q_2$. Therefore,, we have $x = y$, that is, the two maximum eigenvalue are equal. Hence $\|P_{\mathbf{A}}^{(2)}(V)\|_{\text{op}} = \|P_{\mathbf{A}^{\mathrm{t}}}^{(2)}(V)\|_{\text{op}}$. This completes our theorem. □

Remark 4.14. From Theorem 4.13 it follows that if $b \neq |c|$ and $|d_2| \neq 1$, then $\|P_{\mathbf{A}}^{(2)}(V)\|_{\mathrm{op}} \neq \|P_{\mathbf{A}^{\mathrm{t}}}^{(2)}(V)\|_{\mathrm{op}}$ for $V \in \mathcal{M}_2(\mathbf{V_A})$ of the form $(\!(\mathbf{v}_{ij})\!), \mathbf{v}_{ij} \in \mathbf{V_A}, \mathbf{v}_{ij} = 0$ if $i > 1$. Thus $P_{\mathbf{A}}$ and $P_{\mathbf{A}^{\mathrm{t}}}$ induce different operator space structure on $\mathbf{V_A}$. Equivalently, there is a contractive homomorphism on $\mathcal{O}(\Omega_{\mathbf{A}})$, which is not completely contractive. Let $A_1 \in \{A_{11}, A_{12}\}$ and $A_2 \in \{A_{21}, A_{22}\}$, where $A_{11} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & d_2 \end{smallmatrix}\right), A_{12} = \left(\begin{smallmatrix} d_1 & 0 \\ 0 & 1 \end{smallmatrix}\right); A_{21} = \left(\begin{smallmatrix} 1 & b \\ c & 0 \end{smallmatrix}\right), A_{22} = \left(\begin{smallmatrix} 0 & b \\ c & 1 \end{smallmatrix}\right)$. We have proved the theorem for $A_1 = A_{11}$ and $A_2 = A_{21}$. The proof in the remaining cases, namely, $A_1 = A_{11}$ and $A_2 = A_{22}$; $A_1 = A_{12}$ and $A_2 = A_{21}$ and $A_1 = A_{12}$ and $A_2 = A_{22}$ follow similarly.

If we consider the case $\mathbf{A} = \left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & d_2 \end{smallmatrix}\right), A_2 = \left(\begin{smallmatrix} 0 & b \\ c & 0 \end{smallmatrix}\right)\right)$, then we have $P_{\mathbf{A}}^{(k)}(V) = \left(\begin{smallmatrix} V_1 & bV_2 \\ cV_2 & d_2V_1 \end{smallmatrix}\right)$ and $P_{\mathbf{A}^{\mathrm{t}}}^{(k)}(V) = \left(\begin{smallmatrix} V_1 & cV_2 \\ bV_2 & d_2V_1 \end{smallmatrix}\right)$. The following theorem is similar to Theorem 4.13.

Theorem 4.15. *Suppose* $V_1 = \left(\begin{smallmatrix} v_{11} & v_{12} \\ 0 & 0 \end{smallmatrix}\right)$ *and* $V_2 = \left(\begin{smallmatrix} v_{21} & v_{22} \\ 0 & 0 \end{smallmatrix}\right)$. *Then* $\|P_{\mathbf{A}}^{(2)}(V)\|_{\mathrm{op}} = \|P_{\mathbf{A}^{\mathrm{t}}}^{(2)}(V)\|_{\mathrm{op}}$ *if and only if either* $1 = |d_2|$ *or* $b = |c|$.

Proof. Note that

$$\|P_{\mathbf{A}}^{(2)}(V)\|_{\mathrm{op}}^2 = \left\|\left(\begin{smallmatrix} \|\mathbf{v}_1\|^2+b^2\|\mathbf{v}_2\|^2 & \bar{c}\langle \mathbf{v}_1,\mathbf{v}_2\rangle+b\bar{d}_2\langle \mathbf{v}_2,\mathbf{v}_1\rangle \\ c\langle \mathbf{v}_2,\mathbf{v}_1\rangle+bd_2\langle \mathbf{v}_1,\mathbf{v}_2\rangle & |c|^2\|\mathbf{v}_2\|^2+|\bar{d}_2|^2\|\mathbf{v}_1\|^2 \end{smallmatrix}\right)\right\|_{\mathrm{op}}$$

and

$$\|P_{\mathbf{A}^{\mathrm{t}}}^{(2)}(V)\|_{\mathrm{op}}^2 = \left\|\left(\begin{smallmatrix} \|\mathbf{v}_1\|^2+|c|^2\|\mathbf{v}_2\|^2 & b\langle \mathbf{v}_1,\mathbf{v}_2\rangle+c\bar{d}_2\langle \mathbf{v}_2,\mathbf{v}_1\rangle \\ b\langle \mathbf{v}_2,\mathbf{v}_1\rangle+\bar{c}d_2\langle \mathbf{v}_1,\mathbf{v}_2\rangle & b^2\|\mathbf{v}_2\|^2+|\bar{d}_2|^2\|\mathbf{v}_1\|^2 \end{smallmatrix}\right)\right\|_{\mathrm{op}}.$$

Let x, y the maximum eigen value of $\|P_{\mathbf{A}}^{(2)}(V)\|_{\mathrm{op}}^2, \|P_{\mathbf{A}^{\mathrm{t}}}^{(2)}(V)\|_{\mathrm{op}}^2$ respectively. Since we are assuming $x = y$, it follows that

$$(|c|^2 - b^2)\{\|\mathbf{v}_1\|^2\|\mathbf{v}_2\|^2 - |\langle \mathbf{v}_1, \mathbf{v}_2\rangle|^2 - |\bar{d}_2|^2(\|\mathbf{v}_2\|^2\|\mathbf{v}_1\|^2 - |\langle \mathbf{v}_2, \mathbf{v}_1\rangle|^2\} = 0. \tag{4.8}$$

Therefore, we have either $1 = |d_2|$ or $b = |c|$. The converse is also easy to verify. $\square$

Remark 4.16. As before we conclude that if $b \neq |c|$ and $1 \neq |d_2|$, then the operator structures induced by $P_{\mathbf{A}}$ and $P_{\mathbf{A}^{\mathrm{t}}}$ are not isomorphic. Let $A_1 \in \{A_{11}, A_{12}\}$ and $A_2 = \left(\begin{smallmatrix} 0 & b \\ c & 0 \end{smallmatrix}\right)$, where $A_{11} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & d_2 \end{smallmatrix}\right), A_{12} = \left(\begin{smallmatrix} d_1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. We have proved the theorem for $A_1 = A_{11}$ and $A_2 = \left(\begin{smallmatrix} 0 & b \\ c & 0 \end{smallmatrix}\right)$. The proof in the remaining case, namely, $A_1 = A_{12}$ and $A_2 = \left(\begin{smallmatrix} 0 & b \\ c & 0 \end{smallmatrix}\right)$ follows similarly. Equivalently, we also say that there exists a contractive linear map from $(\mathbb{C}^2, \|\cdot\|_{\Omega_{\mathbf{A}}})$ to $\mathcal{M}_n(\mathbb{C})$ which is not completely contractive.

This phenomenon also occurs for the Euclidean Ball as the following example.

Example 4.17. Let $\mathbf{A} = \left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)\right)$. Then $\Omega_{\mathbf{A}}$ defines Euclidean ball. As before we have $P_{\mathbf{A}}^{(k)}(V) = \left(\begin{smallmatrix} V_1 & V_2 \\ 0 & 0 \end{smallmatrix}\right)$ and $P_{\mathbf{A}^{\mathrm{t}}}^{(k)}(V) = \left(\begin{smallmatrix} V_1 & 0 \\ V_2 & 0 \end{smallmatrix}\right)$. For $V_1 = (v_{11}\ v_{12}), V_2 = (v_{21}\ v_{22})$ and $V = (V_1\ V_2)$, it is easy to verify that $\|P_{\mathbf{A}}^{(k)}(V)\|_{\mathrm{op}} \neq \|P_{\mathbf{A}^{\mathrm{t}}}^{(k)}(V)\|_{\mathrm{op}}$. Hence two embedding of $(\mathbf{V_A}, \|\cdot\|_2)$ into $\mathcal{M}_2(\mathbb{C})$ give two distinct operator space structure.

Chapter 5

Bergman kernel

We recall the definition of the well known class of operators $\mathcal{P}_n(\Omega)$ which was introduced in the foundational paper of Cowen and Douglas [10]. An alternative point of view was discussed in the paper of Curto and Salinas [12].

Definition 5.1. The class $\mathcal{P}_n(\Omega)$ consists of m-tuples of commuting bounded operators $T = (T_1, T_2, \dots T_m)$ on a Hilbert space $\mathcal{H}$ satisfying the following conditions:

- the operators $T_1, T_2, \dots, T_m$ commute,
- for $w = (w_1, \dots, w_m) \in \Omega$, the dimension of the joint kernel $\bigcap_{k=1}^m \ker(T_k - w_k)$ is n,
- for $w \in \Omega$, the operator $D_{T-w} : \mathcal{H} \to \mathcal{H} \oplus \cdots \oplus \mathcal{H}$ has closed range, where the operator D_T is defined by $D_T h = \oplus_{k=1}^m T_k h$, $h \in \mathcal{H}$,
- closed span$\{\bigcap_{k=1}^m \ker(T_k - w_k) : w \in \Omega\} = \mathcal{H}$.

Here we relate the contractivity of the homomorphism $\rho_T(w)$ naturally induced by the localization $N_T(w), w \in \Omega$, of an m-tuple of operator T in $\mathcal{P}_1(\Omega)$ to that of its curvature $\mathcal{K}(w)$ corresponding to the holomorphic Hermtian bundle corresponding to the commuting tuple T.

For an m-tuple of operators T in $\mathcal{P}_n(\Omega)$, Cowen and Douglas establish the existence of a non-zero holomorphic map $\gamma : \Omega_0 \to \mathcal{H}$ with $\gamma(w)$ in $\bigcap_{k=1}^m \ker(T_k - w_k)$, w in some open subset Ω_0 of Ω. We fix such an open set and call it Ω. The map γ defines a holomorphic Hermitian vector bundle, say E_T, on Ω. They show that the equivalence class of the vector bundle E_T determines the equivalence class (with respect to unitary equivalence) of the operator T and conversely. The determination of the equivalence class of the operator T in $\mathcal{P}_1(\Omega)$ then is particularly simple since the curvature of the line bundle E_T

$$-\mathsf{K}(w) = \sum_{i,j=1}^m \frac{\partial^2}{\partial w_i \partial \bar{w}_j} \log \|\gamma(w)\|^2 dw_i \wedge d\bar{w}_j$$

is a complete invariant. We reproduce the well-known proof of this fact for the sake of completeness.

Suppose that E is a holomorphic Hermitian line bundle over a bounded domain $\Omega \subseteq \mathbb{C}^m$. Pick a holomorphic frame γ for the line bundle E and let $\Gamma(w) = \langle \gamma_w, \gamma_w \rangle$ be the Hermitian metric. The curvature $(1,1)$ form $\mathsf{K}(w) \equiv 0$ on an open subset $\Omega_0 \subseteq \Omega$ if and only if $\log \Gamma$ is harmonic on Ω_0. Let F be a second line bundle over the same domain Ω with the metric Λ with respect to a holomorphic frame η. Suppose that the two curvatures K_E and K_F are equal on the open subset Ω_0. It then follows that $u = \log(\Gamma/\Lambda)$ is harmonic on this open subset. Thus there exists a harmonic conjugate v of u on Ω_0, which we assume without loss of generality to be simply connected. For $w \in \Omega_0$, define $\tilde{\eta}_w = e^{(u(w)+iv(w))/2}\eta_w$. Then clearly, $\tilde{\eta}_w$ is a new holomorphic frame for F. Consequently, we have the metric $\Lambda(w) = \langle \tilde{\eta}_w, \tilde{\eta}_w \rangle$ for F and we see that

$$\begin{aligned}
\Lambda(w) &= \langle \tilde{\eta}_w, \tilde{\eta}_w \rangle \\
&= \langle e^{(u(w)+iv(w))/2}\eta_w, e^{(u(w)+iv(w))/2}\eta_w \rangle \\
&= e^{u(w)} \langle \eta_w, \eta_w \rangle \\
&= \Gamma(w).
\end{aligned}$$

This calculation shows that the map $U : \eta_w \mapsto \gamma_w$ defines an isometric holomorphic bundle map between E and F. The map, as shown in (cf. [11, Theorem 1]),

$$U\Big(\sum_{|I| \leq n} \alpha_I (\bar{\partial}^I \eta)(w_0) \Big) = \sum_{|I| \leq n} \alpha_I (\bar{\partial}^I \gamma)(w_0), \ \alpha_I \in \mathbb{C}, \tag{5.1}$$

where w_0 is a fixed point in Ω and I is a multi-index of length n, is well-defined, extends to a unitary operator on the Hilbert space spanned by the vectors $(\bar{\partial}^I \eta)(w_0)$ and intertwines the two m-tuples of operators in $\mathcal{P}_1(\Omega)$ corresponding to the vector bundles E and F.

It is natural to ask what other properties of T are directly reflected in the curvature K. One such property that we explore here is the contractivity and complete contractivity of the homomorphism induced by the m-tuple T via the map $\rho_T : f \to f(T)$, $f \in \mathcal{O}(\Omega)$, where $\mathcal{O}(\Omega)$ is the set of all holomorphic function in the neighborhood of $\overline{\Omega}$.

It will be useful for us to work with the matrix of the co-efficient of the $(1,1)$ - form defining the curvature K, namely,

$$\mathcal{K}(w) := -\left(\left(\frac{\partial^2}{\partial w_i \partial \bar{w}_j} \log \|\gamma(w)\|^2 \right) \right)_{i,j=1}^{m}$$

We recall the curvature inequality from Misra and Sastry cf. [24, Theorem 5.2] and produce a large family of examples to show that the "curvature inequality" does not imply contractivity of the homomorphism ρ_T.

5.1 Localization of Cowen-Douglas operators

For T in $\mathcal{P}_1(\Omega)$, we define $\mathcal{N}(w)$ to be the subspace $\bigcap_{j,k=1}^m \ker\big((T_j - w_j)(T_k - w_k)\big)$ of $\mathcal{H}$. The localization $N(w)$ of T at w is the m-tuple $N(w) = (N_1(w), N_2(w), \ldots, N_m(w))$, where $N_k(w) = T_k - w_k|_{\mathcal{N}(w)}$. The subspace $\mathcal{N}(w)$ is easily seen to be spanned by the vectors

$$\{\gamma(w), \bar{\partial}_1\gamma(w), \ldots, \bar{\partial}_m\gamma(w)\}.$$

The localization $N(w)$ of T_k at w then has the matrix representation (recall $(T_i - w_i)\gamma(w) = 0$ and $(T_i - w_i)(\partial_j\gamma)(w) = \delta_{ij}\gamma(w)$ for $1 \le i, j \le m$) $N_k(w) = \left(\begin{smallmatrix} 0 & e_k \\ 0 & \mathbf{0} \end{smallmatrix}\right)$, $k = 1, \ldots, m$. Here $\{e_k\}_{k=1}^m$ is the standard basis of $\mathbb{C}^m$. Representing $N_k(w)$ with respect to an orthonormal basis in $\mathcal{N}(w)$, it is possible to read off the curvature of T at w using the relationship:

$$-\big(\mathcal{K}(w)^{\mathrm{t}}\big)^{-1} = \Big(\!\!\Big(\mathrm{tr}\big(N_k(w)\overline{N_j(w)}^{\mathrm{t}}\big)\Big)\!\!\Big)_{kj=1}^m = A(w)^{\mathrm{t}}\overline{A(w)}, \tag{5.2}$$

where the k^{th}-column of $A(w)$ is the vector $\boldsymbol{\alpha}_k$ (depending on w) which appears in the matrix representation of $N_k(w)$ with respect to any choice of an orthonormal basis in $\mathcal{N}(w)$.

This formula is established for a pair of operators in $\mathcal{P}_1(\Omega)$ (cf. [11, Theorem 7]). However, it is easy to verify it for an m-tuple T in $\mathcal{P}_1(\Omega)$.

Fix w_0 in Ω. We may assume without loss of generality that $\|\gamma(w_0)\| = 1$. The function $\langle\gamma(w), \gamma(w_0)\rangle$ is invertible in some neighborhood of w_0. Then setting $\hat{\gamma}(w) := \langle\gamma(w), \gamma(w_0)\rangle^{-1}\gamma(w)$, we see that

$$\langle\partial_k\hat{\gamma}(w_0), \gamma(w_0)\rangle = 0,\ \ k = 1, 2, \ldots, m.$$

Thus $\hat{\gamma}$ is another holomorphic section of E. The norms of the two sections γ and $\hat{\gamma}$ differ by the absolute square of a holomorphic function, that is, $\frac{\|\hat{\gamma}(w)\|}{\|\gamma(w)\|} = |\langle\gamma(w), \gamma(w_0)\rangle|$. Hence the curvature is independent of the choice of the holomorphic section. Therefore, without loss of generality, we will prove the claim assuming: for a fixed but arbitrary w_0 in Ω,

(i) $\|\gamma(w_0)\| = 1$,

(ii) $\gamma(w_0)$ is orthogonal to $(\partial_k\gamma)(w_0)$, $k = 1, 2, \ldots, m$.

Let G be the Grammian corresponding to the $m+1$ dimensional space spanned by the vectors

$$\{\gamma(w_0), (\partial_1\gamma)(w_0), \ldots, (\partial_m\gamma)(w_0)\}.$$

This is just the space $\mathcal{N}(w_0)$. Let v, w be any two vectors in $\mathcal{N}(w_0)$. Find $\boldsymbol{c} = (c_0, \ldots, c_m)$, $\boldsymbol{d} = (d_0, \ldots, d_m)$ in $\mathbb{C}^{m+1}$ such that $v = \sum_{i=0}^m c_i\partial_i\gamma(w_0)$ and $w = \sum_{j=0}^m d_j\partial_j\gamma(w_0)$. Here $(\partial_0\gamma)(w_0) =$

$\gamma(w_0)$. We have

$$\begin{aligned}\langle v,w\rangle_G &= \langle \sum_{i=0}^m c_i\partial_i\gamma(w_0), \sum_{j=0}^m d_j\partial_j\gamma(w_0)\rangle\\ &= \langle G^{\mathrm{t}}(w_0)\boldsymbol{c},\boldsymbol{d}\rangle_{\mathbb{C}^{m+1}}\\ &= \langle (G^{\mathrm{t}})^{\frac12}(w_0)\boldsymbol{c},(G^{\mathrm{t}})^{\frac12}(w_0)\boldsymbol{d}\rangle_{\mathbb{C}^{m+1}}.\end{aligned}$$

Let $\{e_i\}_{i=0}^m$ be the standard orthonormal basis for $\mathbb{C}^{m+1}$. Also, let $(G^{\mathrm{t}})^{-\frac12}(w_0)e_i := \boldsymbol{\alpha}_i(w_0)$, where $\boldsymbol{\alpha}_i(j)(w_0) = \alpha_{ji}(w_0)$, $i = 0,1,\dots,m$. We see that the vectors $\varepsilon_i := \sum_{j=0}^m \alpha_{ji}(\partial_j\gamma)(w_0)$, $i = 0,1,\dots,m$, form an orthonormal basis in $\mathcal{N}(w_0)$:

$$\begin{aligned}\langle \varepsilon_i,\varepsilon_l\rangle &= \Big\langle \sum_{j=0}^m \alpha_{ij}\partial_j\gamma(w_0), \sum_{p=0}^m \alpha_{lp}\partial_p\gamma(w_0)\Big\rangle\\ &= \langle (G^{\mathrm{t}})^{-\frac12}\boldsymbol{\alpha}_i,(G^{\mathrm{t}})^{-\frac12}(w_0)\boldsymbol{\alpha}_l\rangle_G(w_0)\\ &= \delta_{il},\end{aligned}$$

where δ_{il} is the Kornecker delta. Since $N_k\big((\partial_j\gamma)(w_0)\big) = \gamma(w_0)$ for $j = k$ and 0 otherwise, we have $N_k(\varepsilon_i) = \left(\begin{smallmatrix} 0 & \boldsymbol{\alpha}_k^{\mathrm{t}}\\ 0 & 0\end{smallmatrix}\right)$. Hence

$$\begin{aligned}\operatorname{tr}\big(N_i(w_0)N_j^*(w_0)\big) &= \boldsymbol{\alpha_i}(w_0)^{\mathrm{t}}\overline{\boldsymbol{\alpha}_j}(w_0)\\ &= \big((G^{\mathrm{t}})^{-\frac12}(w_0)e_i\big)^{\mathrm{t}}\overline{\big((G^{\mathrm{t}})^{-\frac12}(w_0)e_j\big)}\\ &= \langle G^{-\frac12}(w_0)e_i, G^{-\frac12}e_j(w_0)\rangle = (G^{\mathrm{t}})^{-1}(w_0)_{ij}.\end{aligned}$$

Since the curvature, computed with respect to the holomorphic section γ satisfying the conditions *(i)* and *(ii)*, is of the form

$$\begin{aligned}\mathcal{K}(w_0)_{ij} &= \frac{\partial^2}{\partial w_i\bar{\partial}w_j}\log\|\gamma(w)\|^2_{|w=w_0}\\ &= \Big(\frac{\|\gamma(w)\|^2\big(\frac{\partial^2\gamma}{\partial w_i\partial\bar{w}_j}\big)(w) - \big(\frac{\partial\gamma}{\partial w_i}\big)(w)\big(\frac{\partial\gamma}{\partial\bar{w}_j}\big)(w)}{\|\gamma(w)\|^4}\Big)_{|w=w_0}\\ &= \big(\frac{\partial^2\gamma}{\partial w_i\partial\bar{w}_j}\big)(w_0) = G(w_0)_{ij},\end{aligned}$$

we have verified the claim (5.2).

The following theorem was proved for $m = 2$ in (cf. [11, Theorem 7]). However, for any natural number m, the proof is evident from the preceding discussion.

Theorem 5.2. *Two m-tuples of operators T and $\tilde{T}$ in $\mathcal{P}_1(\Omega)$ are unitarily equivalent if and only if $N_k(w)$ and $\tilde{N}_k(w)$ are simultaneously unitarily equivalent for w in some open subset of Ω.*

Proof. Let us fix an arbitrary point w in Ω. In what follows, the dependence on this w is implicit. Suppose that there exists a unitary operator $U : \mathcal{N} \to \widetilde{\mathcal{N}}$ such that $UN_i = \tilde{N}_i U$, $i = 1, \ldots, m$. For $1 \leq i, j \leq m$, we have

$$\begin{aligned}\operatorname{tr}\big(\tilde{N}_i \tilde{N}_j^{\,*}\big) &= \operatorname{tr}\big(\big(UN_iU^*\big)\big(UN_jU^*\big)^*\big)\\ &= \operatorname{tr}\big(UN_iN_j^*U^*\big)\\ &= \operatorname{tr}\big(N_iN_j^*U^*U\big)\\ &= \operatorname{tr}\big(N_iN_j^*\big).\end{aligned}$$

Thus the curvature of the operators T and $\tilde{T}$ coincide making them unitarily equivalent proving the Theorem in one direction. In the other direction, we simply have to observe that if the operators T and $\tilde{T}$ are unitarily equivalent then the unitary U given in (5.1) evidently maps $\mathcal{N}$ to $\tilde{\mathcal{N}}$. Thus the restriction of U to the subspace $\mathcal{N}$ intertwines N_k and $\tilde{N}_k$ simultaneously for $k = 1, \cdots, m$. □

As is well-known (cf. [12] and [10]), the m-tuple T in $\mathcal{P}_1(\Omega)$ can be represented as the adjoint of the m-tuple of multiplications M by the co-ordinate functions on a Hilbert space $\mathcal{H}$ of holomorphic functions defined on $\Omega^* = \{\bar{w} \in \mathbb{C}^m : w \in \Omega\}$ possessing a reproducing kernel $K : \Omega^* \times \Omega^* \to \mathbb{C}$ which is holomorphic in the first variable and anti-holomorphic in the second.

