COMMUTING TUPLE OF MULTIPLICATION OPERATORS HOMOGENEOUS UNDER THE UNITARY GROUP

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Abstract. Let $\mathcal{U}(d)$ be the group of $d \times d$ unitary matrices. We find conditions to ensure that a $\mathcal{U}(d)$-homogeneous d-tuple $\boldsymbol{T}$ is unitarily equivalent to multiplication by the coordinate functions on some reproducing kernel Hilbert space $\mathcal{H}_K(\mathbb{B}_d, \mathbb{C}^n) \subseteq \mathrm{Hol}(\mathbb{B}_d, \mathbb{C}^n)$, $n = \dim \cap_{j=1}^d \ker T_j^*$. We describe this class of $\mathcal{U}(d)$-homogeneous operators, equivalently, non-negative kernels K quasi-invariant under the action of $\mathcal{U}(d)$. We classify quasi-invariant kernels K transforming under $\mathcal{U}(d)$ with two specific choice of multipliers. A crucial ingredient of the proof is that the group $SU(d)$ has exactly two inequivalent irreducible unitary representations of dimension d and none in dimensions $2, \ldots, d-1$, $d \geq 3$. We obtain explicit criterion for boundedness, reducibility and mutual unitary equivalence among these operators.

1. Introduction

Let Ω be an irreducible bounded symmetric domain of rank r in $\mathbb{C}^d$ and $\mathrm{Aut}(\Omega)$ be the group of bi-holomorphic automorphisms on Ω. Let G be the connected component of identity in $\mathrm{Aut}(\Omega)$. It is well known that G acts transitively on Ω. Let $\mathbb{K}$ be the subgroup of linear automorphisms in G. By Cartan's theorem [14, Proposition 2, pp. 67], $\mathbb{K} = \{\phi \in G : \phi(0) = 0\}$. The group $\mathbb{K}$ is known to be a maximal compact subgroup of G and Ω is isomorphic to $G/\mathbb{K}$. There is a natural action of $\mathbb{K}$ on Ω given by

$$k \cdot \boldsymbol{z} := \big(k_1(\boldsymbol{z}), \ldots, k_d(\boldsymbol{z})\big), \qquad k \in \mathbb{K} \text{ and } \boldsymbol{z} \in \Omega,$$

where $k_1(\boldsymbol{z}), \ldots, k_d(\boldsymbol{z})$ are linear polynomials. The group $\mathbb{K}$ also acts on a d-tuple $\boldsymbol{T} = (T_1, \ldots, T_d)$ of commuting bounded linear operators defined on a complex separable Hilbert space $\mathcal{H}$, naturally, via the map

$$k \cdot \boldsymbol{T} := \big(k_1(T_1, \ldots, T_d), \ldots, k_d(T_1, \ldots, T_d)\big), \; k \in \mathbb{K}.$$

Definition 1.1 ([10])**.** A d-tuple $\boldsymbol{T} = (T_1, \ldots, T_d)$ of commuting bounded linear operators on $\mathcal{H}$ is said to be $\mathbb{K}$-homogeneous if for all k in $\mathbb{K}$ the operators $\boldsymbol{T}$ and $k \cdot \boldsymbol{T}$ are unitarily equivalent, that is, for all k in $\mathbb{K}$ there exists a unitary operator $\Gamma(k)$ on $\mathcal{H}$ such that

$$T_j \Gamma(k) = \Gamma(k) k_j(T_1, \ldots, T_d), \qquad j = 1, 2, \ldots, d.$$

In particular, when Ω is the Euclidean ball $\mathbb{B}_d$ in $\mathbb{C}^d$, then $\mathbb{K}$ is the group of unitary linear transformations on $\mathbb{C}^d$ and the *spherical* tuples defined in [5] are nothing but $\mathcal{U}(d)$-homogeneous d-tuples. In this paper we would be discussing $\mathcal{U}(d)$-homogeneous commuting d-tuple $\boldsymbol{M}$ of multiplication by coordinate functions $z_1, \ldots, z_d$ on a reproducing kernel Hilbert space $\mathcal{H}_K(\mathbb{B}_d, \mathbb{C}^n)$. This Hilbert space consists of holomorphic functions defined on $\mathbb{B}_d$ and taking values in $\mathbb{C}^n$. We consider in some detail

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the case of $n = d$. However, without any additional effort, we set up the machinery in the much more general context of a bounded symmetric domain Ω and the maximal compact subgroup $\mathbb{K}$ of its bi-holomorphic automorphism group. A detailed study of $\mathbb{K}$-homogeneous operator is underway.

Now, let $D_{\boldsymbol{T}} : \mathcal{H} \to \mathcal{H} \oplus \cdots \oplus \mathcal{H}$ be the operator

$$D_{\boldsymbol{T}} h := (T_1 h, \ldots, T_d h), \qquad h \in \mathcal{H}.$$

We note that $\ker D_{\boldsymbol{T}} = \cap_{i=1}^{d} \ker T_i$ is the *joint kernel* and $\sigma_p(\boldsymbol{T}) = \{\boldsymbol{w} \in \mathbb{C}^d : \ker D_{\boldsymbol{T}-\boldsymbol{w}I} \neq \boldsymbol{0}\}$ is the *joint point spectrum* of the d-tuple $\boldsymbol{T}$. The class $\mathcal{A}\mathbb{K}(\Omega)$ consisting of $\mathbb{K}$-homogeneous d-tuples of operators with the property:

(1) $\dim \ker D_{\boldsymbol{T}^*} = 1$,
(2) $\ker D_{\boldsymbol{T}^*}$ is cyclic for $\boldsymbol{T}$, and
(3) $\Omega \subseteq \sigma_p(\boldsymbol{T}^*)$;

was introduced in the recent paper [10], see also [19]. Among other things, it is shown in [10, Theorem 2.3] that any d-tuple $\boldsymbol{T}$ in $\mathcal{A}\mathbb{K}(\Omega)$ must be unitarily equivalent to the d-tuple $\boldsymbol{M}$ of multiplication by the coordinate functions on a reproducing kernel Hilbert space $\mathcal{H}_K(\Omega) \subseteq \mathrm{Hol}(\Omega, \mathbb{C})$ for some $\mathbb{K}$-invariant kernel K. Recall that the Hilbert space $\mathcal{H}_K(\Omega)$ has a direct sum decomposition $\oplus_{\underline{s} \in \vec{\mathbb{Z}}_+^r} \mathcal{P}_{\underline{s}}$, where $\vec{\mathbb{Z}}_+^r$ is the set of signatures: $\underline{s} := (s_1, \ldots, s_r) \in \mathbb{Z}_+^r$, $s_1 \geq s_2 \geq \cdots \geq s_r \geq 0$ and $\mathcal{P}_{\underline{s}}$ are the irreducible components under the action of $\mathbb{K}$. The invariant kernel K is then of the form: $K_{\boldsymbol{a}}(\boldsymbol{z}, \boldsymbol{w}) = \sum_{\underline{s} \in \vec{\mathbb{Z}}_+^r} a_{\underline{s}} E_{\underline{s}}(\boldsymbol{z}, \boldsymbol{w})$, where $E_{\underline{s}}$ is the reproducing kernel of $\mathcal{P}_{\underline{s}}$ equipped with the Fischer-Fock inner product defined by $\langle p, q \rangle_{\mathcal{F}} := \frac{1}{\pi^d} \int_{\mathbb{C}^d} p(\boldsymbol{z}) \overline{q(\boldsymbol{z})} e^{-\|\boldsymbol{z}\|_2^2} dm(\boldsymbol{z})$. Here $dm(z)$ denotes the Lebesgue measure on $\mathbb{C}^d$.

The results of [10] also show that the properties of $\boldsymbol{M}$ like boundedness, membership in the Cowen-Douglas class $B_1(\Omega)$, unitary and similarity orbit etc. can be determined from the properties of the sequence $\boldsymbol{a} := \{a_{\underline{s}}\}_{\underline{s} \in \vec{\mathbb{Z}}_+^r}$. It is therefore natural to investigate the much larger class of d-tuples of homogeneous operators by assuming only that $\dim \ker D_{\boldsymbol{T}^*}$ is finite rather than 1, which is the main feature of the class defined below. As one might expect, we obtain a model theorem in this case also with the major difference that the kernel K need not be invariant under the action of the group $\mathbb{K}$, instead it is *quasi-invariant!*

Assume that $\ker D_{\boldsymbol{T}^*}$ is a cyclic subspace for $\boldsymbol{T}$ of dimension n. Let $\mathcal{H}^{(0)}$ be the linear space $\{p(\boldsymbol{T})\gamma | \ \gamma \in \ker D_{\boldsymbol{T}^*}, p \in \mathcal{P}\}$, where $\mathcal{P}$ is the space of complex-valued polynomials in d-variables. Fix an orthonormal basis $\{\gamma_1, \ldots, \gamma_n\}$ in $\ker D_{\boldsymbol{T}^*}$. For $\boldsymbol{w} \in \mathbb{C}^d$, the point evaluation $\mathrm{ev}_{\boldsymbol{w}} : \mathcal{H}^{(0)} \to \mathbb{C}^n$ is defined to be the map

$$\mathrm{ev}_{\boldsymbol{w}}\big(\sum_{i=1}^{n} p_i(\boldsymbol{T})(\gamma_i)\big) := \sum_{i=1}^{n} p_i(\boldsymbol{w}) \boldsymbol{e}_i,$$

where $p_1, \ldots, p_n$ are in $\mathcal{P}$ and $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ are the standard unit vectors in $\mathbb{C}^n$. Let $\mathrm{bpe}(\boldsymbol{T})$ be the set $\{\boldsymbol{w} \in \mathbb{C}^d : \mathrm{ev}_{\boldsymbol{w}} \text{ is bounded}\}$ (see [17, Definition 2.1]).

Definition 1.2. Let Ω be an irreducible bounded symmetric domain. A $\mathbb{K}$-homogeneous d-tuple $\boldsymbol{T}$ possessing the following properties

(i) $\dim \ker D_{\boldsymbol{T}^*} = n$,
(ii) the space $\ker D_{\boldsymbol{T}^*}$ is cyclic for $\boldsymbol{T}$,
(iii) $\Omega \subseteq \mathrm{bpe}(\boldsymbol{T})$, and the evaluation maps $\mathrm{ev}_{\boldsymbol{w}}$ are locally uniformly bounded for $\boldsymbol{w} \in \Omega$,

is said to be in the class $\mathcal{A}_n\mathbb{K}(\Omega)$.

The local uniform boundedness of the evaluation functionals might appear to be a strong requirement but is necessary for constructing a model for d-tuples in $\mathcal{A}_n\mathbb{K}(\Omega)$ with $n > 1$ (see proof of Theorem 2.1). This notion appears in the definition of quasi-free modules introduced in [8]. The notion of sharp kernels (see [2]) and generalized Bergman kernels (see [6]) occurring in the work of Agrawal-Salinas and Curto-Salinas are closely related to the kernels implicit in Definition 1.2.

It follows from [10, Theorem 2.3] that the d-tuples in the class $\mathcal{A}\mathbb{K}(\Omega)$ introduced earlier in [10] coincides with to the class $\mathcal{A}_1\mathbb{K}(\Omega)$. It would be convenient for us to let $\mathcal{A}\mathbb{K}(\Omega)$ denote the class $\mathcal{A}_1\mathbb{K}(\Omega)$. In this paper, we continue the investigation initiated in [10], now for the class $\mathcal{A}_n\mathbb{K}(\Omega)$, $n > 1$.

Definition 1.3. Let $K : \Omega \times \Omega \to \mathcal{M}_n(\mathbb{C})$ be a sesqui-analytic Hermitian function and $c : \mathbb{K} \times \Omega \to \mathrm{GL}_n(\mathbb{C}))$ be a function holomorphic in the second variable for each fixed $k \in \mathbb{K}$. The function K is said to be quasi-invariant under the group $\mathbb{K}$ with multiplier c if

$$K(\boldsymbol{z}, \boldsymbol{w}) = c(k, \boldsymbol{z})K(k^{-1} \cdot \boldsymbol{z}, k^{-1} \cdot \boldsymbol{w})c(k, \boldsymbol{w})^*,\ k \in \mathbb{K}.$$

We point out that if the function K is quasi-invariant and non-negative definite, then the map $\Gamma(k)$, $k \in \mathbb{K}$ defined by the rule: $\Gamma(k)(f) = c(k, \boldsymbol{z})f \circ k^{-1}$ is unitary on the reproducing kernel Hilbert space $\mathcal{H}_K(\Omega, \mathbb{C}^n)$. Also, the map $k \to \Gamma(k)$ is a homomorphism if and only if c is a cocycle, that is,

$$c(k_1k_2, \boldsymbol{z}) = c(k_1, k_2 \cdot \boldsymbol{z})c(k_2, \boldsymbol{z}),\ k_1, k_2 \in \mathbb{K}.$$

In the explicit examples we discuss, the map $c : \mathbb{K} \times \Omega \to \mathrm{GL}_n(\mathbb{C})$ is constant in the second variable and therefore defines a unitary representation of the group $\mathbb{K}$. These examples consist of $\Omega = \mathbb{B}_d$ and $c(k) = k$ or $c(k) = \bar{k}$, $k \in \mathbb{K}$, which in this case is $\mathcal{U}(d)$. Consequently, the intertwining operator $\Gamma(k)$ defines a unitary representation $k \to \Gamma(k)$ of the group $\mathbb{K}$. Indeed, if there is a unitary $\Gamma(k)$, $k \in \mathbb{K}$, intertwining $\boldsymbol{M}$ and $k \cdot \boldsymbol{M}$, then the reproducing kernel K must be quasi-invariant. A familiar argument using the very useful notion of "normalized kernel", see Remark 2.2, then shows that the function c must be actually independent of $\boldsymbol{z}$. What is more, it is also shown that $c(k)$ is unitary for each $k \in \mathbb{K}$.

If the d-tuple $\boldsymbol{M}$ on some Hilbert space $\mathcal{H}_K(\Omega)$ is in $\mathcal{A}\mathbb{K}(\Omega)$, then the kernel K is invariant under the action of the group $\mathbb{K}$, that is, $K(\boldsymbol{z}, \boldsymbol{w}) = \sum_{\underline{s} \in \vec{\mathbb{Z}}^r_+} a_{\underline{s}} E_{\underline{s}}(\boldsymbol{z}, \boldsymbol{w})$ with $a_0 = 1$, see [1, Proposition 3.4] and [10, Theorem 2.3]. But if $n > 1$ and the d-tuple $\boldsymbol{M}$ acting on $\mathcal{H}_K(\Omega, \mathbb{C}^n)$ is in $\mathcal{A}_n\mathbb{K}(\Omega)$, then we can only assume that the kernel K is merely quasi-invariant, not necessarily invariant. How do we construct, if there is any, an example of a kernel $K : \Omega \times \Omega \to \mathcal{M}_n(\mathbb{C})$ which is quasi-invariant but not invariant. Equivalently, we are asking: If $\boldsymbol{M}$ is in $\mathcal{A}_n\mathbb{K}(\Omega)$ acting on the Hilbert space $\mathcal{H}_K(\Omega, \mathbb{C}^n)$ $(n > 1)$, then does it follow that the quasi-invariant kernel K must be necessarily invariant? Consider, for example, the kernel

$$\mathcal{K}_{\boldsymbol{a}}(\boldsymbol{w}, \boldsymbol{w}) := K^2_{\boldsymbol{a}}(\boldsymbol{w}, \boldsymbol{w})\Big(\Big(\frac{\partial^2}{\partial w_i \partial \bar{w}_j} \log K_{\boldsymbol{a}}(\boldsymbol{w}, \boldsymbol{w})\Big)\Big),$$

where $K_{\boldsymbol{a}} : \Omega\times\Omega \to \mathbb{C}$ is an invariant positive definite kernel of the form $K_{\boldsymbol{a}}(\boldsymbol{z}, \boldsymbol{w}) = \sum_{\underline{s} \in \vec{\mathbb{Z}}^r_+} a_{\underline{s}} E_{\underline{s}}(\boldsymbol{z}, \boldsymbol{w})$. It is known that $\mathcal{K}_{\boldsymbol{a}}$ is not only a positive definite kernel but also quasi-invariant under $\mathbb{K}$, see [11, Proposition 2.3 and Proposition 6.2]. Indeed, $\mathcal{K}_{\boldsymbol{a}}$ transforms according to the rule:

$${k^{-1}}^{\dagger}\mathcal{K}_{\boldsymbol{a}}(k^{-1} \cdot \boldsymbol{z}, k^{-1} \cdot \boldsymbol{w})\overline{k^{-1}} = \mathcal{K}_{\boldsymbol{a}}(\boldsymbol{z}, \boldsymbol{w}),\ k \in \mathbb{K},$$

where $\dagger$ denotes the transpose of a matrix. The multiplier $c : \mathbb{K} \times \Omega \to \mathrm{GL}_d(\mathbb{C})$ for the quasi-invariant kernel $\mathcal{K}_{\boldsymbol{a}}$ is given by $c(k, \boldsymbol{z}) = \overline{k}$, $k \in \mathbb{K}$, $z \in \Omega$. It is not hard to see that $\mathcal{K}_{\boldsymbol{a}}$ is *not* invariant under $\mathbb{K}$, see Proposition 2.8. Thus, we have many examples of quasi-invariant kernels taking values in $\mathcal{M}_n(\mathbb{C})$ that are not invariant when $n = d$. We briefly describe below the results of this paper.

In Section 2, we find a concrete model for a d-tuple $\boldsymbol{T}$ in $\mathcal{A}_n\mathbb{K}(\Omega)$ as the d-tuple $\boldsymbol{M}$ of multiplication by the coordinate functions $z_1, \ldots, z_d$ on some Hilbert space $\mathcal{H}_K(\Omega, \mathbb{C}^n) \subseteq \mathrm{Hol}(\Omega, \mathbb{C}^n)$ possessing a reproducing kernel $K : \Omega \times \Omega \to \mathcal{M}_n(\mathbb{C})$. This is Theorem 2.1. We prove, see Theorem 2.7, that a quasi-invariant kernel K is a sum (with positive coefficients) of certain quasi-invariant kernels in the Peter-Weyl decomposition of the Hilbert space $\mathcal{H}_K(\Omega, \mathbb{C}^n)$ with respect to the action of the group $\mathbb{K}$.

In Section 3, we restrict to the case of the Euclidean ball $\mathbb{B}_d \subseteq \mathbb{C}^d$. Designating π_ℓ the natural action of $\mathcal{U}(d)$ on the homogeneous polynomials of degree ℓ in d variables equipped with the Fisher-Fock inner product. We prove that $\pi_1 \otimes \pi_\ell$ is reducible and identify an irreducible component in the decomposition of $\pi_1 \otimes \pi_\ell$. We obtain a similar result for $\bar{\pi}_1 \otimes \pi_\ell$, where $\bar{\pi}_1$ is the contragredient of π_1. Choosing the cocycles $c(u, z) = \pi_1(u)$, its contragredient $c(u, z) = \bar{\pi}_1(u)$, $u \in \mathcal{U}(d)$, we describe all the sesqui-analaytic Hermitian quasi-invariant function that transform as in Definition 1.3. Among these, the non-negative definite functions are identified explicitly. We conclude by discussing two sets of examples of d-tuples in $\mathcal{A}_d\mathcal{U}(\mathbb{B}_d)$.

In the first half of Section 4, we find conditions for boundedness and irreducibility of the d-tuple $\boldsymbol{M}$. The second half is devoted to study of quasi-invariant diagonal kernels $K : \mathbb{B}_d \times \mathbb{B}_d \to \mathcal{M}_n(\mathbb{C})$. In this case, such a kernel must be invariant and we prove that it is of the form: $\sum_{\ell=0}^\infty A_\ell \langle \boldsymbol{z}, \boldsymbol{w}\rangle^\ell, \boldsymbol{z}, \boldsymbol{w} \in \mathbb{B}_d$, see Corollary 4.11.

In the concluding Section 5, first, we identify the two components in the decomposition of $\pi_1 \otimes \pi_\ell$ (respectively, $\bar{\pi}_1 \otimes \pi_\ell$) explicitly and show that these components themselves are irreducible. Secondly, we prove that if a kernel K is quasi-invariant under $\mathcal{U}(d)$ taking values in $\mathcal{M}_d(\mathbb{C})$, transforms as in Definition 1.3 with $c : \mathcal{U}(d) \to \mathrm{GL}_d(\mathbb{C})$, and c is assumed to be an irreducible representation of $\mathcal{U}(d)$, then these kernels fall into two classes explicitly described in Theorem 5.7. To prove this result, we first establish that, up to unitary equivalence, there are only two irreducible unitary representations of $SU(d)$, the standard one and its contragredient. We also prove that $SU(d)$ does not have any irreducible unitary representation of dimension ℓ, $2 \le \ell \le d - 1$. We were not able to locate these results that might be of independent interest. Therefore, we have included detailed proofs of these results.

For now, we have complete results only in the particular case of the cocycles $c(u, z) = u$ or $\bar{u}$ of the group $\mathcal{U}(d)$, $d \in \mathbb{N}$. We are hopeful of obtaining similar results for an arbitrary cocycle in the case of the group $\mathcal{U}(2)$.

2. Decomposition of a quasi-invariant kernel

We begin by providing a model for a d-tuple of operator $\boldsymbol{T}$ in the class $\mathcal{A}_n\mathbb{K}(\Omega)$ acting on some Hilbert space $\mathcal{H}$. The proof involves transplanting the inner product of $\mathcal{H}$ on the subspace $\mathbb{C}^n \otimes \mathcal{P}$ of $\mathbb{C}^n$-valued polynomials in the space of holomorphic functions $\mathrm{Hol}(\Omega, \mathbb{C}^n)$. The proof amounts to constructing a unitary operator intertwining $\boldsymbol{T}$ and the d-tuple of multiplication operators defined on the completion of the subspace $\mathbb{C}^n \otimes \mathcal{P}$ in $\mathrm{Hol}(\Omega, \mathbb{C}^n)$.

