ON A VARIANT OF THE GROTHENDIECK INEQUALITY AND ESTIMATES ON TENSOR PRODUCT NORMS

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ABSTRACT. In this paper we propose a generalization of the Grothendieck inequality for pairs of Banach spaces E and F with E being finite dimensional and investigate the behaviour of the Grothendieck constant $K_G(E,F)$ implicit in such an inequality. We show that if $\sup\{K_G(E_n,F) : n \geqslant 1\}$ is finite for some sequence of finite dimensional Banach spaces $(E_n)_{n\geqslant 1}$ with $\dim E_n = n$, and an infinite dimensional Banach space F, then both F and F^* must have finite cotype. In addition to that if F has the bounded approximation property, we conclude that $(E_n^*)_{n\geqslant 1}$ satisfies G.T. uniformly by assuming the validity of a conjecture due to Pisier. We also show that $K_G(E,F)$ is closely related to the constant $\rho(E,F)$, introduced recently, comparing the projective and injective norms on the tensor product of two finite dimensional Banach spaces E and F. We also study, analogously, these constants by computing the supremum only on non-negative tensors.

1. Introduction

An extraordinary theorem of Grothendieck from [10] that he called the "Fundamental theorem of metric theory of tensor products", is now referred to as Grothendieck's theorem (in short G.T.). It has been an useful tool in several applications in geometry of Banach spaces, operator theory and operator algebras, harmonic analysis, theoretical computer science, quantum information theory and other fields. The reader may consult the recent survey article [25] for a lot more information on this topic. The Grothendieck's theorem takes many different forms. Perhaps the simplest equivalent formulation of G.T. is by Lindenstrauss and Pełczyński [15] saying: There is a universal constant K_G such that

$$\sup\left\{\Big|\sum_{i,j=1}^{n} a_{ij}\langle v_i, w_j\rangle\Big| : \|v_i\|_2 = \|w_j\|_2 = 1\right\} \leqslant K_G, \tag{1.1}$$

where the supremum is taken over every real $n\times n$ matrix $A = (a_{ij})_{i,j=1}^n$, $n\in\mathbb{N}$, with

$$\sup\left\{\Big|\sum_{i,j=1}^{n} a_{ij}s_i t_j\Big| : |s_i| = |t_j| = 1\right\} \leqslant 1.$$

The inequality (1.1) is famously called the Grothendieck inequality. Another equivalent formulation (a very similar statement appears in [13]) of the Grothendieck inequality, among many others is in [30]: For any $n\in\mathbb{N}$, there exists positive constant K, independent of n, such that

$$\|A\otimes \mathrm{id}_{\ell_2}\|_{\ell_\infty^n\check{\otimes}\ell_2\to\ell_1^n\hat{\otimes}\ell_2} \leqslant K\|A\|_{\ell_\infty^n\to\ell_1^n}. \tag{1.2}$$

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The infimum of all the admissible constants K in (1.2) is the Grothendieck constant K_G. To prove this version, note that for any $n \times n$ scalar matrix $A = (a_{ij})_{i,j=1}^n$, we have

$$\|A\|_{\ell_\infty^n \to \ell_1^n} = \sup\Big\{\Big|\sum_{i,j=1}^n a_{ij}s_it_j\Big| : |s_i| = |t_j| = 1\Big\}.$$

Since $\ell_\infty^n \check{\otimes} \ell_2 \cong \ell_\infty^n(\ell_2)$, it follows that $\|\sum_{i=1}^n \mathbf{e}_i \otimes x_i\|_{\ell_\infty^n \check{\otimes} \ell_2} = \sup_{1\leqslant i\leqslant n} \|x_i\|_2$, where $\mathbf{e}_i$, $1 \leqslant i \leqslant n$, is the standard basis of $\mathbb{R}^n$. Finally, by duality we have that $\|\sum_{i=1}^n \mathbf{e}_i \otimes x_i\|_{\ell_1^n \hat{\otimes} \ell_2} = \sum_{i=1}^n \|x_i\|_2$. The equality

$$\sup\Big\{\sum_{j=1}^n \Big\|\sum_{i=1}^n a_{ij}v_i\Big\|_2 \;:\; \|v_i\|_2 \;=\; 1\Big\} \;=\; \sup\Big\{\Big|\sum_{i,j=1}^n a_{ij}\langle v_i, w_j\rangle\Big| \;:\; \|v_i\|_2 \;=\; \|w_j\|_2 \;=\; 1\Big\}$$

is easy to verify and the proof of the inequality (1.2) follows from it.

In this article, we discuss a generalization of the Grothendieck inequality which is prompted by the equivalent form of the inequality we have just verified. All the Banach spaces are assumed to be real if not mentioned otherwise. However, most of the results below also make sense for Banach spaces over the complex field. These can be proved with little or no change in the corresponding proof for the real case. We define Grothendieck constant for a pair of finite dimensional Banach spaces with the first one being finite dimenisonal as follows.

Definition 1.1. *Let (E, F) be a pair of Banach spaces with E being finite dimensional. Define the Grothendieck constant $K_G(E, F)$ to be the supremum*

$$K_G(E, F) := \sup\left\{\|A \otimes \mathrm{id}_F\|_{E\check{\otimes}F \to E^*\hat{\otimes}F} : \|A\|_{E\to E^*} \leqslant 1\right\},$$

where id_F is the identity operator on F.

Let E be a finite dimensional Banach space. In what follows, we employ the natural notion of positivity in $E \otimes E$ namely, $A \in E \otimes E$ is *non-negative* ($A \geqslant 0$) if it is in the convex hull of the set of symmetric tensors $e \otimes e$, $e \in E$. In other words, $A \geqslant 0$ if $A = BB^*$ for some $B \in E \otimes \ell_2^k$, $k \in \mathbb{N}$. If the supremum defining $K_G(E, F)$ is taken only over non-negative matrices A, then the constant $K_G^+(E, F)$ obtained this way is referred to as the *positive Grothendieck constant* corresponding to the pair (E, F). Thus,

$$K_G^+(E, F) := \sup\left\{\|A \otimes \mathrm{id}_F\|_{E\check{\otimes}F \to E^*\hat{\otimes}F} : A \geqslant 0, \|A\|_{E\to E^*} \leqslant 1\right\}.$$

Evidently, for any pair of Banach spaces E and F, we have that $K_G^+(E, F) \leqslant K_G(E, F)$.

The positive Grothendieck constant $K_G^+ := \sup_{n\geqslant 1} K_G^+(\ell_\infty^n, \ell_2)$ has occurred very early in the literature, see for instance, [32, Theorem 4]. In the paper [22], the relationship of K_G^+ with the existence of orthogonally scattered dilations of vector measures taking values in a Hilbert space is discussed. More recently, in the PhD thesis of Briët [7], many variants of positive Grothendieck inequality and applications have been investigated. It is well-known that the equality $\Pi_2(L_\infty, \mathcal{H}) = B(L_\infty, \mathcal{H})$ for any Hilbert space $\mathcal{H}$ via an equivalent norm is a manifestation of the finiteness of K_G^+. This equivalence is called "little G.T.". Like the Grothendieck constant, the positive Grothendieck constant, or equivalently, the "little G.T." has been studied vigorously by many authors. Our motivation for defining the Grothendieck constant in the greater generality, as above, is manifold including but not limited to the following.

- Recall that the Grothendieck's theorem is the equality $B(L_1, \ell_2) = \Pi_1(L_1, \ell_2)$. It is natural to ask which other Banach spaces possess such a property. A Banach space E is said to be a G.T. space if $B(E, \ell_2) = \Pi_1(E, \ell_2)$ (see [26]) via an equivalent norm. The Grothendieck inequality (1.2) relates three fundamental Banach spaces ℓ_∞^n, ℓ_1^n and ℓ_2 in a non-trivial way. Thus a different question occurs if we replace ℓ_∞^n, ℓ_1^n and ℓ_2, in the Grothendieck inequality (1.2) by Banach spaces E_n, E_n^* and F, respectively, with $(E_n)_{n\geqslant 1}$ being a sequence of finite dimensional

Banach spaces. Moreover, if E_n, $n \in \mathbb{N}$, and F are taken to be finite dimensional Banach spaces, then the "quantitative" information of the constant $K_G(E_n, F)$, now also depending on n, leads to useful asymptotics.

- Recently, constants like $K_G(E, F)$ have appeared in quantum information theory and in particular XOR games in general probabilistic theories. For instance, similar constants have been studied in Proposition A.1 of [2] to estimate the bias of a XOR game over a bipartite GPT under local strategies.
- Like the Grothendieck inequality and equivalently G.T., a variant involving only non-negative definite matrices in (1.2), and equivalently the "little G.T." has been studied vigorously by many authors. We refer to [26, Chapter 5] for more on this topic. This inequality also appears in questions involving contractivity versus complete contractivity of linear maps. A finite dimensional complex Banach space E is said to possess "Property P", introduced in [3], if

$$\langle A, B\rangle \leqslant \|A\|_{\vee}\|B\|_{\vee},\ A \in E \otimes E,\ B \in E^* \otimes E^*,$$

where $\langle \cdot, \cdot \rangle$ is the Hilbert-Schmidt inner product, $A \geqslant 0$, $B \geqslant 0$.

Any non-negative A is of the form $\left(\langle v_j, v_i\rangle\right)_{i,j=1}^n$, for complex vectors $v_j \in \ell_2^k$, $1 \leqslant j \leqslant n$, and it induces a linear map $L_V : E^* \to \ell_2^k$ by setting $\beta \mapsto \beta_1 v_1 + \cdots + \beta_n v_n$, $\beta \in E^*$. Theorem 1.9 of [3] shows that Property P is equivalent to saying that every contractivity linear map L_V is completely contractivity. The linear maps of the form L_V come from homomorphisms introduced by Parrott and coincide with the localization of commuting tuples of operators from the Cowen-Douglas class. We refer [18], [19], [20], [21] and [31] for more on this topic. For a real or complex Banach space E, let us also set

$$\gamma(E) := \sup\left\{\langle A, B\rangle : A \geqslant 0,\ B \geqslant 0,\ \|A\|_{E\to E^*} \leqslant 1,\ \|B\|_{E^*\to E} \leqslant 1\right\}.$$

Therefore, a complex Banach space E has Property P if and only if $\gamma(E) \leqslant 1$. In this article we note that E has Property P if and only if $\sup\limits_{m\geq 1} K_G^+(E, \ell_2^m) \leqslant 1$. We refer Proposition 5.1 for a proof of this fact in the real case. The proof for the complex case is the same. Moreover, it is known (see [31]) that $\sqrt{\gamma(E)}$ equals the supremum of the *complete norm* $\|L_V\|_{\mathrm{cb}}$ over all the contractive linear operators L_V. Indeed, the asymptotic behaviour of the constant $K_G^+(E, \ell_2^m)$ as $\dim E$ goes to infinity might lead to new examples of polynomially bounded operators that are not completely polynomially bounded. Therefore, it is natural to define $K_G^+(E, F)$ for arbitrary finite dimensional Banach spaces E and F.

To facilitate the study of the Grothendieck constant $K_G(E, F)$ and its positive variant, we introduce the notion of a *Grothendieck pair*.

Definition 1.2 (Grothendieck pair)**.** *Let $\mathcal{E} = (E_n)_{n\geqslant 1}$ be a sequence of finite dimensional Banach spaces such that $\dim E_n = n$. Let F be a Banach space. Then $(\mathcal{E}, F)$ is called a Grothendieck pair if there exists a constant $C > 0$ such that $\|A \otimes \mathrm{id}_F\|_{E_n\check{\otimes}F\to E_n^*\hat{\otimes}F} \leqslant C\|A\|_{E_n\to E_n^*}$ for all $n \in \mathbb{N}$.*

For a Banach space X, set $p(X) := \sup\{p : X \text{ is of type } p\}$ and $q(X) := \inf\{q : X \text{ is of cotype } q\}$. The main result of this paper is the following theorem.

