Which homogeneous operators are subnormal

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The group $\mathsf{SU}(1,1)$

- Let $G := \mathsf{SU}(1,1)$ be the group of complex 2×2 matrices of the form $g := \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}$ with $|\alpha|^2 - |\beta|^2 = 1$. It acts on the unit disc by the rule $g(z) = \frac{\alpha+\beta z}{\bar{\beta}+\bar{\alpha}z}$.

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- It is well known that all biholomorphic automorphisms of the unit disc $\mathbb{D}$ arise in this way.
- Clearly, the element g and $-g$ give rise to the same action. One may say that the kernel of the $\mathsf{SU}(1,1)$ action on the unit disc is the normal subgroup $\{I, -I\}$.

Möb, the Möbius group

- If one already knows that (the Möbius group, to be denoted Möb in what follows, consisting of) the bi-holomorphic automorphisms of $\mathbb{D}$ of the form $\varphi_{\theta,a}(z) = e^{i\theta}\frac{z-a}{1-\bar{a}z}$, $0 \le \theta < 2\pi$, and a in the unit disc $\mathbb{D}$, then it comes from the action of a $g \in \mathsf{SU}(1,1)$, where g is of the form

$$g = \frac{1}{\sqrt{1-|a|^2}} \begin{pmatrix} \exp(i\frac{\theta}{2}) & 0 \\ 0 & \exp(-i\frac{\theta}{2}) \end{pmatrix} \left(\begin{smallmatrix} 1 & -a \\ -\bar{a} & 1 \end{smallmatrix}\right).$$

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  and also from replacing g by changing θ to $\theta + 2\pi$.
- Now, the map $g \mapsto (e^{2i\theta}, a)$ with $a = -\frac{\beta}{\alpha}$ is a two to one smooth homomorphism.

homogeneous operators

- A bounded linear operator T on a complex separable Hilbert space $\mathcal{H}$ with $\sigma(T) \subseteq \mathbb{D}$ is said to be homogeneous if
$$g(T) := e^{i\theta}(T - aI)(I - \bar{a}T)^{-1}$$
is unitarily equivalent to T for all g in $\mathsf{SU}(1,1)$.

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- Of course, for a $g \in \mathrm{SU}(1,1)$, the operator $g(T)$ is the same as the operator $\varphi(T)$, where φ is the element in Möb determined by g.
- All irreducible homogeneous operators in the **Cowen-Douglas class** of the unit disc $\mathbb{D}$ modulo unitary equivalence have been described recently.

The Bergman space

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- The operator $U_g : \mathbb{A}^2(\mathbb{D}) \to \mathbb{A}^2(\mathbb{D})$ given by the formula:

$$(U_{g^{-1}} f)(z) = g'(z)(f \circ g)(z), \ f \in \mathbb{A}^2(\mathbb{D}), \ g \in SU(1,1),$$

is therefore isometric.

unitary homomorphism

- The map $J_{[\cdot]}(\cdot) : SU(1,1) \times \mathbb{D} \to \mathbb{C}^\times$ given by the formula $J_g(z) = g'(z)$ is a cocycle, that is,

$$J_{g_1 g_2}(z) = g_1'(g_2(z))g_2'(z),\ g_1, g_2 \in SU(1,1),\ z \in \mathbb{D}.$$

This is just the chain rule for the derivative, it ensures that $g \mapsto U_g$ is a homomorphism:

$$\begin{aligned} U_{g_2^{-1} g_1^{-1}} f &= (g_1 \circ g_2)' f \circ (g_1 \circ g_2) \\ &= g_2'(z g_1'(g_2(z))(f \circ g_1)(g_2(z)) \\ &= U_{g_2^{-1}} U_{g_1^{-1}} \end{aligned}$$

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- Finally, we note that $g^{-1}(M_z) = M_{g^{-1}(z)}$, which implies $M_z U_g = U_g\, g^{-1}(M_z)$ providing the first example of a homogeneous operator!

how to make more examples?

- First observe that taking a positive power λ of the cocycle $J_g(z)$ is legitimate since it is a non-zero holomorphic function defined on the simply connected set $\mathbb{D}$. More importantly, the function $J_g(z)^\lambda$ continues to satisfy the properties required of a cocycle.