In this representation, if we set $\gamma(w) = K(\cdot, \bar{w})$, $w \in \Omega$, then we obtain a natural non-vanishing "holomorphic" map into the Hilbert space $\mathcal{H}$ defined on Ω.

The localization $N(w)$ obtained from the commuting tuple of operators T defines a a homomorphism ρ_T on the algebra $\mathcal{O}(\Omega)$ of functions, holomorphic in some neighborhood of the closed set $\bar{\Omega}$, by the rule

$$\rho_T(f) = \begin{pmatrix} f(w) & \nabla f(w) A(w)^{\mathrm{t}} \\ 0 & f(w) I_m \end{pmatrix}, \ f \in \mathcal{O}(\Omega). \tag{5.3}$$

We recall from (cf. [24, Theorem5.2]) that the contractivity of the homomorphism implies the curvature inequality $\|\big(\mathcal{K}(w)^{\mathrm{t}}\big)^{-1}\| \leq 1$. Here $\mathcal{K}(w)$ is thought of as a linear transformation from the normed linear space $(\mathbb{C}^m, \mathcal{C}_{\Omega,w})^*$ to the normed linear space $(\mathbb{C}^m, \mathcal{C}_{\Omega,w})$. The operator norm is computed accordingly with respect to these norms.

5.1.1 Infinite divisibility

Let K be a positive definite kernel defined on the domain Ω and let $\lambda > 0$ be arbitrary. Since K^λ is a real analytic function defined on Ω, it admits a power series representation of the form

$$K^\lambda(w, w) = \sum_{I\,J} a_{I\,J}(\lambda)(w - w_0)^I \overline{(w - w_0)}^J$$

in a small neighborhood of a fixed but arbitrary $w_0 \in \Omega$. The polarization $K^\lambda(z,w)$ is the function represented by the power series

$$K^\lambda(z,w) = \sum_{I\,J} a_{I\,J}(\lambda)(z-w_0)^I\overline{(w-w_0)}^J,\ w_0 \in \Omega.$$

It follows that the polarization $K^\lambda(z,w)$ of the function $K(w,w)^\lambda$ defines a Hermitian kernel, that is, $K^\lambda(z,w) = \overline{K(w,z)^\lambda}$. Schur's Lemma (cf. [9]) ensures the positive definiteness of K^λ whenever λ is a natural number. However, it is not necessary that K^λ must be positive definite for all real $\lambda > 0$. Indeed a positive definite kernel K with the property that K^λ is positive definite for all $\lambda > 0$ is called infinitely divisible and plays an important role in studying curvature inequalities (cf. [6, Theorem 3.3]).

Although, K^λ need not be positive definite for all $\lambda > 0$, in general, a related question raised here is relevant to the study of localization of the Cowen-Douglas operators.

Let w_0 in Ω be fixed but arbitrary. Also, fix a $\lambda > 0$. Define the mutual inner product of the vectors

$$\{(\bar{\boldsymbol{\partial}}^I K^\lambda)(\cdot,w_0) : I = (i_1,\ldots,i_m)\},$$

by the formula

$$\langle(\bar{\boldsymbol{\partial}}^J K^\lambda)(\cdot,w_0),(\bar{\boldsymbol{\partial}}^I K^\lambda)(\cdot,w_0)\rangle = \big(\boldsymbol{\partial}^I\bar{\boldsymbol{\partial}}^J K^\lambda\big)(w_0,w_0).$$

Now, if K^λ were positive definite, for the λ we have picked, then this formula would extend to an inner product on the linear span of these vectors. The completion of this inner product space is then a Hilbert space, which we denote by $\mathcal{H}^{(\lambda)}$. The reproducing kernel for the Hilbert space $\mathcal{H}^{(\lambda)}$ is easily verified to be the original kernel K^λ. The Hilbert space $\mathcal{H}^{(\lambda)}$ is independent of the choice of w_0.

Now, even if K^λ is not necessarily positive definite, we may ask whether this formula defines an inner product on the $(m+1)$ dimensional space $\mathcal{N}^{(\lambda)}(w)$ spanned by the vectors

$$\{K^\lambda(\cdot,w),(\bar\partial_1 K^\lambda)(\cdot,w),\ldots,(\bar\partial_m K^\lambda)(\cdot,w)\}.$$

An affirmative answer to this question is equivalent to the positive definiteness of the matrix

$$\big(\!\!\big(\,(\partial_i\bar\partial_j K^\lambda)(w,w)\big)\!\!\big)_{i\,j=0}^m.$$

Let $\bar{\boldsymbol{\partial}}_m^{\mathrm{t}} = \Big(1,\partial_1,\ldots,\partial_m\Big)$ and $\boldsymbol{\partial}_m$ be its conjugate transpose. Now,

$$\big(\boldsymbol{\partial}_m\bar{\boldsymbol{\partial}}_m^{\mathrm{t}} K^\lambda\big)(w,w) := \big(\!\!\big(\,(\partial_j\bar\partial_i K^\lambda)(w,w)\big)\!\!\big)_{i\,j=0}^m,\ w \in \Omega \subseteq \mathbb{C}^m.$$

Lemma 5.3. *For a fixed but arbitrary w in Ω, the $(m+1)\times(m+1)$ matrix $\big(\boldsymbol{\partial}_m\bar{\boldsymbol{\partial}}_m^{\mathrm{t}} K^\lambda\big)(w,w)$ is positive definite.*

Proof. The proof is by induction on m. For $m=1$ and any positive λ, a direct verification, which follows, shows that

$$\big(\boldsymbol{\partial}_1\bar{\boldsymbol{\partial}}_1^{\mathrm{t}}K^\lambda\big)(w,w) := \left(\begin{smallmatrix} K^\lambda(w,w) & \partial_1 K^\lambda(w,w) \\ \bar\partial_1 K^\lambda(w,w) & \partial_1\bar\partial_1 K^\lambda(w,w)\end{smallmatrix}\right)$$

is positive.

Since $K^\lambda(w,w)>0$ for any $\lambda>0$, the verification that $\big(\boldsymbol{\partial}_1\bar{\boldsymbol{\partial}}_1^{\mathrm{t}}K^\lambda\big)(w,w)$ is positive definite amounts to showing that $\det\big(\boldsymbol{\partial}_1\bar{\boldsymbol{\partial}}_1^{\mathrm{t}}K^\lambda\big)(w,w)>0$. An easy computation gives

$$\begin{aligned}\det\big(\boldsymbol{\partial}_1\bar{\boldsymbol{\partial}}_1^{\mathrm{t}}K^\lambda\big)(w,w) &= \lambda K^{2\lambda-2}(w,w)\big\{K(w,w)(\bar\partial_1\partial_1 K)(w,w)-|\partial_1 K(w,w)|^2\big\}\\ &= \lambda K^{2\lambda}(w,w)\frac{\|K(\cdot,w)\|^2\|(\bar\partial_1 K)(\cdot,w)\|^2-|\langle K(\cdot,w),(\bar\partial_1 K)(\cdot,w)\rangle|^2}{\|K(\cdot,w)\|^4},\end{aligned}$$

which is clearly positive since $K(\cdot,w)$ and $(\bar\partial_1 K)(\cdot,w)$ are linearly independent.

Now assume that $\big(\boldsymbol{\partial}_{m-1}\bar{\boldsymbol{\partial}}_{m-1}^{\mathrm{t}}K^\lambda\big)(w,w)$ is positive definite. We note that

$$\big(\boldsymbol{\partial}_m\bar{\boldsymbol{\partial}}_m^{\mathrm{t}}K^\lambda\big)(w,w) = \left(\begin{smallmatrix} \big(\boldsymbol{\partial}_{m-1}\bar{\boldsymbol{\partial}}_{m-1}^{\mathrm{t}}K^\lambda\big)(w,w) & \big(\partial_m\bar{\boldsymbol{\partial}}_{m-1}^{\mathrm{t}}K^\lambda\big)(w,w) \\ \big(\boldsymbol{\partial}_{m-1}\bar\partial_m K^\lambda\big)(w,w) & (\partial_m\bar\partial_m K^\lambda)(w,w)\end{smallmatrix}\right).$$

Since $\big(\boldsymbol{\partial}_{m-1}\bar{\boldsymbol{\partial}}_{m-1}^{\mathrm{t}}K^\lambda\big)(w,w)$ is positive definite by the induction hypothesis and for $\lambda>0$, we have

$$(\partial_m\bar\partial_m K^\lambda)(w,w)=\lambda K(w,w)^{\lambda-2}\big\{K(w,w)(\partial_m\bar\partial_m K)w,w)+(\lambda-1)|(\bar\partial_m K)(w,w)|^2\big\}>0,$$

it follows that $\big(\boldsymbol{\partial}_m\bar{\boldsymbol{\partial}}_m^{\mathrm{t}}K^\lambda\big)(w,w)$ is positive definite if and only if $\det\Big(\big(\boldsymbol{\partial}_m\bar{\boldsymbol{\partial}}_m^{\mathrm{t}}K^\lambda\big)(w,w)\Big)>0$ (cf. [6]). To verify this claim, we note

$$\big(\boldsymbol{\partial}_m\bar{\boldsymbol{\partial}}_m^{\mathrm{t}}K^\lambda\big)(w,w) = \left(\begin{smallmatrix} K^\lambda(w,w) & B(w,w) \\ B(w,w)^* & D(w,w)\end{smallmatrix}\right),$$

where $D=\big(\!\!\big((\partial_j\bar\partial_i K^\lambda)(w,w)\big)\!\!\big)_{i\,j=1}^m$ and $B=\big(\partial_1 K^\lambda(w,w),\ldots,\partial_m K^\lambda(w,w)\big)$. Recall that (cf. [17])

$$\det\big(\boldsymbol{\partial}_m\bar{\boldsymbol{\partial}}_m^{\mathrm{t}}K^\lambda\big)(w,w)=\det\Big(D(w,w)-\frac{B^*(w,w)B(w,w)}{K^\lambda(w,w)}\Big)\det K^\lambda(w,w).$$

Now, following (cf. [6, proposition 2.1(second proof)]), we see that

$$D(w,w)-\frac{B^*(w,w)B(w,w)}{K^\lambda(w,w)}=\lambda K^{2\lambda-2}(w,w)\big(\!\!\big(K^2(w,w)(\partial_j\,\bar\partial_i\log K)(w,w)\big)\!\!\big)_{i\,j=1}^m,$$

which was shown to be a Grammian. Thus $D(w,w)-\frac{B^*(w,w)B(w,w)}{K^\lambda(w,w)}$ is a positive definite matrix and hence its determinant is positive. $\square$

5.2 Explicit formulae

For any bounded open connected subset Ω of $\mathbb{C}^m$, let $\boldsymbol{B}_\Omega$ denote the Bergman kernel of Ω. This is the reproducing kernel of the Bergman space $\mathbb{A}^2(\Omega)$ consisting of square integrable holomorphic functions on Ω with respect to the volume measure. Consequently, it has a representation of the form

$$\boldsymbol{B}_\Omega(z,w) = \sum_k \varphi_k(z)\overline{\varphi_k(w)}, \tag{5.4}$$

where $\{\varphi_k\}_{k=0}^\infty$ is any orthonormal basis of $\mathbb{A}^2(\Omega)$. This series is uniformly convergent on compact subsets of $\Omega \times \Omega$.

We now exclusively study the case of the Bergman kernel on the unit ball $\mathcal{D}$ (with respect to the usual operator norm) in the linear space of all $r \times s$ matrices $\mathcal{M}_{rs}(\mathbb{C})$. The unit ball $\mathcal{D}$ may be also described as

$$\mathcal{D} = \{Z \in \mathcal{M}_{rs}(\mathbb{C}) : I - ZZ^* \geq 0\}.$$

The Bergman Kernel for the domain $\mathcal{D}$ is $\boldsymbol{B}_\mathcal{D}(Z,Z) = \det(I - ZZ^*)^{-p}$, where $p = r+s$. In what follows we give a simple proof of this.

As an immediate consequence of the change of variable formula for integration, we have the transformation rule for the Bergman kernel. We provide the straightforward proof.

Lemma 5.4. *Let Ω and $\tilde\Omega$ be two domains in $\mathbb{C}^m$ and $\varphi : \Omega \to \tilde\Omega$ be a bi-holomorphic map. Then*

$$\boldsymbol{B}_\Omega(z,w) = J_\mathbb{C}\varphi(z)\overline{J_\mathbb{C}\varphi(w)}\boldsymbol{B}_{\tilde\Omega}(\varphi(z),\varphi(w))$$

for all $z, w \in \Omega$, where $J_\mathbb{C}\varphi(w)$ is the determinant of the derivative $D\varphi(w)$.

Proof. Suppose $\{\tilde\phi_n\}$ be an orthonormal basis for $\mathbb{A}^2(\tilde\Omega)$. By change of variable formula, it follows easily that $\phi_n = \{J_\mathbb{C}\varphi(w)\tilde\phi_n \circ \varphi\}$, form an orthonormal basis for $\mathbb{A}^2(\Omega)$. Hence,

$$\begin{aligned}\boldsymbol{B}_\Omega(z,w) = \sum_{n=0}^\infty \phi_n(z)\overline{\phi_n(w)} &= \sum_{n=0}^\infty J_\mathbb{C}\varphi(w)(\tilde\phi_n \circ \varphi)(z)\overline{J_\mathbb{C}\varphi(w)(\tilde\phi_n \circ \varphi)(w)} \\ &= J_\mathbb{C}\varphi(w)\overline{J_\mathbb{C}\varphi(w)}\sum_{n=0}^\infty \tilde\phi_n(\varphi(z))\overline{\tilde\phi_n(\varphi(w))} \\ &= J_\mathbb{C}\varphi(w)\overline{J_\mathbb{C}\varphi(w)}\boldsymbol{B}_{\tilde\Omega}(\varphi(z),\varphi(w))\end{aligned}$$

completing our proof. □

If Ω is a domain in $\mathbb{C}^m$ and the bi-holomorphic automorphism group, $\mathrm{Aut}(\Omega)$ is transitive, then we can determine the Bergman kernel as well as its curvature from its value at 0! A domain with this property is called homogeneous. For instance, the unit ball $\mathcal{D}$ in the linear space of $r \times s$ matrices are homogeneous. If Ω is homogeneous, then for any $w \in \Omega$, there exists an bi-holomorphic automorphism φ_w with the property $\varphi_w(w) = 0$. The following Corollary is an immediate consequence of Lemma 5.4.

Corollary 5.5. *For any homogeneous domain Ω in $\mathbb{C}^m$, we have*

$$\boldsymbol{B}_\Omega(w,w) = J_{\mathbb{C}}\varphi_w(w)\overline{J_{\mathbb{C}}\varphi_w(w)}\boldsymbol{B}_\Omega(0,0),\ w \in \Omega.$$

We recall from (cf. [18, Theorem 2]) that for Z, W in the matrix ball $\mathcal{D}$ (of size $r \times s$) and $\mathbf{u} \in \mathbb{C}^{r\times s}$, we have

$$D\varphi_W(Z) \cdot \mathbf{u} = (I - WW^*)^{\frac{1}{2}}(I - ZW^*)^{-1}\mathbf{u}(I - W^*Z)^{-1}(I - W^*W)^{\frac{1}{2}}.$$

In particular, $D\varphi_W(W) \cdot u = (I - WW^*)^{-\frac{1}{2}}\mathbf{u}(I - W^*W)^{-\frac{1}{2}}$. Thus $D\varphi_W(W) = (I - WW^*)^{-\frac{1}{2}} \otimes (I - W^*W)^{-\frac{1}{2}}$. We therefore (cf. [16, exercise 8] [15]) have

$$\begin{aligned}\det D\varphi_W(W) &= \big(\det(I - WW^*)^{-\frac{1}{2}}\big)^s\big(\det(I - W^*W)^{-\frac{1}{2}}\big)^r\\ &= \big(\det(I - WW^*)^{-\frac{1}{2}}\big)^{r+s}.\end{aligned}$$

It then follows that

$$J_{\mathbb{C}}\varphi_W(W)\overline{J_{\mathbb{C}}\varphi_W(W)} = \det(I - WW^*)^{-(r+s)},\ W \in \mathcal{D}.$$

With a suitable normalization of the volume measure, we may assume that $\boldsymbol{B}_\Omega(0,0) = 1$. With this normalization, we have

$$\boldsymbol{B}_{\mathcal{D}}(W,W) = \det(I - WW^*)^{-(r+s)},\ W \in \mathcal{D}. \tag{5.5}$$

The Bergman kernel $\boldsymbol{B}_\Omega$, where $\Omega = \{(z_1, z_2) : |z_2| \leq (1 - |z_1|^2)\} \subset \mathbb{C}^2$ is known (cf. [19, Example 6.1.6]):

$$\boldsymbol{B}_\Omega(z,w) = \frac{3(1 - z_1\bar{w}_1)^2 + z_2\bar{w}_2}{\{(1 - z_1\bar{w}_1)^2 - z_2\bar{w}_2\}^3},\ z, w \in \Omega. \tag{5.6}$$

The domain Ω is not homogeneous. However, it is a Reinhadt domain. Consequently, an orthonormal basis consisting of monomials exists in the Bergman space of this domain. We give a very similar example below to show that computing the Bergman kernel in a closed form may not be easy even for very simple Reinhadt domains. We take Ω to be the domain

$$\{(z_1, z_2, z_3) : |z_2|^2 \leq (1 - |z_1|^2)(1 - |z_3|^2), 1 - |z_3|^2 \geq 0\} \subset \mathbb{C}^3.$$

Lemma 5.6. *The Bergman kernel $\boldsymbol{B}_\Omega(z,w)$ for the domain Ω is given by the formula*

$$\sum_{p,m,n=0}^{\infty} \frac{m+1}{4\beta(n+1,m+2)\beta(p+1,m+2)}(z_1\bar{w}_1)^n(z_2\bar{w}_2)^m(z_3\bar{w}_3)^p,$$

where $\beta(m,n)$ is the Beta function.

Proof. Let $\{(z_1)^n(z_2)^m(z_3)^p\}_{n,m,p=1}^{\infty}$ be the orthonormal basis for the Bergman space $\mathbb{A}^2(\Omega)$. Now,

$$\begin{aligned}
\|(z_1)^n(z_2)^m(z_3)^p\|^2 &= \int_0^{2\pi} d\theta_1 d\theta_2 d\theta_3 \int_0^1 r_1^{(2n+1)}dr_1 \int_0^1 r_3^{(2p+1)}dr_3 \int_0^{\sqrt{(1-r_1^2)(1-r_3^2)}} r_2^{(2m+1)}dr_2 \\
&= 8\pi^3 \int_0^1 r_1^{(2n+1)}dr_1 \int_0^1 r_3^{(2p+1)}dr_3 \frac{(1-r_1^2)^{(m+1)}(1-r_2^2)^{(m+1)}}{2m+2} \\
&= \frac{\pi^3}{m+1}\int_0^1 s_1^n(1-s_1)^{(m+1)}ds_1 \int_0^1 s_2^p(1-s_2)^{(m+1)}ds_2 \qquad (5.7)
\end{aligned}$$

where $r_1^2 = s_1$ and $r_2^2 = s_2$. Since $\beta(n,m) = \int_0^1 r^{(n-1)}(1-r)^{(m-1)}dr$, therefore equation (5.7)is equal to

$$\|(z_1)^n(z_2)^m(z_3)^p\|^2 = \frac{\pi^3}{m+1}\beta(n+1,m+2)\beta(p+1,m+2).$$

From equation (5.7), it follows that $\|1\|^2\pi^3\beta(1,2)\beta(1,2) = \frac{\pi^3}{4}$. We normalize the volume measure in an appropriate manner to ensure

$$\|(z_1)^n(z_2)^m(z_3)^p\|^2 = \frac{4}{m+1}\beta(n+1,m+2)\beta(p+1,m+2).$$

Having computed an orthonormal basis for the Bergman space, we can complete the the computation of the Bergman kernel using the infinite expansion (5.4). $\square$

The Proposition following the Lemma (a change of variable formula from (cf. [31, The chain rule 1.3.3]) given below provides the transformation rule for the Bergman metric (cf. [20, proposition 1.4.12]).

Lemma 5.7. *Suppose Ω is in $\mathbb{C}^m, F = (f_1, \ldots, f_n)$ maps Ω into $\mathbb{C}^n, g$ maps the range of F into $\mathbb{C}$, and $f_1, \ldots, f_n, g$ are of class $\mathcal{C}^2$. If*

$$h = g \circ F = g(f_1, \ldots, f_n)$$

then, for $1 \le i, j \le m$ and $z \in \Omega$,

$$\big(\overline{D}_j D_i h\big)(z) = \sum_{k=1}^{n}\sum_{l=1}^{n} \big(\overline{D}_l D_k h\big)(w)\overline{D_j f_l}(z) D_i f_k(z),$$

where $\overline{D}_j \bar{f}_l = \overline{D_j f_l}(z)$.

Proposition 5.8. *Let Ω and $\tilde{\Omega}$ be two domain in $\mathbb{C}^m$ and $\varphi : \Omega \to \tilde{\Omega}$ is bi-holomorphic map. Then*
$$\mathcal{K}_{\boldsymbol{B}_\Omega}(w) = \big(D\varphi\big)(w)^{\mathrm{t}} \mathcal{K}_{\boldsymbol{B}_{\widetilde{\Omega}}}(\varphi(w)) \overline{\big(D\varphi\big)(w)}$$
for all $w \in \Omega$.