Theorem 2.1. *Suppose that $\boldsymbol{T}$ is a d-tuple of commuting operators in $\mathcal{A}_n\mathbb{K}(\Omega)$. Then $\boldsymbol{T}$ is unitarily equivalent to the d-tuple $\boldsymbol{M}$ of multiplication by the coordinate functions $z_1, \ldots, z_d$ on a reproducing kernel Hilbert space $\mathcal{H}_K(\Omega, \mathbb{C}^n) \subset \mathrm{Hol}(\Omega, \mathbb{C}^n)$, for some kernel function K quasi-invariant under $\mathbb{K}$.*

Proof. Since $\boldsymbol{T}$ is $\mathbb{K}$-homogeneous, for each $k \in \mathbb{K}$ there exists a unitary operator $\Gamma(k)$ on $\mathcal{H}$ such that

$$T_j\Gamma(k) = \Gamma(k)k_j(\boldsymbol{T}), \qquad j = 1, \ldots, d.$$

Pick an orthonormal basis $\{\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_n\} \subseteq \ker D_{\boldsymbol{T}^*}$. Let $\iota : \ker D_{\boldsymbol{T}^*} \to \mathbb{C}^n$ be a unitary identifying $\boldsymbol{\xi} = \sum_{i=1}^n x_i\boldsymbol{\xi}_i$ with the vector $\boldsymbol{x} = \sum_{i=1}^n x_i\boldsymbol{e}_i$, where $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ are the standard unit vectors in $\mathbb{C}^n$. We define a semi-inner product on $\mathbb{C}^n \otimes \mathcal{P}$ by extending

$$\langle \boldsymbol{e}_i \otimes p,\, \boldsymbol{e}_j \otimes q\rangle := \langle p(\boldsymbol{T})\boldsymbol{\xi}_i,\, q(\boldsymbol{T})\boldsymbol{\xi}_j\rangle_{\mathcal{H}}, p, q \in \mathcal{P}, \tag{2.1}$$

to $\mathbb{C}^n \otimes \mathcal{P}$ by linearity. Suppose that $\big\| \sum_{i=1}^n \boldsymbol{e}_i \otimes p_i\big\| = 0$, then we claim that $\sum_{i=1}^n \boldsymbol{e}_i \otimes p_i = 0$. Pick any $w \in \Omega \subseteq \mathrm{bpe}(\boldsymbol{T})$ and note that

$$\big\| \sum_{i=1}^n p_i(w)\boldsymbol{e}_i\big\|_2 \le C_w\big\| \sum_{i=1}^n p_i(\boldsymbol{T})\boldsymbol{\xi}_i\big\|_{\mathcal{H}} = 0.$$

For $1 \le i \le n$, it follows that $p_i(\boldsymbol{w}) = 0$ for all $\boldsymbol{w} \in \Omega$. Consequently each p_i, $1 \le i \le n$, is the zero polynomial. Therefore, the semi-inner product given by the formula (2.1) defines an inner product on $\mathbb{C}^n \otimes \mathcal{P}$.

Let $\mathscr{H}$ be the completion of $\mathbb{C}^n \otimes \mathcal{P}$ with respect to this inner product. Since we have assumed that the set $\operatorname{bpe}(\boldsymbol{T})$ contains Ω, it follows that the Hilbert space $\mathscr{H}$ is a reproducing kernel Hilbert space consisting of functions defined on Ω. Let $K : \Omega \times \Omega \to \mathcal{M}_n(\mathbb{C})$ be the kernel function given by $K(\boldsymbol{z}, \boldsymbol{w}) = \operatorname{ev}_{\boldsymbol{z}} \operatorname{ev}_{\boldsymbol{w}}^*$, that is,

(1) $K(\cdot, \boldsymbol{w})\boldsymbol{x}$ is in $\mathscr{H}$ for every vector $\boldsymbol{x} \in \mathbb{C}^n$ and every point $\boldsymbol{w} \in \Omega$,
(2) $\langle f, K(\cdot, \boldsymbol{w})\boldsymbol{x}\rangle_{\mathscr{H}} = \langle f(\boldsymbol{w}), \boldsymbol{x}\rangle_2$.

Given any function $f \in \mathscr{H}$, we can find polynomials $p_j \in \mathbb{C}^n \otimes \mathcal{P}$ such that $\|f - p_j\|_{\mathscr{H}} \to 0$ as $j \to \infty$ by assumption. Moreover, since the point evaluations are assumed to be locally uniformly bounded on Ω, it follows that for any fixed but arbitrary $\boldsymbol{w} \in \Omega$, there is an open set $\mathcal{O} \subseteq \Omega$ containing $\boldsymbol{w}$ such that $\sup_{\boldsymbol{z} \in \mathcal{O}} \|K(\boldsymbol{z}, \boldsymbol{z})\| = N_{\mathcal{O},\boldsymbol{w}} < \infty$. For any compact set $X \subseteq \mathcal{O}$, and $\boldsymbol{z} \in X$, we have

$$|\langle f(\boldsymbol{z}) - p_j(\boldsymbol{z}), \boldsymbol{e}_i\rangle| \le \|f(\boldsymbol{z}) - p_j(\boldsymbol{z})\|_2 \le N_{\mathcal{O},\boldsymbol{w}}^{1/2} \|f - p_j\|_{\mathscr{H}} \tag{2.2}$$

proving that f is holomorphic at $\boldsymbol{w}$. Consequently, K is holomorphic in the first variable and anti-holomorphic in the second.

Now for any $k \in \mathbb{K}$, since $\ker D_{\boldsymbol{T}^*}$ is invariant under the unitary map $\Gamma(k)^*$, we have

$$\begin{aligned}
\langle \boldsymbol{e}_i \otimes p, \boldsymbol{e}_j \otimes q\rangle_{\mathbb{C}^n \otimes \mathcal{P}} &= \langle p(\boldsymbol{T})\boldsymbol{\xi}_i, q(\boldsymbol{T})\boldsymbol{\xi}_j\rangle_{\mathcal{H}} \\
&= \langle \Gamma(k)p(k \cdot \boldsymbol{T})\Gamma(k)^*\boldsymbol{\xi}_i, \Gamma(k)q(k \cdot \boldsymbol{T})\Gamma(k)^*\boldsymbol{\xi}_j\rangle_{\mathcal{H}} \\
&= \langle p(k \cdot \boldsymbol{T})\Gamma(k)^*\boldsymbol{\xi}_i, q(k \cdot \boldsymbol{T})\Gamma(k)^*\boldsymbol{\xi}_j\rangle_{\mathcal{H}} \\
&= \langle \iota\Gamma(k)^*\iota^*\boldsymbol{e}_i \otimes p \circ k, \iota\Gamma(k)^*\iota^*\boldsymbol{e}_j \otimes q \circ k\rangle_{\mathbb{C}^n \otimes \mathcal{P}}.
\end{aligned}$$

Therefore, the reproducing kernel K of the Hilbert space $\mathscr{H}$ is quasi-invariant under $\mathbb{K}$ with multiplier $\iota\Gamma(k)^*\iota^*$. Finally, the unitary taking $\boldsymbol{e}_i \otimes p$ to $p(\boldsymbol{T})\boldsymbol{\xi}_i$ extends to a unitary from the Hilbert space $\mathcal{H}$ to the Hilbert space $\mathscr{H}$. This unitary intertwines the commuting d-tuple $\boldsymbol{T}$ on $\mathcal{H}$ with the d-tuple $\boldsymbol{M}$ of multiplication by the coordinate functions z_i, $1 \le i \le d$, on $\mathscr{H}$. ☐

Now we gather a few properties of d-tuples in the class $\mathcal{A}_n\mathbb{K}(\Omega)$. In particular, we prove that if the d-tuple $\boldsymbol{M}$ on $\mathcal{H}_K(\Omega, \mathbb{C}^n)$ is in $\mathcal{A}_n\mathbb{K}(\Omega)$, then the intertwining unitary between $\boldsymbol{M}$ and $k \cdot \boldsymbol{M}$ for each $k \in \mathbb{K}$ must be of the form $f \to c(k)(f \circ k^{-1})$, $c(k) \in \mathcal{U}(n)$.

Remark 2.2. We recall that any non-negative definite kernel $K : \Omega \times \Omega \to \mathcal{M}_n(\mathbb{C})$ admits a normalization K_0 at $\boldsymbol{w}_0 \in \Omega$. The normalized kernel K_0 is characterized by the requirement $K_0(\boldsymbol{z}, \boldsymbol{w}_0) = \mathrm{Id}_n$ for all $\boldsymbol{z} \in \Omega$. The point $\boldsymbol{w}_0$ is arbitrary but fixed. The first two of the three statements below can be found in [6] and the last one is from [7, p. 285, Remark].

(1) The d-tuple $\boldsymbol{M}$ on $\mathcal{H}_K(\Omega, \mathbb{C}^n)$ and $\mathcal{H}_{K_0}(\Omega, \mathbb{C}^n)$ are unitarily equivalent.
(2) If K_1 and K_2 be the kernels normalized at some fixed $\boldsymbol{w}_0 \in \Omega$, then the multiplication d-tuples on $\mathcal{H}_{K_1}(\Omega, \mathbb{C}^n)$ and $\mathcal{H}_{K_2}(\Omega, \mathbb{C}^n)$ are unitarily equivalent if and only if there is a unitary $U \in \mathcal{U}(n)$ such that $U^*K_1(\boldsymbol{z}, \boldsymbol{w})U = K_2(\boldsymbol{z}, \boldsymbol{w})$ for all $\boldsymbol{z}, \boldsymbol{w} \in \Omega$.
(3) Suppose that $\mathbb{C}^n \otimes \mathcal{P}$ is densely contained in $\mathcal{H}_K(\Omega, \mathbb{C}^n)$ and that the multiplication by the coordinate functions are bounded on $\mathcal{H}_K(\Omega, \mathbb{C}^n)$. Then
$$\cap_{i=1}^n \ker(M_i - w_i)^* = \{K(\cdot, \boldsymbol{w})\boldsymbol{x} : \boldsymbol{x} \in \mathbb{C}^n\}.$$
Moreover, the dimension of the joint kernel at $\boldsymbol{w}$ is n for all $\boldsymbol{w} \in \Omega$.

Lemma 2.3. *Let $\mathcal{H}_K(\Omega, \mathbb{C}^n)$ be a reproducing kernel Hilbert space consisting of holomorphic functions on Ω taking values in $\mathbb{C}^n$. Assume that $\mathbb{C}^n \otimes \mathcal{P}$ is densely contained in $\mathcal{H}_K(\Omega, \mathbb{C}^n)$, the d-tuple $\boldsymbol{M}$ on $\mathcal{H}_K(\Omega, \mathbb{C}^n)$ is bounded and the kernel K is normalized at 0. Then the following statements are equivalent.*

(1) The d-tuple $\boldsymbol{M}$ is $\mathbb{K}$-homogeneous, that is, there is a unitary operator $\Gamma(k)$ on $\mathcal{H}_K(\Omega, \mathbb{C}^n)$ with
$$\Gamma(k)(k \cdot \boldsymbol{M})\Gamma(k)^* = \boldsymbol{M}, \ k \in \mathbb{K}.$$

(2) *The kernel K is quasi-invariant under $\mathbb{K}$ with multiplier $c : \mathbb{K} \times \Omega \to \mathcal{U}(n)$, $c(k, \boldsymbol{z})$ is independent of $\boldsymbol{z}$.*
(3) *There is a map $c : \mathbb{K} \to \mathcal{U}(n)$ such that $(\Gamma(k)f)(\boldsymbol{z}) := c(k)f(k^{-1} \cdot \boldsymbol{z})$, is unitary on $\mathcal{H}_K(\Omega, \mathbb{C}^n)$.*

Proof. Since $\mathbb{C}^n \otimes \mathcal{P}$ is densely contained in $\mathcal{H}_K(\Omega, \mathbb{C}^n)$, it follows that the dimension of the joint kernels $\cap_{i=1}^d \ker D_{(\boldsymbol{M}-w)^*}$, $w \in \Omega$, as shown in [7, p. 285, Remark], is n. Therefore, the methods of [6] applies.

First, it is not hard to see that the d-tuple of operators $k \cdot \boldsymbol{M}$ acting on the Hilbert space $\mathcal{H}_K(\Omega, \mathbb{C}^n)$ is unitarily equivalent to the d-tuple $\boldsymbol{M}$ acting on $\mathcal{H}_{\hat{K}}(\Omega, \mathbb{C}^n)$, where $\hat{K}(\boldsymbol{z}, \boldsymbol{w}) := K(k^{-1} \cdot \boldsymbol{z}, k^{-1} \cdot \boldsymbol{w})$ via the unitary operator $f \to f \circ k^{-1}$, $f \in \mathcal{H}_K(\Omega, \mathbb{C}^n)$. Since K is assumed to be normalized at 0 and k is linear, it follows that $\hat{K}$ is also normalized at 0. Second, for a fixed $k \in \mathbb{K}$, any intertwining unitary operator between the d-tuple $\boldsymbol{M}$ on $\mathcal{H}_{\hat{K}}(\Omega, \mathbb{C}^n)$ and $\mathcal{H}_K(\Omega, \mathbb{C}^n)$ must be of the form $\hat{f} \to c(k)\hat{f}$, where $(c(k)\hat{f})(\boldsymbol{z}) = c(k)\hat{f}(\boldsymbol{z})$ for some unitary $c(k) \in \mathcal{U}(n)$. Finally, these two unitaries combine to give a unitary operator $\Gamma(k) : \mathcal{H}_K(\Omega, \mathbb{C}^n) \to \mathcal{H}_K(\Omega, \mathbb{C}^n)$ of the form: $\Gamma(k)f(\boldsymbol{z}) = c(k)(f \circ k^{-1})(\boldsymbol{z})$. Thus we have proved that the statement (1) implies (3).

Moreover, the unitarity of the map Γ in the statement (3) is equivalent to the quasi-invariance of the kernel K, namely, $K(\boldsymbol{z}, \boldsymbol{w}) = c(k)K(k^{-1} \cdot \boldsymbol{z}, k^{-1} \cdot \boldsymbol{w})c(k)^*$. This proves the equivalence of the statements (2) and (3).

The statement (3) clearly implies (1) completing the proof. □

Remark 2.4. Choosing the multiplier $c : \mathbb{K} \to \mathrm{GL}_n(\mathbb{C})$ to be unitary without loss of generality and assuming that c is a homomorphism, we see that the map $f \to c(k)(f \circ k^{-1})$ is a unitary representation of $\mathbb{K}$ on the Hilbert space $\mathcal{H}_K(\Omega, \mathbb{C}^n)$.

The group $\mathbb{K}$ acts on $\mathcal{P}$ naturally by the rule $p \to p \circ k^{-1}$. This action, as is well known, decomposes into irreducible components $\mathcal{P}_{\underline{s}}$ parameterized by the signatures $\underline{s}$ in $\vec{\mathbb{Z}}_+^r$. It is pointed out in [1, Proposition 3.4], that any $\mathbb{K}$-invariant inner product on $\mathcal{P}$ must be of the form

$$\langle p, q \rangle = \sum_{\ell=0}^{\deg\, p} \sum_{\substack{|\underline{s}|=\ell \\ \underline{s} \in \vec{\mathbb{Z}}_+^r}} a_{\underline{s}} \langle p_{\underline{s}}, q_{\underline{s}} \rangle_{\mathcal{F}},$$

where $\deg p$ is the degree of p and $p_{\underline{s}}$, $q_{\underline{s}}$ are the components of $p, q \in \mathcal{P}$ in the Peter-Weyl decomposition of $\mathcal{P}$ into irreducible subspaces $\mathcal{P}_{\underline{s}}$. In this paper, what we study amounts to finding $\mathbb{K}$ quasi-invariant inner products on the space $\mathbb{C}^n \otimes \mathcal{P}$. We do this by obtaining a generalization of the description of an invariant inner product from the scalar case given above. This is Proposition 2.6. For the proof, we need the following elementary lemma (compare with Lemma 2.8 of [5]).

Lemma 2.5. *Let $\mathcal{H}_1 := (\mathcal{H}, \langle \cdot, \cdot \rangle_1)$ and $\mathcal{H}_2 := (\mathcal{H}, \langle \cdot, \cdot \rangle_2)$ be two Hilbert spaces. Let $\rho : \mathbb{K} \to \mathcal{U}(\mathcal{H}_i)$ be an irreducible unitary representation for $i = 1, 2$. Then there exists a positive scalar δ such that $\langle \cdot, \cdot \rangle_1 = \delta \langle \cdot, \cdot \rangle_2$.*

Proof. Let A be the linear map from $\mathcal{H}$ to $\mathcal{H}$ such that $\langle f, g \rangle_{\mathcal{H}_1} = \langle Af, g \rangle_{\mathcal{H}_2}$. Now, note that,

$$\begin{aligned} \langle \rho(k)Af, g \rangle_{\mathcal{H}_2} &= \langle Af, \rho(k^{-1})g \rangle_{\mathcal{H}_2} \\ &= \langle f, \rho(k^{-1})g \rangle_{\mathcal{H}_1} \\ &= \langle \rho(k)f, g \rangle_{\mathcal{H}_1} \\ &= \langle A\rho(k)f, g \rangle_{\mathcal{H}_2} \end{aligned}$$

Thus it follows that $\rho(k)A = A\rho(k)$. An application of Schur's lemma completes the proof. □

Let π be a unitary representation of the compact group $\mathbb{K}$ on a Hilbert space $\mathcal{H}$ containing $\mathbb{C}^n \otimes \mathcal{P}$ as a dense subspace. By the Peter-Weyl theorem, $\mathcal{H}$ is the direct sum of irreducible representations of $\mathbb{K}$ acting on finite dimensional subspaces $\mathcal{H}_\lambda$, $\lambda \in \Lambda$. Let π_λ be the restriction of π to $\mathcal{H}_\lambda$, that is,

$\pi = \oplus_{\lambda\in\Lambda}\pi_\lambda$ is the Peter-Weyl decomposition relative to the direct sum $\mathcal{H} = \oplus_{\lambda\in\Lambda}\mathcal{H}_\lambda$ into reducing subspaces of π. For the complete statement of the Peter-Weyl theorem one may consult [12, Theorem, 1.12, p. 17].

Let us transplant the Fischer-Fock inner product on $\mathcal{P}$ and the Euclidean inner product on $\mathbb{C}^n$ to the tensor product $\mathbb{C}^n \otimes \mathcal{P}$. We let $\langle\cdot,\cdot\rangle_{\mathcal{F}}$ denote the inner product on this tensor product space by a slight abuse of notation. Let P_λ be the linear subspace of $\mathbb{C}^n \otimes \mathcal{P}$ identified with $\mathcal{H}_\lambda$. Now, each of the subspaces $P_\lambda \subset \mathbb{C}^n \otimes \mathcal{P}$ inherits the inner product from that of $(\mathbb{C}^n \otimes \mathcal{P}, \langle\cdot,\cdot\rangle_{\mathcal{F}})$ to be denoted by $(P_\lambda, \langle\cdot,\cdot\rangle_{\mathcal{F}_\lambda})$, $\lambda \in \Lambda$. The hypothesis in the following proposition might appear to be restrictive but for the applications in this paper, they appear naturally.

Proposition 2.6. *Fix an action π of the compact group $\mathbb{K}$ on a Hilbert space $\mathcal{H}$. Let $[\cdot,\cdot]$ denote the inner product of $\mathcal{H}$. Assume that $\mathbb{C}^n \otimes \mathcal{P}$ is a dense subspace of $\mathcal{H}$. Furthermore, we assume that (a) $[p,q] = [\pi(k)p, \pi(k)q]$, that is, π is a unitary representation of $\mathbb{K}$ on $\mathcal{H}$ (b) $\langle p_\lambda, q_\lambda\rangle_{\mathcal{F}_\lambda} = \langle \pi_\lambda(k)p_\lambda, \pi_\lambda(k)q_\lambda\rangle_{\mathcal{F}_\lambda}$, $k \in \mathbb{K}$, (c) π_λ and $\pi_{\lambda'}$ are inequivalent whenever $\lambda \neq \lambda'$. Then there exists positive scalars a_λ such that $[p,q] = \sum_{\lambda\in\Lambda} a_\lambda \langle p_\lambda, q_\lambda\rangle_{\mathcal{F}_\lambda}$, where $p = \sum_{\lambda\in\Lambda} p_\lambda$ and $q = \sum_{\lambda\in\Lambda} q_\lambda$, $p, q \in \mathbb{C}^n \otimes \mathcal{P}$.*

Proof. Let $p, q \in \mathbb{C}^n \otimes \mathcal{P}$ be of the form $\sum_{\lambda\in\Lambda} p_\lambda$, $p_\lambda \in P_\lambda$, and $\sum_{\lambda\in\Lambda} q_\lambda$, $q_\lambda \in P_\lambda$, respectively. For any pair $\lambda \neq \lambda'$, by hypothesis, π_λ and $\pi_{\lambda'}$ are inequivalent, therefore the subspaces P_λ and $P_{\lambda'}$ of the inner product space $(\mathbb{C}^n \otimes \mathcal{P}, [\cdot,\cdot])$ are orthogonal. Therefore, we have

$$[p,q] = \sum_{\lambda\in\Lambda} [p_\lambda, q_\lambda].$$

The representation π_λ of $\mathbb{K}$ on $(P_\lambda, [\cdot,\cdot])$ is unitary and irreducible. It is also unitary and irreducible on the space $(P_\lambda, \langle\cdot,\cdot\rangle_{\mathcal{F}_\lambda})$. The proof of the theorem is completed by applying Lemma 2.5. ☐

As an application of Proposition 2.6, we obtain a description of all the quasi-invariant kernels K with a multiplier c assuming that c is a unitary representation of $\mathbb{K}$.