Theorem 1.3. *Suppose that $(\mathcal{E}, F)$ is a Grothendieck pair. Then $\dim F < \infty$ or both F and F^* are of non-trivial cotype.*

Suppose that $(\mathcal{E}, F)$ is a Grothendieck pair and F is an infinite dimensional GL - space (see [9, pp. 350]). Then combining Theorem 1.3 with [9, Theorem 17.13], it follows that $p(F) > 1$. Moreover, following the proof of Corollary 4.6 one concludes that $(\mathcal{E}, \ell_2)$ is then also a Grothendieck pair.

It is also natural to ask what are all the Grothendieck pairs. After communicating this question to Pisier, he made the following conjecture [24].

Conjecture 1.4. *Suppose that $(\mathcal{E}, F)$ is a Grothendieck pair for a fixed but arbitrary Banach space F with the bounded approximation property. Then either $\dim F < \infty$ or $(\mathcal{E}, \ell_2)$ is also a Grothendieck pair.*

Clearly, by the previous discussion, Conjecture 1.4 is true when F is a GL-space. Surprisingly, Conjecture 1.4 is related to one of his older conjectures, see [28, Final remarks (i)].

Conjecture 1.5. *If X is an infinite dimensional Banach space with bounded approximation property such that $q(X) < \infty$ and $q(X^*) < \infty$, then X is K - convex.*

At the end of the paper [27], under "Added in proof", the existence of a Banach space X such that both X and X^* are of cotype 2, although X is not K-convex (so that $p(X) = 1$) was asserted. Furthermore, it was noted that such a space (which necessarily fails the approximation property) contains uniformly complemented ℓ_p^n's for no p such that $1 \leqslant p \leqslant \infty$.

In a private communication [24], Pisier had hinted that an affirmative answer to Conjecture 1.5 might establish Conjecture 1.4. Corollary 4.6 verifies this implication. This verification relies on Theorem 1.3. The proof of Theorem 1.3 depends on a deep '$\ell_1/\ell_2/\ell_\infty$' trichotomy theorem recently proved in [2]. In Proposition 4.1, several equivalent conditions for a sequence of finite dimensional Banach spaces $\mathcal{E} := (E_n)_{n\geqslant 1}$ such that $(\mathcal{E}, \ell_2)$ is a Grothendieck pair are given. One of them says that $(E_n^*)_{n\geqslant 1}$ will have to satisfy G.T. uniformly. Proposition 4.1 as well as its proof were communicated to one of the authors by G. Pisier in an email message [24], 2017. In a remarkable paper [6] Bourgain proved that L_1/H^1 is a G.T. space. Therefore, if we take any sequence of finite dimensional subspaces of L_1/H^1, the corresponding sequence of finite dimensional dual Banach spaces denoted by $\mathcal{E}$ will have the property that $(\mathcal{E}, \ell_2)$ is a Grothendieck pair. Many other examples of G.T. spaces are discussed in [26]. Along the way we obtain asymptotic bounds for the constant $K_G(E, F)$ associated to several finite dimensional Banach spaces, namely, ℓ_p^n and $S_p^{n,\text{sa}}$.

In the same vein, we also study the positive Grothendieck constant $K_G^+(E, F)$ for various finite dimensional Banach spaces. The imposition of this additional condition makes the computation of $K_G^+(E, F)$ somewhat more difficult. However, the behaviour of $K_G^+(E, F)$ is quite different from that of $K_G(E, F)$. For example, $K_G(\ell_1^n, \ell_\infty^n) = o(\sqrt{n})$, whereas $K_G^+(\ell_1^n, \ell_\infty^n)$ is uniformly bounded by the Grothendieck constant, see Proposition 5.11. Interestingly, the former estimate involves real part of the Discrete Fourier Transform matrix. One of our major tool is the constant $\rho(E, F)$, recently introduced in [2], is defined to be maximum of the ratio of projective norm of an element with the injective norm of it in $E \otimes F$:

$$\rho(E, F) := \|\mathrm{id}_E \otimes \mathrm{id}_F\|_{E\check{\otimes}F \to E\hat{\otimes}F}.$$

Taking a finite dimensional Banach space E $(= F)$, in the definition of ρ and restricting to non-negative tensors of $E \otimes E$, in the definition of ρ, we get a variant of the original ρ, and denote it by $\rho^+(E)$. Property Q for a real Banach space E introduced earlier in [3] is equivalent to requiring $\rho^+(E) \leqslant 1$. Moreover, Property Q implies Property P [3]. In this paper along with the Grothendieck constants, K_G and K_G^+, we also study the constants ρ and ρ^+. The constants ρ and ρ^+ come in very handy while dealing with $K_G(E, F)$ and $K_G^+(E, F)$ respectively.

The paper ends with an Addendum, where we provide a simple proof of the assertion: If $n \leqslant 3$, then $\sup_{m\geqslant 1} K_G^+(\ell_\infty^n, \ell_2^m) = 1$ using some bounds on the rank of the extreme points of correlation matrices obtained recently in [16]. We conclude with a straightforward existential proof of [8, Theorem 2.1].

2. Preliminaries

Let E and F be Banach spaces. The norm of an operator $u : E \to F$ is denoted by $\|u\|_{E\to F}$ or $\|u\|$ whenever the meaning is clear from the context. We let $B(E, F)$ denote the linear space of all bounded linear maps from E to F. The closed unit ball of E is denoted by $(E)_1$. The Banach-Mazur distance $d(E,F)$ between two isomorphic Banach spaces E and F is defined as follows:

$$d(E,F) := \inf \big\{ \|u\|\|u^{-1}\| \mid u : E \to F \text{ bounded invertible}\big\}.$$

If $d(E,F) \leqslant \lambda$ for some $\lambda > 0$, then E is said to be λ-isomorphic to F. The *factorization constant* of a Banach space E through another Banach space F is defined to be

$$f(E,F) := \inf \big\{\|u\|\|v\| \mid u : E \to F,\, v : F \to E, \text{and}\, vu = \mathrm{id}_E\big\},$$

whenever it exists. Evidently, $d(E,F) = f(E,F)$ whenever E and F are finite dimensional Banach spaces with $\dim E = \dim F$. Moreover, $f(E^*,F^*) \leqslant f(E,F)$ with equality if both E and F are finite dimensional.

Let E be a Banach space, and assume that $\lambda \geqslant 1$. We say that E contains ℓ_p^n's λ-uniformly if there exists a sequence of subspaces $(E_n)_{n\geqslant 1}$ of E such that $\sup_{n\geqslant 1} d(\ell_p^n, E_n) \leqslant \lambda$. Dvoretzky's theorem asserts that any infinite dimensional Banach space E contains ℓ_2^n's $(1+\epsilon)$-uniformly for some $\epsilon > 0$.

2.1. **Norms in tensor product of two Banach spaces:** In what follows, as usual, we identify the algebraic tensor product $E \otimes F$ with a subspace of $B(E^*,F)$: Any tensor u of the form $\sum_{j=1}^n e_j \otimes f_j$, with $e_j \in E$ and $f_j \in F$, defines a linear map $u : E^* \to F$ by setting $u(e^*) = \sum_{j=1}^n e^*(e_j) f_j$, $e^* \in E^*$. The injective tensor norm $\|u\|_\vee$ is the operator norm $\|u\|_{E^*\to F}$. Moreover, the projective norm $\|u\|_\wedge$ is defined to be

$$\|u\|_\wedge = \inf \Big\{ \sum_{j=1}^n \|e_j\|_E \|f_j\|_F \mid u = \sum_{j=1}^n e_j \otimes f_j \Big\}.$$

We let $E\check{\otimes}F$ and $E\hat{\otimes}F$ denote the completion of the linear space $E\otimes F$ equipped with the *injective* and *projective* tensor norms, respectively. If E is finite dimensional, we have the remarkable duality $(E\check{\otimes}F)^* \cong E^*\hat{\otimes}F^*$, and $(E\hat{\otimes}F)^* \cong E^*\check{\otimes}F^*$ via the equality $\langle e\otimes f, e^*\otimes f^*\rangle = e^*(e)f^*(f)$. Note that the canonical operator $J : E^*\hat{\otimes}F \to B(E,F)$ defined as $J(e^*\otimes f)(e) = e^*(e)f$, is an isomorphism when E and F are finite dimensional Banach spaces. When E and F are finite dimensional, the nuclear norm of $u \in B(E,F)$ is defined to be $\|J^{-1}(u)\|_{E^*\hat{\otimes}F}$ and is denoted by $\|u\|_{N(E,F)}$, where $N(E,F)$ is the linear space $B(E,F)$ equipped with the nuclear norm. We recall a very useful property of the nuclear norm, namely, Let $C \in B(X,E)$, $B \in B(F,Y)$ and $A \in N(E,F)$. Then $CAB \in N(X,Y)$ and $\|CAB\|_{N(X,Y)} \leqslant \|C\|_{X\to E}\|A\|_{N(E,F)}\|B\|_{F\to Y}$. Our main reference for norms in tensor product of two Banach spaces and their properties is [33]. The following theorem due to S. Chevet, see [14, Theorem 3.20], is useful for estimating injective norm of random tensors.

Theorem 2.1 (Chevet's theorem)**.** *Let X and Y be real finite dimensional Banach spaces. Define the Gaussian random tensor $z = \sum_{i=1}^m \sum_{j=1}^n g_{ij} x_i \otimes y_j \in X \otimes Y$, where (g_{ij}) are iid $N(0,1)$ Gaussian random variables and $(x_i)_{i=1}^m \subseteq X$, $(y_j)_{j=1}^n \subseteq Y$. Let $(g_i)_{i=1}^n$ be a sequence of iid $N(0,1)$ Gaussian random variables. Then*

$$\mathbb{E}\|z\|_{X\check{\otimes}Y} \leqslant \|T\|_{\ell_2^m\to X}\mathbb{E}\Big\|\sum_{i=1}^m g_i y_i\Big\|_Y + \|S\|_{\ell_2^n\to Y}\mathbb{E}\Big\|\sum_{i=1}^n g_i x_i\Big\|_X$$

where $T(\mathbf{e}_i) := x_i$ and $S(\mathbf{e}_j) := y_j$ for $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$, and $(\mathbf{e}_i)_{i\geqslant 1}$ is the canonical basis of ℓ_2.

Definition 2.2 (p-summing operator)**.** *Let $u : E \to F$ be a linear operator between two Banach spaces and $p \in [1,\infty)$. We say u is p-summing if there exists a constant $C > 0$ such that for any finite sequence (x_i) in E we have that $(\sum \|ux_i\|^p)^{\frac{1}{p}} \leqslant C \sup\{(\sum |x^*(x_i)|^p)^{\frac{1}{p}} : x^* \in (E^*)_1\}$.*

Moreover, the best constant C in the above inequality is denoted by $\pi_p(u)$ and is said to be the p-summing norm of u. The set of all p-summing operator from E to F is denoted by $\Pi_p(E,F)$.

A linear operator $u : E \to F$ is said to factor through a Hilbert space if there is a Hilbert space $\mathcal{H}$ and linear operator $B : E \to \mathcal{H}$ and $A : \mathcal{H} \to F$ such that $u = BA$. We then define $\gamma_2(u) = \inf \|A\|\|B\|$ where the infimum runs through all possible factorization. The space of linear operators which factors through a Hilbert space becomes a Banach space equipped with the γ_2-norm and is denoted by $\Gamma_2(X,Y)$.

2.2. **Type, cotype and related notions:** Let $(\epsilon_i)_{i=1}^n$ be a sequence of iid. Bernoulli random variables taking values in $\{+1,-1\}$ with equal probabilities.