Proof. For any holomorphic function φ defined on Ω, we have $\frac{\partial}{\partial w_i \partial \bar{w}_j} \log |J_{\mathbb{C}}\varphi(w)|^2 = 0$. Combining this with Lemma 5.4, we get
$$\begin{aligned} \frac{\partial^2}{\partial w_i \partial \bar{w}_j} \log \boldsymbol{B}_{\widetilde{\Omega}}(\varphi(w), \varphi(w)) &= \frac{\partial^2}{\partial w_i \partial \bar{w}_j} \log |J_{\mathbb{C}}\varphi(w)|^{-2} \boldsymbol{B}_\Omega(w, w) \\ &= -\frac{\partial^2}{\partial w_i \partial \bar{w}_j} \log |J_{\mathbb{C}}\varphi(w)|^2 + \frac{\partial^2}{\partial w_i \partial \bar{w}_j} \boldsymbol{B}_\Omega(w, w) \\ &= \frac{\partial^2}{\partial w_i \partial \bar{w}_j} \boldsymbol{B}_\Omega(w, w). \end{aligned}$$
Also by Lemma 5.7 with $g(z) = \log \boldsymbol{B}_{\widetilde{\Omega}}(z, z)$ and $F = f$ we have,
$$\frac{\partial^2}{\partial w_i \partial \bar{w}_j} \log \boldsymbol{B}_{\widetilde{\Omega}}(\varphi(w), \varphi(w)) = \sum_{k,l=1}^{n} \frac{\partial \varphi_k}{\partial w_i}(w) \frac{\partial^2}{\partial w_k \partial \bar{z}_l} \log \boldsymbol{B}_{\widetilde{\Omega}}(z, z)(\varphi(w), \varphi(w)) \frac{\partial \varphi_l}{\partial w_j}(w).$$
Hence
$$\begin{aligned} &\left(\!\!\left(\frac{\partial^2}{\partial w_i \partial \bar{w}_j} \log \boldsymbol{B}_{\widetilde{\Omega}}(\varphi(w), \varphi(w) \right)\!\!\right)_{ij} \\ &\quad = \left(\!\!\left(\frac{\partial \varphi_k}{\partial w_i}(w) \right)\!\!\right)_{ik} \left(\!\!\left(\frac{\partial^2}{\partial z_k \partial \bar{z}_l} \log \boldsymbol{B}_{\widetilde{\Omega}}(z, z)(\varphi(w), \varphi(w) \right)\!\!\right)_{kl} \left(\!\!\left(\frac{\partial \varphi_l}{\partial w_j}(w) \right)\!\!\right)_{lj} \\ &\quad = \big(D\varphi\big)(w)^{\mathrm{t}} \mathcal{K}_{\boldsymbol{B}_{\widetilde{\Omega}}}(\varphi(w)) \overline{\big(D\varphi\big)(w)}. \end{aligned}$$
Therefore we have the desired transformation rule for the Bergman metric, namely,
$$\mathcal{K}_{\boldsymbol{B}_\Omega}(w) = \big(D\varphi\big)(w)^{\mathrm{t}} \mathcal{K}_{\boldsymbol{B}_{\widetilde{\Omega}}}(\varphi(w)) \overline{\big(D\varphi\big)(w)}, \; w \in \Omega.$$
□

As a consequence of this transformation rule, a formula for the Bergman metric at an arbitrary w in Ω is obtained from its value at 0. The proof follows from the transitivity of the automorphism group.

Corollary 5.9. *For a homogeneous domain Ω, pick a a bi-holomorphic automorphism φ_w of Ω with $\varphi_w(w) = 0$, $w \in \Omega$, we have*
$$\mathcal{K}_{\boldsymbol{B}_\Omega}(w) = \big(D\varphi_w(w)\big)^{\mathrm{t}} \mathcal{K}_{\boldsymbol{B}_\Omega}(0) \overline{D\varphi_w(w)}$$
for all $w \in \Omega$.

For the matrix ball $\mathcal{D}$, as is well-known, $\boldsymbol{B}_{\mathcal{D}}^{\lambda}$ is not necessarily positive definite for all $\lambda > 0$. However, as we have pointed out the space $\mathcal{N}^{(\lambda)}(w)$ has a natural inner product induced by $\boldsymbol{B}_{\mathcal{D}}^{\lambda}$. Thus we explore properties of $\boldsymbol{B}_{\mathcal{D}}^{\lambda}$ for all $\lambda > 0$. In what follows, we will repeatedly use the transformation rule for $\boldsymbol{B}_{\Omega}^{\lambda}$ which is an immediate consequence of the transformation rule for $\boldsymbol{B}_{\Omega}$, namely,

$$\mathcal{K}_{\boldsymbol{B}_{\Omega}^{\lambda}}(w) = \lambda \mathcal{K}_{\boldsymbol{B}_{\Omega}}(w) = \lambda D\varphi_w(w)^{\mathrm{t}} \mathcal{K}_{\boldsymbol{B}_{\Omega}}(0) D\varphi_w(w) \tag{5.8}$$

for $w \in \Omega$ and $\lambda > 0$.

To compute the Bergman metric, we begin with a Lemma on the Taylor expansion of the determinant. To facilitate its proof, for Z in $\mathcal{M}_{rs}(\mathbb{C})$, we write $Z = \left(\begin{smallmatrix} Z_1 \\ \vdots \\ Z_r \end{smallmatrix}\right)$, with $Z_i = (z_{i1}, \ldots, z_{is})$, $i = 1, \ldots, r$. In this notation,

$$I - ZZ^* = \left(\begin{smallmatrix} 1-\|Z_1\|^2 & -\langle Z_1, Z_2\rangle & \cdots & -\langle Z_1, Z_r\rangle \\ \vdots & \vdots & \vdots & \vdots \\ -\langle Z_r, Z_1\rangle & -\langle Z_r, Z_2\rangle & \cdots & 1-\|Z_r\|^2 \end{smallmatrix}\right),$$

where $\|Z_i\|^2 = \sum_{j=1}^{s} |z_{ij}|^2, \langle Z_i, Z_j\rangle = \sum_{k=1}^{s} z_{ik}\bar{z}_{jk}$. Set $X_{ij} = \langle Z_i, Z_j\rangle, 1 \le i, j \le r$.

The curvature $\mathcal{K}_{\boldsymbol{B}_{\mathcal{D}}}(0)$ of the Bergman kernel, which is often called the Bergman metric, is easily seen to be p times the $rs \times rs$ identity as a consequence of the following Lemma. The value of the curvature $\mathcal{K}_{\boldsymbol{B}_{\mathcal{D}}}(W)$ at an arbitrary point W is then easy to write down using the homogeneity of the unit ball $\mathcal{D}$.

Lemma 5.10. *The determinant* $\det(I - ZZ^*) = 1 - \sum_{i=1}^{r} \|Z_i\|^2 + P(X)$, *where* $P(X) = \sum_{|\ell| \ge 2} p_\ell X^\ell$ *with*

$$X^\ell := X_{11}^{\ell_{11}} \ldots X_{1r}^{\ell_{1r}} \ldots X_{r1}^{\ell_{r1}} \ldots X_{rr}^{\ell_{rr}}.$$

Proof. The proof is by induction on r. For $r = 1$ we have $\det(I - ZZ^*) = 1 - \|Z\|^2$. Therefore in this case , $P = 0$ and we are done. For $r = 2$, we have

$$\det(I - ZZ^*) = \det \left(\begin{smallmatrix} 1-\|Z_1\|^2 & -\langle Z_1, Z_2\rangle \\ -\langle Z_2, Z_1\rangle & 1-\|Z_2\|^2 \end{smallmatrix}\right).$$

For $r = 2$, a direct verification shows that the $\det(I - ZZ^*)$ is equal to $1 - \sum_{i=1}^{2} \|Z_i\|^2 + P(X)$, where $P(X) = X_{11}X_{22} - |X_{12}|^2$. The decomposition

$$I - ZZ^* = \left(\begin{array}{cccc|c} 1-\|Z_1\|^2 & -\langle Z_1, Z_2\rangle & \cdots & -\langle Z_1, Z_{r-1}\rangle & -\langle Z_1, Z_r\rangle \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -\langle Z_{r-1}, Z_1\rangle & -\langle Z_{r-1}, Z_2\rangle & \cdots & 1-\|Z_{r-1}\|^2 & -\langle Z_{r-1}, Z_r\rangle \\ \hline -\langle Z_r, Z_1\rangle & -\langle Z_r, Z_2\rangle & \cdots & -\langle Z_r, Z_{r-1}\rangle & 1-\|Z_r\|^2 \end{array}\right)$$

is crucial to our induction argument. Let A_{ij}, $i, j = 1, 2$, denote the blocks in this decomposition. By induction hypothesis, we have

$$\det A_{11} = 1 - \sum_{i=2}^{r} \|Z_i\|^2 + Q(X),$$

where $Q(X) = \sum_{|\ell| \geq 2} q_\ell X^\ell$. Since $\det(A_{22} - A_{21} A_{11}^{-1} A_{12})$ is a scalar, it follows that

$$\begin{aligned}
\det(I - ZZ^*) &= (A_{22} - A_{21} A_{11}^{-1} A_{12}) \det A_{11} \\
&= A_{22} \det A_{11} - A_{21} \big(\det A_{11}\big) A_{11}^{-1} A_{12} \\
&= A_{22} \det A_{11} - A_{21} \big(\mathrm{Adj}(A_{11})\big) A_{12},
\end{aligned}$$

where, as usual, $\mathrm{Adj}(A_{11})$ denotes the transpose of the matrix of co-factors of A_{11}. Clearly, $A_{21}\big(\mathrm{Adj}(A_{11})\big)A_{12}$ is a sum of $(r-1)^2$ terms. Each of these is of the form $X_{k1} a_{jk} X_{1j}$, where a_{jk} denotes the (j, k) entry of $\mathrm{Adj}(A_{11})$. It follows that any one term in the sum $A_{21}\big(\mathrm{Adj}(A_{11})\big)A_{12}$ is some constant multiple of X^ℓ with $|\ell| \geq 2$. Furthermore,

$$A_{22} \det A_{11} = 1 - \sum_{i=1}^{r} \|Z_i\|^2 + \|Z_r\|^2 \sum_{i=1}^{r-1} \|Z_i\|^2 + Q(X)(1 - \|Z_r\|^2).$$

Putting these together, we see that

$$\det(I - ZZ^*) = 1 - \sum_{i=1}^{r} \|Z_i\|^2 + P(X),$$

where $P(X) = X_{rr} \sum_{i=1}^{r-1} X_{ii} + Q(X)(1 - X_{rr}) - A_{21}\big(\mathrm{Adj}(A_{11}) A_{12}$ completing the proof. □

Let $\mathcal{K}_{\boldsymbol{B}_\mathcal{D}}(Z)$ be the curvature (some times also called the Bergman metric) of the Bergman Kernel $\boldsymbol{B}_\mathcal{D}(Z, Z)$. Set $w_1 = z_{11}, \ldots, w_s = z_{1s}, \ldots, w_{rs-s+1} = z_{r1}, \ldots, w_{rs} = z_{rs}$. The formula for the Bergman metric given below is due to Koranyi (cf. [21]).

Theorem 5.11. *$\mathcal{K}_{\boldsymbol{B}_\mathcal{D}}(0) = pI$, where I is the $rs \times rs$ identity matrix.*

Proof. Lemma 5.10 says that

$$\log \boldsymbol{B}_\mathcal{D}(Z) = -p \log \big(1 - \sum_{i=1}^{r} \|Z_i\|^2 + P(X)\big).$$

It now follows that $\big(\frac{\partial^2}{\partial w_i \partial \bar{w}_j} \log \boldsymbol{B}_\mathcal{D}\big)(0) = 0$, $i \neq j$. On the other hand, $\big(\frac{\partial^2}{\partial w_i \partial \bar{w}_i} \log \boldsymbol{B}_\mathcal{D}\big)(0) = p$, $i = 1, \ldots, rs$. □

In consequence, for the matrix ball $\mathcal{D}$, which is a homogeneous domain in $\mathbb{C}^{r\times s}$, we record separately, the transformation rule:

$$\begin{aligned}\big(\mathcal{K}_{\boldsymbol{B}_{\mathcal{D}}}(W)^{\rm t}\big)^{-1} &= \big(D\varphi_W(W)\big)^{-1}\big(\mathcal{K}_{\boldsymbol{B}_\Omega}(0)^{\rm t}\big)^{-1}\big(\overline{D\varphi_W(W)}^{\,\rm t}\big)^{-1}\\ &= \frac{1}{p}\big(\overline{D\varphi_W(W)}^{\,\rm t} D\varphi_W(W)\big)^{-1},\ W\in\mathcal{D},\end{aligned} \tag{5.9}$$

where $p = r + s$.

5.3 Curvature inequalities

5.3.1 The Euclidean Ball

Let Ω be a homogeneous domain and $\theta_w : \Omega \to \Omega$ be a bi-holomorphic automorphism of Ω with $\theta_w(w) = 0$. The linear map $D\theta_w(w) : (\mathbb{C}^m, \mathcal{C}_{\Omega,w}) \to (\mathbb{C}^m, \mathcal{C}_{\Omega,0})$ is a contraction by definition. Since θ_w is invertible, $D\theta_w^{-1}(0) : (\mathbb{C}^m, \mathcal{C}_{\Omega,0}) \to (\mathbb{C}^m, \mathcal{C}_{\Omega,w})$ is also a contraction. However, since $D\theta_w^{-1}(0) = D\theta_w(w)^{-1}$, it follows that $D\theta_w(w)$ must be an isometry. We paraphrase the Theorem from (cf. [24, Theorem 5.2]) slightly.

Lemma 5.12. *If Ω is a homogeneous domain and θ_w is a bi-holomorphic automorphism with $\theta_w(w) = 0$, then $\|A(w)^{\rm t}\|_{\ell^2\to\mathcal{C}_{\Omega,w}} \le 1$ if and only if $\|A(0)^{\rm t}\|_{\ell^2\to\mathcal{C}_{\Omega,0}} \le 1$.*

Proof. As before, let $\boldsymbol{D}_w\Omega := \{Df(w) : f \in \mathrm{Hol}_w(\Omega,\mathbb{D})\}$. The map $\varphi \mapsto \varphi\circ\theta_w(w)$ is injective from $\mathrm{Hol}_0(\Omega,\mathbb{D})$ onto $\mathrm{Hol}_w(\Omega,\mathbb{D})$. Therefore,

$$\begin{aligned}\boldsymbol{D}_w\Omega &= \{D(f\circ\theta_w)(w) : f\in\mathrm{Hol}_0(\Omega,\mathbb{D})\}\\ &= \{Df(0)D\theta_w(w) : f\in\mathrm{Hol}_0(\Omega,\mathbb{D})\}\\ &= \{u\cdot D\theta_w(w) : u\in\boldsymbol{D}_0\Omega\}\end{aligned}$$

This is another way of saying that $D\theta_w(w)$ is an isometry.

$$\begin{aligned}\sup_{v\in\boldsymbol{D}_w\Omega}\|A(w)^{\rm t}v\| &= \sup_{u\in\boldsymbol{D}_0\Omega}\|A(w)^{\rm t}D\theta_w(w)u\|\\ &= \sup_{u\in\boldsymbol{D}_0\Omega}\|A(0)^{\rm t}u\|,\end{aligned}$$

where we have set $A(0)^{\rm t} := A(w)^{\rm t}D\theta_w(w)$. Thus we have shown

$$\begin{aligned}\{A(w)^{\rm t} : \|A(w)^{\rm t}\|_{\ell^2\to\mathcal{C}_{\Omega,w}}\le 1\} &= \{A(0)^{\rm t}D\theta_w(w)^{-1} : \|A(0)^{\rm t}\|_{\ell^2\to\mathcal{C}_{\Omega,w}}\}\\ &= \{A(0)^{\rm t}D\theta_w^{-1}(0) : \|A(0)^{\rm t}\|_{\ell^2\to\mathcal{C}_{\Omega,w}}\}.\end{aligned}$$

The proof is now complete since $D\theta_w(w)$ is an isometry. $\square$

We note that if $\|A(w)^{\mathrm{t}}\|_{\ell^2\to\mathcal{C}_{\Omega,w}} \le 1$, then

$$\begin{aligned}\|\big(\mathcal{K}(w)^{\mathrm{t}}\big)^{-1}\|_{\mathcal{C}^*_{\Omega,w}\to\mathcal{C}_{\Omega,w}} &= \|A(w)^{\mathrm{t}}\overline{A(w)}\|_{\mathcal{C}^*_{\Omega,w}\to\mathcal{C}_{\Omega,w}}\\ &\le \|A(w)^{\mathrm{t}}\|_{\ell^2\to\mathcal{C}_{\Omega,w}}\|\overline{A(w)}\|_{\mathcal{C}^*_{\Omega,w}\to\ell^2}\\ &= \|A(w)^{\mathrm{t}}\|^2_{\ell^2\to\mathcal{C}_{\Omega,w}} \le 1, \end{aligned} \tag{5.10}$$

which is the curvature inequality of (cf. [24, Theorem 5.2]). For a homogeneous domain Ω, using the transformation rules in Corollary 5.9 and the equation (5.9), for the curvature $\mathcal{K}$ of the Bergman kernel $\boldsymbol{B}_\Omega$, we have

$$\begin{aligned}\|\big(\mathcal{K}(w)^{\mathrm{t}}\big)^{-1}\|_{\mathcal{C}^*_{\Omega,w}\to\mathcal{C}_{\Omega,w}} &= \Big\|\big(D\theta_w(w)^{\mathrm{t}}\mathcal{K}(0)\overline{D\theta_w(w)}\big)^{\mathrm{t}-1}\Big\|_{\mathcal{C}^*_{\Omega,w}\to\mathcal{C}_{\Omega,w}}\\ &= \Big\|D\theta_w(w)^{-1}\big(\mathcal{K}(0)^{\mathrm{t}}\big)^{-1}\overline{D\theta_w(w)^{-1}}^{\mathrm{t}}\|_{\mathcal{C}^*_{\Omega,w}\to\mathcal{C}_{\Omega,w}}\\ &= \Big\|D\theta_w(w)^{-1}A(0)^{\mathrm{t}}\overline{A(0)D\theta_w(w)^{-1}}^{\mathrm{t}}\|_{\mathcal{C}^*_{\Omega,w}\to\mathcal{C}_{\Omega,w}}\\ &\le \Big\|D\theta_w(w)^{-1}A(0)^{\mathrm{t}}\|^2_{\ell^2\to\to\mathcal{C}_{\Omega,w}} = \Big\|A(0)^{\mathrm{t}}\|^2_{\ell^2\to\to\mathcal{C}_{\Omega,0}}\end{aligned} \tag{5.11}$$

since $D\theta_w(w)^{-1}$ is an isometry. For the Euclidean ball $\mathbb{B} := \mathbb{B}^n$, the inequality for the curvature is more explicit. In the following, we set $\mathfrak{B}(w,w) := \big(\boldsymbol{B}_{\mathbb{B}}(w,w)\big)^{-\frac{1}{n+1}}$. Thus polarizing $\mathfrak{B}$, we have $\mathfrak{B}(z,w) = \big(1-\langle z,w\rangle\big)^{-1}$, $z,w\in\mathbb{B}$. The inequality appearing below (cf. [24]) is a point-wise inequality with respect to the usual ordering of Hermitian matrices.

Theorem 5.13. *Let θ_w is a bi-holomorphic automorphism of $\mathbb{B}$ such that $\theta_w(w)=0$. If ρ is contractive homomorphism of $\mathcal{O}(\mathbb{B})$ induced by the localization $N(w)$, $T\in\mathcal{P}_1(\mathbb{B})$, then*

$$\mathcal{K}(w) \le -\overline{D\theta_w(w)}^{t} D\theta_w(w) = \mathcal{K}_{\mathfrak{B}}(w),\ \ w\in\mathbb{B}$$

Proof. The equation (5.10) combined with the equality $\mathcal{C}_{\mathbb{B},0} = \|\cdot\|_{\ell^2}$ and the contractivity of ρ_T implies that $\|D\theta_w(w)A(w)^t\|_{\ell_2\to\ell_2}\le 1$. Hence

$$\begin{aligned} I - D\theta_w(w)A(w)^t\overline{A(w)}\,\overline{D\theta_w(w)}^{\mathrm{t}} \ge 0 \ &\Leftrightarrow\ (D\theta_w(w))^{-1}\big(\overline{D\theta_w(w)}^{\mathrm{t}}\big)^{-1} - A(w)^t\overline{A(w)} \ge 0\\ &\Leftrightarrow\ A(w)^t\overline{A(w)} \le (D\theta_w(w))^{-1}\big(\overline{D\theta_w(w)}^{\mathrm{t}}\big)^{-1}\\ &\Leftrightarrow\ \big(-\mathcal{K}(w)^{\mathrm{t}}\big)^{-1} \le \big(\overline{D\varphi_w(w)}^{\mathrm{t}}D\varphi_w(w)\big)^{-1}. \end{aligned}$$

Since $-\big(\mathcal{K}(w)^{\mathrm{t}}\big)^{-1}$ and $\big(\overline{D\theta_w(w)}^{\mathrm{t}}D\theta_w(w)\big)^{-1}$ are positive definite matrices, it follows (cf. [8]) that $\mathcal{K}(w) \le -\overline{D\theta_w(w)}^{\mathrm{t}}D\theta_w(w) = \mathcal{K}_{\mathfrak{B}}(w)$. □

This inequality generalizes the curvature inequality obtained in (cf. [22]) for the unit disc. However, assuming that $\mathcal{K}_{\mathfrak{B}^{-1}K}(w)$ is a non-negative Kernel defined on the ball $\mathbb{B}$

implies $(\mathfrak{B}(w))^{-1}K(w)$ is a non-negative kernel on $\mathbb{B}$ (cf. [6, Theorem 4.1]), indeed, it must be infinitely divisible. This stronger assumption on the curvature amounts to the factorization of the kernel $K(z,w) = \mathfrak{B}(z,w)\tilde{K}(z,w)$ for some positive definite kernel $\tilde{K}$ on the ball $\mathbb{B}$ with the property: $\big(\mathfrak{B}(z,w)\tilde{K}(z,w)\big)^{\lambda}$ is non-negative definite for all $\lambda > 0$.

For $\lambda > 0$, the polarization of the function $\boldsymbol{B}(w,w)^{\lambda}$ defines a positive definite kernel $\boldsymbol{B}^{\lambda}(z,w)$ on the ball $\mathbb{B}$ (cf. [5, Proposition 5.5]). We note that $\mathcal{K}_{\boldsymbol{B}^{\lambda}}(w) \leq \mathcal{K}_{\mathfrak{B}}(w)$ if and only if $\mathcal{K}_{\boldsymbol{B}^{\lambda}}(0) \leq \mathcal{K}_{\mathfrak{B}}(0) = -I$. Since $\mathcal{K}_{\boldsymbol{B}^{\lambda}}(0) = -\lambda(n+1)I$, it follows that $\mathcal{K}_{\boldsymbol{B}^{\lambda}}(w) \leq \mathcal{K}_{\mathfrak{B}}(w)$ if and only if $\lambda \geq \frac{1}{n+1}$. Thus whenever $\lambda \geq \frac{1}{n+1}$, we have the point-wise curvature inequality for $\boldsymbol{B}^{\lambda}(w,w)$. However, since the operator of multiplication by the co-ordinate functions on the Hilbert space corresponding to the kernel $\boldsymbol{B}^{\lambda}(w,w)$, is not even a contraction for $\frac{1}{n+1} \leq \lambda < \frac{n}{n+1}$, the induced homomorphism can't be contractive. We therefore conclude that the curvature inequality does not imply the contractivity of ρ_T whenever $n > 1$. For $n = 1$, an example illustrating this (for the unit disc) was given in (cf. [6, page2]). Thus the contractivity of the homomorphism induced by the commuting tuple of the local operators $N(W)$, for $T \in \mathcal{P}_1(\mathbb{B})$ does not imply the contractivity of the homomorphism induced by the commuting tuple of operators T.