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- A closer examination of the change of variable formula reveals a little more:

$$\int_{\mathbb{D}} |f\circ g(w)|^2 |g'(w)|^{2\lambda} \varrho(w)\, dA(w) = \int_{\mathbb{D}} |f(z)|^2 \varrho(z)\, dA(z)$$

$$= \int_{\mathbb{D}} |f\circ g(w)|^2 |g'(w)|^2 \varrho(g(w))\, dA(w),\ f\in \mathbb{A}^2(\mathbb{D})$$

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- For a $\lambda > 0$, If we can find a ϱ with the property $\varrho(g(w)) = |g'(w)|^{2(\lambda-1)}\varrho(w)$, then the map

$$U_g^{(\lambda)} : \mathbb{A}^{(\lambda)} \to \mathbb{A}^{(\lambda)}$$

given by the formula $(U_g^{(\lambda)} f)(z) = (g')^{\lambda}(z) f(g(z))$ would be a unitary homomorphism as before.

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  given by the formula $(U_g^{(\lambda)} f)(z) = (g')^{\lambda}(z) f(g(z))$ would be a unitary homomorphism as before.
- To find ϱ with the desired property, given any $a \in \mathbb{D}$, pick a g such that $a = g(0)$. Normalizing $\varrho(0)$, such that $\int 1\,\varrho\, dA = 1$, we must have $\varrho(a) = |g'(0)|^{2(\lambda-1)} = (1-|a|^2)^{2(\lambda-1)}$, $a \in \mathbb{D}$.

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- Question: Are these all? More precisely, are these the only homogeneous operators in the Cowen-Douglas class of rank 1 over $\mathbb{D}$? Yet another formulation is to ask if these are the only homogeneous holomorphic Hermitian vector bundles.

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- Question: Are these all? More precisely, are these the only homogeneous operators in the Cowen-Douglas class of rank 1 over $\mathbb{D}$? Yet another formulation is to ask if these are the only homogeneous holomorphic Hermitian vector bundles.
- Well, there are others and we find them next.

quasi invariant

- The Bergman kernel $B : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$ of the disc $\mathbb{D}$ is the reproducing kernel of the Bergman space $\mathbb{A}^2(\mathbb{D})$. It is uniquely determined by two properties:

$$\text{The vector } B_w \text{ is in } \mathbb{A}^2(\mathbb{D}) \quad \text{for all } w \in \mathbb{D};$$

$$\text{reproducing property, } \langle f, B_w \rangle = f(w) \quad \text{for all } f \in \mathbb{A}^2(\mathbb{D}).$$

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- From the reproducing property, it follows that the Bergman kernel is quasi-invariant, that is, it transforms according to the rule:

$$g'(z)B(g(z),g(w))\overline{g'(w)} = B(z,w),\ g\in SU(1,1).$$

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- For any $z\in\mathbb{D}$, picking a $g_z\in SU(1,1)$ such that $g_z(z)=0$, we see that $B(z,z)=|g_z'(z)|^2B(0,0)$.

The Bergman kernel

- Normalizing the measure dA to ensure that $\|1\| = 1$, we see that $B(0,0) = 1$. It follows from the quasi-invariance that the Bergman kernel must be of the form $B(z,w) = (1 - z\bar{w})^{-2}$, $z, w \in \mathbb{D}$.

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- The quasi-invariance of B is equivalent to saying that the map $U_\varphi : \mathbb{A}^2(\mathbb{D}) \to \mathbb{A}^2(\mathbb{D})$ defined by the formula:

$$(U_{g^{-1}} f)(z) = g'(z)(f \circ g)(z), f \in \mathbb{A}^2(\mathbb{D}), z \in \mathbb{D},$$

  is an isometry.
  This is just another proof of what we have seen before.

the connection

- First, note that $B(z,z)$ is a positive real analytic function on the disc $\mathbb{D} \times \mathbb{D}$. It has a power series expansion of the form

$$B(z,z) = \sum_{n=0}^{\infty} (-1)^n \binom{-2}{n} (z\bar{z})^n,$$

where $(-1)^n \binom{-2}{n} = n+1$. By a polarization technique, one obtains, the Bergman kernel of two variables.

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- Now, we see a direct connection between the kernel B and the Hilbert space $\mathbb{A}^2(\mathbb{D})$, namely, the set of vectors $\{\sqrt{n+1}z^n\}_{n=0}^{\infty}$ serves as an orthonormal basis of the Hilbert space $\mathbb{A}^2(\mathbb{D})$.

power of the Bergman kernel

- What about the power of the Bergman kernel? For any $\nu > 0$, the function $B(z,z)^\nu$ admits a convergent power series expansion of the form:

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where $\binom{-\nu}{n} := (-1)^n \frac{\nu(\nu+1)\cdots(\nu+n-1)}{n!}$.