Definition 2.3. *A Banach space E has Rademacher type p (in short, type p) for some $1 \leqslant p \leqslant 2$ if there is a constant $C > 0$ such that for all $n \geqslant 1$ and $e_1, \ldots, e_n \in E$*

$$\Big(\mathbb{E}\Big\|\sum_{i=1}^n \epsilon_i e_i\Big\|_E^2\Big)^{\frac{1}{2}} \leqslant C\Big(\sum_{i=1}^n \|e_i\|^p\Big)^{\frac{1}{p}}.$$

The best constant in the inequality above is denoted by $p(E)$. Analogously, a Banach space E is said to have Rademacher cotype q (in short, cotype q) if for some $2 \leqslant q \leqslant \infty$ there is a constant $C > 0$ such that for all $n \geqslant 1$ and $e_1, \ldots, e_n \in E$

$$\Big(\mathbb{E}\Big\|\sum_{i=1}^n \epsilon_i e_i\Big\|_E^2\Big)^{\frac{1}{2}} \geqslant C^{-1}\Big(\sum_{i=1}^n \|e_i\|^q\Big)^{\frac{1}{q}}.$$

The best constant C in the inequality above is denoted by $q(E)$.

Definition 2.4. *For each $n \in \mathbb{N}$, let G_n be the compact abelian group $\{+1,-1\}^n$ with the normalized Haar measure. Let X be a Banach space. For any $f : G_n \to X$, suppose $f = \sum_{\gamma \in \widehat{G_n}} \widehat{f}(\gamma)\gamma$ is its Fourier-Walsh expansion, where $\widehat{G_n}$ is the Pontryagin dual of G_n. Let $R_n : L_2(G_n;X) \to L_2(G_n;X)$ be the projection defined by $R_n(f) := \sum_{i=1}^n \widehat{f}(\epsilon_i)\epsilon_i$. We say X is K-convex if $K(X) := \sup_{n\geqslant 1} \|R_n\|_{L_2(G_n;X)\to L_2(G_n;X)} < \infty$.*

In Theorems 2.5, 2.6 and 2.7, F is an infinite dimensional Banach space. The following theorem is due to Maurey and Pisier.

Theorem 2.5 (Theorem 3.3(ii) [29])**.** *A Banach space F has finite cotype if and only if F does not contain ℓ_∞^n's λ - uniformly for any $\lambda \geqslant 1$.*

Theorem 5.4 of [29] says that a Banach space F is K-convex if (and only if) it does not contain ℓ_1^n's uniformly. Combining this with Theorem 3.1(i) of the same paper [29], we infer the following.

Theorem 2.6. *A Banach space F is K- convex if and only if $p(F) > 1$.*

A Banach space F is said to be locally π-euclidean if there exists a constant $C > 0$ such that for each $\epsilon > 0$ and each integer n, there is an integer $N(n,\epsilon)$ such that every subspace $E \subseteq F$ with $\dim E \geqslant N$ contains an n-dimensional subspace $G \subseteq E$ such that $d(G,\ell_2^n) \leqslant 1+\epsilon$ and there is a projection from F onto G with norm less than some C depending only on $K(F)$.

Theorem 2.7 (Theorem 5.10, [29])**.** *A Banach space F is locally π-Euclidean if and only if F is K-convex.*

We need the following remarkable '$\ell_1/\ell_2/\ell_\infty$' trichotomy theorem.

Theorem 2.8 (Theorem 20, [2])**.** *Suppose that E is a Banach space of dimension n. Then for every $1 \leqslant A \leqslant \sqrt{n}$ one of the following is true. There exist constants $c > 0$ and $C > 0$, independent of n, such that* *(i)* $f(\ell_\infty^{c\sqrt{n}}, E) \leqslant CA\sqrt{\log n}$;

(ii) $f(\ell_1^{c\sqrt{n}}, E) \leqslant CA\sqrt{\log n}$; *or*

(iii) $f(\ell_2^{\frac{cA^2}{\log n}}, E) \leqslant C \log n$.

Let M_n be the algebra of $n \times n$ complex matrices. For $1 \leqslant p < \infty$, define $\|A\|_{S_p^n} := (\mathrm{tr}(|A|^p))^{\frac{1}{p}}$. This makes M_n a complex Banach space denoted by S_p^n. We denote S_∞^n to be M_n equipped with the usual operator norm. Note that the space $S_p^{n,\mathrm{sa}}$ of all self-adjoint elements of S_p^n is a real Banach space equipped with the norm of S_p^n. It is well known that $(S_p^n)^*$ is isometrically isomorphic to S_q^n where $\frac{1}{p}+\frac{1}{q} = 1$. The duality relation is given by $\langle A, B\rangle = \mathrm{tr}(AB)$ where $A \in S_p^n$ and $B \in S_q^n$. A similar result holds true for $S_p^{n,\mathrm{sa}}$. We need the following non-commutative L_p-Grothendieck theorem.

We recall below, in the form of a theorem, Equation (0.2) of [36] that is stated for L_p spaces. However, since $L_p(M) = S_p$ with $M = B(l_2)$, we have stated it for S_p.

Theorem 2.9 (pp. 527, [36])**.** *Let $1 < p, q < \infty$. For any bounded bilinear form $B : S_{2p} \times S_{2q} \to \mathbb{C}$ there are positive unit functionals ϕ and ψ on S_p and S_q respectively, such that*

$$|B(x, y)| \leqslant K\|B\|\Big(\phi\big(\frac{x^*x + xx^*}{2}\big)\Big)^{\frac{1}{2}}\Big(\psi\big(\frac{y^*y + yy^*}{2}\big)\Big)^{\frac{1}{2}}, \ \forall\ x \in S_{2p},\ y \in S_{2q},$$

where K is a constant which depends only on the cotype constants of S_{2p}, S_{2p}^*, S_{2q}, and S_{2q}^*.

The paper [36] also has an extension of Theorem 2.9 to the case of non-commutative L_p-spaces as well as the case of operator spaces.

3. A number of preparatory lemmas

In this section we prove various results which will be useful for later section. We start with the following useful equivalent description of $K_G(E, F)$ for finite dimensional Banach spaces E and F.

Lemma 3.1. *Suppose that E and F are finite dimensional Banach spaces. Then the following statements are equivalent.*

1. *For all linear maps $A : E \to E^*$, there exists a constant $C > 0$, independent of A such that*
$$\|A \otimes \mathrm{id}_F\|_{E\check{\otimes}F \to E^*\hat{\otimes}F} \leqslant C\|A\|_\vee. \tag{3.1}$$
2. *For all linear maps $A : E \to E^*$ and linear maps $B : E^* \to F$, there exists a constant $C > 0$, independent of A and B such that*
$$N(BA) \leqslant C\|A\|_\vee\|B\|_\vee. \tag{3.2}$$

Moreover, the best constants in (3.1) *and* (3.2) *are equal.*

Proof. Fix bases as $(e_i)_i$ of E and $(f_j)_j$ of F. Let the dual bases be $(e_i^*)_i$ for the dual space E^*. Let $\sum_{i,j} b_{ij}e_i \otimes f_j \in E\check{\otimes}F$ be an arbitrary element and let $A \in E^*\check{\otimes}E^*$. Suppose A is represented by $Ae_i = \sum_{k=1}^n a_{ki}e_k^*$. Note that

$$(A \otimes \mathrm{id}_{F_m})\Big(\sum_{i,j} b_{ij}e_i \otimes f_j\Big) = \sum_{i,j} b_{ij}Ae_i \otimes f_j = \sum_{i,j} b_{ij}\sum_k a_{ki}e_k^* \otimes f_j.$$

Hence

$$\Big\|(A \otimes \mathrm{id}_{F_m})\Big(\sum_{i,j} b_{ij}e_i \otimes f_j\Big)\Big\|_{E_n^*\hat{\otimes}F_m} = N(BA^t),$$

where A^t denotes the transpose of A. Thus (3.1) is equivalent to

$$N(BA^t) \leqslant C\|A\|_\vee \|B\|_\vee = C\|A^t\|_\vee \|B\|_\vee.$$

This proves the equivalence of (3.1) and (3.2). □

Remark 3.2. *Let E and F be finite dimensional Banach spaces. Since π_1 is a cross norm and projective norm is the largest cross norm, in view of Lemma 3.1, it follows that for any pair of maps $A : E \to E^*$ and $B : E^* \to F$ we have*

$$\pi_1(BA) \leqslant K_G(E,F)\|A\|\|B\|. \tag{3.3}$$

Lemma 3.3. *Let E and F be Banach spaces. Assume that E is finite dimensional. Then the following are true:*

(i) $K_G(E,F^*) = K_G(E,F)$.
(ii) If F is also finite dimensional, then $K_G(E,F) \leqslant \min\{\rho(E,F),\ \rho(E,F^*)\}$.
(iii) Let F be finite dimensional. Then $K_G(\ell_2^n, F) = \rho(\ell_2^n, F)$.

Proof. Let $A : E \to E^*$ be a linear map.

(i) From the duality of projective and injective norm, we know that

$$\|A \otimes \mathrm{id}_F\|_{E\check{\otimes}F \to E^*\hat{\otimes}F} = \|A^* \otimes \mathrm{id}_{F^*}\|_{E\check{\otimes}F^* \to E^*\hat{\otimes}F^*}$$

Since $\|A\| = \|A^*\|$, it follows that $K_G(E,F) = K_G(E,F^*)$.

(ii) Note that $A \otimes \mathrm{id}_F = (A \otimes \mathrm{id}_F)(\mathrm{id}_E \otimes \mathrm{id}_F)$. Therefore, we have that

$$\begin{aligned}\|(A \otimes \mathrm{id}_F)\|_{E\check{\otimes}F \to E^*\hat{\otimes}F} &= \|(A \otimes \mathrm{id}_F) \circ (\mathrm{id}_E \otimes \mathrm{id}_F)\|_{E\check{\otimes}F \to E^*\hat{\otimes}F} \\ &\leqslant \|A \otimes \mathrm{id}_F\|_{E\hat{\otimes}F \to E^*\hat{\otimes}F}\|\mathrm{id}_E \otimes \mathrm{id}_F\|_{E\check{\otimes}F \to E\hat{\otimes}F}.\end{aligned}$$

Since $\|A \otimes \mathrm{id}_F\|_{E\hat{\otimes}F \to E^*\hat{\otimes}F} = \|A\|$, we have

$$\|A \otimes \mathrm{id}_F\|_{E\check{\otimes}F \to E^*\hat{\otimes}F} \leqslant \rho(E,F)\|A\|.$$

Therefore, $K_G(E,F) \leqslant \rho(E,F)$. The result follows from part (i).

(iii) Let us choose $A = \mathrm{id}_{\ell_2^n}$. From the definition of $K_G(\ell_2^n, F)$, we get that

$$K_G(\ell_2^n, F) \geqslant \|\mathrm{id}_{\ell_2^n} \otimes \mathrm{id}_F\|_{\ell_2^n\check{\otimes}F \to \ell_2^n\hat{\otimes}F} = \rho(\ell_2^n, F).$$

The inequality on the other side follows from (ii). This completes the proof of the lemma. □

Lemma 3.4. *Let E and F be Banach spaces. Let X and Y be another pair of Banach spaces. Assume that E and X are finite dimensional and that $f(Y,F)$ and $f(Y,F^*)$ exist. Then the following are true:*

(i) $K_G(E,Y) \leqslant \min\{f(Y,F), f(Y,F^*)\}K_G(E,F)$. Moreover,

$$K_G^+(E,Y) \leqslant \min\{f(Y,F), f(Y,F^*)\}K_G^+(E,F).$$

(ii) if $\dim X \leqslant \dim E$ then $K_G(X,F) \leqslant f(X,E)^2 K_G(E,F)$. Moreover,

$$K_G^+(X,F) \leqslant f(X,E)^2 K_G^+(E,F).$$

Proof. (i) Suppose $A : E \to E^*$ is a contraction. Let (u,v) be a pair of operators such that $vu = \mathrm{id}_Y$ with $u : Y \to F$ and $v : F \to Y$. Note that

$$A \otimes \mathrm{id}_Y = A \otimes vu = (\mathrm{id}_{E^*} \otimes v) \circ (A \otimes \mathrm{id}_F) \circ (\mathrm{id}_E \otimes u).$$

Therefore, we have that

$$\begin{aligned}\|A \otimes \mathrm{id}_Y\|_{E\check{\otimes}Y \to E^*\hat{\otimes}Y} &\leqslant \|\mathrm{id}_{E^*} \otimes v\|_{E^*\hat{\otimes}F \to E^*\hat{\otimes}Y}\|A \otimes \mathrm{id}_F\|_{E\check{\otimes}F \to E^*\hat{\otimes}F}\|\mathrm{id}_E \otimes u\|_{E\check{\otimes}Y \to E\check{\otimes}F} \\ &\leqslant \|v\|\|u\|K_G(E,F).\end{aligned}$$

Now taking infimum over the pair (u,v) such that $vu = \mathrm{id}_Y$ in the above computation, we get $K_G(E,Y) \leqslant f(Y,F)K_G(E,F)$. By Lemma 3.3, we have $K_G(E,F^*) = K_G(E,F)$, and hence

$$K_G(E,Y) \leqslant \min\{f(Y,F), f(Y,F^*)\}K_G(E,F)$$

The proof of the 'positive' case is the same as the above.