Theorem 2.7. *Let $\mathcal{H}_K(\Omega, \mathbb{C}^n)$ be a reproducing kernel Hilbert space densely containing $\mathbb{C}^n \otimes \mathcal{P}$ as subspace. Assume that K is quasi-invariant with multiplier c, where $c : \mathbb{K} \to \mathcal{U}(n)$ is a representation of the group $\mathbb{K}$. Let π denote the action of the group $\mathbb{K}$ on $\mathcal{H}_K(\Omega, \mathbb{C}^n)$ given by the rule $\pi(k)f = c(k)(f \circ k^{-1})$. In the Peter-Weyl decomposition $\pi = \oplus_{\lambda\in\Lambda}\pi_\lambda$, assume that the unitary representations π_λ are inequivalent. Then there exists positive scalars $b_\lambda, \lambda \in \Lambda$, such that*

$$K(\boldsymbol{z}, \boldsymbol{w}) = \sum_{\lambda\in\Lambda} b_\lambda K_\lambda(\boldsymbol{z}, \boldsymbol{w}),\ \boldsymbol{z}, \boldsymbol{w} \in \Omega,$$

where K_λ is the reproducing kernel of $(P_\lambda, \langle\cdot,\cdot\rangle_{\mathcal{F}_\lambda})$, and $\mathcal{H}_K(\Omega, \mathbb{C}^n) = \oplus_{\lambda\in\Lambda} P_\lambda$.

Proof. From Lemma 2.3, it follows that the action π of the group $\mathbb{K}$ on $\mathcal{H}_K(\Omega, \mathbb{C}^n)$ is unitary. This verifies the assumption (a) of Proposition 2.6. The inner product space $(\mathbb{C}^n \otimes \mathcal{P}, \langle\cdot,\cdot\rangle_{\mathcal{F}})$ is the tensor product $(\mathbb{C}^n, \langle\cdot,\cdot\rangle_2) \otimes (\mathcal{P}_\lambda, \langle\cdot,\cdot\rangle_\lambda)$. Consequently, since $c(k)$ is unitary for each $k \in \mathbb{K}$ verifying assumption (b) of Proposition 2.6. Finally, the assumption that π_λ, $\lambda \in \Lambda$, are inequivalent is the assumption (c) of Proposition 2.6. Therefore the proof is completed by applying Proposition 2.6. ☐

We show that a non-scalar kernel K, quasi-invariant under $\mathcal{U}(d)$ associated with a multiplier c that is irreducible, can not be invariant.

Proposition 2.8. *Let $K : \Omega \times \Omega \to \mathcal{M}_n(\mathbb{C})$ be a non-negative definite kernel. Suppose that $c : \mathbb{K} \to \mathcal{M}_n(\mathbb{C})$ is an irreducible unitary representation and K is quasi-invariant under $\mathbb{K}$ with multiplier c. If the kernel K is also invariant under $\mathbb{K}$, then there exists a non-negative definite scalar valued kernel κ on $\Omega \times \Omega$ invariant under $\mathbb{K}$ such that $K(\boldsymbol{z}, \boldsymbol{w}) = \kappa(\boldsymbol{z}, \boldsymbol{w}) I_n$, $\boldsymbol{z}, \boldsymbol{w} \in \Omega$.*

Proof. Suppose that K is quasi-invariant with multiplier $c : \mathbb{K} \to \mathcal{M}_n(\mathbb{C})$, that is,

$$K(\boldsymbol{z}, \boldsymbol{w}) = c(k) K(k^{-1} \cdot \boldsymbol{z}, k^{-1} \cdot \boldsymbol{w}) c(k)^*,\ k \in \mathbb{K},\ \boldsymbol{z}, \boldsymbol{w} \in \Omega,$$

where c is an irreducible unitary representation. If the kernel K is also invariant under $\mathbb{K}$, it follows that, $K(\boldsymbol{z},\boldsymbol{w}) = c(k)K(\boldsymbol{z},\boldsymbol{w})c(k)^*$, that is, $K(\boldsymbol{z},\boldsymbol{w})c(k) = c(k)K(\boldsymbol{z},\boldsymbol{w})$ for all $k \in \mathbb{K}$. By Schur's Lemma, $K(\boldsymbol{z},\boldsymbol{w}) = \kappa(\boldsymbol{z},\boldsymbol{w})I_n$ for some scalar $\kappa(\boldsymbol{z},\boldsymbol{w})$. The kernel $K(\boldsymbol{z},\boldsymbol{w})$ is non-negative definite, therefore $\kappa(\boldsymbol{z},\boldsymbol{w})$ is non-negative definite also. Moreover, since $K(\boldsymbol{z},\boldsymbol{w})$ is invariant under $\mathbb{K}$, it follows that $\kappa(\boldsymbol{z},\boldsymbol{w})$ is invariant under $\mathbb{K}$ as well. This completes the proof. $\square$

Remark 2.9. As we have pointed out earlier, under some additional assumptions, any scalar-valued non-negative definite kernel K on $\Omega \times \Omega$ quasi-invariant under $\mathbb{K}$ can be shown to be of the form $\sum_{\underline{s}\in\vec{\mathbb{Z}}_+^r} a_{\underline{s}}E_{\underline{s}}$ for some sequence $\{a_{\underline{s}}\}_{\underline{s}\in\vec{\mathbb{Z}}_+^r}$ of non-negative real numbers.

3. A class of quasi-invariant kernels

Let $(\mathcal{P}, \langle\cdot,\cdot\rangle_{\mathcal{F}})$ denote the linear space of all polynomials in d-variables equipped with the Fischer-Fock inner product and let $(\mathbb{C}^d, \langle\cdot,\,\cdot\rangle_2)$ denote the Euclidean inner product space. We have

$$(\mathbb{C}^d, \langle\cdot,\,\cdot\rangle_2) \otimes (\mathcal{P}, \langle\cdot,\cdot\rangle_{\mathcal{F}}) = \bigoplus_{\ell=0}^{\infty}(\mathbb{C}^d \otimes \mathcal{P}_\ell, \langle\cdot,\,\cdot\rangle_{\mathcal{F}_\ell}),$$

where the linear space $(\mathbb{C}^d \otimes \mathcal{P}_\ell, \langle\cdot,\,\cdot\rangle_{\mathcal{F}_\ell})$ denotes the subspace of $(\mathbb{C}^d, \langle\cdot,\,\cdot\rangle_2) \otimes (\mathcal{P}, \langle\cdot,\cdot\rangle_{\mathcal{F}})$ consisting of homogeneous polynomials of degree ℓ. Thus the reproducing kernel of $(\mathbb{C}^d \otimes \mathcal{P}_\ell, \langle\cdot,\,\cdot\rangle_{\mathcal{F}_\ell})$ is of the form $\frac{\langle z,w\rangle^\ell}{\ell!}I_d$.

Recall that the unitary group $\mathcal{U}(d)$ acts on $\mathcal{P}$ by $(\pi(u)(p))(\boldsymbol{z}) = p(u^{-1}\cdot\boldsymbol{z})$, $p \in \mathcal{P}$. Therefore, the map given by the formula:

$$\big(\hat{\pi}(u)(p)\big)(\boldsymbol{z}) := u(p(u^{-1}\cdot\boldsymbol{z})),\ p \in \mathbb{C}^d \otimes \mathcal{P},\ u \in \mathcal{U}(d) \tag{3.1}$$

is an unitary homomorphism. Let $\pi_\ell(u)$ denote the restriction of $\pi(u)$ to $\mathcal{P}_\ell$ and $\hat{\pi}_\ell(u)$ be the restriction of $\hat{\pi}(u)$ to $\mathbb{C}^d \otimes \mathcal{P}_\ell$. Evidently, the subspaces $\mathbb{C}^d \otimes \mathcal{P}_\ell$, $\ell \in \mathbb{Z}_+$, are not only invariant under the action $\hat{\pi}$ of $\mathcal{U}(d)$ but also the restriction of $\hat{\pi}_\ell$ to these subspaces is unitary.

There is a second action $\tilde{\pi}$ of the unitary group $\mathcal{U}(d)$ on $\mathbb{C}^d \otimes \mathcal{P}$ given by the formula:

$$\big(\tilde{\pi}(u)(p)\big)(\boldsymbol{z}) = \overline{u}(p(u^{-1}\cdot\boldsymbol{z})),\ p \in \mathbb{C}^d \otimes \mathcal{P}. \tag{3.2}$$

Like before, the restriction $\tilde{\pi}_\ell(u)$ of $\tilde{\pi}(u)$ to the space $\mathbb{C}^d \otimes \mathcal{P}_\ell$ is unitary.

3.1. **Decomposition of $\tilde{\pi}_\ell$.** Let $\boldsymbol{A} = (A_1,\ldots,A_n)$ be an n-tuple of bounded linear operators (not necessarily commuting) $A_i : \mathcal{H}_1 \to \mathcal{H}_2$, $1 \le i \le n$, where the Hilbert space $\mathcal{H}_1$ is possibly different from $\mathcal{H}_2$. The operators $D_{\boldsymbol{A}} : \mathcal{H}_1 \to \mathcal{H}_2 \oplus\cdots\oplus \mathcal{H}_2$ and $D^{\boldsymbol{A}} : \mathcal{H}_1 \oplus\cdots\oplus \mathcal{H}_1 \to \mathcal{H}_2$ are defined by the rule

$$D_{\boldsymbol{A}}(h) = (A_1h,\ldots,A_nh),\ h \in \mathcal{H}_1 \text{ and}$$
$$D^{\boldsymbol{A}}\begin{pmatrix} h_1\\ \vdots\\ h_n\end{pmatrix} = A_1h_1 + \cdots + A_nh_n,\ h_1,\ldots,h_n \in \mathcal{H}_1.$$

It is easy to verify that $(D^{\boldsymbol{A}})^* = D_{\boldsymbol{A}^*}$.

For any $u \in \mathcal{U}(d)$, $f = \begin{pmatrix} f_1\\ \vdots\\ f_d\end{pmatrix} \in \mathbb{C}^d \otimes \mathcal{P}_\ell$ and $\boldsymbol{z} \in \mathbb{C}^d$, we have

$$\sum_{i=1}^d z_i(u^\dagger(f\circ u))_i(\boldsymbol{z}) = \langle u^\dagger(f\circ u)(\boldsymbol{z}),\, \overline{\boldsymbol{z}}\rangle_{\mathbb{C}^d} = \langle (f\circ u)(\boldsymbol{z}),\, \overline{u\cdot\boldsymbol{z}}\rangle_{\mathbb{C}^d} = \sum_{i=1}^d (u\cdot\boldsymbol{z})_i f_i(u\cdot\boldsymbol{z}).$$

Thus, $\tilde{\pi}$ leaves the subspace $\tilde{\mathcal{V}}_\ell \subseteq (\mathbb{C}^d \otimes \mathcal{P}_\ell, \langle\cdot,\,\cdot\rangle_{\mathcal{F}_\ell})$ invariant, where

$$\tilde{\mathcal{V}}_\ell = \left\{\begin{pmatrix} f_1\\ \vdots\\ f_d\end{pmatrix} \in \mathbb{C}^d \otimes \mathcal{P}_\ell : z_1f_1 + \cdots + z_df_d = 0\right\}.$$

We claim that the subspace $\tilde{\mathcal{V}}_\ell^\perp \subseteq (\mathbb{C}^d \otimes \mathcal{P}_\ell, \langle\cdot,\,\cdot\rangle_{\mathcal{F}_\ell})$ is $\left\{ \begin{pmatrix} \partial_1 g \\ \vdots \\ \partial_d g \end{pmatrix} : g \in \mathcal{P}_{\ell+1} \right\}$.

To verify the claim, let $M_{z_i}^{(\ell)} : \mathcal{P}_\ell \to \mathcal{P}_{\ell+1}$ be the linear map $M_{z_i}^{(\ell)}(p) = z_i p,\ p \in \mathcal{P}_\ell$. Setting $\boldsymbol{M}^{(\ell)} = (M_{z_1}^{(\ell)}, \ldots, M_{z_d}^{(\ell)})$, we have $\tilde{\mathcal{V}}_\ell = \ker D^{\boldsymbol{M}^{(\ell)}}$. Thus $\tilde{\mathcal{V}}_\ell^\perp = \operatorname{ran}\,(D^{\boldsymbol{M}^{(\ell)}})^* = \operatorname{ran}\, D_{\boldsymbol{M}^{(\ell)*}}$. From the identity $\langle p,\, z_i q\rangle_{\mathcal{F}} = \langle \partial_i p,\, q\rangle_{\mathcal{F}}$ for any pair of polynomials proved in [18], Proposition 4.11.36, it follows that $M_{z_i}^{(\ell)^*} = \partial_i$ completing the verification of the claim.

Lemma 3.1. *The reproducing kernel $\tilde{K}_\ell$ of the inner product space $\tilde{\mathcal{V}}_\ell$ is given by the formula:*

$$\tilde{K}_\ell(\boldsymbol{z}, \boldsymbol{w}) = \frac{\ell}{(\ell+1)\ell!}\langle \boldsymbol{z},\, \boldsymbol{w}\rangle^{\ell-1}\left(\langle \boldsymbol{z},\, \boldsymbol{w}\rangle I_d - \overline{\boldsymbol{w}}\boldsymbol{z}^\dagger\right).$$

The reproducing kernel $\tilde{K}_\ell^\perp$ of $\tilde{\mathcal{V}}_\ell^\perp$ is given by the formula:

$$\tilde{K}_\ell^\perp(\boldsymbol{z}, \boldsymbol{w}) = \frac{\langle \boldsymbol{z},\, \boldsymbol{w}\rangle^{\ell-1}}{(\ell+1)\ell!}\left(\langle \boldsymbol{z},\, \boldsymbol{w}\rangle I_d + \overline{\boldsymbol{w}}\boldsymbol{z}^\dagger\right).$$

Here, $\overline{\boldsymbol{w}}\boldsymbol{z}^\dagger$ is the matrix product of the column vector $\overline{\boldsymbol{w}}$ and the row vector $\boldsymbol{z}^\dagger$.

Proof. Let $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_d)$ be an arbitrary vector in $\mathbb{C}^d$. First note that

$$\begin{aligned}
\sum_{i=1}^d z_i \langle \tilde{K}_\ell(\boldsymbol{z}, \boldsymbol{w})\boldsymbol{\zeta},\, \boldsymbol{e}_i\rangle &= \frac{\ell}{(\ell+1)\ell!}\langle \boldsymbol{z},\, \boldsymbol{w}\rangle^{\ell-1}\left(\sum_{i=1}^d z_i \big\langle \langle \boldsymbol{z},\, \boldsymbol{w}\rangle\boldsymbol{\zeta} - \overline{\boldsymbol{w}}\langle \boldsymbol{z},\, \bar{\boldsymbol{\zeta}}\rangle, \boldsymbol{e}_i\big\rangle\right) \\
&= \frac{\ell}{(\ell+1)\ell!}\langle \boldsymbol{z},\, \boldsymbol{w}\rangle^{\ell-1}\sum_{i=1}^d \left(\langle \boldsymbol{z},\, \boldsymbol{w}\rangle z_i\zeta_i - z_i\bar{w}_i\langle \boldsymbol{z},\, \bar{\boldsymbol{\zeta}}\rangle\right) \\
&= \frac{\ell}{(\ell+1)\ell!}\langle \boldsymbol{z},\, \boldsymbol{w}\rangle^{\ell-1}\left(\langle \boldsymbol{z},\, \boldsymbol{w}\rangle\langle \boldsymbol{z},\, \bar{\boldsymbol{\zeta}}\rangle - \langle \boldsymbol{z},\, \boldsymbol{w}\rangle\langle \boldsymbol{z},\, \bar{\boldsymbol{\zeta}}\rangle\right) \\
&= 0.
\end{aligned}$$

It follows that the vector $\tilde{K}_\ell(\cdot, \boldsymbol{w})\boldsymbol{\zeta} \in \tilde{\mathcal{V}}_\ell$. In order to complete the proof of the first part it suffices to show that for any f in $\tilde{\mathcal{V}}_\ell$, $\boldsymbol{w}, \boldsymbol{\zeta} \in \mathbb{C}^d$, and $i = 1, \ldots, d$ $\langle f,\, \tilde{K}_\ell(\cdot, \boldsymbol{w})\boldsymbol{e}_i\rangle_{\mathcal{F}_\ell} = \langle f(\boldsymbol{w}),\, \boldsymbol{e}_i\rangle_{\mathbb{C}^d}$. Note that

$$\begin{aligned}
\langle f,\, \langle \boldsymbol{z},\, \boldsymbol{w}\rangle^{\ell-1}\overline{\boldsymbol{w}}\boldsymbol{z}^\dagger\boldsymbol{e}_i\rangle_{\mathcal{F}_\ell} &= \sum_{j=1}^d \langle f_j,\, \langle \boldsymbol{z},\, \boldsymbol{w}\rangle^{\ell-1}\overline{w}_j z_i\rangle_{\mathcal{F}} \\
&= \sum_{j=1}^d w_j\langle \partial_i f_j,\, \langle \boldsymbol{z},\, \boldsymbol{w}\rangle^{\ell-1}\rangle_{\mathcal{F}} \\
&= (\ell-1)!\sum_{j=1}^d w_j(\partial_i f_j)(\boldsymbol{w}) \\
&= (\ell-1)!\Big(\partial_i\Big(\sum_{j=1}^d z_j f_j\Big)(\boldsymbol{w}) - f_i(\boldsymbol{w})\Big) \\
&= -(\ell-1)! f_i(\boldsymbol{w}).
\end{aligned}$$

Hence we have

$$\langle f,\, \langle \boldsymbol{z},\, \boldsymbol{w}\rangle^{\ell-1}\overline{\boldsymbol{w}}\boldsymbol{z}^\dagger\boldsymbol{e}_i\rangle_{\mathcal{F}_\ell} = -(\ell-1)!\langle f(\boldsymbol{w}),\, \boldsymbol{e}_i\rangle_{\mathbb{C}^d}. \tag{3.3}$$

Here the second equality follows since $\langle p,\, z_i q\rangle_{\mathcal{F}} = \langle \partial_i p,\, q\rangle_{\mathcal{F}}$ for any pair of polynomials p, q (see [18, Proposition 4.11.36]), and the third equality from the reproducing property of the kernel function of

$\mathcal{P}_{\ell-1}$. Now, using (3.3), we see that

$$\begin{aligned}\langle f,\, \tilde{K}_\ell(\cdot,\boldsymbol{w})\boldsymbol{e}_i\rangle_{\mathcal{F}_\ell} &= \frac{\ell}{(\ell+1)\ell!}\langle f,\, \langle \boldsymbol{z},\, \boldsymbol{w}\rangle^{\ell-1}\Big(\langle \boldsymbol{z},\, \boldsymbol{w}\rangle e_i - \overline{\boldsymbol{w}}\boldsymbol{z}^\dagger \boldsymbol{e}_i\Big)\rangle_{\mathcal{F}_\ell}\\ &= \frac{\ell}{(\ell+1)}\Big(1+\frac{1}{\ell}\Big)\langle f(\boldsymbol{w}),\, \boldsymbol{e}_i\rangle\\ &= \langle f(\boldsymbol{w}),\, \boldsymbol{e}_i\rangle.\end{aligned}$$

This verifies the formula for $\tilde{K}_\ell$.

We note that the reproducing kernel of $(\mathbb{C}^d\otimes\mathcal{P}_\ell, \langle\cdot,\,\cdot\rangle_{\mathcal{F}_\ell})$ is $\frac{\langle \boldsymbol{z},\boldsymbol{w}\rangle^\ell}{\ell!}I_d$. Now, the verification of the formula for $\tilde{K}_\ell^\perp$ follows from part (1) and the equality:

$$\frac{\langle \boldsymbol{z},\, \boldsymbol{w}\rangle^\ell}{\ell!}I_d = \tilde{K}_\ell(\boldsymbol{z},\boldsymbol{w}) + \tilde{K}_\ell^\perp(\boldsymbol{z},\boldsymbol{w}),$$

which follows from general theory of reproducing kernel Hilbert spaces. ☐

The proof of Proposition 3.6 giving an explicit description of a quasi-invariant kernel under $\mathcal{U}(d)$ transforming as in Definition (1.3) with $c(u)=\bar{u}$ is facilitated by the set of three lemmas proved below.

Lemma 3.2. *Let A be a $d\times d$ complex matrix such that $uA = Au$ for all unitary matrices u with $u(\boldsymbol{e}_1)=\boldsymbol{e}_1$. Then A is of the form $\begin{pmatrix} a_1 & 0\\ 0 & a_2 I_{d-1}\end{pmatrix}$ for some complex numbers a_1 and a_2.*

Proof. Let $A=\begin{pmatrix} A_1 & A_3^\dagger\\ A_4 & A_2\end{pmatrix}$, where A_3 and A_4 are column vectors in $\mathbb{C}^{d-1}$ and A_2 is in $\mathcal{M}_{d-1}(\mathbb{C})$. By hypothesis, we get $A_3=A_4=0$ and $vA_2=A_2v$ for all $v\in\mathcal{U}(d-1)$. Now the conclusion follows by an application of the Schur's lemma. ☐

Lemma 3.3. *Suppose that $K:\mathbb{B}_d\times\mathbb{B}_d\to\mathcal{M}_n(\mathbb{C})$ is a sesqui-analytic Hermitian function satisfying the rule $K(\lambda\cdot\boldsymbol{z},\lambda\cdot\boldsymbol{w})=K(\boldsymbol{z},\boldsymbol{w})$ for all λ on the unit circle $\mathbb{T}$. Then $K(\boldsymbol{z},\boldsymbol{w})$ is of the form*

$$\sum_{\ell=0}^\infty \sum_{\substack{\alpha,\beta\in\mathbb{Z}_+^d\\ |\alpha|=|\beta|=\ell}} A_{\alpha,\beta}\boldsymbol{z}^\alpha\overline{\boldsymbol{w}}^\beta, \boldsymbol{z},\boldsymbol{w}\in\mathbb{B}_d,$$

where $A_{\alpha,\beta}$ are $n\times n$ complex matrices.