5.3.2 The matrix ball

We recall that the positive function $\boldsymbol{B}_{\mathcal{D}}^{\lambda}, \lambda > 0$, defines an inner product on the finite dimensional space $\mathcal{N}^{(\lambda)}(w)$ for all $\lambda > 0$ irrespective of whether $\boldsymbol{B}_{\mathcal{D}}^{\lambda}$ is positive definite on the matrix ball $\mathcal{D}$ or not. In this section, we exclusively study the curvature inequality and contrativity of the homomorphism $\rho_{N^{(\lambda)}(w)}^{(\lambda)}(w)$ induced by the commuting tuples $N^{(\lambda)}(w)$ on the finite dimensional Hilbert subspace $\mathcal{N}^{(\lambda)}(w)$, $\lambda > 0$. We set $\mathcal{K}^{(\lambda)}(w) := \mathcal{K}_{\boldsymbol{B}_{\mathcal{D}}^{\lambda}}(w)$, $w \in \mathcal{D}$. If the homomorphism $\rho_{N^{(\lambda)}(w)}^{(\lambda)}(w)$ is contractive for some $\lambda > 0$, then for this λ, we have:$\|\big(\mathcal{K}^{(\lambda)^{\mathrm{t}}}\big)^{-1}(0)\| \leq 1$. Like the Euclidian Ball, we study several implications of the curvature inequality in this case.

Theorem 5.14. *For $\lambda > 0$, we have $\|\big(\mathcal{K}^{(\lambda)^{\mathrm{t}}}\big)^{-1}(0)\|_{\mathcal{C}_{\mathcal{D},0}^{*} \to \mathcal{C}_{\mathcal{D},0}} = \frac{1}{\lambda p}$, $p = r + s$.*

Proof. We have shown that $\big(\mathcal{K}^{\mathrm{t}}\big)^{-1}(0) = \frac{1}{p}I_{rs}$. Since $\mathcal{C}_{\mathcal{D},0}$ is the operator norm on $(\mathcal{M})_{rs}$ and consequently $\mathcal{C}_{\mathcal{D},0}^{*}$ is the trace norm, it follows that $\|I_{rs}\|_{\mathcal{C}_{\mathcal{D},0}^{*} \to \mathcal{C}_{\mathcal{D},0}} \leq 1$. This completes the proof. □

The following Theorem provides a necessary condition for the contractivity of the homomorphism induced by the commuting tuple of the local operators $N^{(\lambda)}(w)$.

Theorem 5.15. *If the homomorphism $\rho_{N^{(\lambda)}(w)}^{(\lambda)}$ is contractive, then $\nu \geq 1$, where $\nu = \lambda p$.*

Proof. The matrix unit ball $\mathcal{D}$ is homogenous. Let $\theta_w(w)$ be the bi-holomorphic automorphism of $\mathcal{D}$ with $\theta_w(w) = 0$. We have seen that $A(w)^{\mathrm{t}} = A(0)^{\mathrm{t}} D\theta_w^{-1}(0)$. Since $D\theta_w^{-1}(0)$ is an isometry, therefore the contractivity of $\rho^{(\lambda)}_{N^{(\lambda)}(w)}(0)$ implies that contractivity of $\rho^{(\lambda)}_{N^{(\lambda)}(w)}(w)$, $w \in \Omega$, see Lemma 5.12. The contractivity of $\rho^{(\lambda)}_{N^{(\lambda)}(w)}(w)$ is equivalent to $\|A(0)^{\mathrm{t}}\|_{\ell^2 \to \mathcal{C}_{\mathcal{D},0}} \leq 1$. Therefore the contractivity of $\rho^{(\lambda)}_{N^{(\lambda)}(w)}(w)$, for some $w \in \mathcal{D}$, implies $\|\big(\mathcal{K}^{(\lambda)^{\mathrm{t}}}\big)^{-\frac{1}{2}}(0)\|_{\mathcal{C}^*_{\mathcal{D},0} \to \mathcal{C}_{\mathcal{D},0}} \leq 1$. Theorem 5.14 shows that $\nu \geq 1$.

□

If $\lambda > 0$ is picked such that $\boldsymbol{B}^{\lambda}_{\mathcal{D}}$ is positive definite, then Arazy and Zhang (cf. [5, Proposition 5.5]) prove that the homomorphism induced by the commuting tuple of multiplication operators on the twisted Bergman space $\mathbb{A}^{(\lambda)}(\mathcal{D})$ is bounded (k-spectral) if and only if $\nu \geq s$.

It follows that if $1 \leq \nu < s$, then the homomorphism induced by the commuting tuple of multiplication operators is not contractive on twisted Bergman space $\mathbb{A}^{(\lambda)}(\mathcal{D})$. While the homomorphism $\rho^{(\lambda)}_{N^{(\lambda)}(w)}(w)$, $w \in \Omega$, is contractive on the finite dimensional Hilbert space $\mathcal{N}^{\lambda}(w)$. This is equivalent to the curvature inequality for $\nu \geq 1$. However, for $1 \leq \nu < s$, the rs-tuple of multiplication operators on twisted Bergman space $\mathbb{A}^{(\lambda)}(\mathcal{D})$ is not contractive .

The localization of $N^{(\lambda)}(w)$ of any commuting tuple of operators T in $\mathcal{P}_1(\mathcal{D})$ induces a homomorphism $\rho^{(\lambda)}_{N^{(\lambda)}(w)}(w) : \mathcal{A}(\mathcal{D}) \to \mathcal{L}(\mathbb{C}^{rs+1})$ as described in the equation (5.3). Therefore $\rho^{(\lambda)}_{N^{(\lambda)}(w)}(w) \otimes I_{rs} : \mathcal{A}(\mathcal{D}) \otimes \mathcal{M}_{rs} \to \mathcal{L}(\mathcal{N}(w)) \otimes \mathcal{M}_{rs}$ is given by the formula

$$\rho^{(\lambda)}_{N^{(\lambda)}(w)}(w) \otimes I_{rs}(P) := \begin{pmatrix} P(w)\otimes I_{rs} & DP(w)\cdot N(w) \\ 0 & P(w)\otimes I_{rs} \end{pmatrix},$$

where

$$DP(w) \cdot N(w) = \partial_1 P(w) \otimes N_1(w) + \ldots + \partial_d P(w) \otimes N_{rs}(w). \tag{5.12}$$

The contractivity of $\rho^{(\lambda)}_{N^{(\lambda)}(w)}(w) \otimes I_{rs}$, as shown in (cf. [25, Theorem 1.7], [30, Theorm 4.2]) is equivalent to the contractivity of the operator

$$\|\partial_1 P(w) \otimes N_1(w) + \ldots + \partial_d P(w) \otimes N_{rs}(w)\|_{\mathrm{op}} \leq 1.$$

Let $P_{\mathbf{A}}$ be the matrix valued polynomial in rs variables:

$$P_{\mathbf{A}}(z) = \sum_{i=1}^{r} \sum_{j=1}^{s} z_{ij} E_{ij},$$

where E_{ij} be the $r \times s$ matrices whose (i,j) entries are 1 and other entries are 0. Let $V = (V_1^{\mathrm{t}}, \ldots, V_{rs}^{\mathrm{t}})$ be the $rs \times rs$ matrix, where

$$V_1 = (v_{11}, 0, \ldots, 0_{sr}), \ldots, V_{sr} = (0, \ldots, 0, \ldots, v_{sr}).$$

We compute the norm of $\rho^{(\lambda)}_{N^{(\lambda)}(w)}(w) \otimes I_{rs}(P_{\mathbf{A}})$.

Theorem 5.16. *For $\rho^{(\lambda)}_{N^{(\lambda)}(w)}(w) \otimes I_{rs}$ as above, we have*

$$\|\rho^{(\lambda)}_{N^{(\lambda)}(w)}(w) \otimes I_{rs}(P_{\mathbf{A}})\|^2 = \max\{\sum_{i=1}^{s} |v_{1i}|^2, \ldots, \sum_{i=1}^{s} |v_{ri}|^2\}.$$

Proof. We have

$$\begin{aligned}\|\big(\rho^{(\lambda)}_{N^{(\lambda)}(w)}(w) \otimes I_{rs}\big)(P_{\mathbf{A}})\|^2 &= \|V_1 \otimes E_{11} + \ldots + V_s \otimes E_{1s} + V_{s+1} \otimes E_{21} + \ldots + V_{rs} \otimes E_{rs}\|^2 \\ &= \left\| \begin{pmatrix} V_1 & \ldots & V_s \\ \vdots & \vdots & \vdots \\ V_{rs-s+1} & \ldots & V_{rs} \end{pmatrix} \right\|^2 = \left\| \begin{pmatrix} W_1 \\ \vdots \\ W_r \end{pmatrix} \right\|^2,\end{aligned}$$

where $W_i = \big(V_{is-s+1}, \ldots, V_{is}\big)$. It is easy to see that $W_i W_j^* = 0$ for $i \neq j$. Furthermore, $W_i W_i^* = \sum_{j=1}^{s} |v_{ij}|^2$. Hence we have

$$\|\rho^{(\lambda)}_{N^{(\lambda)}(w)}(w) \otimes I_{rs}(P_{\mathbf{A}})\|^2 = \max\{\sum_{i=1}^{s} |v_{1i}|^2, \ldots, \sum_{i=1}^{s} |v_{ri}|^2\}$$

completing the proof of the theorem. □

Even for the small class of the form discussed here, homomorphisms finding the cb norm of $\rho^{(\lambda)}_{N^{(\lambda)}(w)}(w)$ is not easy. However, we determine when $\|\rho^{(\lambda)}_{N^{(\lambda)}(w)}(w) \otimes I_{rs}(P_{\mathbf{A}})\|^2 \leq 1$. This gives a necessary condition for the complete contractivity of $\rho^{(\lambda)}_{N^{(\lambda)}(w)}(w)$.

Theorem 5.17. *If $\|\rho^{(\lambda)}_{N^{(\lambda)}(w)}(w) \otimes I_{rs}(P_{\mathbf{A}})\|^2 \leq 1$, then $\nu \geq s$.*

Proof. By Theorem 5.16 we have

$$\|\rho^{(\lambda)}_{N^{(\lambda)}(w)}(w) \otimes I_{rs}(P_{\mathbf{A}})\|^2 = \max\{\sum_{i=1}^{s} |v_{1i}|^2, \ldots, \sum_{i=1}^{s} |v_{ri}|^2\}.$$

Since $|v_{ij}|^2 = \frac{1}{\nu}, 1 \leq i \leq r, 1 \leq j \leq s$, it is immediate that $\|\rho^{(\lambda)}_{N^{(\lambda)}(w)}(w) \otimes I_{rs}(P_{\mathbf{A}})\|^2 \leq 1$ implies $\nu \geq s$ completing the proof of the theorem. □

As a consequence, it follows that if $1 \leq \nu < s$, then the homomorphism induced by the commuting tuple of the local operators $N^{(\lambda)}(w)$ is not completely contractive.

5.3.3 More examples

We have discussed the Bergman kernel $\boldsymbol{B}_\Omega(w,w)$ for the domain $\Omega = \{(z_1,z_2) : |z_2| \leq (1-|z_1|^2)\} \subset \mathbb{C}^2$. The curvature $\mathcal{K}_{\boldsymbol{B}_\Omega}(w) = \sum_{i,j=1}^2 T_{ij}(w) dw_i \wedge d\bar{w}_j$ of the Bergman Kernel $\boldsymbol{B}_\Omega(w,w)$ is (cf. [19, Example 6.2.1]):

$$T_{11}(w) = 6\big(\frac{1}{C(w)} - \frac{1}{D(w)}\big) + 12|w_1|^2|w_2|^2\big(\frac{1}{C^2(w)} + \frac{1}{D^2(w)}\big),$$

$$T_{12}(w) = \bar{T}_{21}(w) = 6w_1\bar{w}_2(1-|w_1|^2)\big(\frac{1}{C^2(w)} + \frac{1}{D^2(w)}\big),$$

$$T_{22}(w) = 3(1-|w_1|^2)^2\big(\frac{1}{C^2(w)} + \frac{1}{D^2(w)}\big),$$

where $C(w) := (1-|w_1|^2)^2 - |w_2|^2$ and $D(w) := 3(1-|w_1|^2)^2 + |w_2|^2$. We have seen that the polarization $\boldsymbol{B}_\Omega^\lambda(z,w)$ of the function $\boldsymbol{B}_\Omega(w,w)^\lambda$ defines a Hermitian structure for $\mathcal{N}^{(\lambda)}(w)$. Specializing to $w=0$, since $-\big(\mathcal{K}(0)^{\mathrm{t}}\big)^{-1} = A(0)^{\mathrm{t}}\overline{A(0)}$, we have $a_{11}^\lambda(0) = \frac{1}{\sqrt{T_{11}(0)}}$ and $a_{22}^\lambda(0) = \frac{1}{\sqrt{T_{22}(0)}}$, where $(A^\lambda(0))^{\mathrm{t}} = \begin{pmatrix} a_{11}^\lambda(0) & 0 \\ 0 & a_{22}^\lambda(0) \end{pmatrix}$.

Theorem 5.18. *The contractivity of the homomorphism $\rho_{N^{(\lambda)}}^{(\lambda)}(0)$ implies $16\lambda \geq 5$.*

Proof. We have $a_{11}^\lambda(0) = \frac{1}{2\sqrt{\lambda}}, a_{12}^\lambda(0) = 0, a_{22}^\lambda(0) = \frac{3}{\sqrt{10\lambda}}$. Contractivity of homomorphism $\rho_{N^{(\lambda)}(0)}^{(\lambda)}$ is equivalent to $\|(A^\lambda(0))^{\mathrm{t}}\|_{\ell^2 \to \mathcal{C}_{\Omega,0}} \leq 1$. This is equivalent to $(2(a_{11}^\lambda(0))^2 - 1)^2 \leq (1-(a_{22}^\lambda(0))^2)$. Hence $16\lambda \geq 5$ completing our proof. $\square$

The bi-holomorphic automorphism group of Ω is not transitive. So the contractivity of the homomorphism $\rho_{N^{(\lambda)}}^{(\lambda)}(0)$ does not necessarily imply the contractivity of the homomorphism $\rho_{N^{(\lambda)}}^{(\lambda)}(w), w \in \Omega$. Determining which of the homomorphism $\rho_{N^{(\lambda)}}^{(\lambda)}(w)$ is contractive, appears to be a hard problem.

Let $P_{\mathbf{A}} : \Omega \to (\mathcal{M}_2)_1$ be the matrix valued polynomial on Ω defined by $P_{\mathbf{A}}(z) = z_1A_1 + z_2A_2$ where $A_1 = I_2$ and $A_2 = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. It is natural to ask when $\rho_{N^{(\lambda)}}^{(\lambda)}(w)$ is completely contractive. As before, we only obtain a necessary condition using the polynomial $P_{\mathbf{A}}$.

Theorem 5.19. $\|\rho_{N^{(\lambda)}}^{(2)}(0)(P_{\mathbf{A}})\| \leq 1$ *if and only if* $\lambda \geq \frac{11}{20}$.

Proof. Suppose that $\|\rho_{N^{(\lambda)}}^{(2)}(0)(P_{\mathbf{A}})\| \leq 1$. Then we have $(a_{11}^\lambda(0))^2 + (a_{22}^\lambda(0))^2 \leq 1$. Hence $\lambda \geq \frac{11}{20}$. The converse verification is also equally easy. $\square$

We conclude that if $\frac{5}{16} \leq \lambda < \frac{11}{20}$, the homomorphism $\rho_{N^{(\lambda)}}^{(\lambda)}(0)$ is contractive but not completely contractive. An explicit description of the set

$$\{\lambda : \|\rho_{N^{(\lambda)}}^{(\lambda)}(w)(P_{\mathbf{A}})\|_{\mathrm{op}} \leq 1, w \in \Omega\}$$

would certainly provide greater insight. However, it appears to be quite intractable, at least for now.

The formula for the Bergman kernel for the domain

$$\Omega := \{(z_1, z_2, z_3) : |z_2|^2 \le (1-|z_1|^2)(1-|z_3|^2), 1-|z_3|^2 \ge 0\} \subset \mathbb{C}^3.$$

is given in Lemma 5.6. From Lemma 5.6 we have $\boldsymbol{B}_\Omega^\lambda(z,0) = 1$ and $\partial_i \boldsymbol{B}_\Omega^\lambda(z,0) = 0$ for $i = 1,2,3$. Hence the desired curvature matrix is of the form

$$\left(\!\!\left((\partial_i \bar{\partial}_j \log \boldsymbol{B}_\Omega^\lambda)(0,0) \right)\!\!\right)_{i\,j=1}^m.$$

Let $T_{ij}(0) = \partial_i\bar{\partial}_j \log \boldsymbol{B}_\Omega^\lambda(0,0)$, that is, $\mathcal{K}_{\boldsymbol{B}_\Omega}(0) = \sum_{i,j=1}^3 T_{ij}(0) dw_i \wedge d\bar{w}_j$. An easy computation shows that $T_{11}(0) = 3\lambda = T_{33}(0), T_{22}(0) = \frac{9\lambda}{2}$ and $T_{ij}(0) = 0$ for $i \ne j$. As before, we have $a_{11}^\lambda(0) = \frac{1}{\sqrt{T_{11}(0)}}, a_{22}^\lambda(0) = \frac{1}{\sqrt{T_{22}(0)}}$ and $a_{33}^\lambda(0) = \frac{1}{\sqrt{T_{33}(0)}}$, where $A(0)^{\mathrm{t}} = \begin{pmatrix} a_{11}^\lambda(0) & 0 & 0 \\ 0 & a_{22}^\lambda(0) & 0 \\ 0 & 0 & a_{33}^\lambda(0) \end{pmatrix}$.

Theorem 5.20. *The contractivity of the homomorphism $\rho_{N^{(\lambda)}}^{(\lambda)}(0)$ implies $\lambda \ge \frac{1}{4}$.*

Proof. From Lemma (5.13) we have $a_{11}^\lambda(0) = \frac{1}{\sqrt{3\lambda}}, a_{12}^\lambda(0) = a_{13}^\lambda(0) = 0, a_{22}^\lambda(0) = \frac{\sqrt{2}}{3\sqrt{\lambda}}, a_{23}^\lambda(0) = 0$ and $a_{33}^\lambda(0) = \frac{1}{\sqrt{3\lambda}}$. The contractivity of the homomorphism $\rho_{N^{(\lambda)}(w)}^{(\lambda)}(w)(0)$ is equivalent to $\left\|A(0)^{\mathrm{t}}\right\|_{\ell^2 \to \mathcal{C}_{\Omega,0}}^2 \le 1$. This is equivalent to $|a_{11}^\lambda(0)|^2(1-|a_{33}^\lambda(0)|^2) \ge (|a_{22}^\lambda(0)|^2 - |a_{33}^\lambda(0)|^2)$. Hence we have $\lambda \ge \frac{1}{4}$. □

For our final example, let $P_{\mathbf{A}} : \Omega \to (\mathcal{M}_2)_1$ be also the matrix valued polynomial on Ω defined by $P_{\mathbf{A}}(z) = z_1A_1 + z_2A_2 + z_3A_3$ where $A_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right), A_2 = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), A_3 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

Theorem 5.21. $\|\rho_{N^{(\lambda)}}^{(2)}(0)(P_{\mathbf{A}})\| \le 1$ *if and only if* $\lambda \ge \frac{5}{9}$.

Proof. Suppose that $\|\rho_{N^{(\lambda)}}^{(2)}(0)(P_{\mathbf{A}})\| \le 1$. Then we have

$$\max\{(a_{11}^\lambda(0))^2 + (a_{22}^\lambda(0))^2, (a_{33}^\lambda(0))^2\} \le 1.$$

Hence $\lambda \ge \frac{5}{9}$. The converse statement is easily verified. □

Thus if $\frac{1}{4} \le \lambda < \frac{5}{9}$, the homomorphism $\rho_{N^{(\lambda)}}^{(\lambda)}(0)$ is contractive but not completely contractive.

Chapter 6

Contractivity vs. complete contractivity

6.1 Homomorphisms induced by m vectors

We now assume that $\mathbf{v}_i = (v_{i1}, \ldots, v_{im}), 1 \le i \le m$, is a vector in $\mathbb{C}^m$. The commuting tuple

$$N(V, w) := \left(\left(\begin{smallmatrix} w_1 & \mathbf{v}_1 \\ 0 & w_1 I_m \end{smallmatrix} \right), \cdots, \left(\begin{smallmatrix} w_m & \mathbf{v}_m \\ 0 & w_m I_m \end{smallmatrix} \right) \right),$$

$w = (w_1, \ldots, w_m) \in \Omega_{\mathbf{A}}$, defines a homomorphism $\rho_V : \mathcal{O}(\Omega_{\mathbf{A}}) \to \mathcal{M}_{m+1}$ which is given by the formula

$$\rho_V(f) = \begin{pmatrix} f(w) & \nabla f(w) V \\ 0 & f(w) I \end{pmatrix}, \; f \in \mathcal{O}(\Omega_{\mathbf{A}}),$$

where $\nabla f(w) V = \partial_1 f(w) \mathbf{v}_1 + \cdots + \partial_m f(w) \mathbf{v}_m$. We derive a criterion for contractivity of the homomorphism ρ_V. We also compute $\|\rho_V^{(n)}(P_{\mathbf{A}})\|$, where $P_{\mathbf{A}}(z_1, \ldots, z_m) = z_1 A_1 + \cdots + z_m A_m$. If $f : \Omega_{\mathbf{A}} \longmapsto \mathbb{D}$ is a holomorphic function with $f(0) = 0$ and $\|f\|_{\infty, \mathbb{D}} \le 1$, then the vector $(\partial_1 f(0), \ldots, \partial_m f(0))$ is in the dual unit ball $(\mathbb{C}^m, \| \cdot \|_{\Omega_{\mathbf{A}}}^*)_1$(see Corollary 2.12). Now,

$$\begin{aligned} \sup\{\|\rho_V(f)\| : \|f\|_{\infty, \mathbb{D}} \le 1\} &= \sup\{\|\rho_V(f)\| : \|f\|_{\infty, \mathbb{D}} \le 1, f(0) = 0\} \\ &= \sup\{\| \partial_1 f(0) \mathbf{v}_1 + \cdots + \partial_m f(0) \mathbf{v}_m \| : \|f\|_{\infty, \mathbb{D}} \le 1, f(0) = 0\} \\ &= \sup\{\| \lambda_1 \mathbf{v}_1 + \cdots + \lambda_m \mathbf{v}_m \| : (\lambda_1, \ldots, \lambda_m) \in (\mathbb{C}^m, \| \cdot \|_{\Omega_{\mathbf{A}}}^*)_1\}. \end{aligned}$$

Let $L_V : (\mathbb{C}^m, \| \cdot \|_{\Omega_{\mathbf{A}}}^*) \to (\mathbb{C}^m, \| \cdot \|_2)$ be the linear map induced by the matrix $(\mathbf{v}_1^{\mathrm{t}}, \ldots, \mathbf{v}_m^{\mathrm{t}})$. The matrix representing L_V also gives a matrix representation of the adjoint

$$L_V^* : (\mathbb{C}^m, \| \cdot \|_2) \to (\mathbb{C}^m, \| \cdot \|_{\Omega_{\mathbf{A}}}).$$

Clearly, $\| L_V \|_{(\mathbb{C}^m, \| \cdot \|_{\Omega_{\mathbf{A}}}^*) \to (\mathbb{C}^m, \| \cdot \|_2)} \le 1$ if and only if $\|L_V^*\|_{(\mathbb{C}^m, \| \cdot \|_2) \to (\mathbb{C}^m, \| \cdot \|_{\Omega_{\mathbf{A}}})} \le 1$ if and only if $\| \rho_V \|_{\mathcal{O}(\Omega_{\mathbf{A}}) \to \mathcal{M}(\mathbb{C}^{m+1})} \le 1$. In characterizing the contractivity of ρ_V, we will often determine if $\|L_V^*\|_{(\mathbb{C}^m,\|\cdot\|_2)\to(\mathbb{C}^m,\|\cdot\|_{\Omega_{\mathbf{A}}})} \leq 1$. The following proposition gives a criterion for the contractivity of ρ_V.