(ii) Suppose $A : X \to X^*$ is a contraction. Choose pair of random operators (u,v) such that $vu = \mathrm{id}_X$. Note that we have $A \otimes \mathrm{id}_F = u^*v^*Avu \otimes \mathrm{id}_F$. Therefore, we have that

$$A \otimes \mathrm{id}_F = (u^* \otimes \mathrm{id}_F) \circ (v^* \otimes \mathrm{id}_F) \circ (A \otimes \mathrm{id}_F) \circ (v \otimes \mathrm{id}_F) \circ (u \otimes \mathrm{id}_F).$$

Therefore, we obtain that

$$\begin{aligned}
&\|A \otimes \mathrm{id}_F\|_{X\check{\otimes}F \to X^*\hat{\otimes}F} \\
&= \|(u^* \otimes \mathrm{id}_F) \circ (v^*Av \otimes \mathrm{id}_F) \circ (u \otimes \mathrm{id}_F)\|_{X\check{\otimes}F \to X^*\hat{\otimes}F} \\
&\leqslant \|u \otimes \mathrm{id}_F\|_{X\check{\otimes}F \to E\check{\otimes}F}\|v^*Av \otimes \mathrm{id}_F\|_{E\check{\otimes}F \to E^*\hat{\otimes}F}\|u^* \otimes \mathrm{id}_F\|_{E^*\hat{\otimes}F \to X^*\hat{\otimes}F} \\
&\leqslant \|u\|^2_{X\to E}K_G(E,F)\|v^*Av\|_{E\to E^*} \\
&\leqslant \|u\|^2_{E_1\to E}\|v\|^2_{E\to E_1}.
\end{aligned}$$

Taking infimum over all admissible (u,v), we obtain the required result. The proof of the 'positive' case also follows. This completes the proof of lemma. ☐

4. Asymptotic behaviour of $K_G(E,F)$

An example of Grothendieck pair is $((\ell_\infty^n)_{n\geqslant 1}, \ell_2)$. Remark 3.2 is precisely the Grothendieck Theorem. In fact, the question of characterizing $(E_n)_{n\geqslant 1}$ such that $((E_n)_{n\geqslant 1}, \ell_2)$ is a Grothendieck pair has been studied in [26]. Indeed, we have the following proposition.

Proposition 4.1. *In the following, statements (1), (2) and (3) are equivalent and the statement (1) implies (4).*

(1) $(E_n)_{n\geqslant 1}$ *is a sequence of finite dimensional Banach spaces with* $\dim E_n = n$, *and*

$$\sup_{n\geqslant 1} K_G(E_n, \ell_2) < \infty.$$

(2) There are positive constants K_1 *and* K_2 *such that for any Hilbert space* $\mathcal{H}$ *and for any* $B : E_n^* \to \mathcal{H}$, *we have*

$$\pi_1(B) \leqslant K_1\|B\|.$$

Moreover for any $A : E_n \to E_n^*$

$$\gamma_2(A) \leqslant K_2\|A\|.$$

(3) There are positive constants K_1 *and* K_2 *such that for any Hilbert space* $\mathcal{H}$

$$\pi_2(B) \leqslant K_1\|B\|$$

for any $B : E_n^* \to \mathcal{H}$, *and for any* $A : E_n \to E_n^*$,

$$\gamma_2(A) \leqslant K_2\|A\|.$$

(4) There is a constant $K > 0$ *such that* $\gamma_2^*(A) \leqslant K\|A\|$ *for all* $A : E_n \to E_n^*$.

Proof. (1) $\implies$ (2): Suppose $C := \sup_{m,n\geqslant 1} K_G(E_n, \ell_2^m)$ is finite. Let $n \in \mathbb{N}$. By Dvoretzky's theorem there exists $(1+\epsilon)$-isometry $j : \ell_2^k \to E_n$ for some $k \in \mathbb{N}$. Let $B : E_n^* \to \ell_2^k$ and $A : E_n \to E_n^*$ be linear operators. From (1), note that

$$|Tr(jBA)| \leqslant N(jBA) \leqslant \|j\|N(BA) \leqslant C\|j\|\|A\|\|B\|, \tag{4.1}$$

Taking supremum over $A \in (E_n^*\check{\otimes}E_n^*)_1$ in (4.1), we get

$$N(jB) \leqslant C(1+\epsilon)\|B\|.$$

Since $\pi_1(A) = \pi_1(jA) \leqslant N(jA)$, we get the first part of (2).

Now for the second part, let $A : E_n \to E_n^*$ be a map and let $v : E_n^* \to E_n$ be a map such that $v = \alpha B$, where $B : E_n^* \to \ell_2^m$ and $\alpha : \ell_2^m \to E_n$. Note that

$$N(vA) = N(\alpha BA) \leqslant \|\alpha\| N(BA) \leqslant C\|\alpha\|\|B\|\|A\|,$$

Now, taking infimum over B and α such that $v = \alpha B$ in the above inequality, we get

$$|tr(vA)| \leqslant N(vA) \leqslant C\gamma_2(v)\|A\|. \tag{4.2}$$

We take supremum over v in (4.2) such that $\gamma_2(v) \leqslant 1$, we get

$$\gamma_2^*(A) \leqslant C\|A\|, \tag{4.3}$$

where γ_2^* is the dual to the norm γ_2. From [26, Chapter 2], we know that

$$\gamma_2^*(A) = \inf\{\pi_2(\phi)\pi_2(\psi^*) : \phi : E_n \to \ell_2,\ \psi : \ell_2 \to E_n^*, A = \psi\phi\}.$$

By definition, $\gamma_2(A) = \inf\{\|\phi\|\|\psi\| : \phi : E_n \to \ell_2,\ \psi : \ell_2 \to E_n^*, A = \psi\phi\}$. Hence from (4.3), we have

$$\gamma_2(A) \leqslant \gamma_2^*(A) \leqslant C\|A\|.$$

This completes the proof of (1) $\implies$ (2).

(2) $\implies$ (3): This follows from the fact that $\pi_2(\cdot) \leqslant \pi_1(\cdot)$.

(3) $\implies$ (1): Assume (3). By [9, pp. 162] or [11, page 36], if $T : \mathcal{H} \to E_n$ is 2-summing then T^* is again 2-summing and $\pi_2(T^*) \leqslant \pi_2(T)$. Since we are dealing with finite dimensional Banach spaces we indeed have $\pi_2(T) = \pi_2(T^*)$. Therefore, for all $T : \mathcal{H} \to E_n$, we have

$$\pi_2(T) = \pi_2(T^*) \leqslant K_1\|T^*\| = K_1\|T\|.$$

Let $S : E_n \to \mathcal{H}$ be a linear map. Let $N \in \mathbb{N}$ be such that $\ell_2^{\dim S(E_n)} \subseteq E_N$ $(1+\epsilon)$-isometrically. We denote the corresponding $(1+\epsilon)$-isometry by j. Then $jS : E_n \to E_N$ and

$$\pi_2(jS) \leqslant \|S\|\pi_2(j) \leqslant K_1\|S\|\|j\|.$$

Clearly $\inf_{\epsilon>0} \pi_2(jS) = \pi_2(S)$. Therefore for any $S : E_n \to \mathcal{H}$, we have $\pi_2(S) \leqslant K_1\|S\|$. Now take $A : E_n \to E_n^*$ such that $A = T^*S$, where $S, T : E_n \to \mathcal{H}$ are linear maps. Then

$$\gamma_2^*(A) \leqslant \pi_2(S)\pi_2(T) \leqslant K_1^2\|S\|\|T\|.$$

Taking infimum over S and T we obtain

$$\gamma_2^*(A) \leqslant K_1^2\gamma_2(A) \leqslant K_1^2 K_2\|A\|. \tag{4.4}$$

Take $B : E_n^* \to \mathcal{H}$ such that $\|B\| \leqslant 1$. Then observe that from trace duality and (4.4), we have

$$\begin{aligned} N(BA) &= \sup\{|\operatorname{tr}(DBA)| : \|D\|_{\mathcal{H}\to E_n} \leqslant 1\} \\ &\leqslant \sup_{\|D\|\leqslant 1} \gamma_2^*(A)\gamma_2(DB) \\ &\leqslant \sup_{\|D\|\leqslant 1} \|B\|\|D\|\gamma_2^*(A) \\ &\leqslant \|B\|K_1^2K_2\|A\|. \end{aligned}$$

This completes the proof of (3) $\Longrightarrow$ (1).

(1) $\implies$ (4): In the proof of the implication (1) $\implies$ (2), we have noted that the inequality in (4.3) follows from the assumptions in the statement (1) proving the assertion of (4). ☐

Lemma 4.2. *Suppose that E is a finite dimensional Banach space. Then $K_G(E, E^*) = K_G(E, E)$ $= \rho(E, E)$.*

Proof. Putting $B = \mathrm{id}_{E^*}$ in (3.3) of Lemma 3.1, it follows that $N(A) \leqslant K_G(E, E^*)\|A\|_{E \to E^*}$ for all A. This shows that $\|A\|_{E^* \hat{\otimes} E^*} \leqslant K_G(E, F)\|A\|_{E^* \check{\otimes} E^*}$ for all A. Therefore, we have that $\rho(E^*, E^*) \leqslant K_G(E, E^*)$. On the other hand, by Lemma 3.3 we know that $K_G(E, E^*) = K_G(E, E) \leqslant \rho(E, E)$. Moreover, $\rho(E^*, E^*) = \rho(E, E)$ [2, Proposition 12]. Therefore, we conclude that $K_G(E, E) = \rho(E, E)$. This completes the proof of the lemma. ☐

The corollary below follows easily from Lemma 4.2.

Corollary 4.3. *Let $1 \leqslant p \leqslant \infty$and p' be such that $\frac{1}{p} + \frac{1}{p'} = 1$. Then, we have that*

(i) $K_G(\ell_p^n, \ell_p^n) = K_G(\ell_p^n, \ell_{p'}^n) = \rho(\ell_p^n, \ell_p^n)$.
(ii) $K_G(S_p^{n,sa}, S_p^{n,sa}) = K_G(S_p^{n,sa}, S_{p'}^{n,sa}) = \rho(S_p^{n,sa}, S_p^{n,sa})$.

Lemma 4.4. *For $n \geqslant 1$, we have*

(i) $c_1\sqrt{n} \leqslant K_G(\ell_\infty^n, \ell_\infty^n) \leqslant c_2\sqrt{n}$ *for some positive constants c_1 and c_2;*
(ii) $K_G(\ell_2^n, \ell_2^n) = \rho(\ell_2^n, \ell_2^n) = n$, *and*
(iii) $c_1\sqrt{n} \leqslant K_G(\ell_1^n, \ell_\infty^n) = \rho(\ell_1^n, \ell_1^n) \leqslant c_2\sqrt{n}$ *for all $n \geqslant 1$.*

Proof. The assertion of part (i) follows from Corollary 4.3 and [2, Page 697]. Also (ii) follows from Corollary 4.3 and [2, Proposition 12]. Part (iii) follows by similar reasoning. This completes the proof of the lemma. ☐

Proof of Theorem 1.3: Suppose that $(\mathcal{E}, F)$ is a Grothendieck pair. Then $\dim F < \infty$ or both F and F^* are of non-trivial cotype.