Proof. Let $K(\boldsymbol{z},\boldsymbol{w})=\sum_{\alpha,\beta\in\mathbb{Z}_+^d}A_{\alpha,\beta}\boldsymbol{z}^\alpha\overline{\boldsymbol{w}}^\beta$, $\boldsymbol{z},,\boldsymbol{w}\in\mathbb{B}_d$. By hypothesis, we have

$$\sum_{\alpha,\beta\in\mathbb{Z}_+^d}A_{\alpha,\beta}\boldsymbol{z}^\alpha\overline{\boldsymbol{w}}^\beta = \sum_{\alpha,\beta\in\mathbb{Z}_+^d}A_{\alpha,\beta}\lambda^{|\alpha|-|\beta|}\boldsymbol{z}^\alpha\overline{\boldsymbol{w}}^\beta,\ \ \boldsymbol{z},,\boldsymbol{w}\in\mathbb{B}_d,\ \ \lambda\in\mathbb{T}.$$

Comparing coefficients in both sides, we get $A_{\alpha,\beta}(1-\lambda^{|\alpha|-|\beta|})=0$ for all $\lambda\in\mathbb{T}$. Hence it follows that $A_{\alpha,\beta}=0$ if $|\alpha|\neq|\beta|$. This completes the proof. ☐

For any $\boldsymbol{z}\in\mathbb{B}_d$, $\|\boldsymbol{z}\|=r$, there is a $u_{\boldsymbol{z}}\in\mathcal{U}(d)$ with the property: $u_{\boldsymbol{z}}(\boldsymbol{z})=r\boldsymbol{e}_1$. The unitary $u_{\boldsymbol{z}}$ can be determined explicitly, namely, $u_{\boldsymbol{z}}^*=\big(\,\frac{\boldsymbol{z}}{r}|\,\bigstar\big)$, where $\boldsymbol{z}$ is the column vector with components $z_1,\ldots,z_d$. For any choice of two sets of complex numbers, $\{a_{m,1}:m\in\mathbb{Z}_+\}$ and $\{a_{m,2}:m\in\mathbb{Z}_+\}$ with $a_{0,1}=a_{0,2}$, set

$$D_i(r,r):=\sum_{m=0}^\infty a_{m,i}r^{2m}, r\in[0,1), i=1,2.$$

Also, for any fixed $z\in\mathbb{B}_d$ with $\|z\|=r$, let $\mathcal{U}_{\boldsymbol{z}}$ be the set $\{u_{\boldsymbol{z}}\in\mathcal{U}(d):u_{\boldsymbol{z}}(\boldsymbol{z})=\|\boldsymbol{z}\|\boldsymbol{e}_1\}$.

Lemma 3.4. *For any $u_{\boldsymbol{z}} \in \mathcal{U}_{\boldsymbol{z}}$, we have*

$$u_{\boldsymbol{z}}^{\dagger}\begin{pmatrix} D_1(r,r) & 0 \\ 0 & D_2(r,r)I_{d-1}\end{pmatrix}\overline{u_{\boldsymbol{z}}} = \big(D_1(r,r)-D_2(r,r)\big)\frac{\overline{\boldsymbol{z}}\boldsymbol{z}^{\dagger}}{r^2} + D_2(r,r)I_d.$$

Proof. For any $u_{\boldsymbol{z}} \in \mathcal{U}_{\boldsymbol{z}}$, we have

$$\begin{aligned} u_{\boldsymbol{z}}^{\dagger}\left(\begin{smallmatrix} D_1(r,r) & 0 \\ 0 & D_2(r,r)I_{d-1}\end{smallmatrix}\right)\overline{u_{\boldsymbol{z}}} &= u_{\boldsymbol{z}}^{\dagger}\left(\begin{smallmatrix} D_1(r,r)-D_2(r,r) & 0 \\ 0 & 0\end{smallmatrix}\right)\overline{u_{\boldsymbol{z}}} + u_{\boldsymbol{z}}^{\dagger}D_2(r,r)I_d\overline{u_{\boldsymbol{z}}} \\ &= D_1(r,r)-D_2(r,r)u_{\boldsymbol{z}}^{\dagger}E_{11}\overline{u_{\boldsymbol{z}}} + D_2(r,r)I_d u_{\boldsymbol{z}}^{\dagger}\overline{u_{\boldsymbol{z}}}\end{aligned}$$

Since $u_{\boldsymbol{z}}(\boldsymbol{z}) = \|\boldsymbol{z}\|e_1$, we get that $u_{\boldsymbol{z}}^{\dagger}e_1 = \frac{\bar{z}}{r}$. Thus,

$$u_{\boldsymbol{z}}^{\dagger}E_{11}\overline{u_{\boldsymbol{z}}} = u_{\boldsymbol{z}}^{\dagger}e_1\overline{e_1}^{\dagger}\overline{u_{\boldsymbol{z}}} = \frac{\bar{\boldsymbol{z}}\boldsymbol{z}^{\dagger}}{r^2}.$$

This completes the proof. □

Remark 3.5. An unitary $u_{\boldsymbol{z}} \in \mathcal{U}_{\boldsymbol{z}}$ such that $u_{\boldsymbol{z}}(\boldsymbol{z}) = \|\boldsymbol{z}\|e_1$ is not uniquely determined. However, if $\boldsymbol{z} \neq 0$, we see that

$$u_{\boldsymbol{z}}^{\dagger}\begin{pmatrix} D_1(r,r) & 0 \\ 0 & D_2(r,r)I_{d-1}\end{pmatrix}\overline{u_{\boldsymbol{z}}}$$

is independent of the choice of $u_{\boldsymbol{z}}$ by Lemma 3.4.

Proposition 3.6. *Suppose that $K : \mathbb{B}_d \times \mathbb{B}_d \to \mathcal{M}_d(\mathbb{C})$ is a sesqui-analytic Hermitian function satisfying the transformation rule with the multiplier $c(u) = \overline{u}$:*

$$(*) \qquad u^{\dagger}K(u\cdot\boldsymbol{z}, u\cdot\boldsymbol{w})\overline{u} = K(z,w), u \in \mathcal{U}(d).$$

Then K must be of the form

$$K(\boldsymbol{z},\boldsymbol{z}) = u_{\boldsymbol{z}}^{\dagger}\left(\begin{smallmatrix} D_1(r,r) & 0 \\ 0 & D_2(r,r)I_{d-1}\end{smallmatrix}\right)\overline{u_{\boldsymbol{z}}},\ u_{\boldsymbol{z}} \in \mathcal{U}_{\boldsymbol{z}},$$

where $D_i(r,r)$, $i = 1,2$ are real analytic function on $[0,1)$ of the form $\sum_{m=0}^{\infty} a_{m,i}r^{2m}$ with $a_{0,1} = a_{0,2}$.

Proof. Note that $u^{\dagger}K(0,0)\bar{u} = K(0,0)$ implying $K(0,0)$ must be a scalar times I_d. Let $\boldsymbol{z} \in \mathbb{B}_d$ and $\boldsymbol{z} \neq 0$. Putting $w = z$ and $u = u_z \in \mathcal{U}_{\boldsymbol{z}}$ in $(*)$ we get that

$$\begin{aligned} K(\boldsymbol{z},\boldsymbol{z}) &= u_{\boldsymbol{z}}^{\dagger}K(u_{\boldsymbol{z}}(\boldsymbol{z}), u_{\boldsymbol{z}}(\boldsymbol{z}))\overline{u_{\boldsymbol{z}}} \\ &= u_{\boldsymbol{z}}^{\dagger}K(\|\boldsymbol{z}\|\boldsymbol{e}_1, \|\boldsymbol{z}\|\boldsymbol{e}_1)\overline{u_{\boldsymbol{z}}}. \end{aligned} \tag{3.4}$$

Using this expression of $K(z,z)$ in $(*)$ we see that

$$u_{\boldsymbol{z}}^{\dagger}K(\|\boldsymbol{z}\|\boldsymbol{e}_1, \|\boldsymbol{z}\|\boldsymbol{e}_1)\overline{u_{\boldsymbol{z}}} = u^{\dagger}u_{u\cdot\boldsymbol{z}}^{\dagger}K(\|u\cdot\boldsymbol{z}\|\boldsymbol{e}_1, \|u\cdot\boldsymbol{z}\|\boldsymbol{e}_1)\overline{u_{u\cdot\boldsymbol{z}}}\ \overline{u}. \tag{3.5}$$

Equivalently, we have

$$\overline{u_{u\cdot\boldsymbol{z}}}\ \overline{u}u_{\boldsymbol{z}}^{\dagger}K(\|\boldsymbol{z}\|\boldsymbol{e}_1, \|\boldsymbol{z}\|\boldsymbol{e}_1) = K(\|\boldsymbol{z}\|\boldsymbol{e}_1, \|\boldsymbol{z}\|\boldsymbol{e}_1)\overline{u_{u\cdot\boldsymbol{z}}}\ \overline{u}u_{\boldsymbol{z}}^{\dagger},\ \text{for all } u \in \mathcal{U}(d), u_{\boldsymbol{z}} \in \mathcal{U}_{\boldsymbol{z}}. \tag{3.6}$$

Note that $\overline{u_{u\cdot\boldsymbol{z}}}\ \overline{u}u_{\boldsymbol{z}}^{\dagger}$ is a unitary and

$$\overline{u_{u\cdot\boldsymbol{z}}}\ \overline{u}u_{\boldsymbol{z}}^{\dagger}(\boldsymbol{e}_1) = \overline{u_{u\cdot\boldsymbol{z}}}\ \overline{u}(\frac{\overline{z}}{\|\boldsymbol{z}\|}) = \frac{\overline{u_{u\cdot\boldsymbol{z}}(u\cdot\boldsymbol{z})}}{\|\boldsymbol{z}\|} = \boldsymbol{e}_1.$$

Moreover, if v is a unitary in $\mathcal{U}(d)$ with $v(\boldsymbol{e}_1) = \boldsymbol{e}_1$, then v can be written as $\overline{u_1}\ \overline{u}u_2^{\dagger}$, where $u = \overline{v}u_{\boldsymbol{z}}$, $u_2 = u_{\boldsymbol{z}}$ and $u_1 = I_d$. Since $\overline{v}u_{\boldsymbol{z}}(\boldsymbol{z}) = \|\boldsymbol{z}\|\overline{v}(\boldsymbol{e}_1) = \|\boldsymbol{z}\|\boldsymbol{e}_1$, we see that $u_1 = I_d \in \mathcal{U}_{u\cdot\boldsymbol{z}}$. Consequently, it follows that the set $\{\overline{u_{u\cdot\boldsymbol{z}}}\ \overline{u}u_{\boldsymbol{z}}^{\dagger} : u \in \mathcal{U}(d), u_{\boldsymbol{z}} \in \mathcal{U}_{\boldsymbol{z}}, u_{u\cdot\boldsymbol{z}} \in \mathcal{U}_{u\cdot\boldsymbol{z}}\}$ coincides with the set $\{v \in \mathcal{U}(d) : v(\boldsymbol{e}_1) = \boldsymbol{e}_1\}$. This together with (3.6) gives

$$vK(\|\boldsymbol{z}\|\boldsymbol{e}_1, \|\boldsymbol{z}\|\boldsymbol{e}_1) = K(\|\boldsymbol{z}\|\boldsymbol{e}_1, \|\boldsymbol{z}\|\boldsymbol{e}_1)v, \tag{3.7}$$

for all $v \in \mathcal{U}(d)$ with $v(\boldsymbol{e}_1) = \boldsymbol{e}_1$. Hence by Lemma 3.2 we get that

$$K(\|\boldsymbol{z}\|\boldsymbol{e}_1, \|\boldsymbol{z}\|\boldsymbol{e}_1) = \begin{pmatrix} K_1(\|\boldsymbol{z}\|\boldsymbol{e}_1,\|\boldsymbol{z}\|\boldsymbol{e}_1) & 0 \\ 0 & K_2(\|\boldsymbol{z}\|\boldsymbol{e}_1,\|\boldsymbol{z}\|\boldsymbol{e}_1)I_{d-1} \end{pmatrix}, \tag{3.8}$$

where K_1 and K_2 are two scalar-valued sesqui-analytic Hermitian functions on $\mathbb{B}_d \times \mathbb{B}_d$. Applying Lemma 3.3, we infer that

$$K(\boldsymbol{z}, \boldsymbol{z}) = \sum_{\ell=0}^{\infty} \sum_{|\alpha|=|\beta|=\ell} a_{\alpha,\beta} \boldsymbol{z}^\alpha \bar{\boldsymbol{z}}^\beta, \;\; a_{\alpha,\beta} \in \mathcal{M}_d(\mathbb{C}).$$

Consequently, we have the equality

$$K(\|\boldsymbol{z}\|\boldsymbol{e}_1, \|\boldsymbol{z}\|\boldsymbol{e}_1) = \sum_{\ell=0}^{\infty} a_{\ell\varepsilon_1,\ell\varepsilon_1} \|\boldsymbol{z}\|^{2\ell}. \tag{3.9}$$

Combining Equation (3.9) with the Equations (3.4) and (3.8), completes the verification of the first of the two equalities claimed for the kernel K. $\square$

Now, we obtain a characterization of the non-negative definite quasi-invariant kernels.

Theorem 3.7. *Any sesqui-analytic Hermitian function quasi-invariant with multiplier $c(u) = \bar{u}$ is of the form*

$$\tilde{K}^{(\alpha,\beta)}(\boldsymbol{z}, \boldsymbol{w}) = \sum_{j=1}^{\infty} \alpha_j \tilde{K}_j(\boldsymbol{z}, \boldsymbol{w}) + \sum_{j=0}^{\infty} \beta_j \tilde{K}_j^{\perp}(\boldsymbol{z}, \boldsymbol{w}), \;\; \alpha_j, \beta_j \in \mathbb{C}.$$

Proof. First, since any sesqui-analytic Hermitian function quasi-invariant with multiplier $c(u) = \bar{u}$, it must be of the form prescribed in Proposition 3.6. Applying Lemma 3.4 to it, and then polarizing the result, we see that it must be of the form

$$\tilde{K}(\boldsymbol{z}, \boldsymbol{w}) = \sum_{\ell=1}^{\infty} \big(a_{\ell,1} - a_{\ell,2}\big)\langle \boldsymbol{z}, \boldsymbol{w}\rangle^{\ell-1} \overline{\boldsymbol{w}} \boldsymbol{z}^\dagger + \sum_{\ell=0}^{\infty} a_{\ell,2} \langle \boldsymbol{z}, \boldsymbol{w}\rangle^{\ell} I_d, \;\; \boldsymbol{z}, \boldsymbol{w} \in \mathbb{B}_d, \tag{$\sharp$}$$

for some choice of complex numbers $a_{\ell,1}$, $\ell \in \mathbb{N}$, and $a_{\ell,2}$, $\ell \in \mathbb{Z}_+$. For any $\ell \geq 1$, by Lemma 3.1, we have

$$\begin{aligned}
\big(a_{\ell,1} - a_{\ell,2}\big)&\langle \boldsymbol{z}, \boldsymbol{w}\rangle^{l-1} \overline{\boldsymbol{w}} \boldsymbol{z}^\dagger + a_{\ell,2} \langle \boldsymbol{z}, \boldsymbol{w}\rangle^{l} I_d \\
&= a_{\ell,1}\langle \boldsymbol{z}, \boldsymbol{w}\rangle^{\ell} I_d - \big(a_{\ell,2} - a_{\ell,1}\big)\langle \boldsymbol{z}, \boldsymbol{w}\rangle^{\ell-1}\big(\langle \boldsymbol{z}, \boldsymbol{w}\rangle - \overline{\boldsymbol{w}} \boldsymbol{z}^\dagger\big) \\
&= a_{\ell,1}\ell!\big(\tilde{K}_\ell + \tilde{K}_\ell^{\perp}\big) - \big(a_{\ell,1} - a_{\ell,2}\big)\frac{(\ell+1)\ell!}{\ell}\tilde{K}_\ell \\
&= a_{\ell,1}\ell!\tilde{K}_\ell^{\perp} + \big((\ell+1)a_{\ell,2} - a_{\ell,1}\big)(\ell-1)!\tilde{K}_\ell.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\tilde{K}(z,w) &= a_{0,2}I_d + \sum_{\ell=1}^{\infty} \big(a_{\ell,1}\ell!\tilde{K}_\ell^{\perp} + \big((\ell+1)a_{\ell,2} - a_{\ell,1}\big)(\ell-1)!\tilde{K}_\ell\big) \\
&= \sum_{j=1}^{\infty} \alpha_j \tilde{K}_j(\boldsymbol{z}, \boldsymbol{w}) + \sum_{j=0}^{\infty} \beta_j \tilde{K}_j^{\perp}(\boldsymbol{z}, \boldsymbol{w}),
\end{aligned}$$

where $\alpha_j = \big((j+1)a_{j,2} - a_{j,1}\big)(j-1)!$, $\beta_j = a_{j,1}j!$.

$\square$

3.2. **Decomposition of $\hat{\pi}$.** Consider the two subspaces $\widehat{\mathcal{V}}_\ell$ and $\widehat{\mathcal{W}}_\ell$ of $(\mathbb{C}^d \otimes \mathcal{P}_\ell, \langle\cdot,\,\cdot\rangle_{\mathcal{F}_\ell})$:

$$\widehat{\mathcal{V}}_\ell = \left\{ f := \begin{pmatrix} f_1 \\ \vdots \\ f_d \end{pmatrix} \in \mathbb{C}^d \otimes \mathcal{P}_\ell : \partial_1 f_1 + \cdots + \partial_d f_d = 0 \right\}$$

and

$$\widehat{\mathcal{W}}_\ell = \Big\{ \begin{pmatrix} z_1 g \\ \vdots \\ z_d g \end{pmatrix} : g \in \mathcal{P}_{\ell-1} \Big\}.$$

Evidently, the subspace $\widehat{\mathcal{W}}_\ell$ is invariant under the unitary representation $\hat{\pi}$. Also, we check that the subspace $\widehat{\mathcal{V}}_\ell^\perp \subseteq (\mathbb{C}^d \otimes \mathcal{P}_\ell, \langle\cdot,\,\cdot\rangle_{\mathcal{F}_\ell})$ is $\widehat{\mathcal{W}}_\ell$.

To verify this, let $M_{z_i}^{(\ell)} : \mathcal{P}_{\ell-1} \to \mathcal{P}_\ell$ be the linear map $M_{z_i}^{(\ell)}(p) = z_i p,\ p \in \mathcal{P}_\ell$. Clearly, setting $\boldsymbol{M}^{(\ell)} = (M_{z_1}^{(\ell)}, \ldots, M_{z_d}^{(\ell)})$, we see that $\widehat{\mathcal{W}}_\ell = \text{ran}\,(D^{\boldsymbol{M}^{(\ell)}})$. Note that for any $\alpha, \beta \in \mathbb{Z}_+^d$, $\langle \boldsymbol{z}^{\alpha+\varepsilon_i}, \boldsymbol{z}^\beta\rangle_{\mathcal{F}} = \beta!\delta_{\alpha+\varepsilon_i,\beta}$. Thus we have

$$\langle z_i p, q\rangle_{\mathcal{F}} = \langle p, \partial_i q\rangle_{\mathcal{F}},\ \ p, q \in \mathcal{P}.$$

Hence it follows that $M_{z_i}^{(\ell)^*} = \partial_i$. Therefore $\widehat{\mathcal{V}}_\ell = \ker D^{\boldsymbol{M}^{(\ell)^*}}$. Since $\left(D^{\boldsymbol{M}^{(\ell)}}\right)^* = D_{\boldsymbol{M}^{(\ell)*}}$, we conclude that

$$\widehat{\mathcal{V}}_\ell^\perp = \text{ran}\ D_{\boldsymbol{M}^{(\ell)}} = \hat{\mathcal{W}}_\ell.$$

Therefore, $\widehat{\mathcal{V}}_\ell$ is also invariant under the representation $\hat{\pi}$.

Lemma 3.8. *Consider the inner product space $(\mathbb{C}^d \otimes \mathcal{P}_\ell, \langle\cdot,\,\cdot\rangle_{\mathcal{F}_\ell})$. Then*

(1) The reproducing kernel $\widehat{K}_\ell$ of $\widehat{\mathcal{V}}_\ell$ is

$$\widehat{K}_\ell(\boldsymbol{z}, \boldsymbol{w}) := \frac{1}{(\ell+d-1)(\ell-1)!}\langle \boldsymbol{z},\, \boldsymbol{w}\rangle^{\ell-1}\left(\frac{(\ell+d-1)}{\ell}\langle \boldsymbol{z},\, \boldsymbol{w}\rangle I_d - \boldsymbol{z}\overline{\boldsymbol{w}}^\dagger\right),$$

where $\boldsymbol{z}\overline{\boldsymbol{w}}^\dagger$ is the matrix product of the column vector $\boldsymbol{z}$ and the row vector $\overline{\boldsymbol{w}}^\dagger$.

(2) The reproducing kernel $\widehat{K}_\ell^\perp$ of $\widehat{\mathcal{V}}_\ell^\perp$ is $\frac{1}{(\ell+d-1)(\ell-1)!}\langle \boldsymbol{z},\, \boldsymbol{w}\rangle^{\ell-1}\boldsymbol{z}\overline{\boldsymbol{w}}^\dagger$.