Proposition 6.1. *The following conditions are equivalent:*

(i) ρ_V *is contractive,*

(ii) $\sup_{\sum_{j=1}^m |x_j|^2\leq 1} \|\sum_{j=1}^m x_jB_j\|^2 \leq 1$, *where* $B_j = \sum_{i=1}^m v_{ij}A_i$,

(iii)

$$B(\beta,\beta) = \begin{pmatrix} 1-\left\langle B_1B_1^*\beta,\beta\right\rangle & -\left\langle B_1B_2^*\beta,\beta\right\rangle & \ldots & -\langle B_1B_m^*\beta,\beta\rangle \\ -\left\langle B_2B_1^*\beta,\beta\right\rangle & 1-\left\langle B_2B_2^*\beta,\beta\right\rangle & \ldots & -\langle B_2B_m^*\beta,\beta\rangle \\ \vdots & \vdots & \vdots & \vdots \\ -\left\langle B_mB_1^*\beta,\beta\right\rangle & -\left\langle B_mB_2^*\beta,\beta\right\rangle & \ldots & 1-\langle B_mB_m^*\beta,\beta\rangle \end{pmatrix} \geq 0, \tag{6.1}$$

where $\sum_{i=1}^n |\beta_i|^2 = 1$.

Proof. First we will prove that (i) and (ii) are equivalent. We have earlier seen that $\|L_V^*\|_{(\mathbb{C}^m,\|\cdot\|_2)\to(\mathbb{C}^m,\|\cdot\|_{\Omega_{\mathbf{A}}})} \leq 1$ if and only if $\|\,\rho_V\,\|_{\mathcal{O}(\Omega_{\mathbf{A}})\to\mathcal{M}(\mathbb{C}^{m+1})} \leq 1$. The matrix representation of L_V^* is of the form $\begin{pmatrix} v_{11} & \ldots & v_{1m} \\ \vdots & \vdots & \vdots \\ v_{m1} & \ldots & v_{mm} \end{pmatrix}$. Since L_V^* maps $(\mathbb{C}^m,\|\cdot\|_2)$ into $(\mathbb{C}^m,\|\cdot\|_{\Omega_{\mathbf{A}}})$, we have $\begin{pmatrix} v_{11} & \ldots & v_{1m} \\ \vdots & \vdots & \vdots \\ v_{m1} & \ldots & v_{mm} \end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} \in (\mathbb{C}^m,\|\cdot\|_{\Omega_{\mathbf{A}}})$. Thus $\|L_V^*\|_{(\mathbb{C}^m,\|\cdot\|_2)\to(\mathbb{C}^m,\|\cdot\|_{\Omega_{\mathbf{A}}})} \leq 1$ if and only if $\sup_{\sum_{j=1}^m |x_j|^2\leq 1} \|\sum_{j=1}^m x_jB_j\|^2 \leq 1$, where $B_j = \sum_{i=1}^m v_{ij}A_i$. Hence (i) and (ii) are equivalent.

To see that (ii) and (iii) are equivalent note that $\sup_{\sum_{j=1}^m |x_j|^2\leq 1} \|\sum_{j=1}^m x_jB_j\|^2 \leq 1$ if and only if

$$I_n - \sum_{j=1}^m |x_j|^2B_jB_j^* - \sum_{i=1}^m\sum_{j,i<j}^m x_i\bar{x}_jB_iB_j^* - \sum_{j=1}^m\sum_{i,j<i}^m \bar{x}_jx_iB_iB_j^* \geq 0$$

which is clearly equivalent to

$$\sum_{j=1}^m |x_j|^2\left\langle (I_n - B_jB_j^*)\alpha,\alpha\right\rangle - \sum_{i=1}^m\sum_{j,i<j}^m x_i\bar{x}_j\left\langle B_iB_j^*\alpha,\alpha\right\rangle - \sum_{j=1}^m\sum_{i,j<i}^m \bar{x}_jx_i\left\langle B_iB_j^*\alpha,\alpha\right\rangle \geq 0.$$

Or, equivalently,

$$\left\langle \begin{pmatrix} \left\langle (I_n-B_1B_1^*)\alpha,\alpha\right\rangle & -\left\langle B_1B_2^*\alpha,\alpha\right\rangle & \ldots & -\langle B_1B_m^*\alpha,\alpha\rangle \\ -\left\langle B_2B_1^*\alpha,\alpha\right\rangle & \left\langle (I_n-B_2B_2^*)\alpha,\alpha\right\rangle & \ldots & -\langle B_2B_m^*\alpha,\alpha\rangle \\ \vdots & \vdots & \vdots & \vdots \\ -\left\langle B_mB_1^*\alpha,\alpha\right\rangle & -\left\langle B_mB_2^*\alpha,\alpha\right\rangle & \ldots & \langle (I_n-B_mB_m^*)\alpha,\alpha\rangle \end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ \vdots \\ x_m \end{pmatrix}, \begin{pmatrix} x_1 \\ \vdots \\ \vdots \\ x_m \end{pmatrix}\right\rangle \geq 0. \tag{6.2}$$

Putting $\beta = \frac{\alpha}{\|\alpha\|}$ in Equation (6.2) we verify Equation (6.1). This completes the proof of the proposition. $\square$

In particular, if $B_1 = v_{11}A_1 + v_{21}A_2$ and $B_2 = v_{12}A_1 + v_{22}A_2$, then we have

$$\sup_{|x|^2+|y|^2\leq 1} \|xB_1 + yB_2\|^2 \leq 1,$$

which is equivalent to one of $1 - \langle B_1B_1^*\beta, \beta\rangle \geq 0$ or $1 - \langle B_2B_2^*\beta, \beta\rangle \geq 0$ and

$$\inf_\beta\{1 - \langle B_1B_1^*\beta, \beta\rangle - \langle B_2B_2^*\beta, \beta\rangle + \langle B_1B_1^*\beta, \beta\rangle \langle B_2B_2^*\beta, \beta\rangle - |\langle B_1B_2^*\beta, \beta\rangle|^2\} \geq 0, \tag{6.3}$$

for all $\beta \in \mathbb{C}^2$ with $\|\beta\|_2 = 1$. Hence $\| \rho_V \|_{\mathcal{O}(\Omega_{\mathbf{A}})\to\mathcal{M}(\mathbb{C}^3)} \leq 1$ if and only if $1-\langle B_1B_1^*\beta, \beta\rangle \geq 0$ and

$$\inf_\beta\{1 - \langle B_1B_1^*\beta, \beta\rangle - \langle B_2B_2^*\beta, \beta\rangle + \langle B_1B_1^*\beta, \beta\rangle \langle B_2B_2^*\beta, \beta\rangle - |\langle B_1B_2^*\beta, \beta\rangle|^2\} \geq 0. \tag{6.4}$$

The following proposition gives a criterion for the contractivity of $\rho_V^{(n)}(P_{\mathbf{A}})$.

Proposition 6.2. *The following conditions are equivalent:*

(i) $\|\rho_V^{(n)}(P_{\mathbf{A}})\| \leq 1$,

(ii) $\|(B_1, \ldots, B_m)\| \leq 1$, *where* $B_i = \sum_{j=1}^m v_{ij}A_j$,

(iii) $1 - \sum_{i=1}^m \sum_{j=1}^m \langle B_iB_j^*\beta, \beta\rangle \geq 0$, *where* $\beta = (\beta_1, \ldots, \beta_m) \in \mathbb{C}^m$ *and* $\sum_{i=1}^m |\beta_i|^2 = 1$.

Proof. First we will prove that (i) and (ii) are equivalent. Since $P_{\mathbf{A}}(0) = 0$, it follows from Proposition 2.17 that $\|\rho_V^{(n)}(P_{\mathbf{A}})\| \leq 1$ if and only if

$$\|A_1 \otimes \mathbf{v}_1 + \ldots + A_m \otimes \mathbf{v}_m\| \leq 1.$$

For $\mathbf{v}_i = (v_{i1}, \ldots, v_{im})$, we have

$$A_1 \otimes \mathbf{v}_1 + \ldots + A_m \otimes \mathbf{v}_m = (B_1, \ldots, B_m).$$

Thus $\rho_V^{(n)}(P_{\mathbf{A}})$ is contractive if and only if $\|(B_1, \ldots, B_m)\| \leq 1$. Hence (i) is equivalent to (ii).

Now, we will prove that (ii) implies (iii). Let $T = (B_1, \ldots, B_m)$. The contractivity of T is equivalent to the positivity of $I - TT^*$, which is equivalent to $\langle (I - TT^*)\alpha, \alpha\rangle \geq 0$ for all α. In particular, putting $\beta = \frac{\alpha}{\|\alpha\|}$, we have (iii). Clearly, (iii) implies (i) completing the proof. □

Example 6.3. If $A_1 = I_2$ and $A_2 = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, then the homomorphism ρ_V is contractive if and only if $|v|^2 \leq 1$ and

$$\inf_\beta\{1 - |v|^2 - |w|^2|\beta_1|^2 + |vw|^2|\beta_1|^4\} \geq 0.$$

Also, $\|P_{\mathbf{A}}(T_1,T_2)\| \leq 1$ if and only if

$$\inf_{\beta}\{1-|v|^2-|w|^2|\beta_1|^2\} \geq 0.$$

If $\mathbf{v}_1 = (\frac{1}{\sqrt{2}},0)$ and $\mathbf{v}_2 = (0,1)$, then it is easy to see that the homomorphism ρ_V is contractive. But for this choice of $\mathbf{v}_1,\mathbf{v}_2$ we have $\|P_{\mathbf{A}}(T_1,T_2)\| > 1$. Hence this contractive homomorphism ρ_V is not completely contractive.

Let $\Omega_{\mathbf{A}} = \{(z_1,z_2) : \|z_1A_1+z_2A_2\|_{\text{op}} < 1\}$ be a domain in $\mathbb{C}^2$, where $A_1 = \left(\begin{smallmatrix}1 & 0\\ 0 & d_2\end{smallmatrix}\right)$ or $\left(\begin{smallmatrix}d_1 & 0\\ 0 & 1\end{smallmatrix}\right)$ and $A_2 = \left(\begin{smallmatrix}0 & b\\ c & 0\end{smallmatrix}\right), \left(\begin{smallmatrix}1 & b\\ c & 0\end{smallmatrix}\right)$ or $\left(\begin{smallmatrix}0 & b\\ c & 1\end{smallmatrix}\right)$ with $b \in \mathbb{R}^+$. Let $P_{\mathbf{A}} : \Omega_{\mathbf{A}} \to (\mathcal{M}_2)_1$ be the matrix valued polynomial of the form $P_{\mathbf{A}}(z) = z_1A_1+z_2A_2$. In particular, if $B_1 = v_{11}A_1+v_{21}A_2$ and $B_2 = v_{12}A_1+v_{22}A_2$, then $\|\rho_V^{(2)}(P_{\mathbf{A}})\| \leq 1$ if and only if

$$\inf_{\beta}\{1-\langle B_1B_1^*\beta,\beta\rangle - \langle B_2B_2^*\beta,\beta\rangle\} \geq 0, \tag{6.5}$$

where $\|\beta\|_2 = 1$. Finding a V such that $\|L_V\|_{(\mathbb{C}^2,\|\cdot\|^*_{\Omega_{\mathbf{A}}})\to(\mathbb{C}^2,\|\cdot\|_2)} \leq 1$ for which $\|\rho_V^{(2)}(P_{\mathbf{A}})\|_{\text{op}} > 1$ produces an example of a contractive homomorphism of $\mathcal{O}(\Omega_{\mathbf{A}})$ which is not completely contractive. Thus it is enough to find $\mathbf{v}_1$ and $\mathbf{v}_2$ for which inequality (6.4) is valid while inequalities (6.5) fails. Luckily for us, to find such an example, it is enough to choose $\mathbf{v}_1 = (v,0)$ and $\mathbf{v}_2 = (0,w)$. In this case, the inequality (6.4) is equivalent to $\inf_\beta\{1-|v|^2\|A_1^*\beta\|^2\} \geq 0$ and

$$\inf_{\beta}\{1-|v|^2\|A_1^*\beta\|^2-|w|^2\|A_2^*\beta\|^2+|vw|^2(\|A_1^*\beta\|^2\|A_2^*\beta\|^2-|\langle A_1A_2^*\beta,\beta\rangle|^2)\} \geq 0. \tag{6.6}$$

and the inequality (6.5) is equivalent to

$$\inf_{\beta}\{1-|v|^2\|A_1^*\beta\|^2-|w|^2\|A_2^*\beta\|^2\} \geq 0. \tag{6.7}$$

Define $g_{(v,w)} : \partial\mathbb{B}^2 \to \mathbb{R}\cup\{0\}$ by

$$g_{(v,w)}(\beta) = 1-|v|^2\|A_1^*\beta\|^2-|w|^2\|A_2^*\beta\|^2+|vw|^2(\|A_1^*\beta\|^2\|A_2^*\beta\|^2-|\langle A_1A_2^*\beta,\beta\rangle|^2),$$

where $\mathbb{B}^2$ is the closed unit ball in $\mathbb{C}^2$ with respect to the ℓ^2 norm. Since g is continuous and $\partial\mathbb{B}^2$ is compact, it follows that $\inf_\beta g_{(v,w)}(\beta)$ exists. Hence $\|\rho_V\| \leq 1$ is equivalent to $|v|^2 \leq \frac{1}{\|A_1^*\|^2}$ and $\inf_\beta g_{(v,w)}(\beta) \geq 0$.

Theorem 6.4. *If A_1 and A_2 are not simultaneously diagonalizable, then there exists a contractive linear map from $(\mathbb{C}^2,\|\cdot\|_{\Omega_{\mathbf{A}}})$ to $\mathcal{M}_n(\mathbb{C})$ which is not completely contractive.*

Proof. Fix $\mathbf{v}_1 = (v,0), \mathbf{v}_2 = (0,w)$. Let $L_{(\mathbf{v}_1,\mathbf{v}_2)} : (\mathbb{C}^2,\|\cdot\|^*_{\Omega_{\mathbf{A}}}) \to (\mathbb{C}^2,\|\cdot\|_2)$ be the linear map $(z_1,z_2)\mapsto(z_1v,z_2w)$. The contractivity of $L_{(\mathbf{v}_1,\mathbf{v}_2)} : (\mathbb{C}^2,\|\cdot\|^*_{\Omega_{\mathbf{A}}}) \to (\mathbb{C}^2,\|\cdot\|_2)$ is equivalent to $|v|^2 \leq \frac{1}{\|A_1^*\|^2}$ and $(v,w) \in \mathcal{E} := \{(v,w) : \inf_{\beta,\|\beta\|_2=1} g_{(v,w)}(\beta) \geq 0\}$ and the contractivity of $L^{(2)}_{(\mathbf{v}_1,\mathbf{v}_2)}(P_{\mathbf{A}})$ is shown to be equivalent to the condition

$$\inf_{\beta}\{1 - |v|^2\|A_1^*\beta\|^2 - |w|^2\|A_2^*\beta\|^2 : \|\beta\|_2 = 1\} \geq 0.$$

Pick (v,w) such that $w = \lambda v, \lambda > 0$. There exists β in $\mathbb{C}^2$ such that either $(A_2^* - \mu A_1^*)\beta = 0$ or $(A_1^* - \nu A_2^*)\beta = 0$ for some $\mu, \nu \in \mathbb{C}$. The set

$$\mathcal{B} := \{\beta : \|\beta\|_2 = 1,\, (A_2^* - \mu A_1^*)\beta = 0 \,\text{or}\, (A_1^* - \nu A_2^*)\beta = 0 \,\text{for some}\, \mu,\, \nu \in \mathbb{C}\}$$

of these vectors is non-empty. The proof of the theorem involves two steps:

Claim 1: We show that there exists a $\lambda > 0$, say λ_0, such that $(v, \lambda_0 v)$ is in $\mathcal{E}$ with the property:

$g_{(v,\lambda_0 v)}(\beta'') > g_{(v,\lambda_0 v)}(\beta') > g_{(v,\lambda_0 v)}(\beta)$ or $g_{(v,\lambda_0 v)}(\beta') > g_{(v,\lambda_0 v)}(\beta'') > g_{(v,\lambda_0 v)}(\beta)$ whenever $\beta', \beta'' \in \mathcal{B}$.

Claim 2: We then prove that there exists a v ($|v| < \frac{1}{\|A_1^*\|}$, this is necessary for contractivity), say v_0, such that $\inf_\beta g_{(v_0,\,\lambda_0 v_0)}(\beta) = 0$, that is,

$$\inf_{\beta}\{1 - |v_0|^2\|A_1^*\beta\|^2 - |\lambda_0 v_0|^2\|A_2^*\beta\|^2 + \lambda_0^2|v_0|^4(\|A_1^*\beta\|^2\|A_2^*\beta\|^2 - |\langle A_1A_2^*\beta, \beta\rangle|^2)\} = 0.$$

Hence there exists a β_0 such that

$$1 - |v_0|^2\|A_1^*\beta_0\|^2 - |\lambda_0 v_0|^2\|A_2^*\beta_0\|^2 + \lambda_0^2|v_0|^4(\|A_1^*\beta_0\|^2\|A_2^*\beta_0\|^2 - |\langle A_1A_2^*\beta_0, \beta_0\rangle|^2) = 0$$

which is equivalent to $\|L^{(2)}_{(\mathbf{v}_1,\mathbf{v}_2)}(P_{\mathbf{A}})\| > 1$. This completes the proof subject to the verification of Claims 1 and 2. $\square$

In the remaining part of this chapter, we will carry out this verification on a case by case basis. This involves four cases, namely, $(i)\, b \neq |c|$ and $|d_2| = 1$, $(ii)\, b = |c|$ and $|d_2| \neq 1$, $(iii)\, b = |c|$ and $|d_2| = 1$ and $(iv)\, b \neq |c|$ and $|d_2| \neq 1$. For case (iv) we have shown in Chapter 3 that there exists a contractive linear map from $(\mathbb{C}^2, \|\cdot\|_{\Omega_{\mathbf{A}}})$ to $\mathcal{M}_n(\mathbb{C})$ which is not completely contractive. We will prove Theorem 6.4 for the remaining cases. The existence of a $\lambda > 0$, say λ_0, such that $(v, \lambda_0 v)$ is in $\mathcal{E}$ with the property:

$g_{(v,\lambda_0 v)}(\beta'') > g_{(v,\lambda_0 v)}(\beta') > g_{(v,\lambda_0 v)}(\beta)$ or $g_{(v,\lambda_0 v)}(\beta') > g_{(v,\lambda_0 v)}(\beta'') > g_{(v,\lambda_0 v)}(\beta)$ whenever $\beta', \beta'' \in \mathcal{B}$ follows from Theorems 6.5 and 6.6.

Theorem 6.5. *Let A_1 be of the form $\left(\begin{smallmatrix} 1 & 0 \\ 0 & d_2 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} d_1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and A_2 be of the form $\left(\begin{smallmatrix} 1 & b \\ c & 0 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 0 & b \\ c & 1 \end{smallmatrix}\right)$ and assume that they are not simultaneously diagonalizable. Then there exists $(v,\, \lambda_0 v)$ in $\mathcal{E}$ such that neither $g_{(v,\,\lambda_0 v)}(\beta')$ nor $g_{(v,\,\lambda_0 v)}(\beta'')$ is equal to $\inf_\beta g_{(v,\,\lambda_0 v)}(\beta)$.*

Proof. Suppose $A_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & d_2 \end{smallmatrix}\right)$, $A_2 = \left(\begin{smallmatrix} 1 & b \\ c & 0 \end{smallmatrix}\right)$. The homomorphism ρ_V is contractive, that is, $\|\rho_V\| \leq 1$ if and only if $\|L_{(\mathbf{v}_1, \mathbf{v}_2)}\|_{(\mathbb{C}^2, \|\cdot\|^*_{\Omega_{\mathbf{A}}}) \to (\mathbb{C}^2, \|\cdot\|_2)} \leq 1$ if and only if $|v|^2 \leq \frac{1}{\|A_1^*\|^2}$ and $(v, w) \in \mathcal{E}$. Now,

$$\begin{aligned}\inf_\beta g_{(v,\lambda v)}(\beta) &= \inf_\beta \{1 - |v|^2 - \lambda^2|v|^2(1+b^2)\}|\beta_1|^2 + \{1 - |v|^2|d_2|^2 - \lambda^2|v|^2|c|^2\}|\beta_2|^2 \\ &\quad -2\lambda^2|v|^2\Re c\beta_1\bar{\beta}_2 + \lambda^2|v|^4|b\bar{\beta}_1{}^2 - d_2\bar{\beta}_2(\bar{\beta}_1 + c\bar{\beta}_2)|^2. \end{aligned} \tag{6.8}$$

The proof of the theorem involves three distinct cases as indicated above. For each case we will follow the following steps:

Step 1: First we show there exists a $\lambda > 0$ say λ_0, such that either $g_{(v,\lambda_0 v)}(\beta'') > g_{(v,\lambda_0 v)}(\beta')$ or $g_{(v,\lambda_0 v)}(\beta'') < g_{(v,\lambda_0 v)}(\beta')$.

Step 2: If $g_{(v,\lambda_0 v)}(\beta'') > g_{(v,\lambda_0 v)}(\beta')$ (resp. $g_{(v,\lambda_0 v)}(\beta') > g_{(v,\lambda_0 v)}(\beta'')$), then we show that there exists a β such that $g_{(v,\lambda_0 v)}(\beta') > g_{(v,\lambda_0 v)}(\beta)$ (resp. $g_{(v,\lambda_0 v)}(\beta'') > g_{(v,\lambda_0 v)}(\beta)$).