For a proof by contradiction, let us assume that $\dim F = \infty$. Note that by Theorem 2.5, it is enough to show that both F and F^* do not contain ℓ_∞^n's uniformly. Moreover, from Lemma 3.3 we only need to show that F does not contain ℓ_∞^n's uniformly. Now we proceed by contradiction. Note that if F contains ℓ_∞^n's uniformly. Then $\sup_{n \geqslant 1} f(\ell_\infty^n, F) < \infty$. Therefore by Theorem 2.8, namely, the '$\ell_1/\ell_2/\ell_\infty$'- trichotomy, and Lemma 3.3, Lemma 3.4, Lemma 4.4, we have the following

(i) $K_G(E_n, F) \gtrsim f(\ell_\infty^{c\sqrt{n}}, E)^{-2} K_G(\ell_\infty^{c\sqrt{n}}, \ell_\infty^{c\sqrt{n}}) \gtrsim \frac{n^{\frac{1}{4}}}{A^2 \log n}$
(ii) $K_G(E_n, F) \gtrsim f(\ell_1^{c\sqrt{n}}, E)^{-2} K_G(\ell_1^{c\sqrt{n}}, \ell_\infty^{c\sqrt{n}}) \gtrsim \frac{n^{\frac{1}{4}}}{A^2 \log n}$
(iii) $K_G(E_n, F) \gtrsim f(\ell_2^{cA^2/\log n}, E)^{-2} K_G(\ell_2^{cA^2/\log n}, \ell_\infty^{cA^2/\log n}) \gtrsim \frac{A^2}{\log^3 n}$

Now taking $A = n^{1/16}$ we see that all the limits that occur above are infinity. contradiction. ☐

As explained in the introduction, we have an immediate corollary of Theorem 1.3.

Corollary 4.5. *Suppose that $(\mathcal{E}, F)$ is a Grothendieck pair and F is an infinite dimensional GL-space. Then $(\mathcal{E}, \ell_2)$ is a Grothendieck pair.*

Proof. Note that by Theorem 1.3 and [9, Theorem 17.13] we have that $p(F) > 1$. Now the implication follows from Theorem 2.6, Theorem 2.7 and Lemma 3.4. ☐

We note that [9, Theorem 17.13] provides many examples Banach spaces, where the assertion of the Conjecture 1.5 is evident.

Corollary 4.6. *An affirmative answer to Conjecture 1.5 verifies the validity of Conjecture 1.4.*

Proof. Assume that $\dim F = \infty$. Note that if Conjecture 1.5 is assumed to have an affirmative answer, then by Theorem 1.3 and Theorem 2.7, F contains uniformly complemented ℓ_2^n's since F is K-convex. The proof is completed by applying Part (ii) of Lemma 3.4. ☐

The corollary stated below follows from Lemma 3.3 and [2, Proposition 12].

Corollary 4.7. *Let (E_n) be a sequence of finite dimensional Banach spaces and F be another Banach space. Suppose that $\sup_{n\geqslant 1} K_G(E_n, F) < \infty$. Then $\sup_{n\geqslant 1} \dim E_n < \infty$ or $\dim F < \infty$ or both F and F^* are of non-trivial cotype.*

Proposition 4.8. *If $n \geqslant 2$ and $\dim F \geqslant 2$, then $K_G(\ell_\infty^n, F) \geqslant \sqrt{2}$.*

Proof. Note that by Lemma 3.4 we have $K_G(\ell_\infty^n, F) \geqslant K_G(\ell_\infty^2, F)$. Since ℓ_∞^2 is isometric to ℓ_1^2, combining with Lemma 3.4, we readily have $K_G(\ell_\infty^2, F) = \rho(\ell_\infty^2, F)$. Now the result follows from [2, Proposition 14]. ☐

Proposition 4.9. *We have the following.*

(i) For $2 \leqslant p < \infty$, we have that $\rho(\ell_1^n, \ell_p^n) = n^{\frac{1}{p'}}$, where p' is conjugate to p.

(ii) For $1 < p \leqslant 2$, we have that $\rho(\ell_1^n, \ell_p^n) = n^{\frac{1}{p}}$.

Proof. Note that $\rho(\ell_1^n, \ell_2^n) = \sqrt{n}$ [2, Equation (59)]. Moreover, by Proposition 12 in [2] we have that
$$\rho(\ell_1^n, \ell_p^n) \leqslant \rho(\ell_1^n, \ell_2^n) d(\ell_2^n, \ell_p^n).$$
Pick p such that $2 \leqslant p < \infty$. By using the fact that $d(\ell_2^n, \ell_p^n) = n^{\frac{1}{2}-\frac{1}{p}}$ (see [13]), we have the estimate $\rho(\ell_1^n, \ell_p^n) \leqslant n^{\frac{1}{p'}}$. Note that for a tensor of the form $z = \sum_{i=1}^n z_i \mathbf{e}_i \otimes \mathbf{e}_i$ we have $\|z\|_{\ell_1^n \check{\otimes} \ell_p^n} = \|(z_i)_{i=1}^n\|_p$ and $\|z\|_{\ell_1^n \hat{\otimes} \ell_p^n} = \|(z_i)_{i=1}^n\|_1$. Thus we have $\rho(\ell_1^n, \ell_p^n) \geqslant n^{\frac{1}{p'}}$. This completes the proof of part (i) of the proposition. The proof of part (ii) is similar and is omitted. ☐

Lemma 4.10. *Let $1 < p < \infty$. Suppose that $z \in S^{n,sa}_{\frac{2p}{2p-1}} \check{\otimes} S^{n,sa}_{\frac{2p}{2p-1}}$ and $\widetilde{z} : S^{n,sa}_{2p} \to S^{n,sa}_{\frac{2p}{2p-1}}$ is the corresponding linear map associated to z. Then there exists a positive linear functional ϕ of norm 1 on S_p^n such that*
$$\|\widetilde{z}(x)\|_{S^{n,sa}_{\frac{2p}{2p-1}}} \leqslant K_p \|z\|_{S^{n,sa}_{\frac{2p}{2p-1}} \check{\otimes} S^{n,sa}_{\frac{2p}{2p-1}}} (\phi(x^2))^{\frac{1}{2}}$$
for all $x \in S^{n,sa}_{2p}$. The constant K_p depends on the cotype constants of S_{2p} and S_{2p}^*.

Proof. By considering z to be also a tensor in $S_p^n \check{\otimes} S_p^n$, we claim that
$$\|z\|_{S^{n,sa}_{\frac{2p}{2p-1}} \check{\otimes} S^{n,sa}_{\frac{2p}{2p-1}}} \leqslant \|z\|_{S^{n}_{\frac{2p}{2p-1}} \check{\otimes} S^{n}_{\frac{2p}{2p-1}}} \leqslant 4 \|z\|_{S^{n,sa}_{\frac{2p}{2p-1}} \check{\otimes} S^{n,sa}_{\frac{2p}{2p-1}}}.$$
Note that
$$\|z\|_{S^{n,sa}_{\frac{2p}{2p-1}} \check{\otimes} S^{n,sa}_{\frac{2p}{2p-1}}} = \sup\{|\operatorname{tr}((a \otimes b)z)| : a, b \in S^{n,sa}_{2p},\ \|a\|_{S^{n,sa}_{2p}} = \|b\|_{S^{n,sa}_{2p}} = 1\}.$$
In a similar way we have
$$\|z\|_{S^{n}_{\frac{2p}{2p-1}} \check{\otimes} S^{n}_{\frac{2p}{2p-1}}} = \sup\{|\operatorname{tr}((a \otimes b)z)| : a, b \in S^{n}_{2p},\ \|a\|_{S^{n}_{2p}} = \|b\|_{S^{n}_{2p}} = 1\}.$$
Also, note that
$$|\operatorname{tr}((a \otimes b)z)| \leqslant$$
$$|\operatorname{tr}((\operatorname{Re} a \otimes \operatorname{Re} b)z)| + |\operatorname{tr}((\operatorname{Re} a \otimes \operatorname{Im} b)z)| + |\operatorname{tr}((\operatorname{Im} a \otimes \operatorname{Re} b)z)| + |\operatorname{tr}((\operatorname{Im} a \otimes \operatorname{Im} b)z)|.$$
Moreover, $\|a\|_{S^n_{2p}} \leqslant 1$ implies $\|\operatorname{Re} a\|_{S^n_{2p}} \leqslant 1$ and $\|\operatorname{Im} a\|_{S^n_{2p}} \leqslant 1$. Therefore, we readily have
$$\|z\|_{S^{n}_{\frac{2p}{2p-1}} \check{\otimes} S^{n}_{\frac{2p}{2p-1}}} \leqslant 4 \|z\|_{S^{n,sa}_{\frac{2p}{2p-1}} \check{\otimes} S^{n,sa}_{\frac{2p}{2p-1}}}.$$
This proves the claim.

Consider the bilinear form $B_z : S_{2p}^n \times S_{2p}^n \to \mathbb{C}$ as $B_z(x,y) := \operatorname{tr}((x \otimes y)z)$. Then

$$\|B_z\| = \|z\|_{S^n_{\frac{2p}{2p-1}} \check{\otimes} S^n_{\frac{2p}{2p-1}}}.$$

Now by Theorem 2.9, we get that for some positive unit norm linear functional ϕ and ψ on S_p^n

$$|\operatorname{tr}(\widetilde{z}(x)y)| = |\operatorname{tr}((x \otimes y)z)| \leqslant K_p \|z\|_{S^n_{\frac{2p}{2p-1}} \check{\otimes} S^n_{\frac{2p}{2p-1}}} \left(\phi(\frac{x^*x + xx^*}{2})\right)^{\frac{1}{2}} \left(\psi(\frac{y^*y + yy^*}{2})\right)^{\frac{1}{2}}$$

for all $x, y \in S_p^n$. Note that if $y \in S_{2p}^{n,sa}$ with $\|y\|_{S_{2p}^{n,sa}} = 1$, then $\|y^*y\|_{S_p^n} = \|yy^*\|_{S_p^n} = 1$. Thus, by taking supremum over $\|y\|_{S_{2p}^{n,sa}} = 1$ we obtain the desired result. $\square$

Proposition 4.11. *For $1 < p < 2$ and $n \in \mathbb{N}$, there is a universal constant C independent of n such that $\rho(S_p^{n,sa}, S_p^{n,sa}) \geqslant Cn^{\frac{5}{2}-\frac{1}{p}}$.*

Proof. We follow the strategy of [2]. Let us fix an orthonormal basis $(x_i)_{i=1}^{n^2}$ in M_n^{sa} with respect to the Hilbert-Schimdt norm. Let us consider the random tensor $z = \sum_{i,j=1}^{n^2} g_{ij} x_i \otimes x_j$ where g_{ij}'s are iid $N(0,1)$. Then by Chevet's theorem (Theorem 2.1), we have that

$$\mathbb{E}\|z\|_{S_p^{n,\mathrm{sa}} \check{\otimes} S_p^{n,\mathrm{sa}}} \leqslant 2\|\mathrm{id}\|_{S_2^{n,\mathrm{sa}} \to S_p^{n,\mathrm{sa}}} \mathbb{E}\Big\| \sum_{i=1}^{n^2} g_i x_i \Big\|_{S_p^{n,\mathrm{sa}}}.$$