Proof. Clearly, part (2) is a direct consequence of part (1) of the Lemma. Therefore, we will prove only part (1), which is similar to the proof of part (1) of Lemma 3.1. Let $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_d)$ be any vector in $\mathbb{C}^d$. As before, we note that

$$\langle \widehat{K}_\ell(\boldsymbol{z}, \boldsymbol{w})\boldsymbol{\zeta},\, \boldsymbol{e}_j\rangle = \frac{1}{(\ell+d-1)(\ell-1)!}\langle \boldsymbol{z},\, \boldsymbol{w}\rangle^{\ell-1}\left(\frac{(\ell+d-1)}{\ell}\langle \boldsymbol{z},\, \boldsymbol{w}\rangle\langle \boldsymbol{\zeta},\, \boldsymbol{e}_j\rangle - \langle \boldsymbol{z},\, \boldsymbol{e}_j\rangle\langle \boldsymbol{\zeta},\, \boldsymbol{w}\rangle\right)$$

A direct verification shows that

$$\sum_{j=1}^d \partial_j \langle \widehat{K}_\ell(\boldsymbol{z}, \boldsymbol{w})\boldsymbol{\zeta},\, \boldsymbol{e}_j\rangle = 0,$$

therefore, it follows that $\widehat{K}_\ell(\cdot,\boldsymbol{w})\boldsymbol{\zeta}\in\widehat{\mathcal{V}}_\ell$. Also,

$$\begin{aligned}\langle f,\,\langle \boldsymbol{z},\,\boldsymbol{w}\rangle^{\ell-1}\boldsymbol{z}\overline{\boldsymbol{w}}^\dagger\boldsymbol{e}_i\rangle_{\mathcal{F}_\ell} &= \sum_{j=1}^d\langle f_j,\,\langle \boldsymbol{z},\,\boldsymbol{w}\rangle^{\ell-1}\overline{w}_i z_j\rangle_{\mathcal{F}}\\ &= \sum_{j=1}^d w_i\langle f_j,\,\langle \boldsymbol{z},\,\boldsymbol{w}\rangle^{\ell-1} z_j\rangle_{\mathcal{F}}\\ &= \sum_{j=1}^d w_i\langle \partial_j f_j,\,\langle \boldsymbol{z},\,\boldsymbol{w}\rangle^{\ell-1}\rangle_{\mathcal{F}}\\ &= (\ell-1)!w_i\sum_{j=1}^d(\partial_j f_j)(\boldsymbol{w})=0.\end{aligned}$$

Thus, $\langle f,\,\widehat{K}_\ell(\cdot,\boldsymbol{w})\boldsymbol{e}_i\rangle_{\mathcal{F}_\ell}=\langle f(\boldsymbol{w}),\,\boldsymbol{e}_i\rangle$. $\square$

The proposition below matching with Proposition 3.6 is obtained by replacing $c(u)=\bar{u}$ by $c(u)=u$ is proved as before.

Proposition 3.9. *Suppose that $K:\mathbb{B}_d\times\mathbb{B}_d\to\mathcal{M}_d(\mathbb{C})$ is a sesqui-analytic Hermitian function satisfying the transformation rule with the multiplier $c(u)=u$:*

$$(**)\qquad uK(u^{-1}\cdot\boldsymbol{z},u^{-1}\cdot\boldsymbol{w})\overline{u}^\dagger=K(\boldsymbol{z},\boldsymbol{w}).$$

Then K must be of the form

$$K(\boldsymbol{z},\boldsymbol{z})=\overline{u_{\boldsymbol{z}}}^\dagger\begin{pmatrix}D_1(r,r)&0\\0&D_2(r,r)I_{d-1}\end{pmatrix}u_{\boldsymbol{z}},\ u_{\boldsymbol{z}}\in\mathcal{U}_{\boldsymbol{z}},$$

where $D_i(r,r)$, $i=1,2$ are real analytic function on $[0,1)$ of the form $\sum_{m=0}^\infty\tilde{a}_{m,i}r^{2m}$ with $\tilde{a}_{0,1}=\tilde{a}_{0,2}$.

We need the following lemma similar to Lemma 3.4 to prove the main theorem describing sesqui-analytic Hermitian function quasi-invariant with multiplier $c(u)=u$.

Lemma 3.10. *For any $u_{\boldsymbol{z}}\in\mathcal{U}_{\boldsymbol{z}}$, we have*

$$\overline{u_{\boldsymbol{z}}}^\dagger\begin{pmatrix}D_1(r,r)&0\\0&D_2(r,r)I_{d-1}\end{pmatrix}u_{\boldsymbol{z}}=\big(D_1(r,r)-D_2(r,r)\big)\frac{\boldsymbol{z}\overline{\boldsymbol{z}}^\dagger}{r^2}+D_2(r,r)I_d.$$

Proof. The proof is similar to the proof of Lemma 3.4 except that we have to use the equality:

$$\overline{u_{\boldsymbol{z}}}^\dagger E_{11}u_{\boldsymbol{z}}=\frac{\boldsymbol{z}\overline{\boldsymbol{z}}^\dagger}{r^2}.$$

$\square$

Theorem 3.11. *Any sesqui-analytic Hermitian function quasi-invariant with multiplier $c(u)=u$ is of the form*

$$\widehat{K}^{(\alpha,\beta)}(\boldsymbol{z},\boldsymbol{w})=\sum_{j=0}^\infty\alpha_j\widehat{K}_j(\boldsymbol{z},\boldsymbol{w})+\sum_{j=1}^\infty\beta_j\widehat{K}_j^\perp(\boldsymbol{z},\boldsymbol{w}),\ \alpha_j,\beta_j\in\mathbb{C}.$$

Proof. As before, since the kernel $\widehat{K}$ is sesqui-analytic Hermitian function quasi-invariant with multiplier $c(u)=u$, it must be of the form prescribed in Proposition 3.9. Now, appealing to Lemma 3.10, we obtain

$$(\sharp\sharp)\qquad \widehat{K}(\boldsymbol{z},\boldsymbol{w})=\sum_{\ell=1}^\infty(\tilde{a}_{\ell,1}-\tilde{a}_{\ell,2})\langle\boldsymbol{z},\,\boldsymbol{w}\rangle^{\ell-1}\boldsymbol{z}\,\overline{\boldsymbol{w}}^\dagger+\sum_{\ell=0}^\infty\tilde{a}_{\ell,2}\langle\boldsymbol{z},\,\boldsymbol{w}\rangle^\ell I_d.$$

The remaining portion of the proof is similar to that of Theorem 3.7, where $\alpha_j=\tilde{a}_{j,2}j!$ and $\beta_j=(\tilde{a}_{j,1}(j+d-1)-\tilde{a}_{j,2}(d-1))(j-1)!$ for all j. $\square$

To determine among the kernels described in Theorem 3.7 (respectively, Theorem 3.11), the ones that are non-negative definite, we recall a slight generalization of the criterion for non-negative definiteness of Farut-Koranyi [9, Lemma 5.4]:

Lemma 3.12 (Lemma 5.1, [4])**.** *Let Ω be a domain in $\mathbb{C}^d$. Let $K : \Omega \times \Omega \to \mathcal{M}_n(\mathbb{C})$ be a non-negative definite kernel and $\mathcal{H}_K(\Omega, \mathbb{C}^n)$ be the reproducing kernel Hilbert space determined by K. Suppose that $\mathcal{H}_K(\Omega, \mathbb{C}^n)$ can be decomposed as an orthogonal direct sum $\oplus_{\ell=0}^{\infty} \mathcal{H}_\ell$ and K_ℓ is the reproducing kernel of $\mathcal{H}_\ell$. Further assume that $\{c_\ell\}_{\ell\in\mathbb{Z}_+}$ is any sequence of complex numbers such that the sum $\sum_{\ell=0}^{\infty} c_\ell K_\ell(z, w)$ converges on $\Omega \times \Omega$. Then $\sum_{\ell=0}^{\infty} c_\ell K_\ell(z, w)$ is non-negative definite if and only if $c_\ell \geq 0$ for all $\ell \in \mathbb{Z}_+$.*

Combining Faraut-Koranyi lemma with Theorem 3.6 and Theorem 3.11, we obtain a condition for a sesqui-analytic Hermitian function to be non-negative.

Theorem 3.13. *Suppose that $\tilde{K}^{(\alpha,\beta)} : \mathbb{B}_d \times \mathbb{B}_d \to \mathcal{M}_d(\mathbb{C})$ is a sesqui-analytic Hermitian function as in Theorem 3.6 (respectively, $\widehat{K}^{(\alpha,\beta)}$ as in Theorem 3.11). Then the kernel $\tilde{K}^{(\alpha,\beta)}$ (respectively, $\widehat{K}^{(\alpha,\beta)}$) is non-negative definite if and only if $\alpha_j \geq 0, \beta_j \geq 0$.*

Proof. In the expansion of $\tilde{K}^{(\alpha,\beta)}$ obtained in Theorem 3.7, the kernels $\tilde{K}_j$ and $\tilde{K}_j^{\perp}$ are non-negative definite. Therefore, by Lemma 3.12, we conclude that $\tilde{K}$ is non-negative definite if and only if $\alpha_j \geq 0, \beta_j \geq 0$. The proof for $\widehat{K}^{(\alpha,\beta)}$ is similar and therefore omitted. ☐

As a corollary of Theorem 3.7 (respectively, Theorem 3.11), we prove that the restriction of the representation $\tilde{\pi}_\ell$ to $\tilde{\mathcal{V}}_\ell$ (respectively, restriction of $\hat{\pi}_\ell$ to $\hat{\mathcal{V}}_\ell$) is irreducible.

Corollary 3.14.
1. *The restriction $\tilde{\pi}_\ell|_{\tilde{\mathcal{V}}_\ell}$ of $\tilde{\pi}_\ell$ to the linear subspace $\tilde{\mathcal{V}}_\ell$ equipped with the restriction of the inner product $\langle\cdot,\cdot\rangle_{\mathcal{F}_\ell}$ from $\mathbb{C}^d \otimes \mathcal{P}_\ell$ is irreducible.*
2. *The restriction $\hat{\pi}_\ell|_{\widehat{\mathcal{V}}_\ell}$ of $\hat{\pi}_\ell$ to the linear subspace $\widehat{\mathcal{V}}_\ell$ equipped with the restriction of the inner product $\langle\cdot,\cdot\rangle_{\mathcal{F}_\ell}$ from $\mathbb{C}^d \otimes \mathcal{P}_\ell$ is irreducible.*

Proof. To prove part (1) of the corollary, suppose that there is a decomposition $\tilde{\mathcal{V}}_\ell = \mathcal{V}_\ell^1 \oplus \mathcal{V}_\ell^2$, where $\mathcal{V}_\ell^1$ and $\mathcal{V}_\ell^2$ are reducing subspaces for $\tilde{\pi}_\ell$. Let K_ℓ^1 and K_ℓ^2 be the kernel functions of $\mathcal{V}_\ell^1$ and $\mathcal{V}_\ell^2$, respectively. Evidently, both K_ℓ^1 and K_ℓ^2 are quasi-invariant with respect to the same multiplier $\bar{u}$. It follows that $\tilde{K}_\ell = K_\ell^1 \oplus K_\ell^2$. If $\ell = 0$, then $\tilde{\mathcal{V}}_0 = \{0\}$ and there is nothing to prove. Fix $\ell \in \mathbb{N}$, it follows from Theorem 3.7 that K_ℓ^1 must be of the form $\sum_j \alpha_j \tilde{K}_j + \beta_j \tilde{K}_j^{\perp}$ for some choice of a set of non-negative numbers $\{\alpha_j\}$ and $\{\beta_j\}$. The Hilbert space determined by $\alpha_j \tilde{K}_j + \beta_j \tilde{K}_j^{\perp}$ contains the Hilbert space determined by $\alpha_j \tilde{K}_j$ as well as the one determined by $\beta_j \tilde{K}_j^{\perp}$. Now, if there is a non-zero α_j with $j \neq \ell$, then $\tilde{\mathcal{V}}_j$ must be a subspace of $\mathcal{V}_\ell^1$. Therefore $\alpha_j = 0$ except for $j = \ell$. A similar argument shows that $\beta_j = 0$ for all j. In consequence, if $\alpha_\ell > 0$, then $\mathcal{V}_\ell^1 = \tilde{\mathcal{V}}_\ell$, otherwise $\mathcal{V}_\ell^1 = \{0\}$.

The proof of part (2) of the Corollary is obtained exactly as in the proof of part (1) using Theorem 3.11. ☐

3.3. **Examples.** The examples discussed below shows that there are many quasi-invariant kernels K on $\mathbb{B}_d$ with multiplier of the form $c(u) = \bar{u}$ (resp. $c(u) = u$). In these examples, the monomials $\{\boldsymbol{z}^\alpha \otimes \boldsymbol{\zeta} : \alpha \in \mathbb{Z}_+^d, \boldsymbol{\zeta} \in \mathbb{C}^d\}$ are no longer orthogonal.

Let $d \geq 2$. Recall that the Bergman kernel B of the unit ball $\mathbb{B}$ is given by $B(\boldsymbol{z}, \boldsymbol{w}) = \frac{1}{(1-\langle \boldsymbol{z}, \boldsymbol{w}\rangle)^{d+1}}$. For $t \in \mathbb{R}$, we set

$$B^{(t)}(\boldsymbol{z}, \boldsymbol{w}) = B^t \Big(\Big(\frac{\partial^2}{\partial \boldsymbol{z}_i \partial \overline{\boldsymbol{w}}_j} \log B\Big)\Big)_{i,j=1}^d (\boldsymbol{z}, \boldsymbol{w}).$$

Clearly $B^{(t)}$ is a sesqui-analytic hermitian function for any real number t. It follows from [11, Lemma 6.1] that $B^{(t)}$ is quasi-invariant with the multiplier $c(u) = \overline{u}$. A direct computation shows that

$$B^{(t)}(\boldsymbol{z},\boldsymbol{w}) = \frac{d+1}{(1-\langle \boldsymbol{z},\boldsymbol{w}\rangle)^{t(d+1)+2}} \begin{pmatrix} 1-\sum_{j\neq 1} z_j\bar{w}_j & z_2\bar{w}_1 & \cdots & z_d\bar{w}_1 \\ z_1\bar{w}_2 & 1-\sum_{j\neq 2} z_j\bar{w}_j & \cdots & z_d\bar{w}_2 \\ \vdots & \vdots & \vdots & \vdots \\ z_1\bar{w}_d & z_2\bar{w}_d & \cdots & 1-\sum_{j\neq d} z_j\bar{w}_j \end{pmatrix}. \tag{3.10}$$

Thus

$$B^{(t)}(re_1,re_1) = \frac{d+1}{(1-r^2)^{t(d+1)+2}} \begin{pmatrix} 1 & 0 \\ 0 & (1-r^2)I_{d-1} \end{pmatrix}, 0\le r<1. \tag{3.11}$$

Note that $B^{(t)}(0,0)=(d+1)I_d$. Thus by Proposition 3.6 we have $B^{(t)}(\boldsymbol{z},\boldsymbol{z}) = u_{\boldsymbol{z}}^{\dagger} B^{(t)}(re_1,re_1)\overline{u_{\boldsymbol{z}}}$, where $r=\|\boldsymbol{z}\|$ and $u_{\boldsymbol{z}}$ is a unitary of the form $u_{\boldsymbol{z}}^* = (\,\frac{\boldsymbol{z}}{r}|\,\bigstar\,)$. Equivalently,

$$B^{(t)}(\boldsymbol{z},\boldsymbol{w}) = \sum_{\ell=1}^{\infty} \big(a_{\ell,1}-a_{\ell,2}\big)\langle \boldsymbol{z},\,\boldsymbol{w}\rangle^{\ell-1}\overline{\boldsymbol{w}}\boldsymbol{z}^{\dagger} + \sum_{\ell=0}^{\infty} a_{\ell,2}\langle \boldsymbol{z},\,\boldsymbol{w}\rangle^{\ell} I_d, \tag{3.12}$$

where $a_{\ell,1}=(d+1)\frac{(t(d+1)+2)_\ell}{\ell!}$ and $a_{\ell,2}=(d+1)\frac{(t(d+1)+1)_\ell}{\ell!}$ for all $\ell\in\mathbb{Z}_+$. In this case it is easy to verify that $a_{\ell,1}\le(\ell+1)a_{\ell,2}$ if and only if $t\ge 0$. Therefore by Theorem 3.13 it follows that $B^{(t)}$ is a non-negative definite kernel if and only if $t\ge 0$.

Since $B^{(t)}$ is quasi-invariant with respect to the multiplier $c(u)=\overline{u}$, it is easy to see that ${B^{(t)}}^{\dagger}$ is quasi-invariant with respect to the multiplier $c(u)=u$. Further, using (3.12) and the identity $\frac{\langle \boldsymbol{z},\boldsymbol{w}\rangle^\ell}{\ell!}I_d = K_\ell + K_\ell^{\perp}$, we obtain

$${B^{(t)}}^{\dagger}(\boldsymbol{z},\boldsymbol{w}) = \sum_{\ell=1}^{\infty}\big((a_{\ell,1}-a_{\ell,2})(\ell+d-1)(\ell-1)!+a_{\ell,2}\;\ell!\big)K_\ell^{\perp}(\boldsymbol{z},\boldsymbol{w}) + \sum_{\ell=0}^{\infty} a_{\ell,2}\;\ell! K_\ell(\boldsymbol{z},\boldsymbol{w}). \tag{3.13}$$

Hence it follows from Theorem 3.13 that the transpose ${B^{(t)}}^{\dagger}$ of the kernel $B^{(t)}$ is a non-negative definite kernel if and only if $t(d+1)+1\ge 0$.

Since $B^{(t)}$, $t\ge 0$, as well as ${B^{(t)}}^{\dagger}$, $t(d+1)+1\ge 0$, are non-negative definite, it follows from Proposition 2.8 that these kernels are quasi-invariant but not invariant.

4. $\mathcal{U}(d)$ Homogeneous operators

4.1. **Boundedness and Irreducibility.** In this subsection, we derive explicit criterion for $\mathcal{U}(d)$-homogeneous d-tuple of multiplication operator $\boldsymbol{M}$ to be (a) bounded and (b) irreducible. This is done separately for the class of kernels of the form appearing in Theorem 3.7 and Theorem 3.11.

Theorem 4.1. *Suppose that $K:\mathbb{B}_d\times\mathbb{B}_d\to\mathcal{M}_d(\mathbb{C})$ is a non-negative definite kernel of the form (♯). Then the d-tuple $\boldsymbol{M}$ on the Hilbert space $\mathcal{H}_K(\mathbb{B}_d,\mathbb{C}^d)$ is bounded if and only if*

$$\sup_\ell \left\{ \frac{(\ell+1)a_{\ell-1,2}-a_{\ell-1,1}}{(\ell+1)a_{\ell,2}-a_{\ell,1}}, \frac{a_{\ell-1,1}}{a_{\ell,1}} \right\} < \infty.$$

Proof. The multiplication d-tuple $\boldsymbol{M}$ on the Hilbert space $\mathcal{H}_K(\mathbb{B}_d,\mathbb{C}^d)$ is bounded if and only if there exists $c>0$ such that $\big(c^2-\langle \boldsymbol{z},\,\boldsymbol{w}\rangle\big)K(\boldsymbol{z},\boldsymbol{w})$ is non-negative definite [11, Lemma 2.7(ii)].

$$\begin{aligned}\big(c^2-\langle \boldsymbol{z},\,\boldsymbol{w}\rangle\big)K(\boldsymbol{z},\boldsymbol{w})|_{\mathrm{res}\ \mathbb{C}^d\otimes\mathcal{P}_\ell} =& \big\{c^2\big(a_{\ell,1}-a_{\ell,2}\big) - \big(a_{\ell-1,1}-a_{\ell-1,2}\big)\big\}\langle \boldsymbol{z},\,\boldsymbol{w}\rangle^{l-1}\overline{\boldsymbol{w}}\boldsymbol{z}^{\dagger} \\ &+ \big(c^2 a_{\ell,2}-a_{\ell-1,2}\big)\langle \boldsymbol{z},\,\boldsymbol{w}\rangle^{l} I_d \\ =& \big\{c^2\big((\ell+1)a_{\ell,2}-a_{\ell,1}\big) - \big((\ell+1)a_{\ell-1,2}-a_{\ell-1,1}\big)\big\}(\ell-1)!K_\ell \\ &+ \big(c^2 a_{\ell,1}-a_{\ell-1,1}\big)\ell! K_\ell^{\perp}.\end{aligned}$$

Hence by Lemma 3.12 $\big(c^2 - \langle \boldsymbol{z}, \boldsymbol{w}\rangle\big)K(\boldsymbol{z}, \boldsymbol{w})$ is non-negative definite if and only if for all $l \in \mathbb{N}$,

$$c^2\big((\ell+1)a_{\ell,2} - a_{\ell,1}\big) - \big((\ell+1)a_{\ell-1,2} - a_{\ell-1,1}\big) \geq 0$$

and

$$c^2 a_{\ell,1} - a_{\ell-1,1} \geq 0.$$

The claim of the theorem is clearly equivalent to these two positivity conditions completing the proof. □

Theorem 4.2. *Suppose that $K : \mathbb{B}_d \times \mathbb{B}_d \to \mathcal{M}_d(\mathbb{C})$ is a non-negative definite kernel function of the form ($\sharp\sharp$). Then the d-tuple $\boldsymbol{M}$ on the Hilbert space $\mathcal{H}_K(\mathbb{B}_d, \mathbb{C}^d)$ is bounded if and only if*

$$\sup_{\ell}\left\{\frac{(\ell+d-1)\tilde{a}_{\ell-1,1} - (d-1)\tilde{a}_{\ell-1,2}}{(\ell+d-1)\tilde{a}_{\ell,1} - (d-1)\tilde{a}_{\ell,2}}, \frac{\tilde{a}_{\ell-1,2}}{\tilde{a}_{\ell,2}}\right\} < \infty.$$

Corollary 4.3. *Let K be a non-negative definite kernel function either of the form ($\sharp$) or ($\sharp\sharp$). Assume that the d-tuple $\boldsymbol{M}$ on the Hilbert space $\mathcal{H}_K(\mathbb{B}_d, \mathbb{C}^d)$ is bounded. Then it is $\mathcal{U}(d)$-homogeneous.*

Proof. Since K is quasi-invariant under $\mathcal{U}(d)$, the conclusion follows from Lemma 2.3. □

Theorem 4.4. *Let $d \geq 2$. Let K be a non-negative definite kernel function either of the form ($\sharp$) or ($\sharp\sharp$). Assume that the d-tuple $\boldsymbol{M}$ on the Hilbert space $\mathcal{H}_K(\mathbb{B}_d, \mathbb{C}^d)$ is bounded. Then $\boldsymbol{M}$ is reducible if and only if $a_{\ell,1} = a_{\ell,2}$ or $\tilde{a}_{\ell,1} = \tilde{a}_{\ell,2}$ according as K is of the form ($\sharp$) or of the form ($\sharp\sharp$), $\ell \in \mathbb{N}$.*