Case (i): Here $b \neq |c|$ and $|d_2| = 1$, that is, $A_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \exp(i\theta) \end{smallmatrix}\right)$ and $A_2 = \left(\begin{smallmatrix} 1 & b \\ c & 0 \end{smallmatrix}\right)$ with $b \neq |c|$. Let $U = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \exp(-i\theta) \end{smallmatrix}\right)$. Then U is a unitary and the pair (A_1U, A_2U) determines the same set $\Omega_{\mathbf{A}}$. So, we may assume without loss of generality that $\mathbf{A}$ is of the form $(I_2, \left(\begin{smallmatrix} 1 & b \\ c & 0 \end{smallmatrix}\right))$ with $b, c \in \mathbb{C}, |b| \neq |c|$. Let W be a unitary such that $WA_2W^* = \left(\begin{smallmatrix} \alpha & \gamma \\ 0 & \delta \end{smallmatrix}\right)$, where α, β are the eigenvalue of A_2 with $|\alpha|^2 \geq |\delta|^2$. Therefore, without loss of generality we may also assume that $A_1 = I_2$ and $A_2 = \left(\begin{smallmatrix} \alpha & \gamma \\ 0 & \delta \end{smallmatrix}\right)$ with $|\alpha|^2 \geq |\delta|^2$. Then Equation (6.8) is equivalent to the condition $|v|^2 \leq 1$ and

$$\begin{aligned}\inf_\beta g_{(v,\lambda v)}(\beta) = \inf_\beta &\{1 - |v|^2 - \lambda^2|v|^2(|\alpha|^2 + |\gamma|^2)\}|\beta_1|^2 + \{1 - |v|^2 - \lambda^2|v|^2|\delta|^2\}|\beta_2|^2 \\ &- 2\lambda^2|v|^2\Re\bar{\gamma}\beta_1\bar{\beta}_2\delta + \lambda^2|v|^4|\bar{\beta}_1(\gamma\bar{\beta}_1 + \delta\bar{\beta}_2) - \bar{\beta}_2\alpha\bar{\beta}_1|^2. \end{aligned} \tag{6.9}$$

The roots of $\det(A_2^* - \mu A_1^*) = 0$ are $\mu_1 = \bar{\alpha}, \mu_2 = \bar{\delta}$. The vectors β', β'' satisfying $(A_2^* - \mu_1 A_1^*)\beta' = 0$ and $(A_2^* - \mu_2 A_1^*)\beta'' = 0$ are

$$\beta' = \Big(\frac{|\delta - \alpha|\exp(i\theta)}{\sqrt{|\delta - \alpha|^2 + |\gamma|^2}}, \frac{-\bar{\gamma}\exp i(\theta - \phi)}{\sqrt{|\delta - \alpha|^2 + |\gamma|^2}}\Big),$$

$\beta'' = (0, \exp(i\psi))$ respectively, where $\bar{\delta} - \bar{\alpha} = |\delta - \alpha|\exp(i\phi)$. From Equation (6.9) it is easy to see that

$$g_{(v,\lambda v)}(\beta') = \{1 - |v|^2 - \lambda^2|v|^2(|\alpha|^2 + |\gamma|^2)\}|\beta_1'|^2 + \{1 - |v|^2 - \lambda^2|v|^2|\delta|^2\}|\beta_2'|^2 - 2\lambda^2|v|^2\Re\bar{\gamma}\beta_1'\bar{\beta}_2'\delta$$

and $g_{(v,\lambda v)}(\beta'') = 1 - |v|^2 - \lambda^2|v|^2|\delta|^2$, where $\beta' = (\beta_1', \beta_2')$. Note that

$$\begin{aligned} g_{(v,\lambda v)}(\beta'') - g_{(v,\lambda v)}(\beta') &= \lambda^2|v|^2(|\alpha|^2 + |\gamma|^2 - |\delta|^2)|\beta_1'|^2 + 2\lambda^2|v|^2\Re\bar{\gamma}\beta_1'\bar{\beta}_2'\delta \\ &= \lambda^2|v|^2(|\alpha|^2 - |\delta|^2). \end{aligned} \tag{6.10}$$

(a) We assume that $|\alpha|^2 > |\delta|^2$. Since $|\alpha|^2 > |\delta|^2$, from Equation (6.10) we have $g_{(v,\lambda v)}(\beta'') > g_{(v,\lambda v)}(\beta')$. Hence we conclude that $\inf_\beta g_{(v,\lambda v)}(\beta) \neq g_{(v,\lambda v)}(\beta'')$.

In order to prove Step 2, it is sufficient to observe that

$$\begin{aligned} g_{(v,\lambda v)}(\beta') - g_{(v,\lambda v)}((1,0)) &= \lambda^2|v|^2(|\alpha|^2+|\gamma|^2-|\delta|^2)|\beta_2'|^2 - 2\lambda^2|v|^2\Re\bar{\gamma}\beta_1'\bar{\beta}_2'\delta - \lambda^2|v|^4|\gamma|^2 \\ &= \lambda^2|v|^2(|\alpha|^2+|\gamma|^2-|\delta|^2)\frac{|\gamma|^2}{|\delta-\alpha|^2+|\gamma|^2} \\ &+ \frac{2\lambda^2|v|^2|\gamma|^2\Re\delta(\bar{\delta}-\bar{\alpha})}{|\delta-\alpha|^2+|\gamma|^2} - \lambda^2|v|^4|\gamma|^2 \\ &= \lambda^2|v|^2|\gamma|^2(1-|v|^2). \end{aligned} \tag{6.11}$$

The Equation (6.11) shows that for all $|v| \in [0,1)$, for all λ, there exists a $\beta = (1,0)$ such that $g_{(v,\lambda v)}(\beta') > g_{(v,\lambda v)}((1,0))$. We therefore conclude that neither $g_{(v,\lambda_0 v)}(\beta')$ nor $g_{(v,\lambda_0 v)}(\beta'')$ is equal to $\inf_\beta g_{(v,\lambda_0 v)}(\beta)$ for any v with $|v| < 1$.

(b) Suppose $|\alpha|^2 = |\delta|^2$. Then from Equation (6.10) we have $g_{(v,\lambda v)}(\beta'') = g_{(v,\lambda v)}(\beta')$. From Equation (6.11) we see that $g_{(v,\lambda v)}(\beta') - g_{(v,\lambda v)}((1,0) = \lambda^2|v|^2|\gamma|^2(1-|v|^2)$. Therefore it follows that for all $|v| \in [0,1)$ and for all λ, there exists a $\beta = (1,0)$ such that $g_{(v,\lambda v)}(\beta') > g_{(v,\lambda v)}((1,0))$. Hence we see that neither $g_{(v,\lambda_0 v)}(\beta')$ nor $g_{(v,\lambda_0 v)}(\beta'')$ is equal to $\inf_\beta g_{(v,\lambda_0 v)}(\beta)$ for any v with $|v| < 1$.

The roots of $\det(\nu A_2^* - A_1^*) = 0$ are $\nu_1 = \frac{1}{\bar{\alpha}}, \nu_2 = \frac{1}{\bar{\beta}}$. The vectors β', β'' satisfying $(\nu_1 A_2^* - A_1^*)\beta' = 0$ and $(\nu_2 A_2^* - A_1^*)\beta'' = 0$ are

$$\beta' = \Big(\frac{|\delta-\alpha|\exp(i\theta)}{\sqrt{|\delta-\alpha|^2+|\gamma|^2}}, \frac{-\bar{\gamma}\exp i(\theta-\phi)}{\sqrt{|\delta-\alpha|^2+|\gamma|^2}}\Big),$$

$\beta'' = (0, \exp(i\psi))$ respectively, where $\bar{\delta} - \bar{\alpha} = |\delta-\alpha|\exp(i\phi)$. Proceeding the same way, as above, we also find that neither $g_{(v,\lambda_0 v)}(\beta')$ nor $g_{(v,\lambda_0 v)}(\beta'')$ is equal to $\inf_\beta g_{(v,\lambda_0 v)}(\beta)$ for any v with $|v| < 1$.

Case (ii): In this case, $A_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & d_2 \end{smallmatrix}\right)$, $A_2 = \left(\begin{smallmatrix} 1 & |c| \\ c & 0 \end{smallmatrix}\right)$ with $|d_2| \neq 1$. The roots of $\det(A_2^* - \mu A_1^*) = 0$ are

$$\mu_1 = \frac{\sqrt{\bar{d}_2} + \sqrt{\bar{d}_2 + 4|c|\bar{c}}}{2\sqrt{\bar{d}_2}}, \mu_2 = \frac{\sqrt{\bar{d}_2} - \sqrt{\bar{d}_2 + 4|c|\bar{c}}}{2\sqrt{\bar{d}_2}}.$$

The vectors β', β'' satisfying $(A_2^* - \mu_1 A_1^*)\beta' = 0$ and $(A_2^* - \mu_2 A_1^*)\beta'' = 0$ are

$$\beta' = \Big(\frac{|c|\exp(i\theta_1)}{\sqrt{|\mu_2|^2+|c|^2}}, \frac{-\mu_2\exp i(\theta_1-\phi_1)}{\sqrt{|\mu_2|^2+|c|^2}}\Big)$$

and

$$\beta'' = \Big(\frac{|c|\exp(i\theta_2)}{\sqrt{|\mu_1|^2+|c|^2}}, \frac{-\mu_1\exp i(\theta_2-\phi_1)}{\sqrt{|\mu_1|^2+|c|^2}}\Big)$$

respectively, where $\bar{c} = |c|\exp(i\phi_1)$. Substituting $\beta = \beta'$ and $\beta = \beta''$ in Equation (6.8) we have

$$g_{(v,\,\lambda v)}(\beta') \;=\; \{1-|v|^2-\lambda^2|v|^2(1+|c|^2)\}|\beta_1'|^2+\{1-|v|^2|d_2|^2-\lambda^2|v|^2|c|^2\}|\beta_2'|^2-2\lambda^2|v|^2\Re c\beta_1'\bar{\beta}_2'$$

and

$$g_{(v,\,\lambda v)}(\beta'') \;=\; \{1-|v|^2-\lambda^2|v|^2(1+|c|^2)\}|\beta_1''|^2+\{1-|v|^2|d_2|^2-\lambda^2|v|^2|c|^2\}|\beta_2''|^2-2\lambda^2|v|^2\Re c\beta_1''\bar{\beta}_2'',$$

where $\beta' = (\beta_1', \beta_2')$ and $\beta'' = (\beta_1'', \beta_2'')$. Now,

$$\begin{aligned} g_{(v,\,\lambda v)}(\beta') - g_{(v,\,\lambda v)}(\beta'') = \{1-|v|^2-\lambda^2|v|^2(1+|c|^2)\}(|\beta_1'|^2-|\beta_1''|^2) \\ + \;\; \{1-|v|^2|d_2|^2-\lambda^2|v|^2|c|^2\}(|\beta_2'|^2-|\beta_2''|^2)-2\lambda^2|v|^2 r, \end{aligned} \tag{6.12}$$

where $r = \Re c(\beta_1'\bar{\beta}_2' - \beta_1''\bar{\beta}_2'')$, depends only on A_1, A_2.

Since A_1 and A_2 are not simultaneously diagonalizable by hypothesis, it follows that $c \neq 0$, or equivalently, $\beta_1' \neq 0$. Similarly we show that $\beta_2'' \neq 0$. Without loss of generality we can assume that $|\beta_1'|^2 \geq |\beta_1''|^2$. This splits into two cases, namely, $(a)|\beta_1'|^2 > |\beta_1''|^2$, which is equivalent to $r \neq 0$. and $(b)|\beta_1'|^2 = |\beta_1''|^2$ which is equivalent to $r = 0$. We now consider these two cases separately.

(a) Suppose $|\beta_1'|^2 > |\beta_1''|^2$ which is equivalent to $r \neq 0$. Since $|\beta_1'|^2 > |\beta_1''|^2$ we have $|\beta_1'|^2 - |\beta_1''|^2 = \delta_1 > 0$. Also, from above relation it follows that $(|\beta_2'|^2 - |\beta_2''|^2) = -\delta_1$. Substituting $(|\beta_1'|^2 - |\beta_1''|^2) = \delta_1$ in Equation (6.12) we have

$$g_{(v,\,\lambda v)}(\beta') - g_{(v,\,\lambda v)}(\beta'') \;=\; \{|v|^2(|d_2|^2-1)-\lambda^2|v|^2)\}\delta_1 - 2\lambda^2|v|^2 r. \tag{6.13}$$

Now, we have several possibilities which are listed below.

- **$1 < |d_2|,\ -1 + 2\Re\bar{\mu}_1 > 0,\ r > 0$:**
From Equation (6.13) we observe that $g_{(v,\,\lambda v)}(\beta') > g_{(v,\,\lambda v)}(\beta'')$ if $\lambda^2 < \frac{(|d_2|^2-1)\delta_1}{(\delta_1+2r)}$.
Hence $\inf_\beta g_{(v,\,\lambda v)}(\beta) \neq g_{(v,\,\lambda v)}(\beta')$ for all λ, $\lambda^2 < \frac{(|d_2|^2-1)\delta_1}{(\delta_1+2r)}$.
Evaluating $g_{(v,\,\lambda v)}$ at $(0,1)$, we have

$$g_{(v,\,\lambda v)}((0,1)) = \{1-|v|^2|d_2|^2-\lambda^2|v|^2|c|^2\} + \lambda^2|v|^4|cd_2|^2$$

which gives

$$\begin{aligned} & g_{(v,\,\lambda v)}(\beta'') - g_{(v,\,\lambda v)}((0,1)) \\ = & \frac{|c|^2|v|^2}{(|\mu_1|^2+|c|^2)}\{(|d_2|^2-1)-\lambda^2+2\lambda^2\Re\bar{\mu}_1-\lambda^2|v|^2|d_2|^2(|\mu_1|^2+|c|^2)\}. \end{aligned} \tag{6.14}$$

Since $|d_2| > 1$, from Equation (6.14), we have $g_{(v,\,\lambda v)}(\beta'') > g_{(v,\,\lambda v)}((0,1))$ for all λ and for all v, $|v|^2 \leq \frac{-1+2\Re\bar{\mu}_1}{|d_2|^2(|\mu_1|^2+|c|^2)}$.

Also, from Equation (6.14), we obtain $g_{(v,\lambda v)}(\beta'') > g_{(v,\lambda v)}((0,1))$ for all λ,

$$\lambda^2 < \frac{(|d_2|^2-1)}{1-2\Re\bar{\mu}_1+|v|^2|d_2|^2(|\mu_1|^2+|c|^2)}$$

and for all v, $|v|^2 > \frac{-1+2\Re\bar{\mu}_1}{|d_2|^2(|\mu_1|^2+|c|^2)}$. Thus we have $g_{(v,\lambda v)}(\beta'') > g_{(v,\lambda v)}((0,1))$ for all λ,

$$\lambda^2 < \frac{(|d_2|^2-1)}{1-2\Re\bar{\mu}_1+|v|^2|d_2|^2(|\mu_1|^2+|c|^2)})$$

for all $v, v > 0$.

We therefore conclude that neither $g_{(v,\lambda v)}(\beta')$ nor $g_{(v,\lambda v)}(\beta'')$ is equal to $\inf_\beta g_{(v,\lambda v)}(\beta)$ for any v with $|v|$ in $(0, \frac{1}{\|A_1^*\|}]$ and for any λ with

$$\lambda^2 < \min\left\{\frac{(|d_2|^2-1)}{1-2\Re\bar{\mu}_1+|v|^2|d_2|^2(|\mu_1|^2+|c|^2)}, \frac{(|d_2|^2-1)\delta_1}{(\delta_1+2r)}\right\}.$$

- $1 < |d_2|$, $-1+2\Re\bar{\mu}_1 < 0$, $r > 0$**:**

  From Equation (6.13) we observe that $g_{(v,\lambda v)}(\beta') > g_{(v,\lambda v)}(\beta'')$ if $\lambda^2 < \frac{(|d_2|^2-1)\delta_1}{(\delta_1+2r)}$. Hence $\inf_\beta g_{(v,\lambda v)}(\beta) \neq g_{(v,\lambda v)}(\beta')$ for all λ, $\lambda^2 < \frac{(|d_2|^2-1)\delta_1}{(\delta_1+2r)}$.

  Since $|d_2| > 1$, from Equation (6.14), we have $g_{(v,\lambda v)}(\beta'') > g_{(v,\lambda v)}((0,1))$ for all λ,

  $$\lambda^2 < \frac{(|d_2|^2-1)}{1-2\Re\bar{\mu}_1+(|\mu_1|^2+|c|^2)}$$

  and for all v, $|v|$ in $(0, \frac{1}{\|A_1^*\|}]$.

  Thus we conclude that neither $g_{(v,\lambda v)}(\beta')$ nor $g_{(v,\lambda v)}(\beta'')$ is equal to $\inf_\beta g_{(v,\lambda v)}(\beta)$ for any v with $|v|$ in $(0, \frac{1}{\|A_1^*\|}]$ and for any λ with

  $$\lambda^2 < \min\left\{\frac{(|d_2|^2-1)}{1-2\Re\bar{\mu}_1+(|\mu_1|^2+|c|^2)}, \frac{(|d_2|^2-1)\delta_1}{(\delta_1+2r)}\right\}.$$

- $|d_2| > 1$, $1-2\Re\bar{\mu}_1 > 0$, $r < 0$, $2r+\delta_1 < 0$**:**

  From Equation (6.13) we have $g_{(v,\lambda v)}(\beta'') < g_{(v,\lambda v)}(\beta')$ for all λ. Hence we have $\inf_\beta g_{(v,\lambda v)}(\beta) \neq g_{(v,\lambda v)}(\beta')$ for all λ.

  Since $|d_2| > 1$, from Equation (6.14) we have $g_{(v,\lambda v)}(\beta'') > g_{(v,\lambda v)}((0,1))$ for all λ and for all v, $|v|^2 \leq \frac{-1+\Re\bar{\mu}_1}{|d_2|^2(|\mu_1|^2+|c|^2)}$.

  Also, from Equation (6.14) we obtain $g_{(v,\lambda v)}(\beta'') > g_{(v,\lambda v)}((0,1))$ for all λ,

  $$\lambda^2 < \frac{(|d_2|^2-1)}{1-2\Re\bar{\mu}_1+|v|^2|d_2|^2(|\mu_1|^2+|c|^2)}$$

and for all v, $|v|^2 > \frac{-1+\Re\bar{\mu}_1}{|d_2|^2(|\mu_1|^2+|c|^2)}$. Thus we have $g_{(v,\lambda v)}(\beta'') > g_{(v,\lambda v)}((0,1))$ for all λ,

$$\lambda^2 < \frac{(|d_2|^2-1)}{1-2\Re\bar{\mu}_1+|v|^2|d_2|^2(|\mu_1|^2+|c|^2)}$$

and for all $v, v > 0$.

We therefore conclude that neither $g_{(v,\lambda v)}(\beta')$ nor $g_{(v,\lambda v)}(\beta'')$ is equal to $\inf_\beta g_{(v,\lambda v)}(\beta)$ for any v with $|v|$ in $(0, \frac{1}{\|A_1^*\|}]$ and for any λ with

$$\lambda^2 < \frac{(|d_2|^2-1)}{1-2\Re\bar{\mu}_1+|d_2|^2|v|^2(|\mu_1|^2+|c|^2)},$$

- $|d_2| > 1$, $1-2\Re\bar{\mu}_1 < 0$, $r < 0$, $2r+\delta_1 < 0$**:**

  From Equation (6.13) we have $g_{(v,\lambda v)}(\beta'') < g_{(v,\lambda v)}(\beta')$ for all λ. Hence we conclude that $\inf_\beta g_{(v,\lambda v)}(\beta) \neq g_{(v,\lambda v)}(\beta')$ for all λ.

  Since $|d_2| > 1$, from Equation (6.14), we have $g_{(v,\lambda v)}(\beta'') > g_{(v,\lambda v)}((0,1))$ for all λ with

  $$\lambda^2 < \frac{(|d_2|^2-1)}{1-2\Re\bar{\mu}_1+(|\mu_1|^2+|c|^2)}$$

  and for all v with $|v|$ in $(0, \frac{1}{\|A^*\|}]$.

  Hence if λ is chosen with

  $$\lambda^2 < \frac{(|d_2|^2-1)}{1-2\Re\bar{\mu}_1+(|\mu_1|^2+|c|^2)},$$

  then it follows that neither $g_{(v,\lambda v)}(\beta')$ nor $g_{(v,\lambda v)}(\beta'')$ is equal to $\inf_\beta g_{(v,\lambda v)}(\beta)$ for any v with $|v|$ in $(0, \frac{1}{\|A_1^*\|}]$.

- $|d_2| > 1$, $1-2\Re\bar{\mu}_1 > 0$, $r < 0$, $2r+\delta_1 > 0$**:**

  From Equation (6.13), we have $g_{(v,\lambda v)}(\beta'') < g_{(v,\lambda v)}(\beta')$ if $\lambda^2 < \frac{(|d_2|^2-1)\delta_1}{(\delta_1+2r)}$. Hence we have $\inf_\beta g_{(v,\lambda v)}(\beta) \neq g_{(v,\lambda v)}(\beta')$ for all λ, $\lambda^2 < \frac{(|d_2|^2-1)\delta_1}{(\delta_1+2r)}$.

  From Equation (6.14) we have $g_{(v,\lambda v)}(\beta'') > g_{(v,\lambda v)}((0,1))$ for all λ and for all v, $|v|^2 \leq \frac{-1+\Re\bar{\mu}_1}{|d_2|^2(|\mu_1|^2+|c|^2)}$.

  Also, from Equation (6.14) we obtain $g_{(v,\lambda v)}(\beta'') > g_{(v,\lambda v)}((0,1))$ for all λ,

  $$\lambda^2 < \frac{(|d_2|^2-1)}{1-2\Re\bar{\mu}_1+|v|^2|d_2|^2(|\mu_1|^2+|c|^2)}$$

  and for all v, $|v|^2 > \frac{-1+\Re\bar{\mu}_1}{|d_2|^2(|\mu_1|^2+|c|^2)}$.

  Thus we have $g_{(v,\lambda v)}(\beta'') > g_{(v,\lambda v)}((0,1))$ for all λ,

  $$\lambda^2 < \frac{(|d_2|^2-1)}{1-2\Re\bar{\mu}_1+|v|^2|d_2|^2(|\mu_1|^2+|c|^2)}$$

and for all $v, v > 0$.

If λ is chosen with

$$\lambda^2 < \min\left\{\frac{(|d_2|^2-1)}{1-2\Re\bar{\mu}_1+|v|^2|d_2|^2(|\mu_1|^2+|c|^2)}, \frac{(|d_2|^2-1)\delta_1}{(\delta_1+2r)}\right\},$$

then neither $g_{(v,\lambda v)}(\beta')$ nor $g_{(v,\lambda v)}(\beta'')$ is equal to $\inf_\beta g_{(v,\lambda v)}(\beta)$ for any v with $|v|$ in $(0, \frac{1}{\|A_1^*\|}]$.

- **$|d_2| > 1$, $1-2\Re\bar{\mu}_1 < 0$, $r < 0$, $2r+\delta_1 > 0$:**

  From Equation (6.13), we have $g_{(v,\lambda v)}(\beta'') < g_{(v,\lambda v)}(\beta')$ if $\lambda^2 < \frac{(|d_2|^2-1)\delta_1}{(\delta_1+2r)}$. Hence we have $\inf_\beta g_{(v,\lambda v)}(\beta) \neq g_{(v,\lambda v)}(\beta')$ for all λ, $\lambda^2 < \frac{(|d_2|^2-1)\delta_1}{(\delta_1+2r)}$.

  From Equation (6.14), we also have $g_{(v,\lambda v)}(\beta'') > g_{(v,\lambda v)}((0,1))$ for all λ,

  $$\lambda^2 < \frac{(|d_2|^2-1)}{1-2\Re\bar{\mu}_1+(|\mu_1|^2+|c|^2)}$$

  and for all v, $|v|$ in $(0, \frac{1}{\|A_1^*\|}]$.

  If λ is chosen with

  $$\lambda^2 < \min\left\{\frac{(|d_2|^2-1)}{1-2\Re\bar{\mu}_1+(|\mu_1|^2+|c|^2)}, \frac{(|d_2|^2-1)\delta_1}{(\delta_1+2r)}\right\},$$

  then neither $g_{(v,\lambda v)}(\beta')$ nor $g_{(v,\lambda v)}(\beta'')$ is equal to $\inf_\beta g_{(v,\lambda v)}(\beta)$ for any v with $|v|$ in $(0, \frac{1}{\|A_1^*\|}]$.