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- Consider the Hilbert space $\mathbb{H}^{(\nu)}(\mathbb{D})$ determined by requiring the set of vectors $\{\binom{-\nu}{n}^{1/2} z^n\}_{n=0}^{\infty}$ to be a complete orthonormal set in it.

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- Then by construction, B_w^{ν} is in $\mathbb{H}^{(\nu)}(\mathbb{D})$ and it has the reproducing property: $\langle f, B_w^{\nu} \rangle = f(w)$, $f \in \mathbb{H}^{(\nu)}(\mathbb{D})$.

repeat

- Now that we have the Hilbert space $\mathbb{H}^{(\nu)}(\mathbb{D})$ and know that its reproducing kernel is $B(z,w)^{\nu}$, it follows that $B(z,w)^{\nu}$, $\nu > 0$, is quasi-invariant as well since the transformation rule clearly carries over to the power. Also, the cocycle in this transformation rule is $(g')^{\nu}(z)$.

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- Now that we have the Hilbert space $\mathbb{H}^{(\nu)}(\mathbb{D})$ and know that its reproducing kernel is $B(z,w)^{\nu}$, it follows that $B(z,w)^{\nu}, \quad \nu > 0,$ is quasi-invariant as well since the transformation rule clearly carries over to the power. Also, the cocycle in this transformation rule is $(g')^{\nu}(z)$.
- We conclude that the operator $\hat{U}_g^{(\nu)} : \mathbb{H}^{(\nu)}(\mathbb{D}) \to \mathbb{H}^{(\nu)}(\mathbb{D})$ given by the formula $(\hat{U}_g^{(\nu)} f)(z) = (g')^{\nu}(z)(f \circ g)(z)$ is a unitary homomorphism. Moreover, we have $M\hat{U}_g^{(\nu)} = g(M)\hat{U}_g^{(\nu)}$ as before.

subnormal operators

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- Two such normal extensions are unitarily equivalent if they are assumed to be minimal, that is, $\mathcal{K}$ is the smallest reducing subspace of N containing $\mathcal{H}$. It follows that if S is homogeneous, then its minimal normal extension N is also homogeneous.

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- Two such normal extensions are unitarily equivalent if they are assumed to be **minimal**, that is, $\mathcal{K}$ is the smallest reducing subspace of N containing $\mathcal{H}$. It follows that if S is homogeneous, then its minimal normal extension N is also homogeneous.
- What does it mean to say that a normal operator N is homogeneous?

imprimitivity

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- Let G be a locally compact second countable (lcsc) topological group and $\mathcal{D}$ be a G-space. Suppose that
- $U : G \to \mathcal{U}(\mathcal{K})$ is a unitary representation of the group G on the on the Hilbert space $\mathcal{K}$ and that $\varrho : \mathbf{C}(\mathcal{D}) \to \mathcal{L}(\mathcal{K})$ is a $*$ - homomorphism of the C^* - algebra of continuous functions $\mathbf{C}(\mathcal{D})$ on the algebra $\mathcal{L}(\mathcal{K})$ of all bounded operators acting on the Hilbert space $\mathcal{K}$.

imprimitivity

- Then the pair (U, ϱ) is a representation of the G -space $\mathcal{D}$ if

$$\varrho(g \cdot f) = U(g)^* \varrho(f) U(g), f \in \mathbf{C}(\mathcal{D}), g \in G,$$

where

$$(g \cdot f)(w) = f(g^{-1} \cdot w), w \in \mathcal{D},$$

which is a generalization of the imprimitivity relation due to Frobenius by Mackey.

quasi-invariant weight

- A normal operator N is homogeneous if and only if it is unitarily equivalent to the multiplication by the coordinate function on the Hilbert space $L^2(X, Q\,dm) \subseteq \mathcal{F}(X, \mathbb{C}^n)$, where X is either $\mathbb{T}$ or $\mathbb{D}$ and dm is the Lebesgue measure. The weight function Q must be quasi-invariant:

$$Q(z) := J_{g_z}(0)^* Q(0) J_{g_z}(0) |g_z'(z)|^{-2},\ g_z(0) = z,$$

where $J_g : SU(1,1) \times D \to \mathbb{C}^{n \times n}$ is a cocycle. Here, for $f \in L^2$, the norm is given by the formula

$$\|f\|^2 := \int_X \langle Q(z) f(z), f(z) \rangle \, dm(z).$$

examples

- We have $\mathbb{A}^{(\lambda)} = \mathbb{H}^{(\lambda)}$, as long as $\lambda > 1/2$. For $\lambda < 1/2$, the space $\mathbb{A}^{(\lambda)} = \{0\}$, while $\mathbb{H}^{(\lambda)}$ is a non-zero Hilbert space for all $\lambda > 0$.