Proof. First, let us consider the case of a kernel of the form ($\sharp$). Assume that $a_{\ell,1} = a_{\ell,2}$, $\ell \in \mathbb{N}$. Then $K(z, w) = \sum_{\ell=0}^\infty a_{\ell,2}\langle \boldsymbol{z}, \boldsymbol{w}\rangle^\ell I_d$. Since $d \geq 2$, it is evident that the multiplication d-tuple $\boldsymbol{M}$ on $\mathcal{H}_K(\mathbb{B}_d, \mathbb{C}^d)$ is reducible. Conversely, assume that $\boldsymbol{M}$ on $\mathcal{H}_K(\mathbb{B}_d, \mathbb{C}^d)$ is reducible. Since $K(\boldsymbol{z}, 0)$ is constant and $\boldsymbol{M}$ is bounded, the discussion following Lemma 5.1 of [13], there exists a non-trivial projection on P on $\mathbb{C}^d$ such that $PK(\boldsymbol{z}, \boldsymbol{w}) = K(\boldsymbol{z}, \boldsymbol{w})P$. In case, K is of the form ($\sharp$), this is equivalent to

$$P\Big(\sum_{\ell=1}^{\infty}\big(a_{\ell,1} - a_{\ell,2}\big)\langle \boldsymbol{z}, \boldsymbol{w}\rangle^{\ell-1}\overline{\boldsymbol{w}}\boldsymbol{z}^\dagger\Big) = \Big(\sum_{\ell=1}^{\infty}\big(a_{\ell,1} - a_{\ell,2}\big)\langle \boldsymbol{z}, \boldsymbol{w}\rangle^{\ell-1}\overline{\boldsymbol{w}}\boldsymbol{z}^\dagger\Big)P. \tag{4.1}$$

Rewriting Equation (4.1), we have

$$\begin{aligned}0 &= \sum_{\ell=1}^{\infty}\big(a_{\ell,1} - a_{\ell,2}\big)\langle \boldsymbol{z}, \boldsymbol{w}\rangle^{\ell-1}\big(P\overline{\boldsymbol{w}}\boldsymbol{z}^\dagger - \overline{\boldsymbol{w}}\boldsymbol{z}^\dagger P\big) \\ &= \sum_{\ell=1}^{\infty}\big(a_{\ell,1} - a_{\ell,2}\big)\sum_{|\alpha|=\ell-1}\frac{|\alpha|!}{\alpha!}\sum_{i,j=1}^{d}(PE_{i,j} - E_{i,j}P)\boldsymbol{z}^{\alpha+\varepsilon_j}\bar{\boldsymbol{w}}^{\alpha+\varepsilon_i}.\end{aligned}$$

Let $\ell \geq 1$ be fixed and choose $\alpha = (\ell-1)\varepsilon_i$, $1 \leq i \leq d$. Then $\alpha + \varepsilon_j$ and $\alpha + \varepsilon_i$ are of the form

$$(\ell-1)\varepsilon_i + \varepsilon_j,\ \ell\varepsilon_i,\ 1 \leq j \leq d,$$

respectively. If we choose any other multi-index $\beta \neq \alpha$ with $|\beta| = \ell - 1$ and a pair of natural numbers m, n, $1 \leq m, n \leq d$, then we can't have $\beta + \varepsilon_m = \ell\varepsilon_i$ and $\beta + \varepsilon_n = (\ell-1)\varepsilon_i + \varepsilon_j$. It follows that the coefficients of $z_i^{\ell-1}z_j\bar{w}_i^\ell$ must be zero. This means that P must commute with all the elementary matrices $E_{i,j}$, $1 \leq i, j \leq d$. Hence P can not be a non-trivial projection contrary to our hypothesis unless $a_{\ell,1} = a_{\ell,2}$.

If K is of the form ($\sharp\sharp$), we have

$$P\Big(\sum_{\ell=1}^{\infty}\big(\tilde{a}_{\ell,1} - \tilde{a}_{\ell,2}\big)\langle \boldsymbol{z}, \boldsymbol{w}\rangle^{\ell-1}\boldsymbol{z}\overline{\boldsymbol{w}}^\dagger\Big) = \Big(\sum_{\ell=1}^{\infty}\big(\tilde{a}_{\ell,1} - \tilde{a}_{\ell,2}\big)\langle \boldsymbol{z}, \boldsymbol{w}\rangle^{\ell-1}\boldsymbol{z}\overline{\boldsymbol{w}}^\dagger\Big)P. \tag{4.2}$$

Again, rewriting Equation (4.2), we have

$$0 = \sum_{\ell=1}^{\infty} \big(\tilde{a}_{\ell,1} - \tilde{a}_{\ell,2}\big)\langle \boldsymbol{z},\, \boldsymbol{w}\rangle^{\ell-1}\big(P\boldsymbol{z}\overline{\boldsymbol{w}}^{\dagger} - \boldsymbol{z}\overline{\boldsymbol{w}}^{\dagger}P\big)$$

$$= \sum_{\ell=1}^{\infty} \big(\tilde{a}_{\ell,1} - \tilde{a}_{\ell,2}\big) \sum_{|\alpha|=\ell-1} \frac{|\alpha|!}{\alpha!} \sum_{i,j=1}^{d} (PE_{i,j} - E_{i,j}P)\boldsymbol{z}^{\alpha+\varepsilon_i}\bar{\boldsymbol{w}}^{\alpha+\varepsilon_j}.$$

Choosing $\alpha = (\ell-1)\varepsilon_i$, as before, we see that P can not be a non-trivial projection contrary to our hypothesis unless $\tilde{a}_{\ell,1} = \tilde{a}_{\ell,2}$. This completes the proof. $\square$

4.2. **Computation of matrix coefficients and unitary equivalence.** We wish to determine when the d-tuple $\boldsymbol{M}$ on the reproducing kernel Hilbert space $\mathcal{H}_K(\mathbb{B}_d, \mathbb{C}^d)$, where K is given by either ($\sharp$) or ($\sharp\sharp$), are unitarily equivalent. For this, we rewrite the kernel K in the form $K(\boldsymbol{z},\boldsymbol{w}) = \sum_{\alpha,\beta} A_{\alpha,\beta}\boldsymbol{z}^{\alpha}\bar{\boldsymbol{w}}^{\beta}$, where $\alpha,\beta \in \mathbb{Z}_+^d$ and $A_{\alpha,\beta}$ are $d\times d$ complex matrices. Since the kernels K given in ($\sharp$) and ($\sharp\sharp$) are normalized, any two d-tuple $\boldsymbol{M}$ acting on $\mathcal{H}_K(\mathbb{B}_d, \mathbb{C}^d)$ and $\mathcal{H}_{K'}(\mathbb{B}_d, \mathbb{C}^d)$ are unitarily equivalent if and only if for all α,β, $A_{\alpha,\beta}$ is unitarily equivalent to $A'_{\alpha,\beta}$ by a fixed unitary U. Here we have taken $K'(\boldsymbol{z},\boldsymbol{w}) = \sum_{\alpha,\beta} A'_{\alpha,\beta}\boldsymbol{z}^{\alpha}\bar{\boldsymbol{w}}^{\beta}$. Therefore, we proceed to find the matrix coefficients $A_{\alpha,\beta}$.

We will first consider a non-negative definite kernel of the form ($\sharp$), that is,

$$\begin{aligned}
K(\boldsymbol{z},\boldsymbol{w}) &= \sum_{\ell=1}^{\infty} \big(a_{\ell,1} - a_{\ell,2}\big)\langle \boldsymbol{z},\, \boldsymbol{w}\rangle^{\ell-1}\overline{\boldsymbol{w}}\cdot\boldsymbol{z}^{\dagger} + \sum_{\ell=0}^{\infty} a_{\ell,2}\langle \boldsymbol{z},\, \boldsymbol{w}\rangle^{\ell} I_d \\
&= \sum_{\ell=0}^{\infty}\sum_{|\alpha|=\ell} \binom{\ell}{\alpha}\Big(P_0(\ell) + \sum_{i,j=1}^{d} P_{i,j}(\ell+1)\boldsymbol{z}_j\bar{\boldsymbol{w}}_i\Big)\boldsymbol{z}^{\alpha}\bar{\boldsymbol{w}}^{\alpha} \\
&= \sum_{\alpha\in\mathbb{Z}_+^d} \binom{|\alpha|}{\alpha} P_0(|\alpha|)\boldsymbol{z}^{\alpha}\bar{\boldsymbol{w}}^{\alpha} + \sum_{\alpha\in\mathbb{Z}_+^d}\sum_{i,j} \binom{|\alpha|}{\alpha} P_{i,j}(|\alpha|+1)\boldsymbol{z}^{\alpha+\varepsilon_j}\bar{\boldsymbol{w}}^{\alpha+\varepsilon_i},
\end{aligned}$$

where $P_0(|\alpha|) = a_{|\alpha|,2}I_d$ and $P_{i,j}(|\alpha|) = (a_{|\alpha|,1} - a_{|\alpha|,2})E_{ij}$. The only monomials that occur in the kernel K are of the form $\boldsymbol{z}^{\alpha}\bar{\boldsymbol{w}}^{\beta}$ with $\alpha - \beta = \varepsilon_j - \varepsilon_i$. To find the coefficient of such a monomial, we consider two cases, namely, $i \neq j$ and $i = j$. If $i \neq j$, then the coefficient $A_{\alpha+\varepsilon_j,\alpha+\varepsilon_i}$ of the monomial $\boldsymbol{z}^{\alpha+\varepsilon_j}\bar{\boldsymbol{w}}^{\beta+\varepsilon_i}$ is

$$A_{\alpha+\varepsilon_j,\alpha+\varepsilon_i} = \binom{|\alpha|}{\alpha} P_{i,j}(|\alpha|+1),\ i \neq j. \tag{4.3}$$

On the other hand if $i = j$, we have

$$A_{\alpha,\alpha} = \binom{|\alpha|}{\alpha} P_0(|\alpha|) + \sum_{i=1}^{d} \binom{|\alpha|-1}{\alpha-\varepsilon_i} P_{i,i}(|\alpha|). \tag{4.4}$$

Replacing $P_0(|\alpha|)$ by $\tilde{P}_0(|\alpha|) := \tilde{a}_{|\alpha|,2}I_d$ and $P_{i,j}(|\alpha|)$ by $\tilde{P}_{i,j}(|\alpha|) := (\tilde{a}_{|\alpha|,1} - \tilde{a}_{|\alpha|,2})E_{ij}^{\dagger}$, we get the matrix coefficients for the kernel K of the form ($\sharp\sharp$).

Theorem 4.5. *Let K and K' be two non-negative definite kernel function either of the form ($\sharp$) or of the form ($\sharp\sharp$). Assume that the d-tuples $\boldsymbol{M}$ on the Hilbert space $\mathcal{H}_K(\mathbb{B}_d, \mathbb{C}^d)$ and $\mathcal{H}_{K'}(\mathbb{B}_d, \mathbb{C}^d)$ are bounded. Then these two d-tuples are unitarily equivalent if and only if the two kernels K and K' are equal.*

Proof. Since the kernels K and K' are normalized at 0, it follows that the d-tuples $\boldsymbol{M}$ on two of these spaces are unitarily equivalent if and only if the matrix coefficients in the expansion of these kernels, as above, are unitarily equivalent via a fixed unitary U of size $d \times d$, see [6, Lemma 4.8 (c)]. To prove the theorem, we first consider two kernels K and K' of the form ($\sharp$), that is,

$$K(\boldsymbol{z},\boldsymbol{w}) = \sum_{\ell=1}^{\infty} \big(a_{\ell,1} - a_{\ell,2}\big)\langle \boldsymbol{z},\, \boldsymbol{w}\rangle^{\ell-1}\overline{\boldsymbol{w}}\boldsymbol{z}^{\dagger} + \sum_{\ell=0}^{\infty} a_{\ell,2}\langle \boldsymbol{z},\, \boldsymbol{w}\rangle^{\ell} I_d$$

and

$$K'(\boldsymbol{z},\boldsymbol{w}) = \sum_{\ell=1}^{\infty} \big(a'_{\ell,1} - a'_{\ell,2}\big)\langle \boldsymbol{z},\, \boldsymbol{w}\rangle^{\ell-1}\overline{\boldsymbol{w}}\boldsymbol{z}^{\dagger} + \sum_{\ell=0}^{\infty} a'_{\ell,2}\langle \boldsymbol{z},\, \boldsymbol{w}\rangle^{\ell} I_d.$$

Assume that the d-tuples $\boldsymbol{M}$ on the Hilbert spaces $\mathcal{H}_K(\mathbb{B}_d, \mathbb{C}^d)$ and $\mathcal{H}_{K'}(\mathbb{B}_d, \mathbb{C}^d)$ are unitarily equivalent. For fixed $\ell \in \mathbb{Z}_+$, set $a_\ell := a_{\ell,1} - a_{\ell,2}$ and $a'_\ell := a'_{\ell,1} - a'_{\ell,2}$. It follows from Equation (4.3) that $a_\ell U E_{i,j} = a'_\ell E_{i,j} U$ for every $i \neq j$, $1 \le i, j \le d$. Therefore we conclude that a_ℓ and a'_ℓ are simultaneously 0 or not. If a_ℓ and a'_ℓ are both zero for all ℓ, then the two kernels K and K' are invariant kernels of the form $\sum_\ell a_{\ell,2} I_d \langle \boldsymbol{z},\, \boldsymbol{w}\rangle^\ell$ and $\sum_\ell a'_{\ell,2} I_d \langle \boldsymbol{z},\, \boldsymbol{w}\rangle^\ell$ respectively. Hence the d-tuples $\boldsymbol{M}$ acting on K and K' are unitarily equivalent if and only if $a_{\ell,2} = a'_{\ell,2}$, for all ℓ.

Assume that $a_{\ell,1} \neq a_{\ell,2}$ for some $\ell \in \mathbb{N}$. Fix one such ℓ and evaluate Equation (4.3) for a fixed pair i, j with $i \neq j$. We then see that every column of the $d \times d$ matrix $a_\ell U E_{i,j}$ is zero except for the jth column. This non-zero column is a_ℓ times the the ith column of U. On the other hand, each row of $d \times d$ matrix $a'_\ell E_{i,j} U$ is zero except for the ith one, which is a'_ℓ times the jth row of U. Since neither a_ℓ nor a'_ℓ is zero, it follows that $U_{k,i} = 0$, $1 \le k \neq i \le d$, similarly, $U_{j,p} = 0$, $1 \le p \neq j \le d$. Hence U must be a diagonal matrix. Moreover, we have that $a_\ell U_{i,i} = a'_\ell U_{j,j}$ for $1 \le i \neq j \le d$. We claim $a_\ell = a'_\ell$. For the proof, start with $a_\ell^2 U_{i,i} = a_\ell(a'_\ell U_{j,j}) = {a'_\ell}^2 U_{i,i}$ and conclude that $a_\ell = a'_\ell$. Hence $U_{i,i} = U_{j,j}$ for $i \neq j$ and it follows that $U_{1,1} = U_{2,2} = U_{3,3} = \cdots = U_{d,d}$. In consequence, U must be a unimodular scalar times identity.

If the kernels K and K' are of the form ($\sharp\sharp$), then the proof is similar and therefore omitted. $\square$

The theorem below answers the question of unitary equivalence between two $\mathcal{U}(d)$-homogeneous multiplication tuples acting on $\mathcal{H}_{K^\sharp}(\mathbb{B}_d, \mathbb{C}^d)$ and $\mathcal{H}_{K^{\sharp\sharp}}(\mathbb{B}_d, \mathbb{C}^d)$.

Theorem 4.6. *Let $K^\sharp$ be a kernel of the form ($\sharp$) and $K^{\sharp\sharp}$ be a kernel of the form ($\sharp\sharp$). Assume that the d-tuples $\boldsymbol{M}$ on the Hilbert space $\mathcal{H}_{K^\sharp}(\mathbb{B}_d, \mathbb{C}^d)$ and $\mathcal{H}_{K^{\sharp\sharp}}(\mathbb{B}_d, \mathbb{C}^d)$ are bounded. Then*

(1) if $d > 2$, these two d-tuples are unitarily equivalent if and only if $a_{\ell,1} = a_{\ell,2} = \tilde{a}_{\ell,1} = \tilde{a}_{\ell,2}$, $\ell \in \mathbb{N}$.
(2) if $d = 2$, these two d-tuples are unitarily equivalent if and only if $a_{\ell,1} = \tilde{a}_{\ell,2}$ and $a_{\ell,2} = \tilde{a}_{\ell,1}$, $\ell \in \mathbb{N}$.

Proof. The idea of the proof of part (1) is the same as that of the proof for Theorem 4.5. As in that proof, expanding $K^\sharp$ and $K^{\sharp\sharp}$ and assume that there is a unitary U intertwining all the coefficients described in (4.3) and (4.4) with the ones described in the comments following these two equations. Assume that $a_{m,1} \neq a_{m,2}$ (and therefore $\tilde{a}_{m,1} \neq \tilde{a}_{m,2}$) for some $m \in \mathbb{N}$. For every fixed but arbitrary pair (i, j), we must have

$$(a_{m,1} - a_{m,2})\Big(\sum_{k,\ell=1}^{d} U_{k,\ell} E_{k,\ell}\Big) E_{i,j} = (\tilde{a}_{m,1} - \tilde{a}_{m,2}) E_{i,j}^{\dagger}\Big(\sum_{k,\ell=1}^{d} U_{k,\ell} E_{k,\ell}\Big).$$

Since $E_{k,\ell}E_{i,j} = \delta_{\ell,i} E_{k,j}$, it follows that $\sum_{k,l} U_{k,\ell} E_{i,j} = \sum_k U_{k,i} E_{k,j}$. Similarly, $E_{i,j}^\dagger \sum_{k,l} U_{k,\ell} = \sum_\ell U_{i,l} E_{j,l}$. Thus for $j \neq i$, we have that $U_{i,j} = \lambda U_{j,i}$, $|\lambda| = 1$. Now, assume that $d > 2$. Moreover, for a fixed $k \neq i$, we have $U_{k,\ell} = 0 = U_{j,\ell}$, and for fixed $\ell \neq j$, we have $U_{j,\ell} = 0 = U_{k,\ell}$. Therefore for $d > 2$, we arrive at a contradiction unless $a_{\ell,1} = a_{\ell,2}$ and $\tilde{a}_{\ell,1} \neq \tilde{a}_{\ell,2}$ for all $\ell \in \mathbb{N}$, or that there is no unitary intertwiner.

The proof of part (2) involves verifying that the unitary $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ intertwines the two kernels whenever $a_{\ell,1} = \tilde{a}_{\ell,2}$ and $a_{\ell,2} = \tilde{a}_{\ell,1}$, $\ell \in \mathbb{N}$. $\square$

4.3. **Quasi-invariant diagonal kernels are invariant.** While there might be a characterization of all the invariant kernels on an arbitrary bounded symmetric domain Ω, unfortunately, we haven't been able to find one. Therefore, we have decided to include a description of all the $\mathcal{U}(d)$-invariant kernels for the special case of $\Omega = \mathbb{B}_d$, the only case that we are able to resolve. We begin by describing the kernels invariant under the group $\mathcal{U}(d)$.

Proposition 4.7. *Let $K : \mathbb{B}_d \times \mathbb{B}_d \to \mathcal{M}_n(\mathbb{C})$ be a non-negative definite kernel. Suppose K is invariant under $\mathcal{U}(d)$. Then K must be of the form $K(\boldsymbol{z}, \boldsymbol{w}) = \sum_{\ell=0}^{\infty} A_\ell \langle \boldsymbol{z},\, \boldsymbol{w}\rangle^\ell$, for some sequence $\{A_\ell\}_{\ell\in\mathbb{Z}_+}$ of positive definite $n \times n$ matrices.*

Proof. Let $K(\boldsymbol{z}, \boldsymbol{w}) = \sum_{\alpha,\beta\in\mathbb{Z}_+^d} A_{\alpha,\beta} \boldsymbol{z}^\alpha \overline{\boldsymbol{w}}^\beta$, $\boldsymbol{z}, \boldsymbol{w} \in \mathbb{B}_d$. Suppose that K is invariant under $\mathcal{U}(d)$, that is, $K(u\cdot \boldsymbol{z}, u\cdot \boldsymbol{w}) = K(\boldsymbol{z}, \boldsymbol{w})$, for all $\boldsymbol{z}, \boldsymbol{w} \in \mathbb{B}_d$ and $u \in \mathcal{U}(d)$. Choosing u to be the diagonal unitary matrices $\operatorname{diag}(e^{i\theta_1}, \ldots, e^{i\theta_d})$, $\theta := (\theta_1, \ldots, \theta_d) \in \mathbb{R}^d$, we get that

$$\sum_{\alpha,\beta\in\mathbb{Z}_+^d} A_{\alpha,\beta} \boldsymbol{z}^\alpha \overline{\boldsymbol{w}}^\beta e^{i(\alpha-\beta)\cdot\theta} = \sum_{\alpha,\beta\in\mathbb{Z}_+^d} A_{\alpha,\beta} \boldsymbol{z}^\alpha \overline{\boldsymbol{w}}^\beta,\, \boldsymbol{z}, \boldsymbol{w} \in \mathbb{B}_d,$$

where $(\alpha - \beta)\cdot\theta := (\alpha_1 - \beta_1)\theta_1 + \cdots + (\alpha_d - \beta_d)\theta_d$. Therefore we have

$$A_{\alpha,\beta}(e^{i((\alpha-\beta)\cdot\theta)} - 1) = 0, \text{ for all } \alpha, \beta \in \mathbb{Z}_+^d,\ \theta \in \mathbb{R}^d. \tag{4.5}$$

Let $\alpha, \beta \in \mathbb{Z}_+^d$ and $\alpha \neq \beta$. Then there exists m, $1 \leq m \leq d$, such that $\alpha_m \neq \beta_m$. Choosing $\theta_j = 0$ for all $j \neq m$ in (4.5), we obtain that $A_{\alpha,\beta} = 0$. Hence $K(\boldsymbol{z}, \boldsymbol{w})$ is of the form $\sum_{\alpha\in\mathbb{Z}_+^d} A_{\alpha,\alpha} \boldsymbol{z}^\alpha \overline{\boldsymbol{w}}^\alpha$. Now choosing u to be $u_{\boldsymbol{z}}$, we see that

$$K(\boldsymbol{z}, \boldsymbol{z}) = K(u_{\boldsymbol{z}}\cdot \boldsymbol{z}, u_{\boldsymbol{z}}\cdot \boldsymbol{z}) = K(\|\boldsymbol{z}\|e_1, \|\boldsymbol{z}\|e_1) = \sum_{\ell=0}^{\infty} A_{\ell\epsilon_1,\ell\epsilon_1} \|\boldsymbol{z}\|^{2\ell}.$$

By polarization, we get that $K(\boldsymbol{z}, \boldsymbol{w}) = \sum_{\ell=0}^{\infty} A_{\ell\epsilon_1,\ell\epsilon_1} \langle \boldsymbol{z},\, \boldsymbol{w}\rangle^\ell = \sum_{\ell=0}^{\infty} \tilde{A}_\ell \langle \boldsymbol{z},\, \boldsymbol{w}\rangle^\ell$, where $\tilde{A}_\ell = A_{\ell\epsilon_1,\ell\epsilon_1}$. Since K is non-negative definite, by [6, Lemma 4.1 (c)], it follows that $\tilde{A}_\ell$ is positive definite, completing the proof. $\square$

For any u in $\mathcal{U}(d)$ and $\alpha \in \mathbb{Z}_+^d$ with $|\alpha| = \ell$, let $X^u_{\alpha,\beta}$, $\beta \in \mathbb{Z}_+^d, |\beta| = \ell$, be the complex numbers given by

$$(u\cdot \boldsymbol{z})^\alpha = \sum_{|\beta|=\ell} X^u_{\alpha,\beta} \boldsymbol{z}^\beta. \tag{4.6}$$

We arrive at the same conclusion as that of Proposition 4.5 even if we assume that K is merely a quasi-invariant diagonal kernel. For the proof, we begin by proving a couple of preparatory lemmas.