- **$|d_2| < 1$, $|\mu_2|^2 + 2|c|^2\Re\bar{\mu}_2 > 0$, $r > 0$:**

  From Equation (6.13), it is easy to see that $g_{(v,\lambda v)}(\beta'') > g_{(v,\lambda v)}(\beta')$ for all λ. Hence $\inf_\beta g_{(v,\lambda v)}(\beta) \neq g_{(v,\lambda v)}(\beta'')$ for all λ.

  Also, note that

  $$\begin{aligned} g_{(v,\lambda v)}(\beta') - g_{(v,\lambda v)}((1,0)) = {} & \frac{|v|^2}{(|\mu_2|^2+|c|^2)}\{(1-|d_2|^2)|\mu_2|^2+\lambda^2|\mu_2|^2 \\ & + \; 2\lambda^2|c|^2\Re\bar{\mu}_2 - \lambda^2|v|^2(|\mu_2|^2+|c|^2)|c|^2\}. \end{aligned} \tag{6.15}$$

  From Equation (6.15), We have $g_{(v,\lambda v)}(\beta') > g_{(v,\lambda v)}((1,0))$ for all λ and for all v, $|v|^2 \leq \frac{(|\mu_2|^2+2|c|^2\Re\bar{\mu}_2)}{|c|^2(|\mu_2|^2+|c|^2)}$.

  Also, from Equation (6.15), we obtain $g_{(v,\lambda v)}(\beta') > g_{(v,\lambda v)}((1,0))$ for all λ,

  $$\lambda^2 < \frac{(1-|d_2|^2)|\mu_2|^2}{|v|^2|c|^2(|\mu_2|^2+|c|^2)-|\mu_2|^2-2|c|^2\Re\bar{\mu}_2}$$

  and for all v, $|v|^2 > \frac{(|\mu_2|^2+2|c|^2\Re\bar{\mu}_2)}{|c|^2(|\mu_2|^2+|c|^2)}$. Thus we have $g_{(v,\lambda v)}(\beta') > g_{(v,\lambda v)}((1,0))$ for all

  $$\lambda^2 < \frac{(1-|d_2|^2)|\mu_2|^2}{|v|^2|c|^2(|\mu_2|^2+|c|^2)-|\mu_2|^2-2|c|^2\Re\bar{\mu}_2}$$

and for all $v, v > 0$.

We therefore conclude that neither $g_{(v,\lambda v)}(\beta')$ nor $g_{(v,\lambda v)}(\beta'')$ is equal to $\inf_\beta g_{(v,\lambda v)}(\beta)$ for any v with $|v|$ in $(0, \frac{1}{\|A_1^*\|}]$ and for any λ with

$$\lambda^2 < \frac{(1-|d_2|^2)|\mu_2|^2}{|v|^2|c|^2(|\mu_2|^2+|c|^2)-|\mu_2|^2-2|c|^2\Re\bar{\mu}_2}.$$

- **$|d_2| < 1$, $|\mu_2|^2 + 2|c|^2\Re\bar{\mu}_2 < 0$, $r > 0$:**
  From Equation (6.13), it is easy to see that $g_{(v,\lambda v)}(\beta'') > g_{(v,\lambda v)}(\beta')$ for all λ. Hence $\inf_\beta g_{(v,\lambda v)}(\beta) \neq g_{(v,\lambda v)}(\beta'')$ for all λ.
  If λ is chosen with
  $$\lambda^2 < \frac{(1-|d_2|^2)|\mu_2|^2}{|c|^2(|\mu_2|^2+|c|^2)-|\mu_2|^2-2|c|^2\Re\bar{\mu}_2},$$
  then from Equation (6.15), we have $g_{(v,\lambda v)}(\beta') > g_{(v,\lambda v)}((1,0))$ for any v with $|v|$ in $(0,1]$.
  Thus we conclude that neither $g_{(v,\lambda v)}(\beta')$ nor $g_{(v,\lambda v)}(\beta'')$ is equal to $\inf_\beta g_{(v,\lambda v)}(\beta)$ for any v with $|v|$ in $(0,1]$ and for any λ with
  $$\lambda^2 < \frac{(1-|d_2|^2)|\mu_2|^2}{|c|^2(|\mu_2|^2+|c|^2)-|\mu_2|^2-2|c|^2\Re\bar{\mu}_2}.$$

- **$|d_2| < 1$, $|\mu_2|^2 + 2|c|^2\Re\bar{\mu}_2 > 0$, $r < 0$, $2r + \delta_1 < 0$:**
  From Equation (6.13), we have $g_{(v,\lambda v)}(\beta'') > g_{(v,\lambda v)}(\beta')$ if $\lambda^2 < \frac{(1-|d_2|^2)\delta_1}{-(\delta_1+2r)}$. Hence $\inf_\beta g_{(v,\lambda v)}(\beta) \neq g_{(v,\lambda v)}(\beta'')$ for all λ, $\lambda^2 < \frac{(1-|d_2|^2)\delta_1}{-(\delta_1+2r)}$.
  From Equation (6.15), we have $g_{(v,\lambda v)}(\beta') > g_{(v,\lambda v)}((1,0))$ for all λ and for all v, $|v|^2 \leq \frac{|\mu_2|^2+2|c|^2\Re\bar{\mu}_2}{(|\mu_2|^2+|c|^2)|c|^2}$.
  Also, from Equation (6.15), we obtain $g_{(v,\lambda v)}(\beta') > g_{(v,\lambda v)}((1,0))$ for all λ,
  $$\lambda^2 < \frac{(1-|d_2|^2)|\mu_2|^2}{|v|^2(|\mu_2|^2+|c|^2)|c|^2-|\mu_2|^2-2|c|^2\Re\bar{\mu}_2}$$
  and for all v, $|v|^2 > \frac{|\mu_2|^2+2|c|^2\Re\bar{\mu}_2}{(|\mu_2|^2+|c|^2)|c|^2}$. Thus we have $g_{(v,\lambda v)}(\beta') > g_{(v,\lambda v)}((1,0))$ for all λ,
  $$\lambda^2 < \frac{(1-|d_2|^2)|\mu_2|^2}{|v|^2(|\mu_2|^2+|c|^2)|c|^2-|\mu_2|^2-2|c|^2\Re\bar{\mu}_2}$$
  and for all $v, v > 0$.
  We therefore conclude that neither $g_{(v,\lambda v)}(\beta')$ nor $g_{(v,\lambda v)}(\beta'')$ is equal to $\inf_\beta g_{(v,\lambda v)}(\beta)$ for any v with $|v|$ in $(0, \frac{1}{\|A_1^*\|}]$ and for any λ with
  $$\lambda^2 < \min\left\{\frac{(1-|d_2|^2)|\mu_2|^2}{|v|^2(|\mu_2|^2+|c|^2)|c|^2-|\mu_2|^2-2|c|^2\Re\bar{\mu}_2}, \frac{(1-|d_2|^2)\delta_1}{-(2r+\delta_1)}\right\}.$$

- $|d_2| < 1,\ |\mu_2|^2 + 2|c|^2\Re\bar{\mu}_2 < 0,\ r < 0,\ 2r + \delta_1 < 0$:

  From Equation (6.13), we have $g_{(v,\,\lambda v)}(\beta'') > g_{(v,\,\lambda v)}(\beta')$ if $\lambda^2 < \frac{(1-|d_2|^2)\delta_1}{-(\delta_1+2r)}$. Hence $\inf_\beta g_{(v,\,\lambda v)}(\beta) \neq g_{(v,\,\lambda v)}(\beta'')$ for all λ, $\lambda^2 < \frac{(1-|d_2|^2)\delta_1}{-(\delta_1+2r)}$.

  From Equation (6.15), we have $g_{(v,\,\lambda v)}(\beta') > g_{(v,\,\lambda v)}((1,0))$ for all λ,

$$\lambda^2 < \frac{(1-|d_2|^2)|\mu_2|^2}{(|\mu_2|^2+|c|^2)|c|^2 - |\mu_2|^2 - 2|c|^2\Re\bar{\mu}_2}$$

  for all v, $|v|$ in $(0, \frac{1}{\|A_1^*\|}]$.

  We therefore conclude that neither $g_{(v,\,\lambda v)}(\beta')$ nor $g_{(v,\,\lambda v)}(\beta'')$ is equal to $\inf_\beta g_{(v,\,\lambda v)}(\beta)$ for any v, with $|v|$ in $(0, \frac{1}{\|A_1^*\|}]$ and for any λ with

$$\lambda^2 < \min\left\{\frac{(1-|d_2|^2)|\mu_2|^2}{(|\mu_2|^2+|c|^2)|c|^2 - |\mu_2|^2 - 2|c|^2\Re\bar{\mu}_2}, \frac{(1-|d_2|^2)\delta_1}{-(2r+\delta_1)}\right\}.$$

- $|d_2| < 1,\ |\mu_2|^2 + 2|c|^2\Re\bar{\mu}_2 > 0,\ r < 0,\ 2r + \delta_1 > 0$:

  From Equation (6.13) it is easy to see that $g_{(v,\,\lambda v)}(\beta'') > g_{(v,\,\lambda v)}(\beta')$ for all λ. Hence $\inf_\beta g_{(v,\,\lambda v)}(\beta) \neq g_{(v,\,\lambda v)}(\beta'')$ for all λ.

  From Equation (6.15) we have $g_{(v,\,\lambda v)}(\beta') > g_{(v,\,\lambda v)}((1,0))$ for all λ and for all v, $|v|^2 \leq \frac{(|\mu_2|^2+2|c|^2\Re\bar{\mu}_2)}{|c|^2(|\mu_2|^2+|c|^2)}$.

  From Equation (6.15) we obtain $g_{(v,\,\lambda v)}(\beta') > g_{(v,\,\lambda v)}((1,0))$ for all λ,

$$\lambda^2 < \frac{(1-|d_2|^2)|\mu_2|^2}{|v|^2|c|^2(|\mu_2|^2+|c|^2) - |\mu_2|^2 - 2|c|^2\Re\bar{\mu}_2}$$

  and for all v, $|v|^2 > \frac{(|\mu_2|^2+2|c|^2\Re\bar{\mu}_2)}{|c|^2(|\mu_2|^2+|c|^2)}$. Thus we have $g_{(v,\,\lambda v)}(\beta') > g_{(v,\,\lambda v)}((1,0))$ for all λ,

$$\lambda^2 < \frac{(1-|d_2|^2)|\mu_2|^2}{|v|^2|c|^2(|\mu_2|^2+|c|^2) - |\mu_2|^2 - 2|c|^2\Re\bar{\mu}_2}$$

  and for all $v, v > 0$.

  We therefore conclude that neither $g_{(v,\,\lambda v)}(\beta')$ nor $g_{(v,\,\lambda v)}(\beta'')$ is equal to $\inf_\beta g_{(v,\,\lambda v)}(\beta)$ for any v with $|v|$ in $(0, \frac{1}{\|A_1^*\|}]$ and for any λ with

$$\lambda^2 < \frac{(1-|d_2|^2)|\mu_2|^2}{|v|^2|c|^2(|\mu_2|^2+|c|^2) - |\mu_2|^2 - 2|c|^2\Re\bar{\mu}_2}.$$

- $|d_2| < 1,\ |\mu_2|^2 + 2|c|^2\Re\bar{\mu}_2 < 0,\ r < 0,\ 2r + \delta_1 > 0$:

  From Equation (6.13) it is easy to see that $g_{(v,\,\lambda v)}(\beta'') > g_{(v,\,\lambda v)}(\beta')$ for all λ. Hence $\inf_\beta g_{(v,\,\lambda v)}(\beta) \neq g_{(v,\,\lambda v)}(\beta'')$ for all λ.

If λ is chosen with

$$\lambda^2 < \frac{(1-|d_2|^2)|\mu_2|^2}{|c|^2(|\mu_2|^2+|c|^2)-|\mu_2|^2-2|c|^2\Re\bar{\mu}_2},$$

then from Equation (6.15), we have $g_{(v,\lambda v)}(\beta') > g_{(v,\lambda v)}((1,0))$ for any v with $|v|$ in $(0,1]$.

Thus we conclude that neither $g_{(v,\lambda v)}(\beta')$ nor $g_{(v,\lambda v)}(\beta'')$ is equal to $\inf_\beta g_{(v,\lambda v)}(\beta)$ for any v with $|v|$ in $(0,1]$ and for any λ with

$$\lambda^2 < \frac{(1-|d_2|^2)|\mu_2|^2}{|c|^2(|\mu_2|^2+|c|^2)-|\mu_2|^2-2|c|^2\Re\bar{\mu}_2}.$$

(b) If $|\beta_1'|^2 = |\beta_1''|^2$, then $|\mu_1|^2 = |\mu_2|^2$. Thus $r=0$. Therefore, from Equation (6.13) we have $g_{(v,\lambda v)}(\beta'') = g_{(v,\lambda v)}(\beta')$. Here we have two possibilities, namely, $|d_2| > 1$ and $|d_2| < 1$.

- **$|d_2| > 1$, $r = 0$:**
  Since $r=0$, we have $1-\Re\bar{\mu}_1 = 0$. From Equation (6.14) we can easily see that $g_{(v,\lambda v)}(\beta'') > g_{(v,\lambda v)}((0,1))$ for any λ with $\lambda^2 < \frac{(|d_2|^2-1)}{(|\mu_1|^2+|c|^2)}$ and for any v with $|v|$ in $(0,\frac{1}{\|A_1^*\|}]$.
  Hence we conclude that neither $g_{(v,\lambda v)}(\beta')$ nor $g_{(v,\lambda v)}(\beta'')$ is equal to $\inf_\beta g_{(v,\lambda v)}(\beta)$ for any v with $|v|$ in $(0,\frac{1}{\|A_1^*\|}]$ and for any λ with $\lambda^2 < \frac{(|d_2|^2-1)}{(|\mu_1|^2+|c|^2)}$.
- **$|d_2| < 1$, $r = 0$, $|\mu_2|^2 + 2|c|^2\Re\bar{\mu}_2 > 0$:**
  From Equation (6.15)we also see that neither $g_{(v,\lambda v)}(\beta')$ nor $g_{(v,\lambda v)}(\beta'')$ is equal to $\inf_\beta g_{(v,\lambda v)}(\beta)$ for any v with $|v|$ in $(0,\frac{1}{\|A_1^*\|}]$ and for any λ with

$$\lambda^2 < \frac{(1-|d_2|^2)|\mu_2|^2}{|v|^2|c|^2(|\mu_2|^2+|c|^2)-|\mu_2|^2-2|c|^2\Re\bar{\mu}_2}.$$

- **$|d_2| < 1$, $r = 0$, $|\mu_2|^2 + 2|c|^2\Re\bar{\mu}_2 < 0$:**
  From Equation (6.15)we also see that neither $g_{(v,\lambda v)}(\beta')$ nor $g_{(v,\lambda v)}(\beta'')$ is equal to $\inf_\beta g_{(v,\lambda v)}(\beta)$ for any v with $|v|$ in $(0,\frac{1}{\|A_1^*\|}]$ and for any λ with

$$\lambda^2 < \frac{(1-|d_2|^2)|\mu_2|^2}{|c|^2(|\mu_2|^2+|c|^2)-|\mu_2|^2-2|c|^2\Re\bar{\mu}_2}.$$

We note that $\nu_1 = \frac{1}{\mu_1}$ and $\nu_2 = \frac{1}{\mu_2}$ are the roots of $\det(\nu A_2^* - A_1^*) = 0$. The vectors β', β'' satisfying $(A_2^* - \mu_1 A_1^*)\beta' = 0$ and $(A_2^* - \mu_2 A_1^*)\beta'' = 0$ are

$$\beta' = \Big(\frac{|c|\exp(i\theta_1)}{\sqrt{|\mu_2|^2+|c|^2}}, \frac{-\mu_2\exp i(\theta_1-\phi_1)}{\sqrt{|\mu_2|^2+|c|^2}}\Big)$$

and
$$\beta'' = \Big(\frac{|c|\exp(i\theta_2)}{\sqrt{|\mu_1|^2+|c|^2}}, \frac{-\mu_1 \exp i(\theta_2-\phi_1)}{\sqrt{|\mu_1|^2+|c|^2}}\Big)$$
respectively, where $\bar{c} = |c|\exp(i\phi_1)$. Proceeding as above, we prove that neither $g_{(v,\,\lambda_0 v)}(\beta')$ nor $g_{(v,\,\lambda_0 v)}(\beta'')$ is equal to $\inf_\beta g_{(v,\,\lambda_0 v)}(\beta)$ for any v with $|v| \in (0, \frac{1}{\|A_1^*\|}]$.

Case (iii): Here we assume that $b = |c|$ and $|d_2| = 1$. The proof is similar to Case (i) and we skip the details.

Let $A_1 \in \{A_{11}, A_{12}\}$ and $A_2 \in \{A_{21}, A_{22}\}$, where $A_{11} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & d_2 \end{smallmatrix}\right), A_{12} = \left(\begin{smallmatrix} d_1 & 0 \\ 0 & 1 \end{smallmatrix}\right); A_{21} = \left(\begin{smallmatrix} 1 & b \\ c & 0 \end{smallmatrix}\right), A_{22} = \left(\begin{smallmatrix} 0 & b \\ c & 1 \end{smallmatrix}\right)$. We have proved the theorem for $A_1 = A_{11}$ and $A_2 = A_{21}$. The proof in the remaining cases, namely, $A_1 = A_{11}$ and $A_2 = A_{22}$; $A_1 = A_{12}$ and $A_2 = A_{21}$ and $A_1 = A_{12}$ and $A_2 = A_{22}$ follow similarly. □

Theorem 6.6. *Let A_1 be of the form $\left(\begin{smallmatrix} 1 & 0 \\ 0 & d_2 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} d_1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and A_2 be of the form $\left(\begin{smallmatrix} 0 & b \\ c & 0 \end{smallmatrix}\right)$ and assume that they are not simultaneously diagonalizable. Then there exists $(v_0, \lambda_0 v_0)$ in $\mathcal{E}$ such that neither $g_{(v_0,\,\lambda_0 v_0)}(\beta')$ nor $g_{(v_0,\,\lambda_0 v_0)}(\beta'')$ is equal to $\inf_\beta g_{(v_0,\,\lambda_0 v_0)}(\beta)$.*

Proof. Suppose $A_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & d_2 \end{smallmatrix}\right)$ and $A_2 = \left(\begin{smallmatrix} 0 & b \\ c & 0 \end{smallmatrix}\right)$. As above we have seen that the homomorphism ρ_V is contractive if and only if $|v|^2 \leq \frac{1}{\|A_1^*\|^2}$ and $\inf_{\beta, \|\beta\|_2=1} g_{(v,\,\lambda v)}(\beta) \geq 0$. Observe that

$$\begin{aligned}\inf_\beta g_{(v,\,\lambda v)}(\beta) &= \inf_\beta \{1 - |v|^2 - \lambda^2|v|^2 b^2\}|\beta_1|^2 + \{1 - |v|^2|d_2|^2 - \lambda^2|v|^2|c|^2\}|\beta_2|^2 \\ &\quad + |v|^4\lambda^2\left(b|\bar{\beta}_1|^2 - |cd_2||\bar{\beta}_2|^2\right)^2. \qquad (6.16)\end{aligned}$$

To complete the proof, we follow exactly the same steps as in Theorem 6.5 except the Case (iii). This is because when $|d_2| = 1$ and $|b| = |c|$, as before without loss of generality, we can take $A_1 = I_2$ and $A_2 = \left(\begin{smallmatrix} 0 & b \\ c & 0 \end{smallmatrix}\right)$ with $b, c \in \mathbb{C}, |b| = |c|$. Since $|b| = |c|$, we can see that A_2 is normal. Therefore, conjugating A_2 by a unitary U we can assume A_2 is a diagonal matrix. This contradicts the fact that A_1 and A_2 are not simultaneously diagonalizable. Hence the proof of the theorem involves two cases.

Case (i): Here $b \neq |c|$ and $|d_2| = 1$, that is, $A_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \exp(i\theta) \end{smallmatrix}\right)$ and $A_2 = \left(\begin{smallmatrix} 0 & b \\ c & 0 \end{smallmatrix}\right)$ with $b \neq |c|$. Let $U = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \exp(-i\theta) \end{smallmatrix}\right)$. Then U is a unitary and the pair (A_1U, A_2U) determines the same set $\Omega_{\mathbf{A}}$. So, we may assume without loss of generality that $\mathbf{A}$ is of the form $(I_2, \left(\begin{smallmatrix} 0 & b \\ c & 0 \end{smallmatrix}\right))$ with $b, c \in \mathbb{C}, |b| \neq |c|$.

(a) Suppose $c = 0$. The roots of $\det(A_2^* - \mu A_1^*) = 0$ are $\mu_1 = \mu_2 = 0$. The vectors β', β'' satisfying $(A_2^* - \mu_1 A_1^*)\beta' = 0$ and $(A_2^* - \mu_2 A_1^*)\beta'' = 0$ are $\beta' = \beta'' = (0, \exp(i\psi))$. Note that

$$g_{(v,\,\lambda v)}((0, \exp(i\psi))) - g_{v,\,\lambda v)}(\beta) = \lambda^2|v|^2|b|^2|\beta_1|^2(1 - |v|^2|\bar{\beta}_1|^2). \qquad (6.17)$$

From Equation (6.17) we have for all v, $|v| \leq 1$, for all λ, there exists a β with $|\beta| < 1$ such that $g_{(v,\lambda v)}((0, \exp(i\psi))) > g_{v,\lambda v)}(\beta)$. We therefore conclude that neither $g_{(v,\lambda v)}(\beta')$ nor $g_{(v,\lambda v)}(\beta'')$ is equal to $\inf_\beta g_{(v,\lambda v)}(\beta)$ for any λ and for any v with $|v| \leq 1$. Also, we observe that $\det(\nu A_2^* - A_1^*) \neq 0$. Thus there is no vector γ with $\|\gamma\| = 1$ satisfying $(\nu A_2^* - A_1^*)\gamma = 0$. We therefore conclude that there exists no vector γ such that $g_{(v,\lambda v)}(\gamma)$ is equal to $\inf_\beta g_{(v,\lambda v)}(\beta)$ for any λ and for any v with $|v| \leq 1$. We arrive at the same conclusion whenever $b = 0$.