It can be deduced from [34, Theorem 1.3], $\mathbb{E}\Big\| \sum_{i=1}^{n^2} g_i x_i \Big\|_{S_p^{n,\mathrm{sa}}} \leqslant Cn^{\frac{1}{2}+\frac{1}{p}}$ for some universal constant $C > 0$. It is easy to see that

$$\|\mathrm{id}\|_{S_2^{n,\mathrm{sa}} \to S_p^{n,\mathrm{sa}}} = \begin{cases} 1, \text{ for } p \geqslant 2; \\ n^{\frac{1}{p}-\frac{1}{2}}, \text{ for } p < 2. \end{cases}$$

Therefore, we have that

$$\mathbb{E}\|z\|_{S_p^{n,\mathrm{sa}} \check{\otimes} S_p^{n,\mathrm{sa}}} \leqslant C \begin{cases} n^{\frac{1}{2}+\frac{1}{p}}, \text{for } p \geqslant 2; \\ n^{\frac{2}{p}}, \text{ for } p < 2. \end{cases}$$

Note that by proceeding, as in [2], we have the following inequality

$$\mathbb{E}(\sum_{i,j=1}^{n^2} g_{ij}^2)^{\frac{1}{2}} \leqslant \sqrt{\mathbb{E}\|z\|_{S_p^{n,\mathrm{sa}} \hat{\otimes} S_p^{n,\mathrm{sa}}}} \sqrt{\mathbb{E}\|z\|_{S_{p'}^{n,\mathrm{sa}} \check{\otimes} S_{p'}^{n,\mathrm{sa}}}}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. Therefore, we have that $\mathbb{E}\|z\|_{S_p^{n,\mathrm{sa}} \hat{\otimes} S_p^{n,\mathrm{sa}}} \geqslant cn^{\frac{7}{2}-\frac{1}{p'}}$. Arguing as in [2] we have that

$$\rho(S_p^{n,\mathrm{sa}}, S_p^{n,\mathrm{sa}}) \geqslant Cn^{\frac{5}{2}-\frac{1}{p}}.$$

This completes the proof of the proposition. $\square$

Proposition 4.12. *For $p > 1$, we have $\rho(S^{n,sa}_{\frac{2p}{2p-1}}, S^{n,sa}_{\frac{2p}{2p-1}}) \leqslant Kn^{\frac{3}{2}+\frac{1}{2p}}$ for some K independent of n.*

Proof. The proof is similar to that of Theorem 8 in [2]. Let $z \in M_n^{\mathrm{sa}} \otimes M_n^{\mathrm{sa}}$ be such that $\|z\|_{S^{n,\mathrm{sa}}_{\frac{2p}{2p-1}} \check{\otimes} S^{n,\mathrm{sa}}_{\frac{2p}{2p-1}}} = 1$. By realizing z as $\widetilde{z} : S_{2p}^{n,\mathrm{sa}} \to S^{n,\mathrm{sa}}_{\frac{2p}{2p-1}}$, by Lemma 4.10 there exists a positive linear functional φ on S_p^n such that

$$\|\widetilde{z}(x)\|_{S^{n,sa}_{\frac{2p}{2p-1}}} \leqslant K(\phi(x^2))^{\frac{1}{2}}$$

for all $x \in S_{2p}^{n,sa}$. The constant K depends on the cotype constants of S_{2p} and S_{2p}^*. There exists o.n.b $(u_j)_{j=1}^n$ such that we have $\varphi = \sum_{j=1}^n \lambda_j P_j$ where $P_j(u) := \langle u, u_j\rangle u_j$ for $1 \leqslant j \leqslant n$. Now define E_{jk} as $E_{jk}(u) := \langle u, u_k\rangle u_j$ for all $1 \leqslant j,k \leqslant n$. Then it is easy to check that $((E_{jk})_{j,k=1}^n, (E_{kj})_{k,j=1}^n)$ is a biorthogonal system. By denoting $F_{jk} = E_{jk} + E_{kj}$ and $H_{jk} = i(E_{jk} - E_{kj})$, we have

$$z = \sum_{j=1}^n E_{jj} \otimes \widetilde{z}(E_{jj}) + \frac{1}{2}\sum_{j<k}\Big(F_{jk} \otimes \widetilde{z}(F_{jk}) + H_{jk} \otimes \widetilde{z}(H_{jk})\Big).$$

Again proceeding as in [2] we must have that

$$\|z\|_{S^{n,\mathrm{sa}}_{\frac{2p}{2p-1}} \hat{\otimes} S^{n,\mathrm{sa}}_{\frac{2p}{2p-1}}} \leqslant K\Big(\sum_{j=1}^n \sqrt{\lambda_j} + \sum_{1\leqslant j<k\leqslant n} \sqrt{\lambda_j + \lambda_k}\Big).$$

Note that we must have $\sum_{j=1}^n \lambda_j^{p'} = 1$ where $\frac{1}{p} + \frac{1}{p'} = 1$. Hence we must have

$$\|z\|_{S^{n,\mathrm{sa}}_{\frac{2p}{2p-1}} \hat{\otimes} S^{n,\mathrm{sa}}_{\frac{2p}{2p-1}}} \leqslant K n^{\frac{3}{2}+\frac{1}{2p}}. \qquad \square$$

This completes the proof.

By using Corollary 4.3, Proposition 4.11 and Proposition 4.12, we have the following corollary.

Corollary 4.13. *For $1 \leqslant p < 2$, we have the inequality:*

$$cn^{\frac{5}{2}-\frac{1}{p}} \leqslant K_G(S_p^{n,sa}, S_p^{n,sa}) \leqslant Cn^{\frac{5}{2}-\frac{1}{p}}$$

for two positive constants c and C depending only on p.

We now try find asymptotic behaviour of $\rho(\ell_p^n, \ell_p^n)$ as $n \to \infty$. The proof follows in a similar way as the above corollary. We sketch only the important points in the proof.

Proposition 4.14. *If $1 < p < 2$ and $n \in \mathbb{N}$, then $\rho(\ell_p^n, \ell_p^n) \geqslant Cn^{\frac{3}{2}-\frac{1}{p}}$ for some constant C independent of n.*

Proof. Consider the random tensor $z = \sum_{i,j==1}^n g_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$ where g_{ij}'s are iid $N(0,1)$. Note that by Kahane's inequality, we have for some constant $C_p > 0$,

$$\mathbb{E}\Big\|\sum_{i=1}^n g_i\mathbf{e}_i\Big\|_{\ell_p^n} \leqslant C_p\Big(\mathbb{E}\Big\|\sum_{i=1}^n g_i\mathbf{e}_i\Big\|_{\ell_p^n}^p\Big)^{\frac{1}{p}} = C_p\Big(\sum_{i=1}^n \mathbb{E}|g_i|^p\Big)^{\frac{1}{p}} = D_p n^{\frac{1}{p}}.$$

In above the last inequality follows from known values for $\mathbb{E}|g|^p$. Then proceeding as in Proposition 4.11 by Chevet's theorem (Theorem 2.1), we have that

$$\mathbb{E}\|z\|_{\ell_p^n \check{\otimes} \ell_p^n} \leqslant D_p \begin{cases} n^{\frac{1}{p}}, \text{for } p \geqslant 2; \\ n^{\frac{2}{p}-\frac{1}{2}}, \text{ for } p < 2. \end{cases}$$

Note that by proceeding as in Proposition 4.11, we have the following inequality

$$\mathbb{E}(\sum_{i,j=1}^n g_{ij}^2)^{\frac{1}{2}} \leqslant \sqrt{\mathbb{E}\|z\|_{\ell_p^n \hat{\otimes} \ell_p^n}}\sqrt{\mathbb{E}\|z\|_{\ell_{p'}^n \check{\otimes} \ell_{p'}^n}}.$$

Therefore, we have that $\mathbb{E}\|z\|_{S_p^{n,\mathrm{sa}} \hat{\otimes} S_p^{n,\mathrm{sa}}} \geqslant cn^{2-\frac{1}{p'}}$. Arguing as before, we have that

$$\rho(\ell_p^n, \ell_p^n) \geqslant Cn^{\frac{3}{2}-\frac{1}{p}}.$$

This completes the proof of the proposition. $\square$

Proposition 4.15. *For $p > 1$, we have $\rho(\ell^n_{\frac{2p}{2p-1}}, \ell^n_{\frac{2p}{2p-1}}) \leqslant Kn^{\frac{1}{2}+\frac{1}{2p}}$ for some K independent of n.*

Proof. Let $z \in \ell^n_{\frac{2p}{2p-1}} \otimes \ell^n_{\frac{2p}{2p-1}}$ be such that $\|z\|_{\ell^n_{\frac{2p}{2p-1}} \check{\otimes} \ell^n_{\frac{2p}{2p-1}}} = 1$. By realizing z as $\widetilde{z} : \ell^n_{2p} \to \ell^n_{\frac{2p}{2p-1}}$, by L_p-Grothendieck theorem [17] there exists a positive linear functional φ on ℓ^n_p such that we have $\|\widetilde{z}(x)\|_{\ell^n_{\frac{2p}{2p-1}}} \leqslant K(\phi(x^2))^{\frac{1}{2}}$ for all $x \in \ell^n_{2p}$. The constant K depends on the cotype constants of ℓ_{2p} and ℓ^*_{2p}. Note that we have an identification $\varphi = \sum_{j=1}^n \lambda_j \mathbf{e}_j$ where $(\mathbf{e}_j)_{j=1}^n$ is standard basis. Since $(\mathbf{e}_j)_{j=1}^n$ is a biorthogonal system we have that $z = \sum_{i=1}^n \mathbf{e}_i \otimes \widetilde{z}(\mathbf{e}_i)$. Therefore, we obtain the estimate

$$\|z\|_{\ell^n_{\frac{2p}{2p-1}} \hat{\otimes} \ell^n_{\frac{2p}{2p-1}}} \leqslant K\Big(\sum_{j=1}^n \sqrt{\lambda_j}\Big).$$

Note that we must have $\sum_{j=1}^n \lambda_j^{p'} = 1$ where $\frac{1}{p} + \frac{1}{p'} = 1$. Hence we must have the following estimate $\|z\|_{\ell^n_{\frac{2p}{2p-1}} \hat{\otimes} \ell^n_{\frac{2p}{2p-1}}} \leqslant K n^{\frac{1}{2}+\frac{1}{2p}}$. This completes the proof. □

Corollary 4.16. *For $1 \leqslant p < 2$, we have the inequality*

$$c n^{\frac{3}{2}-\frac{1}{p}} \leqslant K_G(\ell^n_p, \ell^n_p) \leqslant C n^{\frac{3}{2}-\frac{1}{p}},$$

where c and C are two positive constants that depend only on p.

5. $K_G^+(E, F)$

In [3, Fact 2], it was shown that Property P is equivalent to the 2-summing property. The proposition below shows that more is true.