Lemma 4.8. *For any $u \in \mathcal{U}(d)$, the matrix $\Big(\big((\frac{\beta!}{\alpha!})^{\frac{1}{2}} X^u_{\alpha,\beta}\big)\Big)_{|\alpha|=|\beta|=\ell}$ is unitary.*

Proof. Consider the space of homogeneous polynomials $\mathcal{P}_\ell$ endowed with the Fischer-Fock inner product. Note that $\{\frac{\boldsymbol{z}^\gamma}{(\gamma!)^{\frac{1}{2}}}\}_{|\gamma|=\ell}$ forms an orthonormal basis of $\mathcal{P}_\ell$ and $\Big(\big((\frac{\beta!}{\alpha!})^{\frac{1}{2}} X^u_{\alpha,\beta}\big)\Big)_{|\alpha|=|\beta|=\ell}$ is the matrix representation of the unitary map $p \to p \circ u$ with respect to this orthonormal basis. $\square$

Lemma 4.9. *There exists a unitary $u \in \mathcal{U}(d)$ such that $X^u_{\ell\varepsilon_1,\alpha} \neq 0$ for all $\alpha \in \mathbb{Z}_+^d$ with $|\alpha| = \ell$.*

Proof. Choose a unitary $u = (u_{ij})_{i,j=1}^d$ in $\mathcal{U}(d)$ such that $u_{1j} \neq 0$ for $j = 1, \ldots, d$. Since

$$(u\cdot \boldsymbol{z})^{\ell\varepsilon_1} = (u_{11}z_1 + \cdots + u_{1d}z_d)^\ell = \sum_{|\alpha|=\ell} \frac{\ell!}{\alpha!} u_{11}^{\alpha_1} \ldots u_{1d}^{\alpha_d}\ \boldsymbol{z}^\alpha,\ \alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}_+^d,$$

we get that $X^u_{\ell\varepsilon_1,\alpha} = \frac{\ell!}{\alpha!}u_{11}^{\alpha_1}\ldots u_{1d}^{\alpha_d}$, which is certainly non-zero by our choice of u. □

We now prove the main theorem of this section stated below using Lemma 4.8 and Lemma 4.9.

Theorem 4.10. *Let $\mathcal{H} \subset \mathrm{Hol}(\mathbb{B}_d, \mathbb{C}^n)$ be a reproducing kernel Hilbert space. Suppose that $\mathbb{C}^n$-valued polynomials are dense in $\mathcal{H}$ and $\langle \boldsymbol{z}^\alpha \otimes \boldsymbol{\xi}, \boldsymbol{z}^\beta \otimes \boldsymbol{\eta}\rangle = 0$, for all $\alpha \neq \beta$ in $\mathbb{Z}_+^d$ and $\boldsymbol{\xi}, \boldsymbol{\eta}$ in $\mathbb{C}^n$. If the d-tuple $\boldsymbol{M}$ on $\mathcal{H}$ is $\mathcal{U}(d)$-homogeneous, then there exists a sequence of positive definite $n \times n$ matrices $\{A_\ell\}_{\ell\in\mathbb{Z}_+}$ such that*

$$\|\boldsymbol{z}^\alpha \otimes \boldsymbol{\xi}\|^2 = \alpha!\langle A_{|\alpha|}\boldsymbol{\xi}, \boldsymbol{\xi}\rangle, \qquad \alpha \in \mathbb{Z}_+^d,\ \boldsymbol{\xi} \in \mathbb{C}^n.$$

Proof. Since $\boldsymbol{M}$ on $\mathcal{H}$ is $\mathcal{U}(d)$-homogeneous, by Lemma 2.3, for each $u \in \mathcal{U}(d)$ there exists a unitary $\Gamma(u)$ on $\mathcal{H}$ of the form

$$\Gamma(u)(f) = c(u) f \circ u,\ f \in \mathcal{H},$$

where $c(u) \in \mathcal{U}(n)$ for all $u \in \mathcal{U}(d)$. Let $\ell \in \mathbb{Z}_+$. For $\alpha, \beta \in \mathbb{Z}_+^d$ with $|\alpha| = |\beta| = \ell$, $\alpha \neq \beta$, and $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{C}^n$, we have

$$\begin{aligned}\langle \Gamma(u)(\boldsymbol{z}^\alpha \otimes \boldsymbol{\xi}), \Gamma(u)(\boldsymbol{z}^\beta \otimes \boldsymbol{\eta})\rangle &= \langle (u\cdot\boldsymbol{z})^\alpha \otimes c(u)\boldsymbol{\xi}, (u\cdot\boldsymbol{z})^\beta \otimes c(u)\boldsymbol{\eta}\rangle \\ &= \langle \sum_{|\gamma|=\ell} X^u_{\alpha,\gamma}\boldsymbol{z}^\gamma \otimes c(u)\xi, \sum_{|\delta|=\ell} X^u_{\beta,\delta}\boldsymbol{z}^\delta \otimes c(u)\boldsymbol{\eta}\rangle \\ &= \sum_{|\gamma|=\ell} X^u_{\alpha,\gamma}\overline{X^u_{\beta,\gamma}}\langle \boldsymbol{z}^\gamma \otimes c(u)\boldsymbol{\xi}, \boldsymbol{z}^\gamma \otimes c(u)\boldsymbol{\eta}\rangle. \end{aligned}\tag{4.7}$$

Since $\Gamma(u)$ is unitary and $\langle \boldsymbol{z}^\alpha \otimes \boldsymbol{\xi}, \boldsymbol{z}^\beta \otimes \boldsymbol{\eta}\rangle = 0$, it follows that $\langle \Gamma(u)(\boldsymbol{z}^\alpha \otimes \boldsymbol{\xi}), \Gamma(u)(\boldsymbol{z}^\beta \otimes \boldsymbol{\eta})\rangle = 0$. Hence from (4.7) we obtain

$$\sum_{|\gamma|=\ell} X^u_{\alpha,\gamma}\overline{X^u_{\beta,\gamma}}\langle \boldsymbol{z}^\gamma \otimes c(u)\boldsymbol{\xi}, \boldsymbol{z}^\gamma \otimes c(u)\boldsymbol{\eta}\rangle = 0. \tag{4.8}$$

Since $c(u)$ is unitary and the above equality holds for all $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{C}^n$, we get

$$\sum_{|\gamma|=\ell} X^u_{\alpha,\gamma}\overline{X^u_{\beta,\gamma}}\langle \boldsymbol{z}^\gamma \otimes \boldsymbol{\xi}, \boldsymbol{z}^\gamma \otimes \boldsymbol{\eta}\rangle = 0. \tag{4.9}$$

By Lemma 4.9, there exists a unitary $u_0 \in \mathcal{U}(d)$ such that $X^{u_0}_{\ell\varepsilon_1,\gamma} \neq 0$ for all γ with $|\gamma| = \ell$. Choosing $\alpha = \ell\varepsilon_1$ and $u = u_0$ in (4.9), we get for all $\beta \neq \ell\varepsilon_1$ with $|\beta| = \ell$,

$$\sum_{|\gamma|=\ell} X^{u_0}_{\ell\varepsilon_1,\gamma}\langle \boldsymbol{z}^\gamma \otimes \boldsymbol{\xi}, \boldsymbol{z}^\gamma \otimes \boldsymbol{\eta}\rangle\ \overline{X^{u_0}_{\beta,\gamma}} = 0. \tag{4.10}$$

Hence it follows from Lemma 4.8 that

$$X^{u_0}_{\ell\varepsilon_1,\gamma}\langle \boldsymbol{z}^\gamma \otimes \boldsymbol{\xi}, \boldsymbol{z}^\gamma \otimes \boldsymbol{\eta}\rangle = \chi_{\ell,\boldsymbol{\xi},\boldsymbol{\eta}}\ \gamma! X^{u_0}_{\ell\varepsilon_1,\gamma},$$

that is, $\langle \boldsymbol{z}^\gamma \otimes \boldsymbol{\xi}, \boldsymbol{z}^\gamma \otimes \boldsymbol{\eta}\rangle = \chi_{\ell,\boldsymbol{\xi},\boldsymbol{\eta}}\ \gamma!$, for all γ with $|\gamma| = \ell$ and for some constant $\chi_{\ell,\boldsymbol{\xi},\boldsymbol{\eta}}$. Clearly there exists a $n \times n$ positive definite matrix A_ℓ such that

$$\langle A_\ell\boldsymbol{\xi}, \boldsymbol{\eta}\rangle_{\mathbb{C}^n} = \chi_{\ell,\boldsymbol{\xi},\boldsymbol{\eta}},\ \boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{C}^n.$$

This completes the proof. □

As a corollary, we conclude that a quasi-invariant non-negative definite diagonal kernel defined on the Euclidean ball must necessarily be invariant.

Corollary 4.11. *Let $K : \mathbb{B}_d \times \mathbb{B}_d \to \mathcal{M}_n(\mathbb{C})$ be a non-negative definite kernel such that $\partial^\alpha\bar\partial^\beta K(0,0) = 0$ whenever $\alpha \neq \beta$. Suppose that $\mathbb{C}^n$-valued polynomials are dense in $\mathcal{H}_K(\mathbb{B}_d, \mathbb{C}^n)$. If K is quasi-invariant under $\mathcal{U}(d)$ then it must be of the form $K(\boldsymbol{z}, \boldsymbol{w}) = \sum_\ell A_\ell^{-1}\frac{\langle \boldsymbol{z}, \boldsymbol{w}\rangle^\ell}{\ell!}$, where A_ℓ is a positive invertible $n \times n$ matrix for all $\ell \in \mathbb{Z}_+$.*

Proof. Since K is quasi-invariant under $\mathcal{U}(d)$, by Lemma 2.3, the d-tuple $\boldsymbol{M}$ on $\mathcal{H}_K(\mathbb{B}_d, \mathbb{C}^n)$ is $\mathcal{U}(d)$-homogeneous. It follows from Theorem 4.10 that the set

$$\Big\{\frac{1}{\sqrt{\alpha!}}\boldsymbol{z}^\alpha A_\ell^{-1/2}\varepsilon_i : 1 \leq i \leq n, |\alpha| = \ell\Big\}$$

forms an orthonormal basis for the space of $\mathbb{C}^n$-valued homogeneous polynomial $\mathcal{P}_\ell \otimes \mathbb{C}^n$ in $\mathcal{H}_K(\mathbb{B}_d, \mathbb{C}^n)$, where A_ℓ is a positive definite invertible $n \times n$ matrix for all $\ell \in \mathbb{Z}_+$. Equivalently, the reproducing kernel K_ℓ of the (finite dimensional) Hilbert space $\mathcal{P}_\ell \otimes \mathbb{C}^n$ is given by the formula:

$$K_\ell(\boldsymbol{z}, \boldsymbol{w}) = A_\ell^{-1}\frac{\langle \boldsymbol{z}, \boldsymbol{w}\rangle^\ell}{\ell!}.$$

Thus the kernel K must be of the $K(\boldsymbol{z}, \boldsymbol{w}) = \sum_\ell A_\ell^{-1}\frac{\langle \boldsymbol{z}, \boldsymbol{w}\rangle^\ell}{\ell!}$ for all $\boldsymbol{z}, \boldsymbol{w} \in \mathbb{B}^d$. This proves the result. □

There are several separate equivalent assertions that are implicit in the previous corollary. We list them below.

(1) the inner product on $\mathcal{P}_\ell \otimes \mathbb{C}^n$ is given by the usual Hilbert space tensor product of the two finite dimensional Hilbert spaces, namely, $\big(\mathcal{P}_\ell, \langle\cdot, \cdot\rangle_{\mathcal{F}_\ell}\big)$ and $\big(\mathbb{C}^n, \langle\cdot, \cdot\rangle_{A_\ell}\big)$, where $\langle \boldsymbol{\xi}, \boldsymbol{\eta}\rangle_{A_\ell} = \langle A_\ell\boldsymbol{\xi}, \boldsymbol{\eta}\rangle_{\mathbb{C}^n}$.

(2) The set $\Big\{\frac{1}{\sqrt{\alpha!}}\boldsymbol{z}^\alpha A_\ell^{-1/2}\varepsilon_i : 1 \leq i \leq n, |\alpha| = \ell\Big\}$ form an orthonormal basis for $\mathcal{P}_\ell \otimes \mathbb{C}^n$.

(3) The kernel function K_ℓ on the (finite dimensional) Hilbert space $\big(\mathcal{P}_\ell, \langle\cdot, \cdot\rangle_{\mathcal{F}_\ell}\big) \otimes \big(\mathbb{C}^n, \langle\cdot, \cdot\rangle_{A_\ell}\big)$ is given by the formula:

$$K_\ell(\boldsymbol{z}, \boldsymbol{w}) = A_\ell^{-1}\frac{\langle \boldsymbol{z}, \boldsymbol{w}\rangle^\ell}{\ell!},\ \boldsymbol{z}, \boldsymbol{w} \in \mathbb{B}^d.$$

(4) The kernel function K of the Hilbert space $\mathcal{H}_K(\mathbb{B}_d, \mathbb{C}^n)$ is of the form $K(\boldsymbol{z}, \boldsymbol{w}) = \sum_\ell A_\ell^{-1}\frac{\langle \boldsymbol{z}, \boldsymbol{w}\rangle^\ell}{\ell!}$.

5. Classification

Before we discuss the question of classification of $\mathcal{U}(d)$-homogeneous operators, we note that some of our results exist in the representation theory literature albeit somewhat disguised. We believe unraveling this relationship would serve a useful purpose.

5.1. Decomposition of tensor product of $\pi_1 \otimes \pi_\ell$ and $\bar{\pi}_1 \otimes \pi_\ell$. There is an alternative but equivalent description of the representations $\tilde{\pi}_\ell$ and $\hat{\pi}_\ell$, given below, which is also useful. For this, we identify the space of linear polynomials $\mathcal{P}_1$ as the dual of the linear space $\mathbb{C}^d$. We define $\phi : \mathbb{C}^d \otimes \mathcal{P}_\ell \to \mathcal{P}_1 \otimes \mathcal{P}_\ell$ by setting

$$\phi\Big(\sum_{i=1}^d \boldsymbol{e}_i p_\ell^i\Big)(\boldsymbol{z}, \boldsymbol{w}) = \sum_{i=1}^d z_i p_\ell^i(\boldsymbol{w}),\ \boldsymbol{z},\ \boldsymbol{w} \in \mathbb{B}_d.$$

Therefore we see that $\mathrm{Im}\,(\phi)$ is the space $\mathcal{P}_1 \otimes \mathcal{P}_\ell$ of homogeneous polynomials of degree $\ell + 1$ in $2d$-variables. Since the monomials $z_1, \ldots, z_d$ form an orthonormal basis in $\mathcal{P}_1$ with respect to the Fisher-Fock inner product, it follows that ϕ is unitary. Hence, $\tilde{\pi}_\ell$ is unitarily equivalent, via ϕ, with $\pi_1 \otimes \pi_\ell$, where

$$(\pi_1(u) \otimes \pi_\ell(u))\, p(\boldsymbol{z}, \boldsymbol{w}) = p(u^{-1}\cdot \boldsymbol{z}, u^{-1}\cdot \boldsymbol{w}),\ p \in \mathcal{P}_1 \otimes \mathcal{P}_\ell.$$

The contragredient of the representation π_1 is the defined to be the representation $\overline{\pi}_1(u)p_1(\boldsymbol{z}) := p_1(u^\dagger \cdot \boldsymbol{z})$, $p_1 \in \mathcal{P}_1$, we have

$$(\overline{\pi}_1(u) \otimes \pi_\ell(u))p(\boldsymbol{z}, \boldsymbol{w}) = p(u^\dagger \cdot \boldsymbol{z}, u^{-1}\cdot \boldsymbol{w}), p \in \mathcal{P}_1 \otimes \mathcal{P}_\ell.$$

Again, ϕ intertwines $\hat{\pi}_\ell$ and $\overline{\pi}_1 \otimes \pi_\ell$:

$$\begin{aligned}(\phi\hat{\pi}_\ell(u))f(\boldsymbol{w}) &= \sum_{i=1}^{d} z_i(u(f\circ u^{-1}))_i(\boldsymbol{w}) \\ &= \langle u(f\circ u^{-1})(\boldsymbol{w}),\, \overline{\boldsymbol{z}}\rangle_{\mathbb{C}^d} \\ &= \langle (f\circ u^{-1})(\boldsymbol{w}),\, \overline{u^\dagger\cdot \boldsymbol{z}}\rangle_{\mathbb{C}^d} \\ &= \sum_{i=1}^{d}(u^\dagger\cdot\boldsymbol{z})_i f_i(u^{-1}\cdot\boldsymbol{w}) = (\overline{\pi}_1(u)\otimes\pi_\ell(u))\phi(f)(\boldsymbol{w}),\end{aligned}$$

where $u\in\mathcal{U}(d)$ and $f = \begin{pmatrix} f_1\\ \vdots \\ f_d\end{pmatrix} \in \mathbb{C}^d\otimes\mathcal{P}_\ell$.

Let $\overline{S}_\ell = (\mathcal{P}_\ell, \pi_\ell)$ and $S_1 = (\mathcal{P}_1, \overline{\pi}_1)$. Note that in the standard terminology of representation theory, the representation $\tilde{\pi}_\ell \sim_u \pi_1\otimes\pi_\ell$ is $\bar{S}_1\otimes\bar{S}_\ell$, where $\sim_u$ stands for unitary equivalence of the two representations. Similarly, $\hat{\pi}_\ell \sim_u \overline{\pi}_1\otimes\pi_\ell$ is $S_1\otimes\bar{S}_\ell$. From Equation (23.12) of [16], we see that

$$S_1\otimes\overline{S}_\ell = D_{(1,0,\ldots,0,-\ell)}\oplus D_{(0,\ldots,0,1-\ell)}, \tag{5.1}$$

where $D_{(0,\ldots,0,1-\ell)}\sim_u \overline{S}_{\ell-1}$ and using Proposition 23.3 of [16], it follows that $D_{(1,0,\ldots,0,-\ell)}$ is unitarily equivalent to the restriction of the representation $\hat{\pi}_\ell$ to the subspace $\widehat{\mathcal{V}}_\ell\subset\mathbb{C}^d\otimes\mathcal{P}_\ell$ via the map ϕ. Note that the restriction of $\hat{\pi}_\ell$ to $\widehat{\mathcal{V}}_\ell$ is irreducible (refer to Corollary 3.14) and the representation $\hat{\pi}_\ell$ has exactly two irreducible components, see (5.1). Therefore, we have proved the following theorem.

Theorem 5.1. *The subspaces $\widehat{\mathcal{V}}_\ell$ and $\widehat{\mathcal{V}}_\ell^\perp$ of $\mathbb{C}^d\otimes\mathcal{P}_\ell$ are reducing for the representation $\hat{\pi}_\ell$, moreover, the restriction of $\hat{\pi}_\ell$ to these subspaces are irreducible.*

One would like to obtain a similar decomposition of $\tilde{\pi}_\ell$ into irreducible representations as in Theorem 5.1. However, such a decomposition appears to be not available in any explicit form. This, we provide below. Clearly,

$$\mathrm{Im}\,(\phi) = \phi(\tilde{\mathcal{V}}_\ell)\oplus\phi(\tilde{\mathcal{V}}_\ell^\perp),$$

where

1. $\phi(\tilde{\mathcal{V}}_\ell) = \{p(\boldsymbol{z},\boldsymbol{w}) = \sum_{i=1}^d z_i p_\ell^i(\boldsymbol{w})\in\mathcal{P}_1\otimes\mathcal{P}_\ell : p_{|\mathrm{res}\,\Delta} = 0\}$, where $\Delta := \{(\boldsymbol{z},\boldsymbol{z}) : \boldsymbol{z}\in\mathbb{B}_d\}$,
2. $\phi(\tilde{\mathcal{V}}_\ell^\perp) = \{\sum_{i=1}^d z_i\partial_i q_{\ell+1}(\boldsymbol{w})\in\mathcal{P}_1\otimes\mathcal{P}_\ell : q_{\ell+1}\in\mathcal{P}_{\ell+1}\}$.