(b) Suppose b, c are not simultaneously zero. If we consider $\det(A_2^* - \mu A_1^*) = 0$, then we see that $\mu_1 = \sqrt{\bar{b}\bar{c}}, \mu_2 = -\sqrt{\bar{b}\bar{c}}$ are the roots of $\det(A_2^* - \mu A_1^*) = 0$. The vectors β', β'' satisfying $(A_2^* - \mu_1 A_1^*)\beta' = 0$ and $(A_2^* - \mu_2 A_1^*)\beta'' = 0$ are

$$\beta' = \Big(\frac{\sqrt{|c|}\exp(i\theta_3)}{\sqrt{|c| + |b|}}, \frac{\sqrt{\bar{b}}\exp i(\theta_3 - \phi_2)}{\sqrt{|c| + |b|}}\Big)$$

and

$$\beta'' = \Big(\frac{-\sqrt{|c|}\exp(i\theta_3)}{\sqrt{|c| + |b|}}, \frac{\sqrt{\bar{b}}\exp i(\theta_3 - \phi_2)}{\sqrt{|c| + |b|}}\Big)$$

respectively, where $\bar{b} = |b|\exp i(\phi_2)$. Substituting $\beta = \beta'$ and $\beta = \beta''$ in Equation (6.16) we have

$$\begin{aligned} g_{(v,\lambda v)}(\beta'') &= g_{(v,\lambda v)}(\beta') \\ &= \{1 - |v|^2 - \lambda^2|v|^2|b|^2\}|\beta_1'|^2 + \{1 - |v|^2 - \lambda^2|v|^2|c|^2\}|\beta_2'|^2, \end{aligned}$$

where $\beta' = (\beta_1', \beta_2'), \beta'' = (\beta_1'', \beta_2'')$. Now,

$$\begin{aligned} & g_{(v,\lambda v)}(\beta'') - g_{(v,\lambda v)}(\beta) \\ = \; & \{1 - |v|^2 - \lambda^2|v|^2|b|^2\}(|\beta_1''|^2 - |\beta_1|^2) + \{1 - |v|^2 - \lambda^2|v|^2|c|^2\}(|\beta_2''|^2 - |\beta_2|^2) \\ & -|v|^4\lambda^2\left(|b||\bar{\beta}_1|^2 - |c||\bar{\beta}_2|^2\right)^2. \end{aligned} \tag{6.18}$$

- Assume that $|b|^2 > |c|^2$. Since b, c are not simultaneously zero, it follows that $\beta_1'' \neq 0$, or equivalently $|\beta_1''| \neq 1$. Since $|\beta_1''| \neq 1$, we can choose β_1 such that $(|\beta_1''|^2 - |\beta_1|^2) = -\delta_2$, where $\delta_2 > 0$. Then $(|\beta_2''|^2 - |\beta_2|^2) = \delta_2$. Hence from Equation (6.18) we have

$$g_{(v,\lambda v)}(\beta'') - g_{(v,\lambda v)}(\beta) \;=\; \{\lambda^2|v|^2(|b|^2 - |c|^2)\}\delta_2 - |v|^4\lambda^2(|b| + |c|)^2\delta_2^2. \tag{6.19}$$

If we choose $\delta_2 < \frac{(|b|^2 - |c|^2)}{(|b|+|c|)^2}$, then we have $g_{(v,\lambda v)}(\beta'') > g_{(v,\lambda v)}(\beta)$ for all λ and for all v, $|v|^2 \leq 1$. We therefore conclude that neither $g_{(v,\lambda v)}(\beta')$ nor $g_{(v,\lambda v)}(\beta'')$ is equal to $\inf_\beta g_{(v,\lambda v)}(\beta)$ for any λ and for any v with $|v| \leq 1$.

- Suppose $|b|^2 < |c|^2$. We can also choose β_1 such that $(|\beta_1''|^2 - |\beta_1|^2) = \delta_3, \delta_3 > 0$. Therefore from Equation (6.18) we have

$$g_{(v,\,\lambda v)}(\beta'') - g_{(v,\,\lambda v)}(\beta) \;=\; \{\lambda^2|v|^2(|c|^2 - |b|^2)\}\delta_3 - |v|^4\lambda^2(|b| + |c|)^2\delta_3^2. \quad (6.20)$$

From Equation (6.20) we have $g_{(v,\,\lambda v)}(\beta'') > g_{(v,\,\lambda v)}(\beta)$ for all λ, for all v, $|v|^2 \leq 1$ and for all δ_3, $\delta_3 < \frac{(|c|^2-|b|^2)}{(|b|+|c|)^2}$. Therefore we conclude that neither $g_{(v,\,\lambda v)}(\beta')$ nor $g_{(v,\,\lambda v)}(\beta'')$ is equal to $\inf_\beta g_{(v,\,\lambda v)}(\beta)$ for any λ and for any v with $|v| \leq 1$.

The roots of $\det(\nu A_2^* - A_1^*) = 0$ are $\nu_1 = \frac{1}{\mu_1}, \nu_2 = \frac{1}{\mu_2}$. The vectors β', β'' satisfying $(\nu_1 A_2^* - A_1^*)\beta' = 0$ and $(\nu_2 A_2^* - A_1^*)\beta'' = 0$ are

$$\beta' = \Big(\frac{\sqrt{|c|}\exp(i\theta_3)}{\sqrt{|c| + |b|}}, \frac{\sqrt{\bar{b}}\exp i(\theta_3 - \phi_2)}{\sqrt{|c| + |b|}}\Big)$$

and

$$\beta'' = \Big(\frac{-\sqrt{|c|}\exp(i\theta_3)}{\sqrt{|c| + |b|}}, \frac{\sqrt{\bar{b}}\exp i(\theta_3 - \phi_2)}{\sqrt{|c| + |b|}}\Big)$$

respectively, where $\bar{b} = |b|\exp i(\phi_2)$. Proceeding as above, we also find that neither $g_{(v,\,\lambda_0 v)}(\beta')$ nor $g_{(v,\,\lambda_0 v)}(\beta'')$ is equal to $\inf_\beta g_{(v,\,\lambda_0 v)}(\beta)$ for any v with $|v| < 1$.

Case (ii): In this case, $A_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & d_2 \end{smallmatrix}\right)$, $A_2 = \left(\begin{smallmatrix} 0 & |c| \\ c & 0 \end{smallmatrix}\right)$ with $|d_2| \neq 1$. The roots of $\det(A_2^* - \mu A_1^*) = 0$, are $\mu_1 = \sqrt{\frac{|c|\bar{c}}{d_2}}, \mu_2 = \sqrt{\frac{|c|\bar{c}}{d_2}}$. The vectors β', β'' satisfying $(A_2^* - \mu_1 A_1^*)\beta' = 0$ and $(A_2^* - \mu_2 A_1^*)\beta'' = 0$ are

$$\beta' = \Big(\frac{\sqrt{|c|}\exp(i\theta_4)}{\sqrt{|c| + |\frac{|c|}{d_2}|}}, \frac{\sqrt{\frac{|c|}{\bar{d}_2}}\exp i(\theta_4 - \phi_3)}{\sqrt{|c| + |\frac{|c|}{d_2}|}}\Big)$$

and

$$\beta'' = \Big(\frac{-\sqrt{|c|}\exp(i\theta_4)}{\sqrt{|c| + |\frac{|c|}{d_2}|}}, \frac{\sqrt{\frac{|c|}{\bar{d}_2}}\exp i(\theta_4 - \phi_3)}{\sqrt{|c| + |\frac{|c|}{d_2}|}}\Big)$$

respectively, where $\frac{|c|}{d_2} = \frac{|c|}{|d_2|}\exp i(\phi_3)$.

- Suppose $1 < |d_2|$. From Equation (6.18) we have

$$\begin{aligned} g_{(v,\,\lambda v)}(\beta'') - g_{(v,\,\lambda v)}((0,1)) &= (|d_2|^2 - 1)|\beta_1''|^2 - |v|^2\lambda^2|cd_2|^2 \\ &= |d_2|\{(|d_2| - 1) - |v|^2\lambda^2|c|^2|d_2|\}. \end{aligned} \quad (6.21)$$

From Equation (6.21) it follows the $g_{(v,\lambda v)}(\beta'') > g_{(v,\lambda v)}((0,1))$ for all v, $|v|$ in $(0, \frac{1}{\|A_1^*\|}]$ and for all λ, $\lambda^2 < \frac{(|d_2|-1)|d_2|}{|c|^2}$.

We therefore conclude that neither $g_{(v,\lambda v)}(\beta')$ nor $g_{(v,\lambda v)}(\beta'')$ is equal to $\inf_\beta g_{(v,\lambda v)}(\beta)$ for any λ with $\lambda^2 < \frac{(|d_2|-1)|d_2|}{|c|^2}$ and for any v with $|v|$ in $(0, \frac{1}{\|A_1^*\|}]$.

- Let $1 > |d_2|$. From Equation (6.18) we have

$$\begin{aligned} g_{(v,\lambda v)}(\beta'') - g_{(v,\lambda v)}((1,0)) &= (1-|d_2|^2)|\beta_2''|^2 - |v|^2\lambda^2|c|^2 \\ &= (1-|d_2|) - |v|^2\lambda^2|c|^2. \end{aligned} \tag{6.22}$$

From Equation (6.22) it is also easy to see that $g_{(v,\lambda v)}(\beta'') > g_{(v,\lambda v)}((0,1))$ for all λ, $\lambda^2 < \frac{(|d_2|-1)}{|c|^2}$ and for all v, $|v|$ in $(0, \frac{1}{\|A_1^*\|}]$.

We therefore conclude that neither $g_{(v,\lambda v)}(\beta')$ nor $g_{(v,\lambda v)}(\beta'')$ is equal to $\inf_\beta g_{(v,\lambda v)}(\beta)$ for any λ with $\lambda^2 < \frac{(|d_2|-1)}{|c|^2}$ and for any v with $|v|$ in $(0, \frac{1}{\|A_1^*\|}]$.

The roots of $\det(\nu A_2^* - A_1^*) = 0$, are $\nu_1 = \frac{1}{\mu_1}, \nu_2 = \frac{1}{\mu_2}$. The vectors β', β'' satisfying $(\nu_1 A_2^* - A_1^*)\beta' = 0$ and $(\nu_2 A_2^* - A_1^*)\beta'' = 0$ are

$$\beta' = \Big(\frac{\sqrt{|c|}\exp(i\theta_4)}{\sqrt{|c| + |\frac{|c|}{d_2}|}}, \frac{\sqrt{\frac{|c|}{d_2}}\exp i(\theta_4 - \phi_3)}{\sqrt{|c| + |\frac{|c|}{d_2}|}}\Big)$$

and

$$\beta'' = \Big(\frac{-\sqrt{|c|}\exp(i\theta_4)}{\sqrt{|c| + |\frac{|c|}{d_2}|}}, \frac{\sqrt{\frac{|c|}{d_2}}\exp i(\theta_4 - \phi_3)}{\sqrt{|c| + |\frac{|c|}{d_2}|}}\Big)$$

respectively, where $\frac{|c|}{d_2} = \frac{|c|}{|d_2|}\exp i(\phi_3)$. Proceeding as above, we also find that neither $g_{(v,\lambda v)}(\beta')$ nor $g_{(v,\lambda v)}(\beta'')$ is equal to $\inf_\beta g_{(v,\lambda v)}(\beta)$ for any v with $|v|$ in $(0, \frac{1}{\|A_1^*\|}]$.

Let $A_1 \in \{A_{11}, A_{12}\}$ and $A_2 = \left(\begin{smallmatrix} 0 & b \\ c & 0 \end{smallmatrix}\right)$, where $A_{11} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & d_2 \end{smallmatrix}\right), A_{12} = \left(\begin{smallmatrix} d_1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. We have proved the theorem for $A_1 = A_{11}$ and $A_2 = \left(\begin{smallmatrix} 0 & b \\ c & 0 \end{smallmatrix}\right)$. The proof in the remaining case, namely, $A_1 = A_{12}$ and $A_2 = \left(\begin{smallmatrix} 0 & b \\ c & 0 \end{smallmatrix}\right)$ follows similarly. $\square$

The following theorem gives the existence of a v say v_0, such that $(v_0, \lambda_0 v_0)$ is in $\mathcal{E}_0$.

Theorem 6.7. *If A_1 is either $\left(\begin{smallmatrix} 1 & 0 \\ 0 & d_2 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} d_1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and A_2 is one of $\left(\begin{smallmatrix} 1 & b \\ c & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & b \\ c & 1 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 0 & b \\ c & 0 \end{smallmatrix}\right)$ (A_1 and A_2 are not simultaneously diagonalizable), then there exist a v_0 such that $L_V : (\mathbb{C}^2, \|\cdot\|^*_{\Omega_{\mathbf{A}}}) \to (\mathbb{C}^2, \|\cdot\|_2)$ defines a contractive linear map which is not completely contractive, where $V = \left(\begin{smallmatrix} v_0 & 0 \\ 0 & \lambda_0 v_0 \end{smallmatrix}\right)$.*

Proof. As we have seen in Theorems 6.5 and Theorem 6.6, for all v, $|v|$ in $(0, \frac{1}{\|A_1^*\|})$ there exists a $\lambda > 0$, say λ_0, such that $(v, \lambda_0 v)$ is in $\mathcal{E}$ with the property:

$g_{(v,\lambda_0 v)}(\beta'') > g_{(v,\lambda_0 v)}(\beta') > g_{(v,\lambda_0 v)}(\beta)$ or $g_{(v,\lambda_0 v)}(\beta') > g_{(v,\lambda_0 v)}(\beta'') > g_{(v,\lambda_0 v)}(\beta)$ whenever $\beta', \beta'' \in \mathcal{B}$.

Let $\mathbf{B}$ denote the set $\{|v|^2 : \inf_\beta g_{(v,\,\lambda_0 v)}(\beta) \leq 0\}$. This set is bounded below by $\frac{1}{\|A_1^*\|^2+\lambda_0^2\|A_2^*\|^2}$. Therefore the infimum of $\mathbf{B}$ is positive. Let

$$\alpha = \inf_{|v|}\{|v|^2 : \inf_\beta g_{(v,\,\lambda_0 v)}(\beta) \leq 0\}.$$

Hence there exists a v_0 such that $|v_0|^2 = \alpha$.

We claim that $g_{(v_0,\,\lambda_0 v_0)}(\beta) \geq 0$ for all β with $\|\beta\|_2 = 1$.

Assume there exists a $\hat{\beta}$ such that $g_{(v_0,\,\lambda_0 v_0)}(\hat{\beta}) < 0$. Then there exists a neighborhood U of v_0 such that $g_{(v,\,\lambda_0 v)}(\hat{\beta}) < 0$ for all $v \in U$. For any $v \in U$, $\inf_\beta g_{(v,\,\lambda_0 v)}(\beta) < 0$, since the function $g_{(v,\,\lambda_0 v)}$ is negative at $\hat{\beta}$ for all $v \in U$. Hence $|v|^2$ is in $\mathbf{B}$ for every $v \in U$. Since U is a neighborhood of v_0 there exists a $v \in U$ such that $|v|^2 < |v_0|^2$. By the previous assertion, this smaller value of $|v|^2$ also lies in $\mathbf{B}$, which is a contradiction. Also, we have $\inf_\beta g_{(v_0,\,\lambda_0 v_0)}(\beta) \leq 0$. Hence $\inf_\beta g_{(v_0,\,\lambda_0 v_0)}(\beta) = 0$.

From all possible choice for λ_0, in accordance with Theorem 6.5 and Theorem 6.6, we further restrict it to satisfy $\lambda_0 \leq \frac{1}{|v_0|\|A_2^*\|}$. This will make L_V contractive. The choice of λ_0, v_0 ensure that the infimum of $g_{(v_0,\lambda_0 v_0)}(\beta)$ is equal to neither $g_{(v_0,\lambda_0 v_0)}(\beta')$ nor $g_{(v_0,\lambda_0 v_0)}(\beta'')$. Thus L_V is not completely contractive. □

Remark 6.8. In chapter 3, we have shown the existence of two distinct operator space on $\mathbf{V_A}$, where $\mathbf{A}$ is of the form $\left(\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}1&0\\c&0\end{smallmatrix}\right)\right), \left(\left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right), \left(\begin{smallmatrix}0&b\\0&1\end{smallmatrix}\right)\right), \left(\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0\\c&0\end{smallmatrix}\right)\right)$ or $\left(\left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right), \left(\begin{smallmatrix}0&b\\0&0\end{smallmatrix}\right)\right)$. In this case, $\det(A_2^*-\mu A_1^*) \equiv 0$ and $\det(\nu A_2^*-A_1^*) \equiv 0$, therefore $\|A_1^*\beta\|^2\|A_2^*\beta\|^2 = |\langle A_1A_2^*\beta, \beta\rangle|^2$ for all β in $\mathbb{C}^2$. Consequently, we have $\|\rho_V\| = \|\rho_V^{(2)}(P_{\mathbf{A}})\|$. To obtain a counter example in this case, following methods of this chapter, one simply use $P_{\mathbf{A}^t}$ instead of $P_{\mathbf{A}}$.

Example 6.9. If $A_1 = I_2$ and $A_2 = \left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$, then the homomorphism ρ_V is contractive if and only if $|v|^2 \leq 1$ and

$$\inf_\beta\{1 - |v|^2 - \lambda^2|v|^2|\beta_1|^2 + \lambda^2|v|^4|\beta_1|^4\} \geq 0.$$

Also, $\|P_{\mathbf{A}}(T_1, T_2)\| \leq 1$ implies that

$$\inf_\beta\{1 - |v|^2 - \lambda^2|v|^2|\beta_1|^2\} \geq 0.$$

The roots of $\det(A_2^* - \mu A_1^*) = 0$ are $\mu_1 = \mu_2 = 0$. The vectors β', β'' satisfying $(A_2^* - \mu_1 A_1^*)\beta' = 0$ and $(A_2^* - \mu_2 A_1^*)\beta'' = 0$ are $\beta' = \beta'' = (0, \exp(i\psi))$ respectively. Note that

$$g_{(v,\,\lambda v)}(\beta'') - g_{(v,\,\lambda v)}(\beta) = \lambda^2|v|^2|\beta|^2(1 - |v|^2|\beta|^2). \tag{6.23}$$

From Equation (6.23) we have for all v, $|v| \leq 1$, for all λ, there exists a β with $|\beta| < 1$ such that $g_{(v,\lambda v)}((0, \exp(i\psi))) > g_{v,\lambda v)}(\beta)$. Hence there exists $(v, \lambda_0 v)$ in $\mathcal{E}$ such that neither $g_{(v,\lambda_0 v)}(\beta')$ nor $g_{(v,\lambda_0 v)}(\beta'')$ is equal to $\inf_\beta g_{(v,\lambda_0 v)}(\beta)$.

Also, we observe that $\det(\nu A_2^* - A_1^*) \neq 0$. Thus there is no vector γ satisfying $(\nu A_2^* - A_1^*)\gamma = 0$. We therefore conclude that there exists no vector γ such that $g_{(v,\lambda v)}(\gamma)$ is equal to $\inf_\beta g_{(v,\lambda v)}(\beta)$ for any choice of λ and v with $|v| \leq 1$.

Now, $\inf_\beta g_{(v,\lambda_0 v)}(\beta) \leq 0$ is equivalent to $4|v|^2 - 4 + \lambda^2 \geq 0$ with $2|v|^2 \geq 1$. If $\lambda^2 = 1, |v|^2 = \frac{3}{4}$, then $\inf_\beta g_{(v,\lambda_0 v)}(\beta) = 0$. Also, we have $\|P_{\mathbf{A}}(T_1, T_2)\| > 1$. Hence this contractive homomorphism ρ_V is not completely contractive.

In chapter 1, we have seen that if A_1 and A_2 are simultaneously diagonalizable, then every contractive linear map from $(\mathbb{C}^2, \|\cdot\|_{\Omega_{\mathbf{A}}})$ to $\mathcal{M}_n(\mathbb{C})$ is completely contractive. We have also seen that the particular matrix valued polynomial $P_{\mathbf{A}}$ plays an important role for constructing a contractive homomorphisms which is not complete contractive. The following theorem says that if A_1 and A_2 are simultaneously diagonalizable, then $\|\rho_V\| \leq 1$ if and only if $\|\rho_V^{(2)}(P_{\mathbf{A}})\| \leq 1$.

Theorem 6.10. *If $A_1 = \left(\begin{smallmatrix} d_1 & 0 \\ 0 & d_2 \end{smallmatrix}\right)$ and $A_2 = \left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right)$, then there exists a $(v, \lambda v)$ in $\mathcal{E}$ such that the infimum is attained at either β' or β''.*

Proof. As above we have seen that the homomorphism ρ_V is contractive if and only if $|v|^2 \leq \frac{1}{\|A_1^*\|^2}$ and $\inf_{\beta, \|\beta\|_2 = 1} g_{(v,\lambda v)}(\beta) \geq 0$. Observe that

$$\begin{aligned}\inf_\beta g_{(v,\lambda v)}(\beta) &= \inf_\beta (1 - |d_1|^2|v|^2 - |a|^2\lambda^2|v|^2)|\beta_1|^2 + (1 - |d_2 v|^2 - |d|^2\lambda^2|v|^2)|\beta_2|^2 \\ &\quad + |v|^4\lambda^2|a\bar{d}_1 - d\bar{d}_2|^2|\beta_1\beta_2|^2 \end{aligned} \tag{6.24}$$

where $|\beta_1|^2 + |\beta_2|^2 = 1$.

The roots of $\det(A_2^* - \mu A_1^*) = 0$ are $\mu_1 = \frac{\bar{c}}{d_1}, \mu_2 = \frac{\bar{d}}{d_2}$. The vectors β', β'' satisfying $(\nu_1 A_2^* - A_1^*)\beta' = 0, (\nu_2 A_2^* - A_1^*)\beta'' = 0$ are $\beta' = (1, 0), \beta'' = (0, 1)$ respectively. In this case, $\beta' = \beta''_\perp, \beta'' = \beta'_\perp$, where $\beta''_\perp, \beta'_\perp$ are orthogonal to β'', β' respectively. Substituting $\beta' = (1, 0)$ and $\beta'' = (0, 1)$ in Equation (6.24) we have

$$g_{(v,\lambda v)}(\beta') = (1 - |d_1|^2|v|^2 - |a|^2\lambda^2|v|^2)$$

and

$$g_{(v,\lambda v)}(\beta'') = (1 - |d_2 v|^2 - |d|^2\lambda^2|v|^2).$$

Without loss generality we can assume that $g_{(v,\lambda v)}(\beta') \leq g_{(v,\lambda v)}(\beta'')$. Note that

$$g_{(v,\lambda v)}(\beta') - g_{(v,\lambda v)}(\beta) = (g_{(v,\lambda v)}(\beta') - g_{(v,\lambda v)}(\beta''))|\beta_2|^2 - |v|^4\lambda^2|a\bar{d}_1 - d\bar{d}_2|^2|\beta_1\beta_2|^2. \tag{6.25}$$

Since $g_{(v,\lambda v)}(\beta') \leq g_{(v,\lambda v)}(\beta'')$, from Equation (6.25) we observe that $g_{(v,\lambda v)}(\beta') \leq g_{(v,\lambda v)}(\beta)$ for all β. Hence we conclude that the infimum is attained at β'. Similarly we also prove that the infimum is attained at β''.

The roots of $\det(\nu A_2^* - A_1^*) = 0$ are $\frac{1}{\mu_1}, \frac{1}{\mu_2}$. The vectors β', β'' satisfying $(\nu_1 A_2^* - A_1^*)\beta' = 0$ and $(\nu_2 A_2^* - A_1^*)\beta'' = 0$ are $\beta' = (1,0)$ and $\beta'' = (0,1)$ respectively. We also therefore conclude that infimum is attained at either β' or β''. This completes the proof. $\square$

The following Corollary is an immediate consequence of Theorem 6.10 and Corollary 1.3.

Corollary 6.11. *Suppose A_1 and A_2 are simultaneously diagonalizable. Then $\|\rho_V\| \leq 1$ if and only if $\|\rho_V^{(2)}(P_{\mathbf{A}})\| \leq 1$.*

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