Proposition 5.1. *For any Banach space E of dimension m, $\gamma(E) = \sup_{n \geqslant 1} K_G^+(E, \ell^n_2)$.*

Proof. Let $\dim E = m$. Let $A : E \to E^*$ and $B : E^* \to E$ be positive operators. Let $B = C^*C$ for some linear map $C : E^* \to \ell^m_2$. Also assume that $A = \beta^*\beta$ for some $\beta : E \to \ell^m_2$. Note that

$$\begin{aligned} |tr(BA)| &= |tr(C^*CA)| \\ &\leqslant N(C^*CA) \\ &\leqslant \|C^*\| N(CA) \\ &\leqslant \|C\|^2 \|A\| K_G^+(E, \ell^m_2) \\ &= \|A\|\|B\| K_G^+(E, \ell^m_2). \end{aligned}$$

Thus we have $\gamma(E) \leqslant K_G^+(E, \ell^m_2)$. For the other side inequality, suppose $A : E \to E^*$ is a positive operator. Then

$$\begin{aligned} \sup\big\{N(BA) : \|B\|_{E^* \to \ell^m_2} \leqslant 1\big\} &= \sup\big\{|tr(CB\beta^*\beta)| : \|B\|_{E^* \to \ell^m_2} \leqslant 1,\ \|C\|_{\ell^m_2 \to E} \leqslant 1\big\} \\ &= \sup\big\{|tr(B\beta^*\beta C)| : \|B\|_{E^* \to \ell^m_2} \leqslant 1,\ \|C\|_{\ell^m_2 \to E} \leqslant 1\big\} \\ &= \sup\big\{|tr(B\beta^*\beta C)| : \|B\|_{E^* \to \ell^m_2} \leqslant 1,\ \|C\|_{\ell^m_2 \to E} \leqslant 1\big\} \\ &= \sup\big\{|tr((\beta B^*)^*(\beta C))| : \|B\|_{E^* \to \ell^m_2} \leqslant 1,\ \|C\|_{\ell^m_2 \to E} \leqslant 1\big\} \\ &\leqslant \sup\big\{\|\beta B^*\|_2 \|\beta C\|_2 : \|B\|_{E^* \to \ell^m_2} \leqslant 1,\ \|C\|_{\ell^m_2 \to E} \leqslant 1\big\} \\ &= \sup\big\{\|\beta B^*\|_2^2 : \|B\|_{E^* \to \ell^m_2} \leqslant 1\big\} \\ &= \sup\{|tr(DA)| : D \geqslant 0,\ \|D\|_{E^* \to E} \leqslant 1\}. \end{aligned}$$

Now, taking supremum over all positive operators $A : E \to E^*$, we get $K_G^+(E, \ell^m_2) \leqslant \gamma(E)$. This completes the proof of the proposition. □

Recall that positive variant of the Grothendieck inequality asserts that $\sup_{n\in\mathbb{N}} \gamma(\ell_1^n) < \infty$. In fact, $\sup_{n\in\mathbb{N}} \gamma(\ell_1^n) = \frac{\pi}{2}$. This motivates the following definition.

Definition 5.2 (Little G. T. flag)**.** *Let $(E_n)_{n\in\mathbb{N}}$ be a sequence of finite dimensional Banach spaces with the property that*

H1: $\dim(E_n) = n$ *for all* $n \in \mathbb{N}$.

H2: there exists an isometry $j_n : E_n \to E_{n+1}$ *for each* $n \in \mathbb{N}$.

We say $(E_n, j_n)_{n\in\mathbb{N}}$ is a Little G.T. flag if the quantity $\sup\{\gamma(E_n) : n \in \mathbb{N}\}$ *is finite.*

Remark 5.3. *Let $(E_n)_{n\geqslant 1}$ be a sequence of Banach spaces satisfying (H1) and (H2). We can talk about the inductive limit of $(j_n, E_n)_{n\geqslant 1}$, which is described as below. Consider the subspace of $\prod_{n\geqslant 1} E_n$ formed by sequences $(x_n)_{n\geqslant 1}$ with $j_n x_n = x_{n+1}$ for all n large. We can set $\|x\| := \lim\limits_{n\to\infty} \|x_n\|_{E_n}$. Clearly, this defines a seminorm. After taking quotient by $\{x : \|x\| = 0\}$ and taking closure, we obtain a Banach space which is denoted by E^{ind}. We have a canonical isometric inclusion of E_n into E^{ind} for all $n \geqslant 1$. Under this identification, we may assume that $E_n \subseteq E_{n+1}$ for all $n \in \mathbb{N}$ and $E^{ind} = \overline{\cup_{n=1}^{\infty} E_n}$.*

Definition 5.4 (Hilbert-Schmidt space, [12])**.** *Let E be a Banach space. We say E is a Hilbert-Schmidt space if any bounded operator $u : E \to \ell_2$ is 2-summing, i.e.,* $\mathrm{id} : \Pi_2(E, \ell_2) \to B(E, \ell_2)$ *is an isomorphism.*

Lemma 5.5. *A sequence $(E_n, j_n)_{n\in\mathbb{N}}$ is a Little G.T. flag if and only if the inductive limit E^{ind} is a Hilbert-Schmidt space.*

Proof. Without loss of any generality we may assume by Remark 5.3 that the maps j_n's are all inclusion maps.

Note that it is enough to show that given any $u \in B(E^{\mathrm{ind}}, \ell_2)$, there is a positive constant C such that for all $n \in \mathbb{N}$ we have $\pi_2(u|_{E_n}) \leqslant C\|u|_{E_n}\|_{E_n\to\ell_2}$. Given $x_1, \ldots, x_n \in E_k$ define an operator $T : \ell_2^n \to E_k$ by $Te_i = x_i$ where $1 \leqslant i \leqslant n$ and e_i's are canonical basis of ℓ_2^n. Note that $T^*(x^*) = \sum_{i=1}^n x^*(x_i)e_i$. Therefore, we have that $\sum_{i=1}^n |x^*(x_i)|^2 \leqslant 1$ for all $x^* \in (E_k^*)_1$ if and only if $\|T^*\|_{E_k^*\to\ell_2^n} \leqslant 1$ if and only if $\|T\|_{\ell_2^n\to E_k} \leqslant 1$. To this end observe that

$$\sum_{i=1}^n \|u|_{E_i} x_i\|_2^2 = \sum_{i=1}^n \|u|_{E_i} Te_i\|_2^2 = \langle u|_{E_k}^* u|_{E_k}, TT^*\rangle.$$

Therefore, by denoting $S = TT^*$, we have that

$$\pi_2(u)^2 = \sup\{\langle u^*u, S\rangle : k \geqslant 1,\ S \geqslant 0,\ \|S\|_{E_k^*\to E_k} \leqslant 1\}.$$

Now comparing with the definition of Hilbert-Schimdt space we have the desired result. ☐

Lemma 5.6. *For any Banach space F, we have $K_G^+(\ell_2^n, F) = \rho(\ell_2^n, F)$.*

Proof. As the supremum in the extremum of $K_G^+(\ell_2^n, F)$ is attained at the identity operator which is also a positive operator, it follows that $K_G^+(\ell_2^n, F) = \rho(\ell_2^n, F)$. ☐

For a finite dimensional Banach space E we define

$$\rho^+(E) := \sup\{\|z\|_{E\hat{\otimes}E} : \|z\|_{E\check{\otimes}E} \leqslant 1,\ z \geqslant 0\}.$$

Lemma 5.7. $K_G^+(\ell_\infty^n, \ell_\infty^n) \geqslant \rho^+(\ell_1^n)$.

Proof. In view of Lemma 3.1, we have by taking $B = \mathrm{id}$, that is the identity matrix,

$$\begin{aligned} K_G^+(\ell_\infty^n, \ell_\infty^n) = K_G^+(\ell_\infty^n, \ell_1^n) &= \sup\{N(BA) : A \geqslant 0,\ \|A\|_{\ell_\infty^n\to\ell_1^n} \leqslant 1,\ \|B\|_{\ell_1^n\to\ell_1^n} \leqslant 1\} \\ &\geqslant \sup\{\|A\|_{\ell_1^n\hat{\otimes}\ell_1^n} : \|A\|_{\ell_\infty^n\to\ell_1^n} \leqslant 1,\ A \geqslant 0\} \\ &= \rho^+(\ell_1^n). \end{aligned}$$

This completes the proof of the lemma. □

Theorem 5.8. *There exists a constant $c > 0$ such that for large n,*

$$\rho^+(\ell_1^n) \geqslant c\sqrt{n}.$$

Proof. Consider the sequence of matrices $B_n = (b_{jk})_{j,k=0}^{n-1}$ with

$$b_{jk} = \sqrt{\frac{1}{n}} cos(\frac{2\pi jk}{n}), \quad j,k = 0,\dots,n-1.$$

Since B_n is nothing but the real part of Discrete Fourier Transform matrix. Each of the operator $B_n : \ell_2^n \to \ell_2^n$ has norm atmost 1 for all $n \in \mathbb{N}$. It follows that $B_n : \ell_\infty^n \to \ell_1^n$ is of norm atmost n for each $n \in \mathbb{N}$. Moreover, for large n, the quantity $\sum_{j,k=1}^n |b_{jk}|$ is of order $n^{3/2}$ [23]. Since B_n is real symmetric matrix and is an operator on ℓ_2^n of norm atmost 1, it follows that spectrum of B_n is contained in $[-1,1]$. Thus $A_n := B_n + I$ is a positive operator and

$$\|A\|_{\ell_\infty^n \to \ell_1^n} \leqslant \|B_n\|_{\ell_\infty^n \to \ell_1^n} + \|I\|_{\ell_\infty^n \to \ell_1^n} \leqslant 2n.$$

On the other hand $\sum_{j,k=1}^n |a_{jk}|$ is still at least of order $n^{3/2}$, where a_{jk} is the (j,k) entry of A_n. Choosing A_n as above, by definition, we have

$$\begin{aligned}
\rho^+(\ell^1(n), \ell^1(n) &= \sup_{X \geqslant 0} \frac{\|X\|_{\ell^1(n)\hat{\otimes}\ell^1(n)}}{\|X\|_{\ell^1(n)\check{\otimes}\ell^1(n)}} \\
&= \sup_{X \geqslant 0} \frac{\|X\|_{\ell^1(n)\hat{\otimes}\ell^1(n)}}{\|X\|_{\ell^\infty(n)\to\ell^1(n)}} \\
&\geqslant \frac{\sum_{j,k=1}^n |a_{jk}|}{\|A\|_{\ell^\infty(n)\to\ell^1(n)}} \\
&\geqslant \frac{o(n^3/2)}{2n} \\
&= o(\sqrt{n}).
\end{aligned}$$

This completes the proof of the lemma. □

Remark 5.9. *The example of the matrix B_n also plays a vital role in the context of Grothendieck inequality* [5]. *Some of the properties this matrix that we have used are also in* [5].

The verification of the useful characterization of a non-negative contraction A from ℓ_1^n to ℓ_∞^n, given below, follows from the observation that $\|A\|_{\ell_1^n \to \ell_\infty^n} = \max_{1\leqslant i,j \leqslant n} |a_{ij}|$.

Fact 5.10. *Suppose that $A \geqslant 0$ is a linear operator from ℓ_1^n to ℓ_∞^n. Then $\|A\|_{\ell_1^n \to \ell_\infty^n} \leqslant 1$ if and only if there exists a finite dimensional Hilbert space $\mathcal{H}$ and $v_1, \dots, v_n \in \mathcal{H}$ with $\|v_i\|_{\mathcal{H}} \leqslant 1$ for all $1 \leqslant i \leqslant n$ such that $a_{ij} = \langle v_i, v_j \rangle$ for all $1 \leqslant i \leqslant n$.*

Proposition 5.11. *For all $n \geqslant 1$, we have $K_G^+(\ell_1^n, \ell_1^n) = K_G^+(\ell_1^n, \ell_\infty^n) \leqslant K_G$.*

Proof. In view of Lemma 3.1, we have by taking $A = \mathrm{id}$, that is the identity matrix,

$$\begin{aligned}
K_G^+(\ell_1^n, \ell_\infty^n) &= \sup\{N(XA) : A \geqslant 0,\ \|A\|_{\ell_1^n \to \ell_\infty^n} \leqslant 1,\ \|X\|_{\ell_\infty^n \to \ell_\infty^n} \leqslant 1\} \\
&= \sup\{|\operatorname{tr}(CXA)| : A \geqslant 0,\ \|A\|_{\ell_1^n \to \ell_\infty^n} \leqslant 1,\ \|X\|_{\ell_\infty^n \to \ell_\infty^n} \leqslant 1,\ \|C\|_{\ell_\infty^n \to \ell_1^n} \leqslant 1\} \\
&= \sup\{|\operatorname{tr}(ZA)| : A \geqslant 0,\ \|A\|_{\ell_1^n \to \ell_\infty^n} \leqslant 1,\ \|Z\|_{\ell_\infty^n \to \ell_1^n} \leqslant 1\} \\
&= \sup\{|\sum_{i,j=1}^n Z_{ij} \langle v_i, v_j \rangle| : \|Z\|_{\ell_\infty^n \to \ell_1^n} \leqslant 1,\ \|v_i\|_2 \leqslant 1\} \leqslant K_G.
\end{aligned}$$

The last equality follows from Fact 5.10. □

5.1. $\gamma(E)$ **for complex Banach spaces.** In this subsection all the Banach spaces are assumed to be over the field of complex numbers. Let $K_G^+(\mathbb{C})$ be the complex positive Grothendieck constant.