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- Why do we make the fuss?
  These simple examples show that among all the homogeneous operators in the CD class of rank 1, some are subnormal and some are not. The previous discussion provides a complete answer.

what about vector valued functions?

- Suppose the Hilbert space consists of holomorphic functions defined on $\mathbb{D}$, taking values in $\mathbb{C}^n$. Then we must first find all the cocycles $J_{[\cdot]}(\cdot) : SU(1,1) \times \mathbb{D} \to \mathbb{C}^{n \times n}$ and then determine if the function $\mathrm{B} : \mathbb{D} \times \mathbb{D} \to \mathbb{C}^{n \times n}$ defined by polarizing the expression

$$\mathrm{B}(z,z) = J_g(z)^{-1}\mathrm{B}(0,0)(J_g(z)^*)^{-1},$$

is positive definite for certain choices of the non-negative $n \times n$ matrix $\mathrm{B}(0,0)$.

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is positive definite for certain choices of the non-negative $n\times n$ matrix $\mathsf{B}(0,0)$.

- Simultaneously, one may also consider the weight function

$$Q(z) := J_{g_z}(0)^* Q(0) J_{g_z}(0)|g_z'(z)|^{-2},\ g_z(0) = z.$$

cocycles, in general

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- A co-cycle $J : SU(1,1) \times \mathbb{D} \to \mathbb{C}^{(n+1)\times(n+1)}$ is given by the formula:

$$J_g(z) = (g')^{2\lambda - \frac{m}{2}}(z) D(g)^{\frac{1}{2}} \quad (c_\varphi S_n) D(g)^{\frac{1}{2}},$$

where S_n is the forward shift with weights $\{1, 2, \ldots, m\}$ and $D(g)$ is a diagonal matrix whose diagonal sequence is $\{(g')^m(z), (g')^{m-1}(z), \ldots, 1\}$.

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where S_n is the forward shift with weights $\{1,2,\ldots,m\}$ and $D(g)$ is a diagonal matrix whose diagonal sequence is $\{(g')^m(z),(g')^{m-1}(z),\ldots,1\}$.
- This cocycle determines a kernel function B on the disc $\mathbb{D}$ by the quasi-invariance except that we have to choose $B(0,0)$.

the two Hilbert spaces

- We now have the Hilbert space $\mathbb{A}^{(\lambda,n)}$ of square integrable holomorphic functions on the unit disc with respect to the measure $Q(z)dV(z)$ described completely in terms of J except for the value of Q at o. For this Hilbert space $\mathcal{H}^{(\lambda,n)}$ to be non-zero, it is necessary and sufficient that $\lambda > \frac{m+1}{2}$.

the two Hilbert spaces

- We now have the Hilbert space $\mathbb{A}^{(\lambda,n)}$ of square integrable holomorphic functions on the unit disc with respect to the measure $Q(z)dV(z)$ described completely in terms of J except for the value of Q at o. For this Hilbert space $\mathcal{H}^{(\lambda,n)}$ to be non-zero, it is necessary and sufficient that $\lambda > \frac{m+1}{2}$.
- The Hilbert spaces $\mathbb{H}^{(\lambda,n)}$ determined by the kernel function $\mathbb{B}$ and $\mathbb{A}^{(\lambda,n)}$ are again related. This relationship is a question of finding $B(0,0)$ in terms of $Q(0)$ and vice-versa.

$n=2$

- Let us work out the special case of $n=2$. In this case,

$$K^{(\lambda,\mu)}(z,w)=\begin{pmatrix} \frac{1}{(1-\bar{w}z)^{2\lambda-1}} & \frac{z}{(1-\bar{w}z)^{2\lambda}} \\ \frac{\bar{w}}{(1-\bar{w}z)^{2\lambda}} & \frac{1}{2\lambda-1}\frac{1+(2\lambda-1)\bar{w}z}{(1-\bar{w}z)^{2\lambda+1}} \end{pmatrix}+ \\ +\mu^2\begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{(1-\bar{w}z)^{2\lambda+1}} \end{pmatrix}.$$

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- The existence of an integral inner product implies that the operator $M^{(\lambda,\mu)}$ is subnormal and isolating these is often very important. The homogeneous operator $M^{(\lambda,\mu)}$ is subnormal if and only if $\lambda\geq 1$ and $\mu^2\geq\frac{\lambda}{(2\lambda-1)(\lambda-1)}$.

Thank You!