Also, we note that $\phi(\tilde{\mathcal{V}}_\ell^\perp) = \{p_{|\mathrm{res}\,\Delta} : p\in\mathcal{P}_1\otimes\mathcal{P}_\ell\}$. Since $\tilde{\mathcal{V}}_\ell$ is invariant under $\tilde{\pi}_\ell$ and ϕ is an intertwining map between $\tilde{\pi}_\ell$ and $\pi_1\otimes\pi_\ell$, it follows that $\phi(\tilde{\mathcal{V}}_\ell)$ is invariant under $\pi_1\otimes\pi_\ell$. Let $R:\mathcal{P}_1\otimes\mathcal{P}_\ell\to\mathcal{P}_{\ell+1}$ be the restriction map, that is, $Rp(\boldsymbol{z},\boldsymbol{w}) := p(\boldsymbol{z},\boldsymbol{z}) = \sum_{i=1}^d z_i p_\ell^i(\boldsymbol{z})$. Thus we have proved the lemma that follows.

Lemma 5.2. *The map R on $\phi(\tilde{\mathcal{V}}_\ell^\perp)$ is onto $\mathcal{P}_{\ell+1}$ and is isometric when $\mathcal{P}_{\ell+1}$ is equipped with the Fischer-Fock inner product. Moreover, $R(\pi_1(u)\otimes\pi_\ell(u))R^* = \pi_{\ell+1}(u)$.*

As before, since $\pi_{\ell+1}$ is an irreducible representation, the proof of the theorem stated below follows from Lemma 5.2.

Theorem 5.3. *The subspaces $\tilde{\mathcal{V}}_\ell$ and $\tilde{\mathcal{V}}_\ell^\perp$ of $\mathbb{C}^d\otimes\mathcal{P}_\ell$ are reducing for the representation $\tilde{\pi}_\ell$, moreover, the restriction of $\tilde{\pi}_\ell$ to these subspaces are irreducible.*

We point out that half of Theorems 5.1 and 5.3 has been already proved in Corollary 3.14. The remaining half can also be proved in a similar manner to that of the proof in Corollary 3.14. However, we believe the proof we have given here is more revealing.

Recall the decomposition of $\tilde{K}^{(\alpha,\beta)}(\boldsymbol{z},\boldsymbol{w})$ given in Theorem 3.7. Let $\Lambda = \mathbb{Z}_+$. For $\lambda \in \Lambda$, choosing

$$b_\lambda = \begin{cases} \alpha_j, \text{ if } \lambda = 2j+1 \\ \beta_j, \text{ if } \lambda = 2j, \end{cases}$$

and setting

$$K_\lambda(\boldsymbol{z},\boldsymbol{w}) = \begin{cases} \tilde{K}_j(\boldsymbol{z},\boldsymbol{w}), \text{ if } \lambda = 2j+1 \\ \tilde{K}_j^{\perp}(\boldsymbol{z},\boldsymbol{w}), \text{ if } \lambda = 2j, \end{cases}$$

we obtain a second decomposition of the kernel $\tilde{K}^{(\alpha,\beta)}$ from Theorem 2.7 that coincides with the previous one from Theorem 3.7. A similar statement can be made about the kernel $\widehat{K}^{(\alpha,\beta)}$ appearing in Theorem 3.11.

5.2. **Classification.** The natural action of the unitary group $\mathcal{U}(d)$ on $\mathbb{C}^d \otimes \mathcal{P}$ associated with the multiplier c is given by $p \to c(u)(p \circ u^{-1})$, $p \in \mathbb{C}^d \otimes \mathcal{P}$ and $u \in \mathcal{U}(d)$. We obtain two classes of $\mathcal{U}(d)$-homogeneous d-tuple of operators with respect to two different multipliers $c(u) = \bar{u}$ (see Theorem 3.7) and $c(u) = u$ (see Theorem 3.11). The map $u \mapsto \bar{u}$ and $u \mapsto u$ are d-dimensional irreducible unitary representations of the group $\mathcal{U}(d)$.

The classification of finite dimensional irreducible unitary representations of the unitary group $\mathcal{U}(n)$ is well studied. The result is summarized in [16, Proposition 22.2] and is reproduced below for ready reference.

Proposition 5.4. *Each irreducible unitary representation of $\mathcal{U}(n)$ restricts to an irreducible unitary representation of* $\mathrm{SU}(n)$*, and all irreducible unitary representations of* $\mathrm{SU}(n)$ *are obtained in this fashion. Furthermore, two irreducible unitary representations π_1 and π_2 of $\mathcal{U}(n)$ restrict to the same representation of* $\mathrm{SU}(n)$ *if and only if, for some $j \in \mathbb{Z}$,*

$$\pi_2(g) = (\det g)^j \pi_1(g), \quad \forall g \in \mathcal{U}(n).$$

Hence the set of equivalence classes of irreducible unitary representations of $\mathrm{SU}(n)$ *is parametrized by*

$$\{(d_1, \ldots, d_{n-1}, 0) \in \mathbb{Z}^n : d_1 \geq d_2 \geq \cdots \geq d_{n-1} \geq 0\}$$

Also, recall the Weyl dimension formula for an irreducible unitary representation π of $\mathcal{U}(n)$ with weights: $w_1 \geq \cdots \geq w_n$, $w_i \in \mathbb{Z}$, [15, Theorem 11.4] (see also [3, Proposition 2.5]),

$$\dim \pi = \prod_{1 \leq j < k \leq n} \frac{w_j - w_k + k - j}{k - j}.$$

Combining Proposition 5.4 with the Weyl dimension formula, we find all the d-dimensional representations of $SU(d)$. The representations of $\mathcal{U}(d)$ can be then made up from the ones for $SU(d)$ using the relationship between these representations prescribed in Proposition 5.4 as follows. The d-dimensional (inequivalent, irreducible and unitary) representations of the group $\mathcal{U}(d)$ are determined by weights of the form: $(\ell+1, \ell, \ldots, \ell)$ and $(m, \ldots, m, m-1)$, $\ell, m \in \mathbb{Z}$. As noted in [16, Proposition 22.2], the representation ρ_ℓ corresponding to the weight $(\ell+1, \ell, \ldots, \ell)$ differs from ρ_0 by a power of the determinant: $\rho_\ell(u) = (\det(u))^\ell \rho_0(u)$, $u \in \mathcal{U}(d)$. The representation $\bar{\rho}_m$ corresponding to $(m, \ldots, m, m-1)$ is similarly related to $\bar{\rho}_0$. We also point out that $\bar{\rho}_0$ is the contragredient of ρ_0. We claim that ρ_ℓ and $\bar{\rho}_m$ are the only d-dimensional irreducible unitary representations of $\mathcal{U}(d)$ up to unitary equivalence (Lemma 5.5). We also claim that $SU(d)$ has no irreducible unitary representation of dimension $2, \ldots, d-1$ (Lemma 5.6).

It might be that both of these results are well-known, although, we are not able to locate them. However, A. Koranyi in private communication to one of the authors, has provided a very short proof of Lemma 5.6 using Lie algebraic machinery. A little more effort gives a proof of Lemma 5.5 as well, thanks to A. Khare, E. K. Narayanan, and C. Varughese. However, here we give, what we consider to be an elementary proof of these assertions.

Lemma 5.5. *Suppose that $c : \mathcal{U}(d) \to \mathrm{GL}_d(\mathbb{C})$ is an irreducible unitary representation of $\mathcal{U}(d)$. Then, up to unitary equivalence, either $c(u) = \det(u)^\ell \bar{u}$ or $c(u) = \det(u)^m u$, $\ell, m \in \mathbb{Z}$.*

Lemma 5.6. *If $n \in \mathbb{N}$: $2 \leq n \leq d-1$, then there is no n-dimensional irreducible unitary representation of $\mathcal{U}(d)$, or that of $SU(d)$.*

B. Bagchi has observed that Lemma 5.5 and 5.6 can be combined into the following assertion.
Let $w_1 \geq \cdots \geq w_d = 0$ be integers. Then, either $w_1 = \cdots = w_d = 0$, or $\prod_{\substack{1\leq j<k\leq d\\ w_d=0}} \big(1 + \frac{w_j - w_k}{k-j}\big) \geq d$. Equality holds in this inequality if and only if either $w_1 = \cdots = w_{d-1} = 1, w_d = 0$ or $w_1 = 1$ and $w_2 = \cdots = w_d = 0$. The proof is then by induction on the dimension d similar to the proofs we give below.

The first half of Theorem 5.7 below describing all the quasi-invariant kernels, which transform as in Definition 1.3 via an irreducible d-dimensional unitary representation c of $\mathcal{U}(d)$, is an immediate consequence of Lemma 5.5 combined with Theorem 3.7 (resp. Theorem 3.11) and Theorem 3.13. The second half follows from Lemma 5.6. We would have liked to prove a similar classification theorem for all the $\mathcal{U}(d)$-homogeneous operators in the class $\mathcal{A}_d\mathcal{U}(\mathbb{B}_d)$. However, unfortunately, such a classification doesn't follow immediately from the theorem below and requires further investigation.

Theorem 5.7. *Let $K : \mathbb{B}_d \times \mathbb{B}_d \to \mathcal{M}_n(\mathbb{C})$ be a non-negative definite kernel.*

(a) *Suppose that $n = d$, and K is quasi-invariant under $\mathcal{U}(d)$ with respect to the multiplier c, where $c : \mathcal{U}(d) \to \mathrm{GL}_d(\mathbb{C})$ is an irreducible unitary representation. Then there exists $U \in \mathcal{U}(d)$ such that $UK(\boldsymbol{z}, \boldsymbol{w})U^*$ is either of the form*

$$\sum_{\ell=1}^{\infty} \big(a_{\ell,1} - a_{\ell,2}\big)\langle \boldsymbol{z}, \boldsymbol{w}\rangle^{\ell-1} \overline{\boldsymbol{w}} \boldsymbol{z}^\dagger + \sum_{\ell=0}^{\infty} a_{\ell,2} \langle \boldsymbol{z}, \boldsymbol{w}\rangle^\ell I_d, \quad \boldsymbol{z}, \boldsymbol{w} \in \mathbb{B}_d,$$

where $a_{\ell,1} \geq 0$ and $a_{\ell,1} \leq (\ell+1)a_{\ell,2}$ for all *$\ell \in \mathbb{Z}_+$, or of the form*

$$\sum_{\ell=1}^{\infty} (\tilde{a}_{\ell,1} - \tilde{a}_{\ell,2})\langle \boldsymbol{z}, \boldsymbol{w}\rangle^{\ell-1} \boldsymbol{z}\overline{\boldsymbol{w}}^\dagger + \sum_{\ell=0}^{\infty} \tilde{a}_{\ell,2} \langle \boldsymbol{z}, \boldsymbol{w}\rangle^\ell I_d, \quad \boldsymbol{z}, \boldsymbol{w} \in \mathbb{B}_d,$$

where $\tilde{a}_{\ell,2} \geq 0$ and $(d-1)\tilde{a}_{\ell,2} \leq (\ell + d - 1)\tilde{a}_{\ell,1}$ for all $\ell \in \mathbb{Z}_+$.

(b) *If $1 < n < d$, then there is no n-dimensional irreducible unitary representation c such that K is quasi-invariant under $\mathcal{U}(d)$ with multiplier $c : \mathcal{U}(d) \to GL_n(\mathbb{C})$.*

5.3. **Elementary proof of Lemma 5.5 and of Lemma 5.6.**

Proof of Lemma 5.5. We begin the proof with the *claim* that any irreducible unitary representation, up to unitary equivalence, of $SU(d)$ acting on $\mathbb{C}^d$ are the ones determined by the weights: $(1, 0, \ldots, 0)$ and $(1, \ldots, 1, 0)$. In other words, we have to show that the only (admissible) weights $w = (w_1, \ldots, w_{d-1}, 0)$ for which

$$\prod_{\substack{1\leq j<k\leq d\\ w_d=0}} \frac{w_j - w_k + k - j}{k-j} = d \tag{5.2}$$

are of the form: $(1, 0, \ldots, 0)$ or $(1, 1, \ldots, 1, 0)$.

For $d = 2$, the claim is evident from the dimension formula. Assume that the claim is valid for $d - 1$, that is, if

$$\prod_{\substack{1\leq j<k\leq d-1\\ w_{d-1}=0}} \frac{w_j - w_k + k - j}{k-j} = d - 1,$$

then there are only two alternatives for w, namely, either $w = (1, 0, \ldots, 0)$, or $w = (1, \ldots, 1, 0)$.

Let $w=(w_1,\ldots,w_{d-1},0)$ be a weight satisfying the equality in the dimension formula (5.2). Splitting the product in (5.2), we have

$$\prod_{\substack{1\le j<k\le d\\ w_d=0}}\frac{w_j-w_k+k-j}{k-j}=\prod_{1\le j<k\le d-1}\frac{w_j-w_k+k-j}{k-j}\prod_{1\le j\le d-1}\frac{w_j+d-j}{d-j}. \tag{5.3}$$

We shall consider three possibilities, namely,

$$\prod_{1\le j<k\le d-1}\frac{w_j-w_k+k-j}{k-j}=d-1 \tag{5.4}$$

and the two other possibilities of being strictly greater than $d-1$ and less than $d-1$. First, consider the case of equality. In this case, the weight $\hat{w}=(w_1,\ldots,w_{d-1})$ satisfying (5.4) determines a irreducible unitary representation of $\mathcal{U}(d-1)$ of dimension $d-1$. But this is also the dimension of the irreducible unitary representation of $SU(d-1)$ determined by $(w_1-w_{d-1},w_2-w_{d-1},\ldots,w_{d-2}-w_{d-1},0)$. Then by the induction hypothesis, we either have $w_1=w_{d-1}+1, w_2=\cdots=w_{d-2}=w_{d-1}$ or $w_1=w_2=\cdots=w_{d-2}=w_{d-1}+1$. Therefore, the weight w of size d must be of the form $(m,m-1,\ldots,m-1,0)$, or $(m,\ldots,m,m-1,0)$, $m\ge 1$. In case of the first alternative, to ensure validity of (5.2), we must also have

$$\frac{d}{d-1}=\prod_{1\le j\le d-1}\frac{w_j+d-j}{d-j}\Big(=\frac{(m+d-1)(m+d-3)\cdots(m+2)\cdot(m+1)\cdot m}{(d-1)(d-2)\cdots 2\cdot 1}\Big).$$

This is possible only if $m=1$ providing one of the two choices in the induction step. In case of the second alternative, $w=(m,\ldots,m,m-1,0)$, and we have

$$\prod_{1\le j\le d-1}\frac{w_j+d-j}{d-j}=\frac{(m+d-1)(m+d-2)\cdots(m+2)\cdot m}{(d-1)(d-2)\cdots 2\cdot 1}.$$

Since $m\ge 1$, it follows that the smallest possible value of this product is $\frac{d}{2}$ and it is achieved at $m=1$. Thus it cannot equal $\frac{d}{d-1}$ unless $d=3$. But if $d=3$, and $m=1$, the weight of size 2 from the induction hypothesis is of the form $(1,0)$. So, we get nothing new when $d=3$.

Now, if possible, suppose that $\prod_{1\le j<k\le d-1}\frac{w_j-w_k+k-j}{k-j}\ge d$. Then we must have

$$\prod_{1\le j\le d-1}\frac{w_j+d-j}{d-j}\le 1,$$

which is evidently false unless $w_j=0$, $1\le j\le d-1$. But if we choose $w=(0,\ldots,0)$, then we can't have equality in Equation (5.2), therefore it is not an admissible choice.

Finally, let us suppose that $1\le\prod_{1\le j<k\le d-1}\frac{w_j-w_k+k-j}{k-j}=\ell\le d-2$. First, if $\ell=1$, the only possible choice of the weight w is $w_1=\cdots=w_{d-1}$. We must then ensure that

$$\prod_{1\le j\le d-1}\frac{w_j+d-j}{d-j}=d,$$

which is possible only if $w_1=\cdots=w_{d-1}=1$. This, together with the choice $w_d=0$ that we have made earlier, proves that $w=(1,\ldots,1,0)$ providing the second choice in the induction step. In particular, the dimension of the representation determined by the weight $(1,1,\ldots,1,0)$ is d. Now, we must establish that there is no other choice of w satisfying (5.2). This follows from Lemma 5.6 proved below. It is also easy to verify directly: If $d=2$ or 3, there is nothing more to be done. If $d>3$, then fix $\ell: 2\le\ell\le d-2$, and pick w such that $\prod_{1\le j<k\le d-1}\frac{w_j-w_k+k-j}{k-j}=d-\ell$. Having picked w, we also need

$$\frac{d}{d-\ell}=\prod_{1\le j\le d-1}\frac{w_j+d-j}{d-j},$$

that is,

$$d! = (w_1 + d - 1)\cdots(w_\ell + d - \ell)(d-\ell)(w_{\ell+1} + d - \ell - 1)\cdots(w_{d-1}+1),$$

which is valid only if w is of the form $(1,\ldots,1,w_\ell = 1,0,\ldots,0)$. For this choice of w, we see that

$$\prod_{1\le j<k\le d-1} \frac{w_j - w_k + k - j}{k-j} = \binom{d-1}{\ell},$$

which can't be equal to ℓ for any $d > 3$. So, there are no more admissible weights in this case. This completes the verification of the induction step and therefore the proof of the claim. Now, the result follows directly from Proposition 5.4. ☐

Proof of Lemma 5.6. The proof is by induction on the dimension d. The base case of $d = 3$ is easily verified. Now, we assume by the induction hypothesis, that there are no irreducible unitary representation such that

$$2 \le t := \prod_{1\le j<k\le d-1} \frac{w_j - w_k + k - j}{k-j} \le d-2.$$

Thus the only choice for t is either $t = 1$, or $t \ge d-1$. To complete the induction step, we have to show that there is no weight $w = (w_1,\ldots,w_{d-1},0)$ such that

$$2 \le \ell := \prod_{\substack{1\le j<k\le d\\ w_d=0}} \frac{w_j - w_k + k - j}{k-j} \le d-1.$$

If $t = 1$, then the only possible choice of the weight w is $w_1 = \cdots = w_{d-1}$, say u. From Equation (5.3), it follows that

$$\prod_{1\le j\le d-1} \frac{u+d-j}{d-j} = \ell.$$

However since the product on the left hand side of the equation above is an increasing function of u and its smallest value is 1, the next possible value is d, it follows that the value $\ell : 2 \le \ell \le d-1$ is not taken. Now, let $t \ge d-1$ for some w. Then from Equation (5.3), we see that

$$\frac{\ell}{t} = \prod_{1\le j\le d-1} \frac{w_j + d - j}{d-j}$$

to ensure the existence of a ℓ-dimensional representation. Since $\frac{\ell}{t} \le 1$ while the product on the right hand side of the equation above is greater or equal to 1, it follows that the two sides can be equal only if $w_1 = \cdots = w_{d-1} = 0$. But then t must be equal to 1 contrary to our hypothesis. ☐

A. Koranyi has pointed out that $SU(d)$ is a simple Lie group with discrete center and its Lie algebra $su(d)$ is simple. Therefore any non-trivial homomorphism of it can have at most a discrete null space, i.e., has to be a local isomorphism. So the image of a representation is a closed subgroup of $\mathcal{U}(n)$, therefore must have the same dimension (as a Lie group) as $SU(d)$. If $d > n$, then this is not possible proving Lemma 5.6.

E. K. Narayanan observed that a proof of Lemma 5.5 follows from the description of the Lie algebra homomorphisms from $su(d)$ to $u(d)$, the Lie algebra of $\mathcal{U}(d)$. A. Khare and C. Varughese independently of each other have provided the following argument proving Lemma 5.5: Since $su(d)$ is simple and $u(d) = su(d) \oplus \mathbb{R}$, it follows that any Lie algebra homomorphism must map $su(d)$ to itself isomorphically. Also, the inequivalent representastions of $su(d)$ are characterized by the outer automorphisms. These are in one to one correspondence with automorphisms of the corresponding Dynkin diagram. The Dynkin diagram of $su(d)$ is $A_{(d-1)}$ consisting of $d-1$ dots connected by single lines. For $d > 2$, the (graph) automorphism group of $A_{(d-1)}$ is of order 2 (identity and a reflection). It follows that there are at most two inequivalent irreducible unitary representations of $SU(d)$, $d \ge 2$.

We believe, it will be interesting to find an answer to the two questions: (a) What possible values $\dim \pi$ can take if d is fixed. (b) If d and $n = \dim \pi$ are fixed, how many n-dimensional inequivalent irreducible unitary representations are there of the group $SU(d)$.

Note added in proof. One of the reviewers has noted the following, and we quote: A.A. Johnson, in "The automorphisms of unitary groups over infinite fields", Amer. J. Math. 95 (1973), has proved the following theorem: Let K be an infinite field and consider the unitary group $U_d(K)$ with respect to some involution $a \to \bar{a}$ of K. Suppose that $d \geqslant 3$. Then any automorphism $\pi : U_d(K) \to U_d(K)$ has the form

$$\pi(u) = \chi(u) g u g^{-1}$$

where χ is a character and $g : K^d \to K^d$ is a semi-linear automorphism. For $K = \mathbf{C}$ the characters are $\chi(u) = \det(u)^n$ for some $n \in \mathbf{Z}$, and semi-linear means $\mathbf{C}$-linear or $\mathbf{C}$ antilinear. In the first case, $g \in U_d(\mathbf{C})$ and we are done. In the second case define $h := g \circ \iota$ where $\iota\xi = \bar{\xi}$ for all $\xi \in \mathbf{C}^d$ is the conjugation. Then h is $\mathbf{C}$-linear and hence $h \in U_d(\mathbf{C})$. Moreover

$$gug^{-1} = (h \circ \iota) \circ u \circ \left(\iota \circ h^{-1}\right) = h \circ (\iota \circ u \circ \iota) \circ h^{-1} = h \circ \bar{u} \circ h^{-1}.$$

This yields the second case in Lemma 5.5. Johnson's proof is in the spirit of projective geometry and is independent of Lie theory.

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