Proposition 5.12. *For $1 \leqslant p \leqslant 2$, we have $n^{\frac{2}{p'}} \leqslant \gamma(\ell_p^n) \leqslant K_G^+(\mathbb{C}) n^{\frac{2}{p'}}$, where p' is the conjugate of p.*

Proof. Since $p \leqslant p'$, we have that $\|x\|_{p'} \geqslant \|x\|_p$. Thus $\|\operatorname{id}\|_{\ell_p^n \to \ell_{p'}^n} \leqslant 1$. On the other hand we have

$$\|\operatorname{id}\|_{\ell_{p'}^n \to \ell_p^n} = n^{1-\frac{2}{p'}}.$$

Therefore, we have that

$$\gamma(\ell_p^n) \geqslant \frac{\langle \mathrm{id}, \mathrm{id} \rangle}{\|\mathrm{id}\|_{\ell_{p'}^n \to \ell_p^n} \|\operatorname{id}\|_{\ell_p^n \to \ell_{p'}^n}} = n^{\frac{2}{p'}}.$$

noting that $\gamma(\ell_p^n) \leqslant d(\ell_p^n, \ell_1^n)^2 \gamma(\ell_1^n) \leqslant n^{\frac{2}{p'}} K_G^+(\mathbb{C})$ and using the known bounds of $d(\ell_1^n, \ell_p^n)$, see [35, Proposition 37.6]. □

Remark 5.13. *We recall from* [3] *that $\gamma(\ell_1^2) = 1$. Consequently, taking $1 \leqslant p \leqslant 2$ and $n = 2$, we have $\gamma(\ell_p^2) = 2^{\frac{2}{p'}}$ since $d(\ell_1^2, \ell_p^2) = 2^{\frac{2}{p'}}$. From the definition of γ, it follows that $\gamma(E) = \gamma(E^*)$. As a result, for $2 \leqslant p \leqslant \infty$, we have $\gamma(\ell_p^2) = 2^{\frac{2}{p}}$.*

Proposition 5.14. *We have the following estimates for Schatten-p classes.*

(i) $n \leqslant \gamma(S_1^n)$.

(ii) *Let $1 < p < 2$. Then $n^{1+\frac{2}{p'}} \leqslant \gamma(S_p^n)$.*

Proof. Note that $\mathrm{id} : S_\infty^n \to S_1^n$ has operator norm n and $\mathrm{id} : S_1^n \to S_\infty^n$ has norm 1. Therefore, we have that

$$\gamma(S_1^n) \geqslant \frac{\langle \mathrm{id}, \mathrm{id} \rangle}{\|\operatorname{id}\|_{S_1^n \to S_\infty^n} \|\operatorname{id}\|_{S_\infty^n \to S_1^n}} = n.$$

Thus $\gamma(S_1^n) \geqslant n$. Part (ii) follows from a similar calculation. □

Remark 5.15. *Propositions 5.12 and 5.14 show the difference between ℓ_p^n and S_p^n through the distinct behaviour of the constants $\gamma(\ell_p^n)$ and $\gamma(S_p^n)$.*

6. Addendum

In what follows, all the Banach spaces are over the field of complex numbers. In this Addendum, we give elementary and short proofs for $\sup_{m \geqslant 1} K_G^+(\ell_\infty^2, \ell_2^m) = \sup_{m \geqslant 1} K_G^+(\ell_\infty^3, \ell_2^m) = 1$. These were proved earlier in [1, Theorem 4.2] and [3, Fact 7]. In [4], the Grothendieck constant in dimensions 2 and 3 was computed using the known list of extreme points (of an appropriate unit ball) in the real case. The computation in the complex case employed different techniques. The proofs here are taken from [30]. The main new ingredient of this proof is an upper bound on the rank of extreme points of correlation matrices given in [16] in the complex case. We also give a simple proof of the inequality $K_G^+(\ell_\infty^4, \ell_2^2) > 1$. From this, an existential proof of [8, Theorem 2.1] follows.

Although, the following assertions are implicit in [32] (and also in [3]), we indicate briefly how to verify these for the sake of completeness.

Fact 6.1. *Suppose that $A = (a_{ij})_{i,j=1}^n \in M_n$ is a non-negative matrix. Then we have*

$$\sup\Big\{\Big|\sum_{i,j=1}^n a_{ij}\langle x_i, x_j\rangle\Big| : \|x_i\|_{\mathcal{H}} \leqslant 1\Big\} = \sup\Big\{\Big|\sum_{i,j=1}^n a_{ij}\langle x_i, x_j\rangle\Big| : \|x_i\|_{\mathcal{H}} = 1\Big\}, \qquad (\dagger)$$

for any Hilbert space $\mathcal{H}$, and

$$\|A\|_{\ell_\infty^n \to \ell_1^n} = \sup_{|z_i|=1, 1\leqslant i\leqslant n} \sum_{i,j=1}^n a_{ij} z_i \overline{z}_j. \tag{‡}$$

Proof. For any non-negative $n \times n$ matrix A, we first prove that

$$\sup\Big\{\Big|\sum_{i,j=1}^n a_{ij}\langle x_i, y_j\rangle\Big| : \|x_i\|_{\mathcal{H}} \leqslant 1,\ \|y_j\|_{\mathcal{H}} \leqslant 1\Big\} = \sup\Big\{\Big|\sum_{i,j=1}^n a_{ij}\langle x_i, x_j\rangle\Big| : \|x_i\|_{\mathcal{H}} \leqslant 1\Big\}.$$

Note that $A = B^*B$ for some $B = (b_{ij})_{i,j=1}^n \in M_n$. Then

$$\begin{aligned}
\Big|\sum_{i,j=1}^n a_{ij}\langle x_i, y_j\rangle\Big| &= \Big|\sum_{i,j=1}^n \sum_{k=1}^n \bar{b}_{ki} b_{kj}\langle x_i, y_j\rangle\Big| \\
&= \Big|\sum_{k=1}^n \langle \sum_{i=1}^n \bar{b}_{ki} x_i, \sum_{j=1}^n \bar{b}_{kj} y_j\rangle\Big| \\
&\leqslant \Big(\sum_{k=1}^n \Big\|\sum_{i=1}^n \bar{b}_{ki} x_i\Big\|_{\mathcal{H}}^2\Big)^{\frac{1}{2}} \Big(\sum_{k=1}^n \Big\|\sum_{j=1}^n \bar{b}_{kj} y_j\Big\|_{\mathcal{H}}^2\Big)^{\frac{1}{2}}.
\end{aligned}$$

Taking supremum over all x_i, y_j with $\|x_i\|_{\mathcal{H}} \leqslant 1$, and $\|y_j\|_{\mathcal{H}} \leqslant 1$, we get

$$\sup\Big\{\Big|\sum_{i,j=1}^n a_{ij}\langle x_i, y_j\rangle\Big| : \|x_i\|_{\mathcal{H}} \leqslant 1,\ \|y_j\|_{\mathcal{H}} \leqslant 1\Big\} = \sup\Big\{\Big|\sum_{i,j=1}^n a_{ij}\langle x_i, x_j\rangle\Big| : \|x_i\|_{\mathcal{H}} \leqslant 1\Big\}. \tag{6.1}$$

A similar equality is evident when we restrict to the unit sphere rather than the unit ball. For any non-negative matrix $A = (a_{ij})_{i,j=1}^n$, and any finite dimensional Hilbert space $\mathcal{H}$, we have

$$\sup\Big\{\Big|\sum_{i,j=1}^n a_{ij}\langle x_i, y_j\rangle\Big| : \|x_i\|_{\mathcal{H}} = \|y_j\|_{\mathcal{H}} = 1\Big\} = \sup\Big\{\Big|\sum_{i,j=1}^n a_{ij}\langle x_i, y_j\rangle\Big| : \|x_i\|_{\mathcal{H}} \leqslant 1,\ \|y_j\|_{\mathcal{H}} \leqslant 1\Big\}. \tag{6.2}$$

Combining (6.1) with (6.2), the equality in (†) is established.

Applying the Cauchy-Schwarz inequality to the right hand side in the equality below

$$\|A\|_{\ell_\infty^n \to \ell_1^n} = \sup_{|z_i|=|w_j|=1} \Big|\sum_{i,j=1}^n a_{ij} z_i \overline{w}_j\Big|,$$

we obtain

$$\|A\|_{\ell_\infty^n \to \ell_1^n} = \sup_{\mathbf{z},\mathbf{w} \in (\ell_\infty^n)_1} |\langle B\mathbf{z}, B\mathbf{w}\rangle| \leqslant \sup_{\mathbf{z} \in (\ell_\infty^n)_1} \|B\mathbf{z}\|_2^2.$$

The reverse equality is evident. This proves the second equality (‡). □

Part (i) of the theorem below has been proved in [1] and also in [3]. Recall that a complex positive semi-definite matrix with all its diagonal elements equal to one is called a Correlation matrix. We denote the set of all $n \times n$ Correlation matrices by $\mathcal{C}(n)$.

Theorem 6.2. *We have the following.*

(i) $\sup_{m\geqslant 1} K_G^+(\ell_\infty^2, \ell_2^m) = \sup_{m\geqslant 1} K_G^+(\ell_\infty^3, \ell_2^m) = 1.$

(ii) $K_G^+(\ell_\infty^4, \ell_2^2) > 1.$

Proof. Given a complex $n \times n$ non-negative matrix A, set

$$\beta(A) = \sup\big\{\langle A, B\rangle \mid B \geqslant 0, \|B\|_{\ell_1^n \to \ell_\infty^n} \leqslant 1\big\}.$$

Note that, Part (i) of Fact 6.1 taken together with Fact 5.10 show that $\beta(A) = \sup_{B\in\mathcal{C}(n)}\langle A, B\rangle$. Now, observing that the quantity $\langle A, B\rangle$ is $\mathbb{C}$-linear in B and $\mathcal{C}(n)$ is a compact convex set, we conclude that $\beta(A) = \sup_{B\in E(\mathcal{C}(n))}\langle A, B\rangle$, where $E(\mathcal{C}(n))$ is the set of all extreme points of $\mathcal{C}(n)$. Since, all the elements of $E(\mathcal{C}(n))$ are of rank less than or equal to $\sqrt{n}$, see [16], in case n is either 2 or 3, we conclude that extreme correlation matrices are of rank one. Now, if the correlation matrix $B = (\langle x_i, x_j\rangle)_{i,j=1}^n$ is of rank 1, then x_i's can be chosen to be one dimensional unit vectors. So for $n = 2, 3$, we obtain the following

$$\beta(A) = \sup_{B\in E(\mathcal{C}(n))}\langle A, B\rangle = \sup_{|z_i|=1}\sum_{i,j=1}^n a_{ij} z_i \bar{z}_j = \|A\|_{\ell_\infty^n \to \ell_1^n}.$$

The last equality follows from Part (ii) of Fact 6.1 completing the proof of Part (i) of the theorem.

The proof of Part (ii) follows by combining the Part (i) of the theorem with Example 2.3 of [1]. See also Fact 8 of [3]. □

Corollary 6.3. *There exists a quadruple of* 3×3 *commuting tuple of matrices which are contractions but they do not coextend to commuting isometries.*

Proof. Note that in view of [3] and Part (ii) of Theorem 6.2, there exist a contractive commuting tuple of the form

$$\begin{pmatrix} 0 & 0 & a_1 & a_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & b_1 & b_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & c_1 & c_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 & d_1 & d_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which does not satisfy matrix-valued von Neumann inequality and equivalently do not extend to commuting isometries but satisfies the von Neumann inequality. This is also true if we delete the first column and last row of each of the matrices. Thus, the proof is complete. □

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